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Some new inequalities for generalized fractional conformable integral operators

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Abstract

The present paper aims to establish certain new classes of integral inequalities for a class of n ($n \in \mathbb{N}$) positive continuous and decreasing functions by utilizing the generalized fractional conformable integral operators (FCIO) recently defined by Khan and Khan. From these results, we also derive several particular cases.

Keywords: Fractional integral; Generalized conformable fractional integral; Integral inequalities

1 Introduction

Fractional calculus earned more recognition due to its applications in diverse domains. Recent research focuses on developing a large number of the fractional integral operators (FIO) and their applications in multiple disciplines of sciences (see [13, 14, 20, 26]). In [15], Liu et al. introduced interesting integral inequalities for continuous functions on $[a,b]$. Later on, Dahmani [8] generalized the work of [15] involving the Riemann–Liouville fractional integral operators. In [9], Dahmani and Tabharit introduced weighted Grüss type inequalities involving fractional integral operators. Dahmani [7] established some new inequalities for fractional integrals. Polya–Szego and Chebyshev type inequalities involving the Riemann–Liouville fractional integral operators are found in [19]. Nisar et al. [16] established some inequalities involving extended gamma and the Kummar confluent hypergeometric k-functions. In [28], Set et al. established generalized Grüss type inequalities for k-fractional integrals and applications. Certain Gronwall inequalities associated with Riemann–Liouville k and Hadamard k-fractional derivatives and their applications are found in the work of Nisar et al. [17]. The (k,s)-fractional integrals and their applications are found in [27]. Rahman et al. [23] presented certain inequalities via (k,ρ)-fractional integral operators. In [11], the authors introduced the idea of fractional conformable derivative operators with a shortcoming that the new derivative operator does not tend to the original function when the order $\rho \to 0$. In [1], the author studied certain various properties of the fractional conformable derivative operators and raised the problem of how to use conformable derivative operators to generate more general types of nonlocal fractional derivative operators, after that the method was demonstrated in [10].

The generalized FCIO defined in [12] is given by

$$ {}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r^+} f(x) = \frac{1}{\Gamma(\beta)} \int_r^x \left(\frac{x^{\alpha+\mu} - \tau^{\alpha+\mu}}{\alpha+\mu} \right)^{\beta-1} \frac{f(\tau)}{\tau^{1-\alpha-\mu}}\, d\tau, \quad x > r, \tag{1} $$

and

$$
{}^{\mu}_{\alpha}\Im^{\beta}_{s^-}f(x) = \frac{1}{\Gamma(\beta)} \int_x^s \left(\frac{\tau^{\alpha+\mu} - x^{\alpha+\mu}}{\alpha+\mu}\right)^{\beta-1} \frac{f(\tau)}{\tau^{1-\alpha-\mu}}\, d\tau, \quad x < s, \tag{2}
$$

where $\beta \in \mathbb{C}$, $\Re(\beta) > 0$, $\alpha \in (0,1]$, $\mu \in \mathbb{R}$, $\alpha + \mu \neq 0$, and Γ is the gamma function [29].

Remark 1 (i) If we set $\mu = 0$ in (1) and (2), then we have the following Riemann–Liouville (R-L) type FCIO:

$$
{}_{\alpha}\Im^{\beta}_{r^+}f(x) = \frac{1}{\Gamma(\beta)} \int_r^x \left(\frac{x^{\alpha} - \tau^{\alpha}}{\alpha}\right)^{\beta-1} \frac{f(\tau)}{\tau^{1-\alpha}}\, d\tau, \quad x > r, \tag{3}
$$

and

$$
{}_{\alpha}\Im^{\beta}_{s^-}f(x) = \frac{1}{\Gamma(\beta)} \int_x^s \left(\frac{\tau^{\alpha} - x^{\alpha}}{\alpha}\right)^{\beta-1} \frac{f(\tau)}{\tau^{1-\alpha}}\, d\tau, \quad x < s, \tag{4}
$$

where $\beta \in \mathbb{C}$, $\Re(\beta) > 0$, $\alpha \in (0,1]$.

(ii) If $\alpha = 1$ in 3 and 4, then we obtain the following R-L FIO:

$$
\Im^{\beta}_{r^+}f(x) = \frac{1}{\Gamma(\beta)} \int_r^x (x-\tau)^{\beta-1} f(\tau)\, d\tau, \quad x > r, \tag{5}
$$

and

$$
\Im^{\beta}_{s^-}f(x) = \frac{1}{\Gamma(\beta)} \int_x^s (\tau-x)^{\beta-1} f(\tau)\, d\tau, \quad x < s, \tag{6}
$$

where $\beta \in \mathbb{C}$, $\Re(\beta) > 0$.

Recently, the researchers [21, 24] established inequalities of Grüss type and Čebyšev type by utilizing fractional conformable integral operators. Rahman et al. [25] established certain Chebyshev type inequalities involving fractional conformable integral operators. In [22], the authors introduced the Minkowski inequalities via generalized proportional fractional integral operators. Some new inequalities involving fractional conformable integrals are found in the work of Nisar et al. [18]. Adjabi et al. [6] presented generalized fractional integral operators and Gronwall type inequalities with applications. In [2], Abdeljawad established a Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel. Abdeljawad et al. [4] introduced Lyapunov type inequalities for mixed nonlinear forced differential equations within conformable derivatives. Fractional operators with exponential kernels and a Lyapunov type inequality are found in [3]. Abdeljawad et al. [5] presented a generalized Lyapunov type inequality in the frame of conformable derivatives.

Our aim in this paper is to generalize the inequalities obtained earlier by [8, 15] by employing the left generalized fractional conformable integral operator (1).

2 Main results

In this section, we employ the left generalized FCIO to establish the generalization of some classical inequalities.

Theorem 1 *Let $(g_i)_{i=1,2,3,\ldots,n}$ be n positive continuous and decreasing functions on the interval $[r,s]$. Let $r < x \le s$, $\vartheta > 0$, $\sigma \ge \gamma_p > 0$ for any fixed $p \in \{1,2,3,\ldots,n\}$. Then, for generalized fractional conformable integral (1), we have*

$$\frac{{}^{\mu}_{\alpha}\Im^{\beta}_{r}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]}{{}^{\mu}_{\alpha}\Im^{\beta}_{r}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)]} \ge \frac{{}^{\mu}_{\alpha}\Im^{\beta}_{r}[(x-r)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]}{{}^{\mu}_{\alpha}\Im^{\beta}_{r}[(x-r)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}, \tag{7}$$

where $\beta \in \mathbb{C}$, $\alpha \in (0,1]$, $\mu \in \mathbb{R}$, $\alpha + \mu \neq 0$, and $\Re(\beta) > 0$.

Proof Since $(g_i)_{i=1,2,3,\ldots,n}$ are n positive continuous and decreasing functions on the interval $[r,s]$. Therefore, we have

$$\bigl((\rho - r)^{\vartheta} - (t-r)^{\vartheta}\bigr)\bigl(g_p^{\sigma-\gamma_p}(t) - g_p^{\sigma-\gamma_p}(\rho)\bigr) \ge 0, \tag{8}$$

where $r < x \le s$, $\vartheta > 0$, $\sigma \ge \gamma_p > 0$, $t,\rho \in [r,x]$ and for any fixed $p \in \{1,2,3,\ldots,n\}$.

Define a function

$$\begin{aligned} {}^{\mu}_{\alpha}\Im^{\beta}_{r}(x,\rho,t) &= \frac{1}{\Gamma(\beta)}\left(\frac{x^{\alpha+\mu} - t^{\alpha+\mu}}{\alpha+\mu}\right)^{\beta-1} \\ &\quad \times \frac{\prod_{i=1}^{n} g_i^{\gamma_i}(t)}{t^{1-\alpha-\mu}}\bigl((\rho - r)^{\vartheta} - (t-r)^{\vartheta}\bigr)\bigl(g_p^{\sigma-\gamma_p}(t) - g_p^{\sigma-\gamma_p}(\rho)\bigr). \end{aligned} \tag{9}$$

We observe that the above function satisfies all the assumptions stated in Theorem 1, and hence the function ${}^{\mu}_{\alpha}\Im^{\beta}_{r}(x,\rho,t)$ is positive for all $t \in (r,x)$ $(x > r)$. Integrating both sides of (9) with respect to t over (r,x), we have

$$\begin{aligned} 0 &\le \int_r^x {}^{\mu}_{\alpha}\Im^{\beta}_{r}(x,\rho,t)\,dt \\ &= \frac{1}{\Gamma(\beta)}\int_r^x \left(\frac{x^{\alpha+\mu} - t^{\alpha+\mu}}{\alpha+\mu}\right)^{\beta-1} \\ &\quad \times \prod_{i=1}^{n} g_i^{\gamma_i}(t)\bigl((\rho - r)^{\vartheta} - (t-r)^{\vartheta}\bigr)\bigl(g_p^{\sigma-\gamma_p}(t) - g_p^{\sigma-\gamma_p}(\rho)\bigr)\frac{dt}{t^{1-\alpha-\mu}} \\ &= \left[(\rho - r)^{\vartheta}\,{}^{\mu}_{\alpha}\Im^{\beta}_{r}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right] + g_p^{\sigma-\gamma_p}(\rho)\,{}^{\mu}_{\alpha}\Im^{\beta}_{r}\left[(x-r)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right] \\ &\quad - (\rho - r)^{\vartheta} g_p^{\sigma-\gamma_p}(\rho)\left[{}^{\mu}_{\alpha}\Im^{\beta}_{r}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right] - {}^{\mu}_{\alpha}\Im^{\beta}_{r}\left[(x-r)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]. \end{aligned} \tag{10}$$

Multiplying (10) by $\frac{1}{\Gamma(\beta)}(\frac{x^{\alpha+\mu}-\rho^{\alpha+\mu}}{\alpha+\mu})^{\beta-1}\frac{\prod_{i=1}^{n}g_i^{\gamma_i}(\rho)}{\rho^{1-\alpha-\mu}}$ and integrating the resultant identity with respect to ρ over (r,x), we have

$$\begin{aligned}0 \leq{}& \left[{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\left[(x-r)^{\vartheta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right]\\&-{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\left[(x-r)^{\vartheta}\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)\right]\left[{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\prod_{i=1}^{n}g_i^{\gamma_i}(x)\right],\end{aligned}$$

which completes the desired inequality (7). □

Corollary 1 *Let $(g_i)_{i=1,2,3,\ldots,n}$ be n positive continuous and decreasing on $[r,s]$. Let $r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ for any fixed $p \in \{1,2,3,\ldots,n\}$. Then, for R-L type fractional conformable integral (3), we have*

$$\frac{{}_{\alpha}\mathfrak{I}^{\beta}_{r}[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)]}{{}_{\alpha}\mathfrak{I}^{\beta}_{r}[\prod_{i=1}^{n}g_i^{\gamma_i}(x)]} \geq \frac{{}_{\alpha}\mathfrak{I}^{\beta}_{r}[(x-r)^{\vartheta}\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)]}{{}_{\alpha}\mathfrak{I}^{\beta}_{r}[(x-r)^{\vartheta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)]}, \tag{11}$$

where $\beta \in \mathbb{C}$, $\alpha \in (0,1]$, and $\Re(\beta) > 0$.

Corollary 2 *Let $(g_i)_{i=1,2,3,\ldots,n}$ be n positive continuous and decreasing on $[r,s]$. Let $r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ for any fixed $p \in \{1,2,3,\ldots,n\}$. Then, for R-L fractional integral (5), we have*

$$\frac{\mathfrak{I}^{\beta}_{r}[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)]}{\mathfrak{I}^{\beta}_{r}[\prod_{i=1}^{n}g_i^{\gamma_i}(x)]} \geq \frac{\mathfrak{I}^{\beta}_{r}[(x-r)^{\vartheta}\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)]}{\mathfrak{I}^{\beta}_{r}[(x-r)^{\vartheta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)]}, \tag{12}$$

where $\beta \in \mathbb{C}$ and $\Re(\beta) > 0$.

Remark 2 The inequality in Theorem 1 will reverse if $(g_i)_{i=1,2,3,\ldots,n}$ are increasing on the interval $[r,s]$. If we let $\alpha = 1$, $\mu = 0$, then Theorem 1 will lead to Theorem 3.1 [8]. Moreover, setting $\mu = 0$, $\alpha = \beta = n = 1$, $x = s$, then Theorem 1 reduces to the well-known Theorem 3 [15].

Theorem 2 *Let $(g_i)_{i=1,2,3,\ldots,n}$ be n positive continuous and decreasing on $[r,s]$. Let $r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ for any fixed $p \in \{1,2,3,\ldots,n\}$. Then, for generalized fractional conformable integral (1), we have*

$$\begin{aligned}&\frac{{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)]{}^{\mu}_{\alpha}\mathfrak{I}^{\lambda}_{r}[(x-r)^{\vartheta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)]+{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}[\prod_{i\neq p}^{n}g_i^{\gamma_i}g_p^{\sigma}(x)]{}^{\mu}_{\alpha}\mathfrak{I}^{\lambda}_{r}[(x-r)^{\vartheta}\prod_{i=1}^{n}g_i^{\gamma_i}(x)]}{{}^{\mu}_{\alpha}\mathfrak{I}^{\lambda}_{r}[(x-r)^{\vartheta}\prod_{i\neq p}^{n}g_i^{\gamma_i}(x)]{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}[\prod_{i=1}^{n}g_i^{\gamma_i}(x)]+{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}[(x-r)^{\vartheta}\prod_{i\neq p}^{n}g_i^{\gamma_i}(x)]{}^{\mu}_{\alpha}\mathfrak{I}^{\lambda}_{r}[\prod_{i=1}^{n}g_i^{\gamma_i}(x)]}\\&\quad\geq 1,\end{aligned} \tag{13}$$

where $\beta,\lambda \in \mathbb{C}$, $\alpha \in (0,1]$, $\mu \in \mathbb{R}$, $\alpha+\mu \neq 0$, $\Re(\beta) > 0$, and $\Re(\lambda) > 0$.

Proof Firstly, multiplying both sides of equation (10) by $\frac{1}{\Gamma(\lambda)}(\frac{x^{\alpha+\mu}-\rho^{\alpha+\mu}}{\alpha+\mu})^{\lambda-1}\frac{\prod_{i=1}^{n}g_i^{\gamma_i}(\rho)}{\rho^{1-\alpha-\mu}}$ and integrating the resultant identity with respect to ρ over (r,x), we have

$$0 \leq \int_{r}^{x}\int_{r}^{x}\frac{1}{\Gamma(\lambda)}\left(\frac{x^{\alpha+\mu}-\rho^{\alpha+\mu}}{\alpha+\mu}\right)^{\lambda-1}\frac{\prod_{i=1}^{n}g_i^{\gamma_i}(\rho)}{\rho^{1-\alpha-\mu}}{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}(x,\rho,t)\,dt\,d\rho$$

$$
\begin{aligned}
&= {}_{\alpha}^{\mu}\mathfrak{I}_r^{\beta}\left[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right] {}_{\alpha}^{\mu}\mathfrak{I}_r^{\lambda}\left[(x-r)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right] \\
&\quad + {}_{\alpha}^{\mu}\mathfrak{I}_r^{\lambda}\left[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right] {}_{\alpha}^{\mu}\mathfrak{I}_r^{\beta}\left[(x-r)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right] \\
&\quad - {}_{\alpha}^{\mu}\mathfrak{I}_r^{\beta}\left[(x-r)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right] {}_{\alpha}^{\mu}\mathfrak{I}_r^{\lambda}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right] \\
&\quad - {}_{\alpha}^{\mu}\mathfrak{I}_r^{\lambda}\left[(x-r)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right] {}_{\alpha}^{\mu}\mathfrak{I}_r^{\beta}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right].
\end{aligned} \tag{14}
$$

Hence, dividing both sides of (14) by

$$
{}_{\alpha}^{\mu}\mathfrak{I}_r^{\beta}\left[(x-r)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right] {}_{\alpha}^{\mu}\mathfrak{I}_r^{\lambda}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right] + {}_{\alpha}^{\mu}\mathfrak{I}_r^{\lambda}\left[(x-r)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right] {}_{\alpha}^{\mu}\mathfrak{I}_r^{\beta}\left[\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right],
$$

we get the desired proof. □

Corollary 3 *Let $(g_i)_{i=1,2,3,\ldots,n}$ be n positive continuous and decreasing on $[r,s]$. Let $r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ for any fixed $p \in \{1,2,3,\ldots,n\}$. Then, for R-L type fractional conformable integral (3), we have*

$$
\begin{aligned}
&\frac{{}_{\alpha}\mathfrak{I}_r^{\beta}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}_{\alpha}\mathfrak{I}_r^{\lambda}[(x-r)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)] + {}_{\alpha}\mathfrak{I}_r^{\beta}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}_{\alpha}\mathfrak{I}_r^{\lambda}[(x-r)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}{{}_{\alpha}\mathfrak{I}_r^{\lambda}[(x-r)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i}(x)]{}_{\alpha}\mathfrak{I}_r^{\beta}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)] + {}_{\alpha}\mathfrak{I}_r^{\beta}[(x-r)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i}(x)]{}_{\alpha}\mathfrak{I}_r^{\lambda}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)]} \\
&\quad \geq 1,
\end{aligned} \tag{15}
$$

where $\beta, \lambda \in \mathbb{C}$, $\alpha \in (0,1]$, $\Re(\beta) > 0$, and $\Re(\lambda) > 0$.

Corollary 4 *Let $(g_i)_{i=1,2,3,\ldots,n}$ be n positive continuous and decreasing on $[r,s]$. Let $r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ for any fixed $p \in \{1,2,3,\ldots,n\}$. Then, for R-L fractional integral (5), we have*

$$
\begin{aligned}
&\frac{\mathfrak{I}_r^{\beta}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]\mathfrak{I}_r^{\lambda}[(x-r)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)] + \mathfrak{I}_r^{\beta}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]\mathfrak{I}_r^{\lambda}[(x-r)^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}{\mathfrak{I}_r^{\lambda}[(x-r)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i}(x)]\mathfrak{I}_r^{\beta}[\prod_{i=1}^{n} g_i^{\gamma_p}(x)] + \mathfrak{I}_r^{\beta}[(x-r)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\gamma_i}(x)]\mathfrak{I}_r^{\lambda}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)]} \\
&\quad \geq 1,
\end{aligned} \tag{16}
$$

where $\beta, \lambda \in \mathbb{C}$, $\Re(\beta) > 0$, and $\Re(\lambda) > 0$.

Remark 3 Applying Theorem 2 for $\beta = \lambda$, we get Theorem 1. Again, the inequality will reverse if $(g_i)_{i=1,2,3,\ldots,n}$ are increasing functions on the interval $[r,s]$. If we let $\alpha = 1$, $\mu = 0$, then Theorem 1 will lead to Theorem 3.4 [8]. Moreover, setting $\mu = 0$, $\alpha = \beta = \lambda = n = 1$, $x = s$, then again Theorem 1 reduces to the well-known Theorem 3 [15].

Theorem 3 *Let $(g_i)_{i=1,2,3,\ldots,n}$ and h be positive continuous on the interval $[r,s]$ such that h is increasing and $(g_i)_{i=1,2,3,\ldots,n}$ are decreasing functions on $[r,s]$. Let $r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$*

for any fixed $p \in \{1,2,3,\dots,n\}$. *Then, for generalized fractional conformable integral* (1), *we have*

$$\frac{{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}{{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)]} \geq 1, \tag{17}$$

where $\beta \in \mathbb{C}$, $\alpha \in (0,1]$, $\mu \in \mathbb{R}$, $\alpha + \mu \neq 0$, *and* $\Re(\beta) > 0$.

Proof Under the conditions stated in Theorem 3, we can write

$$\big(h^{\vartheta}(\rho) - h^{\vartheta}(t)\big)\big(g_p^{\sigma-\gamma_p}(t) - g_p^{\sigma-\gamma_p}(\rho)\big) \geq 0, \tag{18}$$

where $r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$, $t, \rho \in [r,x]$ and for any fixed $p \in \{1,2,3,\dots,n\}$.

Define a function

$$\begin{aligned} {}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}(x,\rho,t) &= \frac{1}{\Gamma(\beta)}\left(\frac{x^{\alpha+\mu} - t^{\alpha+\mu}}{\alpha+\mu}\right)^{\beta-1} \\ &\quad \times \frac{\prod_{i=1}^{n} g_i^{\gamma_i}(t)}{t^{1-\alpha-\mu}}\big(h^{\vartheta}(\rho) - h^{\vartheta}(t)\big)\big(g_p^{\sigma-\gamma_p}(t) - g_p^{\sigma-\gamma_p}(\rho)\big). \end{aligned} \tag{19}$$

We observe that the above function satisfies all the assumptions stated in Theorem 3, and hence the function ${}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}(x,\rho,t)$ is positive for all $t \in (r,x)$ $(x > r)$. Therefore, integrating both sides of (19) with respect to t over (r,x), we have

$$\begin{aligned} 0 &\leq \int_r^x {}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}(x,\rho,t)\,dt \\ &= \frac{1}{\Gamma(\beta)}\int_r^x \left(\frac{x^{\alpha+\mu} - t^{\alpha+\mu}}{\alpha+\mu}\right)^{\beta-1} \\ &\quad \times \prod_{i=1}^{n} g_i^{\gamma_i}(t)\big(h^{\vartheta}(\rho) - h^{\vartheta}(t)\big)\big(g_p^{\sigma-\gamma_p}(t) - g_p^{\sigma-\gamma_p}(\rho)\big)\frac{dt}{t^{1-\alpha-\mu}} \\ &= \left[h(\rho)^{\vartheta}\,{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right] + g_p^{\sigma-\gamma_p}(\rho){}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right] \\ &\quad - h^{\vartheta}(\rho)g_p^{\sigma-\gamma_p}(\rho)\left[{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right] - {}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]. \end{aligned} \tag{20}$$

Multiplying (20) by $\frac{1}{\Gamma(\beta)}(\frac{x^{\alpha+\mu}-\rho^{\alpha+\mu}}{\alpha+\mu})^{\beta-1}\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\rho)}{\rho^{1-\alpha-\mu}}$ and integrating the resultant identity with respect to ρ over (r,x), we have

$$\begin{aligned} 0 &\leq \left[{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right] \\ &\quad - {}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\left[{}^{\mu}_{\alpha}\mathfrak{I}^{\beta}_{r}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right], \end{aligned}$$

which completes the desired inequality (17) of Theorem 3. □

Corollary 5 *Let $(g_i)_{i=1,2,3,\dots,n}$ and h be positive continuous on $[r,s]$ such that h is increasing and $(g_i)_{i=1,2,3,\dots,n}$ are decreasing functions on the interval $[r,s]$. Let $r < x \le s$, $\vartheta > 0$, $\sigma \ge \gamma_p > 0$ for any fixed $p \in \{1,2,3,\dots,n\}$. Then, for R-L type fractional conformable integral (3), we have*

$$\frac{{}_{\alpha}\mathfrak{I}_r^{\beta}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}_{\alpha}\mathfrak{I}_r^{\beta}[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}{{}_{\alpha}\mathfrak{I}_r^{\beta}[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}_{\alpha}\mathfrak{I}_r^{\beta}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)]} \ge 1, \tag{21}$$

where $\beta \in \mathbb{C}$, $\alpha \in (0,1]$, and $\Re(\beta) > 0$.

Corollary 6 *Let $(g_i)_{i=1,2,3,\dots,n}$ and h be positive continuous on $[r,s]$ such that h is increasing and $(g_i)_{i=1,2,3,\dots,n}$ are decreasing functions on the interval $[r,s]$. Let $r < x \le s$, $\vartheta > 0$, $\sigma \ge \gamma_p > 0$ for any fixed $p \in \{1,2,3,\dots,n\}$. Then, for R-L fractional integral (5), we have*

$$\frac{\mathfrak{I}_r^{\beta}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]\mathfrak{I}_r^{\beta}[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}{\mathfrak{I}_r^{\beta}[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]\mathfrak{I}_r^{\beta}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)]} \ge 1, \tag{22}$$

where $\beta \in \mathbb{C}$ and $\Re(\beta) > 0$.

Theorem 4 *Let $(g_i)_{i=1,2,3,\dots,n}$ and h be positive continuous functions on $[r,s]$ such that h is increasing and $(g_i)_{i=1,2,3,\dots,n}$ are decreasing functions on the interval $[r,s]$. Let $r < x \le s$, $\vartheta > 0$, $\sigma \ge \gamma_p > 0$ for any fixed $p \in \{1,2,3,\dots,n\}$. Then, for generalized fractional conformable integral (1), we have*

$$\begin{aligned}&\frac{{}^{\mu}_{\alpha}\mathfrak{I}_r^{\beta}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}^{\mu}_{\alpha}\mathfrak{I}_r^{\lambda}[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)] + {}^{\mu}_{\alpha}\mathfrak{I}_r^{\lambda}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}^{\mu}_{\alpha}\mathfrak{I}_r^{\beta}[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}{{}^{\mu}_{\alpha}\mathfrak{I}_r^{\beta}[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}^{\mu}_{\alpha}\mathfrak{I}_r^{\lambda}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)] + {}^{\mu}_{\alpha}\mathfrak{I}_r^{\lambda}[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}^{\mu}_{\alpha}\mathfrak{I}_r^{\beta}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}\\ &\quad\ge 1,\end{aligned} \tag{23}$$

where $\beta, \lambda \in \mathbb{C}$, $\alpha \in (0,1]$, $\mu \in \mathbb{R}$, $\alpha + \mu \neq 0$, $\Re(\beta) > 0$, and $\Re(\lambda) > 0$.

Proof Multiplying (20) by $\frac{1}{\Gamma(\lambda)}(\frac{x^{\alpha+\mu}-\rho^{\alpha+\mu}}{\alpha+\mu})^{\lambda-1}\frac{\prod_{i=1}^{n} g_i^{\gamma_i}(\rho)}{\rho^{1-\alpha-\mu}}$ then integrating with respect to ρ over (r,x), we have

$$\begin{aligned}0 \le{}& \left[{}^{\mu}_{\alpha}\mathfrak{I}_r^{\beta}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]{}^{\mu}_{\alpha}\mathfrak{I}_r^{\lambda}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]\\ &+\left[{}^{\mu}_{\alpha}\mathfrak{I}_r^{\lambda}\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]{}^{\mu}_{\alpha}\mathfrak{I}_r^{\beta}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]\\ &-{}^{\mu}_{\alpha}\mathfrak{I}_r^{\beta}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\left[{}^{\mu}_{\alpha}\mathfrak{I}_r^{\lambda}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right]\\ &-{}^{\mu}_{\alpha}\mathfrak{I}_r^{\lambda}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)\right]\left[{}^{\mu}_{\alpha}\mathfrak{I}_r^{\beta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\right].\end{aligned}$$

After simplification, we get the desired result. □

Remark 4 Applying Theorem 4 for $\beta = \lambda$, we get Theorem 3.

Corollary 7 *Let $(g_i)_{i=1,2,3,\dots,n}$ and h be positive continuous on $[r,s]$ such that h is increasing and $(g_i)_{i=1,2,3,\dots,n}$ are decreasing functions on the interval $[r,s]$. Let $r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ for any fixed $p \in \{1,2,3,\dots,n\}$. Then, for R-L type fractional conformable integral (3), we have*

$$\begin{aligned}&\frac{{}_{\alpha}\Im_r^{\beta}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}_{\alpha}\Im_r^{\lambda}[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)] + {}_{\alpha}\Im_r^{\lambda}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}_{\alpha}\Im_r^{\beta}[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}{{}_{\alpha}\Im_r^{\beta}[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}_{\alpha}\Im_r^{\lambda}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)] + {}_{\alpha}\Im_r^{\lambda}[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}_{\alpha}\Im_r^{\beta}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}\\ &\quad \geq 1, \end{aligned} \tag{24}$$

where $\beta, \lambda \in \mathbb{C}$, $\alpha \in (0,1]$, $\Re(\beta) > 0$, and $\Re(\lambda) > 0$.

Corollary 8 *Let $(g_i)_{i=1,2,3,\dots,n}$ and h be positive continuous on $[r,s]$ such that h is increasing and $(g_i)_{i=1,2,3,\dots,n}$ are decreasing on the interval $[r,s]$. Let $r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$ for any fixed $p \in \{1,2,3,\dots,n\}$. Then, for R-L fractional integral (5), we have*

$$\begin{aligned}&\frac{\Im_r^{\beta}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]\Im_r^{\lambda}[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)] + \Im_r^{\lambda}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]\Im_r^{\beta}[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}{\Im_r^{\beta}[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]\Im_r^{\lambda}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)] + \Im_r^{\lambda}[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]\Im_r^{\beta}[\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}\\ &\quad \geq 1, \end{aligned} \tag{25}$$

where $\beta, \lambda \in \mathbb{C}$, $\Re(\beta) > 0$, and $\Re(\lambda) > 0$.

Theorem 5 *Let $(g_i)_{i=1,2,3,\dots,n}$ and h be positive continuous on $[r,s]$, and let for any fixed $p \in \{1,2,3,\dots,n\}$,*

$$\bigl(g_p^{\vartheta}(t)h^{\vartheta}(\rho) - g_p^{\vartheta}(\rho)h^{\vartheta}(t)\bigr)\bigl(g_p^{\sigma-\gamma_p}(t) - g_p^{\sigma-\gamma_p}(\rho)\bigr) \geq 0,$$

$r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$, then we have

$$\frac{{}_{\alpha}^{\mu}\Im_r^{\beta}[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma+\vartheta}(x)]{}_{\alpha}^{\mu}\Im_r^{\beta}[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)]}{{}_{\alpha}^{\mu}\Im_r^{\beta}[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma}(x)]{}_{\alpha}^{\mu}\Im_r^{\beta}[g_p^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)]} \geq 1, \tag{26}$$

where $\beta \in \mathbb{C}$, $\alpha \in (0,1]$, $\mu \in \mathbb{R}$, $\alpha + \mu \neq 0$, and $\Re(\beta) > 0$.

Proof The proof of Theorem 5 is similar to the proof of Theorem 3 if we replace $h^{\vartheta}(\rho) - h^{\vartheta}(t)$ by $(g_p^{\vartheta}(t)h^{\vartheta}(\rho) - g_p^{\vartheta}(\rho)h^{\vartheta}(t))$. □

Theorem 6 *Let $(g_i)_{i=1,2,3,\dots,n}$ and h be positive continuous functions on $[r,s]$, and let for any fixed $p \in \{1,2,3,\dots,n\}$,*

$$\bigl(g_p^{\vartheta}(t)h^{\vartheta}(\rho) - g_p^{\vartheta}(\rho)h^{\vartheta}(t)\bigr)\bigl(g_p^{\sigma-\gamma_p}(t) - g_p^{\sigma-\gamma_p}(\rho)\bigr) \geq 0,$$

$r < x \leq s$, $\vartheta > 0$, $\sigma \geq \gamma_p > 0$, *then we have*

$$\begin{aligned}
&\Bigg({}_{\alpha}^{\mu}\mathfrak{I}_r^{\beta}\Bigg[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma+\vartheta}(x)\Bigg] {}_{\alpha}^{\mu}\mathfrak{I}_r^{\lambda}\Bigg[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\Bigg] \\
&\quad + {}_{\alpha}^{\mu}\mathfrak{I}_r^{\lambda}\Bigg[\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma+\vartheta}(x)\Bigg] {}_{\alpha}^{\mu}\mathfrak{I}_r^{\beta}\Bigg[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\gamma_i}(x)\Bigg]\Bigg) \\
&\quad \Bigg/ \Bigg({}_{\alpha}^{\mu}\mathfrak{I}_r^{\beta}\Bigg[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma+\vartheta}(x)\Bigg] {}_{\alpha}^{\mu}\mathfrak{I}_r^{\lambda}\Bigg[g_p^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\Bigg] \\
&\quad + {}_{\alpha}^{\mu}\mathfrak{I}_r^{\lambda}\Bigg[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\gamma_i} g_p^{\sigma+\vartheta}(x)\Bigg] {}_{\alpha}^{\mu}\mathfrak{I}_r^{\beta}\Bigg[g_p^{\vartheta}\prod_{i=1}^{n} g_i^{\gamma_i}(x)\Bigg]\Bigg) \\
&\quad \geq 1,
\end{aligned} \tag{27}$$

where $\beta, \lambda \in \mathbb{C}$, $\alpha \in (0, 1]$, $\mu \in \mathbb{R}$, $\alpha + \mu \neq 0$, $\Re(\beta) > 0$, *and* $\Re(\lambda) > 0$.

Proof The proof of Theorem 6 runs parallel as to the proof of Theorem 4 if we replace $h^{\vartheta}(\rho) - h^{\vartheta}(t)$ by $(g_p^{\vartheta}(t)h^{\vartheta}(\rho) - g_p^{\vartheta}(\rho)h^{\vartheta}(t))$. □

Remark 5 In a similar way, we can get the inequalities for the generalized right FCIO (2) and special cases for integrals (4) and (6).

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References

1. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015). https://doi.org/10.1016/j.cam.2014.10.016
2. Abdeljawad, T.: A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel. J. Inequal. Appl. **2017**, 130 (2017). https://doi.org/10.1186/s13660-017-1400-5
3. Abdeljawad, T.: Fractional operators with exponential kernels and a Lyapunov type inequality. Adv. Differ. Equ. **2017**, 313 (2017). https://doi.org/10.1186/s13662-017-1285-0
4. Abdeljawad, T., Agarwal, R.P., Alzabut, J., Jarad, F., ÖZbekler, A.: Lyapunov-type inequalities for mixed non-linear forced differential equations within conformable derivatives. J. Inequal. Appl. **2018**, 143 (2018). https://doi.org/10.1186/s13660-018-1731-x

5. Abdeljawad, T., Alzabut, J., Jarad, F.: A generalized Lyapunov-type inequality in the frame of conformable derivatives. A generalized Lyapunov-type inequality in the frame of conformable derivatives. Adv. Differ. Equ. **2017**, 321 (2017). https://doi.org/10.1186/s13662-017-1383-z
6. Adjabi, Y., Jarad, F., Abdeljawad, T.: On generalized fractional operators and a Gronwall type inequality with applications. Filomat **31**(17), 5457–5473 (2017). https://doi.org/10.2298/FIL1717457A
7. Dahmani, Z.: New inequalities in fractional integrals. Int. J. Nonlinear Sci. **9**(4), 493–497 (2010)
8. Dahmani, Z.: New classes of integral inequalities of fractional order. Matematiche **69**(1), 237–247 (2014). https://doi.org/10.4418/2014.69.1.18
9. Dahmani, Z., Tabharit, L.: On weighted Gruss type inequalities via fractional integration. J. Adv. Res. Pure Math. **2**, 31–38 (2010)
10. Jarad, F., Ugurlu, E., Abdeljawad, T., Baleanu, D.: On a new class of fractional operators. Adv. Differ. Equ. **2017**, 247 (2017)
11. Khalil, R., Horani, M.A., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**(65), 65–70 (2014)
12. Khan, T.U., Khan, M.A.: Generalized conformable fractional integral operators. J. Comput. Appl. Math. **346**, 378–389 (2019). https://doi.org/10.1016/j.cam.2018.07.018
13. Kilbas, A.A.: Hadamard-type fractional calculus. J. Korean Math. Soc. **38**, 1191–1204 (2001)
14. Kiryakova, V.: Generalized Fractional Calculus and Applications. Pitman Res. Notes Math. Ser., vol. 301. Longman, New York (1994)
15. Liu, W., Ngǒ, Q.A., Huy, V.N.: Several interesting integral inequalities. J. Math. Inequal. **3**(2), 201–212 (2009)
16. Nisar, K.S., Qi, F., Rahman, G., Mubeen, S., Arshad, M.: Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric *k*-function. J. Inequal. Appl. **2018**, 135 (2018)
17. Nisar, K.S., Rahman, G., Choi, V., Mubeen, S., Arshad, M.: Certain Gronwall type inequalities associated with Riemann–Liouville *k*- and Hadamard *k*-fractional derivatives and their applications. East Asian Math. J. **34**(3), 249–263 (2018)
18. Nisar, K.S., Tassaddiq, A., Rahman, G., Khan, A.: Some inequalities via fractional conformable integral operators. J. Inequal. Appl. **2019**, 217 (2019)
19. Ntouyas, K.S., Agarwal, P., Tariboon, J.: On Polya–Szego and Chebyshev types inequalities involving the Riemann–Liouville fractional integral operators. J. Math. Inequal. **10**(2), 491–504 (2016)
20. Podlubny, I.: Fractional Differential Equations. Academic Press, London (1999)
21. Qi, F., Rahman, G., Hussain, S.M., Du, W.S., Nisar, K.S.: Some inequalities of Čebyšev type for conformable *k*-fractional integral operators. Symmetry **10**, 614 (2018). https://doi.org/10.3390/sym10110614
22. Rahman, G., Khan, A., Abdeljawad, T., Nisar, K.S.: The Minkowski inequalities via generalized proportional fractional integral operators. Adv. Differ. Equ. **2019**, 287 (2019). https://doi.org/10.1186/s13662-019-2229-7
23. Rahman, G., Nisar, K.S., Mubeen, S., Choi, J.: Certain inequalities involving the (k,ρ)-fractional integral operator. Far East J. Math. Sci.: FJMS **103**(11), 1879–1888 (2018)
24. Rahman, G., Nisar, K.S., Qi, F.: Some new inequalities of the Gruss type for conformable fractional integrals. AIMS Mathematics **3**(4), 575–583 (2018)
25. Rahman, G., Ullah, Z., Khan, A., Set, E., Nisar, K.S.: Certain Chebyshev type inequalities involving fractional conformable integral operators. Mathematics **7**, 364 (2019). https://doi.org/10.3390/math7040364
26. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, Reading (1993)
27. Sarikaya, M.Z., Dahmani, Z., Kiris, M.E., Ahmad, F.: (k,s)-Riemann–Liouville fractional integral and applications. Hacet. J. Math. Stat. **45**(1), 77–89 (2016)
28. Set, E., Tomar, M., Sarikaya, M.Z.: On generalized Grüss type inequalities for *k*-fractional integrals. Appl. Math. Comput. **269**, 29–34 (2015)
29. Srivastava, H.M., Choi, J.: Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier, Amsterdam (2012)