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Exact dynamical behavior for a dual Kaup–Boussinesq system by symmetry reduction and coupled trial equations method

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Abstract

We propose a coupled trial equation method for a coupled differential equations system. Furthermore, according to the invariant property under the translation, we give the symmetry reduction of a dual Kaup–Boussinesq system, and then we use the proposed trial equation method to construct its exact solutions which describe its dynamical behavior. In particular, we get a cosine function solution with a constant propagation velocity, which shows an important periodic behavior of the system.

Keywords: Trial equation method; Symmetry; The complete discrimination system for polynomial; Exact solution; Kaup–Boussinesq system

1 Introduction

The usual Kaup–Boussinesq system which reads

$$\begin{cases} u_t = u_{xxx} + 2(uv)_x, \\ v_t = u_x + vv_x, \end{cases}$$

describes the motion of water wave, where $u(x,t)$ is the height of the water surface above a horizontal bottom and $v(x,t)$ is the horizontal velocity. The solutions of the nonlinear system have been studied from several aspects, and some interesting phenomena have been discovered [1–6]. For example, Smirnov obtained its real finite-gap regular solution [1], Borisov et al. studied its proliferation scheme [2], Kamchatnov et al. constructed the asymptotic soliton train solutions [3], and so on.

In general, it is difficult to study the exact dynamical behavior for nonlinear evolution problems. Therefore, some powerful methods, such as Ma and Lee's transformed rational function method [7], Liu's canonical-like transformation method [8] and trial equation method [9–14], Ma and Zhu's multiple exp-function method [15], and other direct expansion methods [16], and so forth, have been proposed to solve such problems. On the other hand, we can use the complete discrimination system for polynomial method to classify exact solutions for some nonlinear differential equations [17–25]. These methods have been extensively developed and applied to a lot of nonlinear problems [26–37]. Liu's renormalization method and its applications can be found in [38–45]. Some new methods and results on fractional differential equations can be seen in [46–57] and the references therein.

In the paper, we will study a dual Kaup–Boussinesq system. By proposing a coupled trial equation method and using symmetry reduction and a complete discrimination system for polynomial, we obtain its exact solutions which describe the dynamical behavior of the system. In particular, we find a cosine function solution which shows an important periodic motion.

This paper is organized as follows. In Sect. 2, we propose a coupled trial equation method. In Sect. 3, we give the reduction of the dual Kaup–Boussinesq system according to the symmetry property and the proposed trial equation method. In Sect. 4, we give the exact solutions by using the complete discrimination system for polynomial. In particular, we get an interesting periodic cosine solution. The last section is a short conclusion.

2 Trial equation method for a coupled system

We propose a generalization of Liu's trial equation method [9–14] to coupled differential equations systems as follows. Consider the coupled system:

$$N_1(u, v, u_t, v_t, u_x, v_x, \ldots) = 0, \tag{1}$$

$$N_2(u, v, u_t, v_t, u_x, v_x, \ldots) = 0. \tag{2}$$

Under the traveling wave transformation $u(x, t) = u(\xi)$, $v(x, t) = v(\xi)$, where $\xi = x - ct$, the above system of equations becomes a coupled ordinary differential equations system

$$M_1\bigl(u, v, u', v', \ldots\bigr) = 0, \tag{3}$$

$$M_2\bigl(u, v, u', v', \ldots\bigr) = 0. \tag{4}$$

We take trial equations as follows:

$$u' = H(u), \tag{5}$$

$$v = G(u), \tag{6}$$

or

$$v' = H(v), \tag{7}$$

$$u = G(v), \tag{8}$$

where H and G are two unknown functions which need to be determined. Substituting these trial equations into the coupled system, we solve H and G, and then integrate the trial equation (5) or (7) to give the corresponding exact solutions such as

$$\xi - \xi_0 = \int \frac{\mathrm{d}u}{H(u)}, \qquad v = G(u). \tag{9}$$

If H is a polynomial, we will use the complete discrimination system for polynomial to classify the exact solutions. In the next section, we give the application of the proposed trial equation method to a dual Kaup–Boussinesq system.

3 Symmetry and reduction

The considered new dual Kaup–Boussinesq (for simplicity, KB) system is taken as follows:

$$\begin{cases} u_{xxt} + (uu_{xx} + u_x^2 + u^2 + 2uv)_x = 0, \\ v_t + (uv)_x = 0. \end{cases} \tag{10}$$

By taking the traveling wave transformation $u = u(\xi)$, $v = v(\xi)$, $\xi = x - ct$, we have

$$\begin{cases} -cu''' + (uu'' + (u')^2 + u^2 + 2uv)' = 0. \\ -cv' + (uv)' = 0. \end{cases} \tag{11}$$

It is easy to see that the above equations are invariant under the translational transformation of ξ, so we can reduce the system. Substituting trial equations (5) and (6) into the above system and integrating them yield

$$\begin{cases} (u-c)H'H + H^2 + u^2 + 2uv = c_1, \\ v = G(u) = \frac{c_2}{u-c}, \end{cases} \tag{12}$$

where c_1 and c_2 are two arbitrary constants. And then the above system becomes

$$(u-c)H'H + H^2 + u^2 + \frac{2uc_2}{u-c} = c_1. \tag{13}$$

Furthermore, we take the following transformation:

$$H^2 = W, \tag{14}$$

and get

$$W'(u) + \frac{2}{u-c}W(u) = \frac{2(c_1 - u^2 - \frac{2uc_2}{u-c})}{u-c}. \tag{15}$$

This is a first order linear non-homogeneous differential equation whose general solution is given by

$$\begin{aligned} W(u) &= e^{-\int \frac{2}{u-c}\,\mathrm{d}u}\left\{\int 2\frac{c_1 - u^2 - 2u\frac{c_2}{u-c}}{u-c} e^{\int \frac{2}{u-c}\,\mathrm{d}u}\,\mathrm{d}u + c_3\right\} \\ &= \frac{-\frac{1}{2}u^4 + \frac{2}{3}cu^3 + (2c_1 - 2c_2)u^2 - 2cc_1u + c_3}{(u-c)^2}, \end{aligned} \tag{16}$$

that is,

$$\left(u'\right)^2 = H^2(u) = \frac{-\frac{1}{2}u^4 + \frac{2}{3}cu^3 + (2c_1 - 2c_2)u^2 - 2cc_1u + c_3}{(u-c)^2}, \tag{17}$$

which is just our needed result. Rewrite it as the form of elementary integrals

$$\pm\sqrt{\frac{1}{2}}(\xi - \xi_0) = \int \frac{(u-c)\,\mathrm{d}u}{\sqrt{-(u^4 - \frac{4}{3}cu^3 - (4c_1 - 4c_2)u^2 - 4cc_1u + 2c_3)}}, \tag{18}$$

from which we can give exact solutions. In fact, we can further simplify it by taking

$$\omega = u - \frac{1}{3}c, \tag{19}$$

and hence

$$\pm\sqrt{\frac{1}{2}}(\xi - \xi_0) = \int \frac{(\omega - \frac{2}{3}c)\,\mathrm{d}\omega}{\sqrt{-(\omega^4 + p\omega^2 + q\omega + r)}}, \tag{20}$$

where

$$\begin{aligned} &p = -\frac{7}{9}c^2 - 4c_1 + 4c_2, \qquad q = \frac{1}{9}c^2 - \frac{8}{27}c^3 - \frac{20}{3}c_1c + \frac{8}{3}c_2c, \\ &r = \frac{20}{9}c_1c + \frac{4}{9}c_2c^2 + 2c_3. \end{aligned} \tag{21}$$

In the next section, we give exact solutions for the KB coupled system (10) according to the above integral (20).

Remark 1 If we consider $(u')^2$ as kinetic energy and the right-hand side of (17) as negative potential energy, then (17) gives a first integral, that is, conservation of energy.

4 Exact solutions

Denote $F(\omega) = \omega^4 + p\omega^2 + q\omega + r$. Then its complete discrimination system is given by [25]

$$\begin{cases} D_1 = 4, \\ D_2 = -p, \\ D_3 = -2p^3 + 8pr - 9q^2, \\ D_4 = -p^3q^2 + 4p^4r + 36pq^2r - 32p^2r^2 - \frac{27}{4}q^4 + 64r^3, \\ E_2 = 9q^2 - 32pr. \end{cases} \tag{22}$$

According to the discrimination system [25], and considering the special form of integrand in (20), we have the following four families of solutions.

Family 1 $D_4 = 0$, $D_3 > 0$, $D_2 > 0$. Then we have

$$F(\omega) = (\omega - \alpha)^2(\omega - \beta)(\omega - \gamma), \tag{23}$$

where α, β, γ are real numbers, and $\beta > \gamma$. When $\gamma < \alpha < \beta$, we have

$$\pm\sqrt{\frac{1}{2}}(\xi - \xi_0) = \frac{2\alpha - c}{\sqrt{(\beta - \alpha)(\gamma - \alpha)}} \arctan\sqrt{\frac{\alpha - \beta}{\gamma - \alpha}\frac{u + \frac{c}{3} - \gamma}{\beta - u - \frac{c}{3}}} + 2\arctan\sqrt{\frac{u + \frac{c}{3} - \gamma}{\beta - u - \frac{c}{3}}}; \tag{24}$$

and when $\alpha > \beta$ and $w > \beta$, or when $\alpha < \gamma$ and $w < \gamma$, we have

$$\pm\sqrt{\frac{1}{2}}(\xi - \xi_0) = \frac{2\alpha - c}{\sqrt{(\alpha - \beta)(\gamma - \alpha)}} \ln \frac{\sqrt{\frac{\alpha-\beta}{\gamma-\alpha}\frac{u+\frac{c}{3}-\gamma}{\beta-u-\frac{c}{3}}} + 1}{\sqrt{\frac{\alpha-\beta}{\gamma-\alpha}\frac{u+\frac{c}{3}-\gamma}{\beta-u-\frac{c}{3}}} - 1} + 2\arctan\sqrt{\frac{u + \frac{c}{3} - \gamma}{\beta - u - \frac{c}{3}}}. \tag{25}$$

Family 2 $D_4 = 0$, $D_3 = 0$, $D_2 > 0$, $E_2 = 0$. Then we have

$$F(\omega) = (\omega - \alpha)^3(\omega - \beta), \tag{26}$$

where α, β are real numbers. The solution is given by

$$\pm\sqrt{\frac{1}{2}}(\xi - \xi_0) = \frac{2\alpha - c}{\alpha - \beta}\sqrt{\frac{u + \frac{c}{3} - \gamma}{\beta - u - \frac{c}{3}}} + 2\arctan\sqrt{\frac{u + \frac{c}{3} - \gamma}{\beta - u - \frac{c}{3}}}. \tag{27}$$

Family 3 $D_4 > 0$, $D_3 > 0$, $D_1 > 0$. Then we have

$$F(\omega) = (\omega - \alpha_1)(\omega - \alpha_2)(\omega - \alpha_3)(\omega - \alpha_4), \tag{28}$$

where α_1, α_2, α_3, α_4 are real numbers, and $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. The solutions can be represented in terms of the first and second kinds of elliptic integrals. We can also give more simple forms by taking the transformation

$$\omega_1 = \frac{A^{\frac{1}{3}}}{\omega - \alpha_1}, \tag{29}$$

where $A = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)$. The corresponding integral becomes

$$\begin{aligned}\pm\sqrt{\frac{1}{2}}(\xi - \xi_0) = {}& \int \frac{d\omega_1}{\omega_1\sqrt{-(\omega_1 - \beta_1)(\omega_1 - \beta_2)(\omega_1 - \beta_3)}} \\ & + \int \frac{(\alpha_1 - \frac{2c}{3})A^{-\frac{1}{3}}\,d\omega_1}{\omega_1\sqrt{-(\omega_1 - \beta_1)(\omega_1 - \beta_2)(\omega_1 - \beta_3)}},\end{aligned} \tag{30}$$

where $\beta_1 = \frac{A^{\frac{1}{3}}}{\alpha_4 - \alpha_1}$, $\beta_2 = \frac{A^{\frac{1}{3}}}{\alpha_3 - \alpha_1}$, $\beta_3 = \frac{A^{\frac{1}{3}}}{\alpha_2 - \alpha_1}$.

Family 4 $D_4 < 0$, $D_2D_3 \geq 0$. Then we have

$$F(\omega) = (\omega - \alpha)(\omega - \beta)\big((\omega - l_1)^2 + s_1^2\big), \tag{31}$$

where α, β, l_1, and s_1 are real numbers, and $\alpha > \beta >$, $s_1 > 0$. The solutions can be represented in terms of the first and second kinds of elliptic integrals.

From Remark 1, we know that the above result gives the classification of all solutions of integral (20).

Remark 2 Here we only write the expressions of u, by which v can be given from (4). For simplicity, we omit v.

In particular, we find that there is an interesting periodic cosine function solution although all the above solutions are implicit forms. Indeed, if we take the wave velocity to be a constant $c = 2\alpha$, we get the periodic solution

$$u = \beta - \frac{2\alpha}{3} + \left(\frac{\alpha}{3} - \beta - \gamma\right)\cos^2\left(\frac{\xi - \xi_0}{2\sqrt{2}}\right), \tag{32}$$

where $\xi = x - \frac{2\alpha}{3}t$. This is a periodic traveling wave with constant velocity.

In fact, we only need to take the first term to be zero in solution (24) or (25) or (27), and then we have

$$\pm\sqrt{\frac{1}{2}}(\xi - \xi_0) = 2\arctan\sqrt{\frac{u + \frac{c}{3} - \gamma}{\beta - u - \frac{c}{3}}}, \tag{33}$$

by which we get the above periodic solution.

From solution (32), we know that the dual KB system shows an important periodic dynamical behavior.

5 Conclusion

A dual Kaup–Boussinesq system is solved by symmetry reduction and a coupled trial equation method. The result includes four families of exact single traveling wave solutions for this system. Among those, if we consider ξ as the functions of u or v respectively, the solutions are given by the explicit functions, and reversely, the solutions are represented by implicit functions. In particular, when the wave propagation velocity is taken as a special constant, the KB system has a periodic cosine function solution. This solution shows an important periodic dynamical behavior. In summary, according to these exact solutions, a variety of evolution patterns for the coupled KB system are obtained.

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Authors' contributions
WL completed the whole computation and wrote the draft, YW proposed the topic. Both authors read and approved the final manuscript.

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