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An analytical approach to obtain exact solutions of some space-time conformable fractional differential equations

Nematollah Kadkhoda[1*] and Hossein Jafari[2]

*Correspondence:
kadkhoda@buqaen.ac.ir
[1]Department of Mathematics, Faculty of Basic Sciences, Bozorgmehr University of Qaenat, Qaenat, Iran
Full list of author information is available at the end of the article

Abstract

In this paper, the sine-Gordon expansion method is used to obtain analytical solutions of the conformable space-time generalized reaction Duffing model and conformable space-time Eckhaus equation with the aid of symbolic computation. These equations can be reduced into ordinary differential equations (ODEs) using a suitable wave transformation with a predicted polynomial-type solution.

Keywords: Space-time fractional Eckhaus equation; Sine-Gordon expansion method; Space-time fractional generalized reaction Duffing model; Conformable derivative

1 Introduction

Nonlinear fractional differential equations (FDEs) have an important role in studying various areas of engineering, physics, and applied mathematics [1–5]. Investigation of analytical solutions of nonlinear FDEs is very important in the analysis of some physical phenomena, such as plasma physics, solid-state physics, nonlinear optics, and so on [6–9]. In order to understand the mechanisms of these cases, it is necessary to obtain their exact solutions [10, 11]. Thus, many researchers have tried to obtain analytical solutions of these equations. Therefore, many methods and techniques are found to seek exact solutions of nonlinear FDEs, such as separating variables method [12], homotopy analysis method [13], Adomian decomposition method [14], fractional complex transform [15], variational iteration method [16], Hirota's bilinear method [17], homotopy perturbation pade technique [18], $\frac{G'}{G^2}$-expansion method [19], sub-equation method [20, 21], simplest equation method [22, 23], first integral method [24], and so on. In this letter, the following two FDEs are solved using the sine-Gordon expansion method:

(I) The form of the space-time fractional Eckhaus equation

$$iD_t^{\alpha}\varphi + D_x^{2\alpha}\varphi + 2D_x^{\alpha}|\varphi|^2\varphi + |\varphi|^4\varphi = 0, \tag{1}$$

where $\varphi = \varphi(x,t)$, $\varphi : \mathbb{R}^2 \to \mathbb{C}$. This equation was introduced by Wiktor Eckhaus to describe the propagation of waves in dispersive media [25]. Many of the properties of the Eckhaus were studied in [26].

(II) The form of the space-time fractional generalized reaction Duffing model [27]

$$D_t^{2\alpha}\varphi + aD_x^{2\alpha}\varphi + b\varphi + c\varphi^2 + d\varphi^3 = 0. \tag{2}$$

This model illustrates the motion of a damped oscillator. It is a case of a dynamical system which presents chaotic behavior. Here, D^α is the conformable derivative [28, 29]. The paper is organized as follows. Section 2 is given to introduce the definition of the conformable derivative and its properties. Description of the conformable sine-Gordon expansion method and its application to the space-time FDEs are given in Sect. 3. We obtain exact solutions to the space-time fractional Eckhaus equation and the space-time fractional generalized reaction Duffing model with sine-Gordon expansion method in Sect. 4. Conclusions of this paper are summarized in Sect. 5.

2 Definition of the conformable derivative and its properties

In this section, we illustrate the definition of the conformable derivative and some of its important properties of order α with respect to the independent variable z as follows [28].

Definition 1 For a function $g : [0,\infty] \to \mathbb{R}$, the conformable fractional derivative of g of order α is defined by

$$D^\alpha\{g(z)\} = \lim_{\eta\to 0}\frac{g(z+\eta z^{1-\alpha}) - g(z)}{\eta}. \tag{3}$$

Some well-known properties to this newly defined fractional derivative are as follows. If f and $g \neq 0$ are two functions α-differentiable, $\alpha \in (0,1]$ and $a, b \in \mathbb{R}$, then we have

(1) $D^\alpha\{af(z) + bg(z)\} = aD^\alpha f(z) + bD^\alpha g(z)$,

(2) $D^\alpha\{f(z)g(z)\} = f(z)D^\alpha g(z) + g(z)D^\alpha f(z)$,

(3) $D^\alpha\left\{\dfrac{f(z)}{g(z)}\right\} = \dfrac{g(z)D^\alpha f(z) - f(z)D^\alpha g(z)}{g^2(z)}$,

(4) $D^\alpha C = 0$, for all constant functions $f(z) = C$,

(5) $D^\alpha(f)(z) = z^{1-\alpha}\dfrac{df}{dz}$.

Also, conformable fractional derivatives of some special functions are as follows:

(a) $D^\alpha(z^r) = rz^{r-\alpha}$ for all $r \in \mathbb{R}$,

(b) $D^\alpha(1) = 0$,

(c) $D^\alpha(e^{cz}) = cz^{1-\alpha}e^{cz}$, $c \in \mathbb{R}$,

(d) $D^\alpha(\sin bz) = bz^{1-\alpha}\cos bz$, $b \in \mathbb{R}$,

(e) $D^\alpha(\cos bz) = -bz^{1-\alpha}\sin bz$, $b \in \mathbb{R}$,

(f) $D^\alpha\left(\dfrac{1}{\alpha}z^\alpha\right) = 1$.

The proofs of these properties can be seen in [28].

Definition 2 Let $\alpha \in (n, n+1]$, and g be α-differentiable at $t > 0$. Then the conformable fractional derivative of g of order α is defined as

$$D^{\alpha}\bigl(g(t)\bigr) = \lim_{\eta \to 0} \frac{g^{(\lceil\alpha\rceil-1)}(t + \eta t^{(\lceil\alpha\rceil-\alpha)}) - g^{(\lceil\alpha\rceil-1)}(t)}{\eta}, \tag{4}$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

3 Conformable sine-Gordon expansion method and its applications to the space-time fractional differential equations

The proper fractional form $\varphi(x,t) = \phi(\xi)$ with

$$\xi = k\left(\frac{x^{\alpha}}{\alpha} - 2r\frac{t^{\alpha}}{\alpha}\right) \tag{5}$$

of wave transformation reduces the fractional sine-Gordon equation

$$D_x^{2\alpha}\varphi - D_t^{2\alpha}\varphi = m^2 \sin\varphi, \tag{6}$$

where m is constant, to the ODE

$$\frac{d^2\phi}{d\xi^2} = \frac{m^2}{k^2(1-4r^2)} \sin\phi. \tag{7}$$

Here, r is velocity of the traveling wave. Therefore, we have

$$\left(\frac{d(\frac{\phi}{2})}{d\xi}\right)^2 = \frac{m^2}{k^2(1-4r^2)} \sin^2\frac{\phi}{2}, \tag{8}$$

with $\omega(\xi) = \frac{\phi(\xi)}{2}$ and $b^2 = \frac{m^2}{k^2(1-4r^2)} = 1$, equation (8) is changed to

$$\frac{d\omega}{d\xi} = \sin\omega. \tag{9}$$

The following relations can be obtained from (9):

$$\sin\omega(\xi) = \frac{2\gamma e^{\xi}}{\gamma^2 e^{2\xi} + 1}\bigg|_{\gamma=1} = \operatorname{sech}\xi \tag{10}$$

or

$$\cos\omega(\xi) = \frac{\gamma^2 e^{2\xi} - 1}{\gamma^2 e^{2\xi} + 1}\bigg|_{\gamma=1} = \tanh\xi, \tag{11}$$

where γ is integral constant and $\gamma \neq 0$. Nonlinear FDEs

$$P\bigl(\varphi, D_t^{\alpha}\varphi, D_x^{\alpha}\varphi, D_t^{2\alpha}\varphi, D_x^{2\alpha}\varphi, \ldots\bigr) = 0 \tag{12}$$

with the traveling transformation (5) can be reduced to an ODE

$$\tilde{P}\bigl(\phi, \phi', \phi'', \ldots\bigr) = 0. \tag{13}$$

Here prime denotes the derivative with respect to ξ. Now, the solution to (13) can be written

$$\phi(\xi) = a_0 + \sum_{j=1}^{n} \tanh^{j-1}(\xi)\bigl(a_j \tanh(\xi) + b_j \operatorname{sech}(\xi)\bigr), \tag{14}$$

or

$$\phi(\omega) = a_0 + \sum_{j=1}^{n} \cos^{j-1}(\omega)\bigl(a_j \cos(\omega) + b_j \sin(\omega)\bigr), \tag{15}$$

where a_0, a_j, b_j $(1 \le j \le n)$ are constants to be determined later, and n is fixed by balancing between the highest nonlinear term and highest order derivative in (13). It is worth noting the following derivatives of $\phi(\omega)$ in equation (15).

• The first, second, and third derivatives of $\phi(\omega)$ for $n = 1$ in equation (15) can be written as follows:

$$\phi(\omega) = a_0 + a_1 \cos\omega + b_1 \sin\omega, \tag{16}$$

$$\phi'(\omega) = b_1 \cos\omega \sin\omega - a_1 \sin^2\omega, \tag{17}$$

$$\phi''(\omega) = b_1\bigl[\cos^2\omega \sin\omega - \sin^3\omega\bigr] - 2a_1 \sin^2\omega \cos\omega, \tag{18}$$

$$\phi'''(\omega) = b_1\bigl[\cos^3\omega \sin\omega - 5\sin^3\omega \cos\omega\bigr] - 2a_1 \sin^2\omega\bigl(2\cos^2\omega - \sin^2\omega\bigr). \tag{19}$$

• The first and second derivatives of $\phi(\omega)$ for $n = 2$ in equation (15) can be written as follows:

$$\phi(\omega) = a_0 + a_1 \cos\omega + b_1 \sin\omega + a_2 \cos^2\omega + b_2 \sin\omega \cos\omega, \tag{20}$$

$$\phi'(\omega) = -a_1 \sin^2\omega - 2a_2 \cos\omega \sin^2\omega + b_1 \cos\omega \sin\omega + b_2\bigl(\sin\omega - 2\sin^3\omega\bigr), \tag{21}$$

$$\begin{aligned}\phi''(\omega) = {}& -2a_1 \sin^2\omega \cos\omega + 2a_2 \sin^2\omega\bigl(\sin^2\omega - 2\cos^2\omega\bigr) \\ & + b_1\bigl(\cos^2\omega \sin\omega - \sin^3\omega\bigr) + b_2 \cos\omega\bigl(\sin\omega - 6\sin^3\omega\bigr).\end{aligned} \tag{22}$$

Substituting (15) along (9)–(11) into (13) and collecting all terms with the same powers of $\sin\omega\cos\omega$ together, the left-hand side of (13) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of $a_0, a_1, b_1, \ldots, k$, and r. Solving the system of algebraic equations and then substituting the results into (13), gives solutions of (12).

4 Applications of the sine-Gordon expansion method to the space-time FDEs

• *Applications of the sine-Gordon expansion method to the space-time fractional Eckhaus equation*

Here, we will demonstrate this method to the space-time fractional Eckhaus equation as follows:

$$iD_t^{\alpha}\varphi + D_x^{2\alpha}\varphi + 2D_x^{\alpha}|\varphi|^2\varphi + |\varphi|^4\varphi = 0, \tag{23}$$

by transformations

$$\begin{aligned} &\varphi(x,t) = \phi(\xi)e^{i\theta}, \\ &\xi = k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right), \\ &\theta = \left(r\frac{x^\alpha}{\alpha} + s\frac{t^\alpha}{\alpha}\right), \end{aligned} \tag{24}$$

where θ illustrates the phase component, r is the velocity, s is the frequency, and k is the width of the soliton, equation (23) becomes:

$$k^2\phi_{\xi\xi} - (s + r^2)\phi + 4k\phi_\xi\phi^2 + \phi^5 = 0. \tag{25}$$

Now, we suppose that

$$\phi = \sqrt{\psi}, \tag{26}$$

therefore, equation(25) can be reduced to the following ODE:

$$2k^2\psi\psi_{\xi\xi} - k^2\psi_\xi^2 - 4(s + r^2)\psi^2 + 8k\psi_\xi\psi^2 + 4\psi^4 = 0. \tag{27}$$

Therefore, the solution of (27) can be expressed as follows:

$$\psi(\xi) = a_0 + a_1\tanh(\xi) + b_1\operatorname{sech}(\xi). \tag{28}$$

Substituting (16)–(18) into (27) and equating all terms with the powers of $\sin\omega\cos\omega$ to zero, the following system can be obtained:

$$\begin{aligned} &4a_1^4 - 4a_0^2r^2 - 8a_1a_0r^2 - 4a_1^2r^2 - 4a_0^2s - 8a_1a_0s - 4a_1^2s + 4a_0^4 + 16a_1a_0^3 \\ &\quad + 24a_1^2a_0^2 + 16a_1^3a_0 = 0, \\ &8a_1a_0r^2 - 4a_0^2r^2 - 4a_1^2r^2 - 4a_0^2s + 8a_1a_0s - 4a_1^2s + 4a_0^4 - 16a_1a_0^3 + 24a_1^2a_0^2 \\ &\quad - 16a_1^3a_0 + 4a_1^4 = 0, \\ &4a_0b_1k^2 + 4a_1b_1k^2 + 16a_0^2b_1k + 32a_1a_0b_1k + 16a_1^2b_1k - 16a_0b_1r^2 - 16a_1b_1r^2 \\ &\quad - 16a_0b_1s - 16a_1b_1s + 32a_0^3b_1 + 96a_1a_0^2b_1 + 96a_1^2a_0b_1 + 32a_1^3b_1 = 0, \\ &64b_1^3k - 20a_0b_1k^2 - 44a_1b_1k^2 + 16a_0^2b_1k - 160a_1a_0b_1k - 176a_1^2b_1k - 48a_0b_1r^2 \\ &\quad - 16a_1b_1r^2 - 48a_0b_1s - 16a_1b_1s + 96a_0^3b_1 + 96a_1a_0^2b_1 + 128a_0b_1^3 - 96a_1^2a_0b_1 \\ &\quad + 128a_1b_1^3 - 96a_1^3b_1 = 0, \\ &192a_0^2b_1^2 - 256a_1b_1^2k - 192a_1^2b_1^2 + 16a_1^2k^2 - 64a_1a_0^2k + 64a_1^3k - 24a_0^2r^2 + 8a_1^2r^2 \\ &\quad - 24a_0^2s + 8a_1^2s + 24a_0^4 - 48a_1^2a_0^2 + 24a_1^4 - 40b_1^2k^2 - 32b_1^2r^2 - 32b_1^2s + 64b_1^4 = 0, \\ &44a_1b_1k^2 - 20a_0b_1k^2 - 16a_0^2b_1k - 160a_1a_0b_1k + 176a_1^2b_1k - 48a_0b_1r^2 + 16a_1b_1r^2 \\ &\quad - 48a_0b_1s + 16a_1b_1s + 96a_0^3b_1 - 96a_1a_0^2b_1 + 128a_0b_1^3 - 96a_1^2a_0b_1 - 128a_1b_1^3 \end{aligned}$$

$$+ 96a_1^3b_1 - 64b_1^3k = 0,$$

$$4a_0b_1k^2 - 4a_1b_1k^2 - 16a_0^2b_1k + 32a_1a_0b_1k - 16a_1^2b_1k - 16a_0b_1r^2 + 16a_1b_1r^2$$
$$- 16a_0b_1s + 16a_1b_1s + 32a_0^3b_1 - 96a_1a_0^2b_1 + 96a_1^2a_0b_1 - 32a_1^3b_1 = 0.$$

Solving this system, we find a_0, a_1, and b_1 as follows:

• Case 1:

$$a_0 = -\frac{k}{4}, \qquad a_1 = -\frac{k}{4}, \qquad b_1 = \pm\frac{\sqrt{-r^2-s}}{2}; \quad k = 2\sqrt{r^2+s}. \tag{29}$$

Substituting (29) (case 1) in (28), we obtain solutions of (27) as follows:

$$\psi_1(x,t) = -\frac{\sqrt{r^2+s}}{2}\left(1 + \tanh\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right)\right) \pm \frac{i\sqrt{r^2+s}}{2}\operatorname{sech}\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right).$$

Therefore, solutions of (23) can be written as follows:

$$\varphi_1(x,t) = \left\{-\frac{\sqrt{r^2+s}}{2}\left(1 + \tanh\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right)\right)\right.$$
$$\left.\pm \frac{i\sqrt{r^2+s}}{2}\operatorname{sech}\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right)\right\}^{\frac{1}{2}} e^{i(r\frac{x^\alpha}{\alpha}+s\frac{t^\alpha}{\alpha})}.$$

• Case 2:

$$a_0 = \frac{k}{4}, \qquad a_1 = -\frac{k}{4}, \qquad b_1 = \pm\frac{\sqrt{-r^2-s}}{2}; \quad k = 2\sqrt{r^2+s}. \tag{30}$$

Substituting (30) (case 2) in (28), we obtain solutions of (27) as follows:

$$\psi_2(x,t) = \frac{\sqrt{r^2+s}}{2}\left(1 - \tanh\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right)\right) \pm \frac{i\sqrt{r^2+s}}{2}\operatorname{sech}\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right).$$

Now, solutions of (23) can be written as follows:

$$\varphi_2(x,t) = \left\{\frac{\sqrt{r^2+s}}{2}\left(1 - \tanh\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right)\right)\right.$$
$$\left.\pm \frac{i\sqrt{r^2+s}}{2}\operatorname{sech}\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right)\right\}^{\frac{1}{2}} e^{i(r\frac{x^\alpha}{\alpha}+s\frac{t^\alpha}{\alpha})}.$$

• Case 3:

$$a_0 = -\frac{k}{2}, \qquad a_1 = -\frac{k}{2}, \qquad b_1 = 0; \quad k = -\sqrt{r^2+s}. \tag{31}$$

Substituting (31) (case 3) in (28), we obtain solutions of (27) as follows:

$$\psi_3(x,t) = \frac{\sqrt{r^2+s}e^{(2k(\frac{x^\alpha}{\alpha}-2r\frac{t^\alpha}{\alpha}))}}{e^{(2k(\frac{x^\alpha}{\alpha}-2r\frac{t^\alpha}{\alpha}))}+1}.$$

Also, solutions of (23) can be obtained as follows:

$$\varphi_3(x,t) = \left\{ \frac{\sqrt{r^2+s} e^{(2k(\frac{x^\alpha}{\alpha}-2r\frac{t^\alpha}{\alpha}))}}{e^{(2k(\frac{x^\alpha}{\alpha}-2r\frac{t^\alpha}{\alpha}))}+1} \right\}^{\frac{1}{2}} e^{i(r\frac{x^\alpha}{\alpha}+s\frac{t^\alpha}{\alpha})}.$$

• Case 4:

$$a_0 = \frac{k}{2}, \qquad a_1 = -\frac{k}{2}, \qquad b_1 = 0; \quad k = \sqrt{r^2+s}. \tag{32}$$

Finally, substituting (32) (case 4) in (28), solutions of (27) can be calculated as follows:

$$\psi_4(x,t) = \frac{\sqrt{r^2+s}}{e^{(2k(\frac{x^\alpha}{\alpha}-2r\frac{t^\alpha}{\alpha}))}+1}.$$

Furthermore, we obtain solutions of (27) as follows:

$$\varphi_4(x,t) = \left\{ \frac{\sqrt{r^2+s}}{e^{(2k(\frac{x^\alpha}{\alpha}-2r\frac{t^\alpha}{\alpha}))}+1} \right\}^{\frac{1}{2}} e^{i(r\frac{x^\alpha}{\alpha}+s\frac{t^\alpha}{\alpha})}.$$

• *Applications of the sine-Gordon expansion method to the space-time fractional generalized reaction Duffing model*

Now, we will demonstrate the sine-Gordon expansion method to the space-time fractional generalized reaction Duffing model as follows:

$${}_0^A D_t^{2\alpha}\varphi + a{}_0^A D_x^{2\alpha}\varphi + b\varphi + c\varphi^2 + d\varphi^3 = 0 \tag{33}$$

by transformations

$$\xi = k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right). \tag{34}$$

Equation (33) becomes:

$$\left(ak^2 + 4k^2r^2\right)\phi_{\xi\xi} + b\varphi + c\varphi^2 + d\varphi^3 = 0. \tag{35}$$

With balancing, we get $n = 1$. Therefore, the solution of (35) can be expressed as follows:

$$\varphi(\xi) = a_0 + a_1\tanh(\xi) + b_1\operatorname{sech}(\xi). \tag{36}$$

Substituting (36) along (17) and (18) into (35) and equating all terms with the powers of $\sin\omega\cos\omega$ to zero, the following system can be obtained:

$$a_0b + a_1b + a_0^2c + 2a_1a_0c + a_1^2c + a_0^3d + 3a_1a_0^2d + 3a_1^2a_0d + a_1^3d = 0,$$

$$4a_0b_1c + 4a_1b_1c + 6a_0^2b_1d + 6a_1^2b_1d + 12a_0a_1b_1d + 2ab_1k^2 + 8b_1k^2r^2 + 2bb_1 = 0,$$

$$8a_0b_1c + 12a_0^2b_1d - 12a_1^2b_1d - 12ab_1k^2 + 8b_1^3d - 48b_1k^2r^2 + 4bb_1 = 0,$$

$$4a_0b_1c - 4a_1b_1c + 6a_0^2b_1d + 6a_1^2b_1d - 12a_0a_1b_1d + 2ab_1k^2 + 8b_1k^2r^2 + 2bb_1 = 0,$$

$$a_0b - a_1b + a_0^2c - 2a_1a_0c + a_1^2c + a_0^3d - 3a_1a_0^2d + 3a_1^2a_0d - a_1^3d = 0,$$
$$12a_0b_1^2d - 12a_1b_1^2d + 3a_0b - a_1b + 3a_0^2c - 2a_1a_0c - a_1^2c + 3a_0^3d - 3a_1a_0^2d - 3a_1^2a_0d$$
$$+ 3a_1^3d + 32a_1k^2r^2 + 8aa_1k^2 + 4b_1^2c = 0,$$
$$12a_0b_1^2d + 12a_1b_1^2d + 3a_0b + a_1b + 3a_0^2c + 2a_1a_0c - a_1^2c + 3a_0^3d + 3a_1a_0^2d - 3a_1^2a_0d$$
$$- 3a_1^3d - 32a_1k^2r^2 - 8aa_1k^2 + 4b_1^2c = 0.$$

Solving this system, we obtain a_0, a_1, and b_1 as follows:

- Case 1:

$$a_0 = -\frac{3b}{2c}, \qquad a_1 = \pm\frac{3b}{2c}, \qquad b_1 = 0; \qquad a = \frac{-b - 16k^2r^2}{4k^2}, \qquad b = \frac{2c^2}{9d}. \tag{37}$$

Substituting (37) (case 1) in (35), we obtain solutions of (33) as follows:

$$\varphi_1(x,t) = \frac{3b}{2c}\left(-1 \pm \tanh\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right)\right).$$

- Case 2:

$$a_0 = -\frac{3b}{2c}, \qquad a_1 = 0, \qquad b_1 = \pm\frac{3\sqrt{2}b}{2c}; \qquad a = \frac{b - 8k^2r^2}{2k^2}, \qquad b = \frac{2c^2}{9d}. \tag{38}$$

Now, substituting (38) (case 2) in (35), we obtain solutions of (33) as follows:

$$\varphi_2(x,t) = \frac{3b}{2c}\left(-1 \pm \sqrt{2}\operatorname{sech}\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right)\right).$$

- Case 3

$$a_0 = -\frac{3b}{2c}, \qquad a_1 = \frac{3b}{2c}, \qquad b_1 = \pm\frac{3ib}{2c}; \qquad a = \frac{-b - 4k^2r^2}{k^2}, \qquad b = \frac{2c^2}{9d}. \tag{39}$$

Furthermore, substituting (39) (case 3) in (35), we obtain solutions of (33) as follows:

$$\varphi_3(x,t) = \frac{3b}{2c}\left(-1 + \tanh\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right) \pm i\operatorname{sech}\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right)\right).$$

- Case 4:

$$a_0 = \frac{-2b + ak^2 + 4k^2r^2}{c}, \qquad a_1 = 0, \qquad b_1 = \pm\frac{2k\sqrt{a+4r^2}(-2b + ak^2 + 4k^2r^2)}{c\sqrt{b}};$$
$$a = -4r^2, \qquad b = \frac{c^2}{4d}. \tag{40}$$

Finally, substituting (40) (case 4) in (35), we obtain solutions of (33) as follows:

$$\varphi_4(x,t) = \frac{k^2(a+4r^2) - 2b}{c}\left(1 \pm \frac{2k\sqrt{a+4r^2}}{\sqrt{b}}\operatorname{sech}\left(k\left(\frac{x^\alpha}{\alpha} - 2r\frac{t^\alpha}{\alpha}\right)\right)\right).$$

5 Conclusion

In this paper, the sine-Gordon expansion method has been successfully used to obtain exact solutions of the space-time fractional Eckhaus equation and the space-time fractional generalized reaction Duffing model. For this method, by means of balance equations, we obtained exact solutions of the studied class nonlinear FDEs. These solutions can be useful to describe some physical phenomena. The results show that the sine-Gordon expansion method is accurate and effective. Mathematica has been used for computations and programming in this paper.

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Authors' contributions

The authors read and approved the final version of the manuscript.

Author details

[1]Department of Mathematics, Faculty of Basic Sciences, Bozorgmehr University of Qaenat, Qaenat, Iran. [2]Department of Mathematical Sciences, University of South Africa, Pretoria, South Africa.

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