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Persistence of global well-posedness for the 2D Boussinesq equations with fractional dissipation

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Abstract

In this paper, we study the IBVP for the 2D Boussinesq equations with fractional dissipation in the subcritical case, and prove the persistence of global well-posedness of strong solutions. Moreover, we also prove the long time decay of the solutions, and investigate the existence of the solutions in Sobolev spaces $W^{2,p}(R^2) \times W^{1,p}(R^2)$ for some $p > 2$.

Keywords: Boussinesq equations; Fractional dissipation; Global well-posedness

1 Introduction

In this paper, we study the 2D Boussinesq equations with fractional dissipation. The model reads

$$\begin{aligned}
&u_t + \nu \Lambda^{2\alpha} u + u \cdot \nabla u + \nabla P = \theta e_2, \\
&\operatorname{div} u = 0, \\
&\theta_t + \kappa \Lambda^{2\beta} \theta + u \cdot \nabla \theta = 0, \\
&u(x,0) = u_0(x), \qquad \theta(x,0) = \theta_0(x), \quad x \in \Omega, \\
&u(x,t) = 0, \qquad \theta(x,t) = 0, \quad x \in \partial\Omega,
\end{aligned} \tag{1}$$

where $u = (u_1, u_2)$ is the velocity vector field, $u_i = u_i(x,t)$ $(i = 1,2), (x,t) \in R^2 \times R_+$, $\theta(x,t)$ and $P(x,t)$ denote the scalar temperature and pressure of the fluid, respectively. The constants $\nu \geq 0$ and $\kappa \geq 0$ denote the viscosity and thermal diffusivity; $e_2 = (0,1)$ is the unit vector in the vertical direction, and the unknown function θe_2 is the buoyancy force. For the sake of simplicity, we denote $\Lambda := \sqrt{-\Delta}$, the square root of the negative Laplacian, and obviously $\widehat{\Lambda f}(k) = |k|\hat{f}(k)$, where $k = (k_1, k_2)$ is a tuple consisting two integers, $|k| = \sqrt{k_1^2 + k_2^2}$ and the Fourier transform $\hat{f}$ of a tempered distribution $f(x)$ on Ω is defined as

$$\hat{f}(k) = \frac{1}{(2\pi)^2} \int_\Omega f(x) e^{-ik\cdot x}\, dx. \tag{2}$$

More generally, we will define the fractional Laplacian $\Lambda^s f$ for $s \in R$ with the Fourier series

$$\Lambda^s f := \sum_{k\in Z^2} |k|^s \hat{f}(k) e^{ik\cdot x}. \tag{3}$$

As suggested by Jiu, Miao, Wu and Zhang in [33], we classify the parameters α and β into three categories:

(1) the subcritical case, $\alpha + \beta > \frac{1}{2}$;
(2) the critical case, $\alpha + \beta = \frac{1}{2}$;
(3) the supercritical case, $\alpha + \beta < \frac{1}{2}$.

When $\alpha = \beta = 1$, the Boussinesq equations (1) reduce to the standard Boussinesq equation. So far, there has been a lot of literature about the mathematical theory of the standard Boussinesq equation. In the cases when $\nu, \kappa > 0$, $\nu > 0$ and $\kappa = 0$, as well as $\kappa > 0$ and $\nu = 0$, the global regularity has been studied by many authors (see, e.g., [1, 6, 8, 11, 16, 24, 26–28, 32, 40, 41, 45, 65, 86, 87]). However, in the case of $\nu = \kappa = 0$, we only have the local well-posedness theory (see, e.g., [12, 13, 23]), the global regularity or singularity question is a rather challenging problem in mathematical fluid mechanics. Recently, the 2D incompressible Boussinesq equations with temperature-dependence or anisotropy dissipation have attracted considerable attention. In the case of temperature-dependent dissipation, the global-in-time regularity is well-known (see, e.g., [4–6, 29, 30, 43, 47, 48, 58, 60]). In the case of anisotropy dissipation, many authors have proved the global well-posedness (see, e.g., [2, 3, 9, 17, 42, 44, 61, 80]). For a detailed review on interesting results, we refer the reader to [52, 57].

Our main focus of the research on the 2D Boussinesq equation has been on the global regularity issue when only fractional dissipation is present. Using the Fourier localization method, Fang, Qian, and Zhang [19] obtained the local and global well-posedness and gave some blowup criteria with the velocity or temperature. Hmidi, Keraani, and Rousset [25] proved the global well-posedness results. Jia, Peng, and Li [31] proved that the generalized 2D Boussinesq equation has a global and unique solution. Jiu, Miao, Wu, and Zhang [33, 34] aimed at the global regularity. Jiu, Wu, and Yang [35] studied the solutions in the periodic box. KC, Regmi, Tao, and Wu [38, 39] studied the global (in time) regularity problem. Miao and Xue [49] proved the global well-posedness results for rough initial data. Stefanov and Wu [56] solved the global regularity problem. Wu and Xu [63] were concerned with the global well-posedness and inviscid limits of several systems of Boussinesq equations. Using energy methods, the Fourier localization technique, and Bony's paraproduct decomposition, Xiang and Yan [64] showed the global existence of the classical solutions. Xu [66] has proved the global existence, uniqueness and regularity of the solution. Xu and Xue [67] considered the Yudovich-type solution and gave a refined blowup criterion in the supercritical case. Yang, Jiu, and Wu [70] examined the global regularity issue and established the global well-posedness. Ye and Xu [83] established the global regularity of the smooth solutions, and in [84] they proved the global regularity of the smooth solutions.

There are many papers dealing with the fractional differential equation [10, 14, 20, 22, 50, 53–55, 59, 69, 71, 74–79, 81, 82, 85]. For a recent review of the fractional calculus operators, we refer the reader to [72]. In hydrodynamics, Boussinesq equation is a low-dimensional model of fluid dynamics, which plays a very important role in the study of Raleigh–Bernard convection. Boussinesq equation has many applications in modeling fluids and geophysical fluids [15, 21, 51, 70, 73].

The following is the first main result of this paper, which asserts the global well-posedness of the 2D Boussinesq equations (1).

Theorem 1 *Let $\nu > 0, \kappa > 0$, $\alpha, \beta \in (\frac{2}{3}, 1)$. Assume that $(u_0, \theta_0) \in H^{1+s}(R^2) \times H^{1+s}(R^2)$, $s \in (0,1)$. Then there exists a unique global solution $(u(t), \theta(t))$ of Boussinesq equations (1) such that, for any $T > 0$,*

$$u(t) \in C([0,T]; H^{1+s}(R^2)) \cap L^2([0,T]; H^{1+s+\alpha}(R^2)), \tag{4}$$

$$\theta(t) \in C([0,T]; H^{1+s}(R^2)) \cap L^2([0,T]; H^{1+s+\beta}(R^2)). \tag{5}$$

Moreover, there exist positive constants λ and C independent of t and such that

$$\|\nabla\theta(t)\|^2 \le C, \qquad \|\nabla u(t)\|^2 \le C. \tag{6}$$

And in the case when $\min\{\nu\lambda^{2\alpha}, \kappa\lambda^{2\beta}\} > \frac{1}{2}$, one has

$$\|\Lambda^{1+s}\theta(t)\|^2 \le C, \qquad \|\Lambda^{1+s}u(t)\|^2 \le C. \tag{7}$$

Inspired by the work of [62, 68], the second main result of this paper asserts the existence of the solutions in Sobolev spaces $W^{2,p}(R^2) \times W^{1,p}(R^2)$ for some $p > 2$.

Theorem 2 *Let $\nu > 0, \kappa > 0$, $\alpha \ge \frac{1}{2} + \frac{n}{4}$ (we consider $n = 2$), $\beta \in (0,1)$. For some $p \ge 2$, assume that $(u_0, \theta_0) \in W^{2,p}(R^2) \times W^{1,p}(R^2)$, with $\operatorname{div} u_0 = 0$. Then there exists a global solution $(u(t), \theta(t))$ of Boussinesq equations (1) such that, for any $T > 0$,*

$$(u(t), \theta(t)) \in C([0,T], W^{2,p}(R^2)) \times ([0,T], W^{1,p}(R^2)). \tag{8}$$

Remark 1 The same result holds for the case $n \ge 3$. The persistence of global well-posedness should be true in Sobolev spaces, which is left to a future work.

Remark 2 In the case $\kappa = 0$, our guess is that Theorems 1–2 remain true.

2 Preliminaries

In this section, we first introduce Kato–Ponce inequality from [37] (see also [28, 36]) which is important for the proof of Theorem 1, and give a positive inequality from [46] (see also [40, 66]) and Brezis–Wainger inequality from [7] (see also [18]), which are important for the proof of Theorem 2.

Lemma 1 ([37]) *Suppose that $f, g \in C_c^\infty(\Omega)$. Let $s > 0$ and $1 < r \le p_1, p_2, q_1, q_2 \le +\infty$ be such that $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$ with the restriction $p_1, q_2 \ne +\infty$. Then*

$$\|\Lambda^s(fg)\|_{L^r} \le C(\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{q_1}} \|\Lambda^s g\|_{L^{q_2}}), \tag{9}$$

where $C > 0$ is a constant.

Lemma 2 ([46]) *Suppose that $u \in L^p(R^n)$ is such that $\Lambda^\alpha u \in L^p(R^n)$. Let $0 \le m \le 2$. For all $p > 1$, one has*

$$\frac{4(p-1)}{p^2}\int_{R^n}\left(\Lambda^{\frac{\alpha}{2}}|u|^{\frac{p}{2}}\right)^2 dx \le \int_{R^n}\Lambda^\alpha u \cdot u|u|^{p-2}\,dx. \tag{10}$$

Observe that, if $\alpha = 2$, integrating (10) by parts, we obtain

$$\int_{R^n}\left(\Lambda|u|^{\frac{p}{2}}\right)^2 dx = \int_{R^n}\Lambda^2 u \cdot u|u|^{p-2}\,dx. \tag{11}$$

Lemma 2 is well-known in the theory of sub-Markovian operators, its statement and the proof are given in [46].

Lemma 3 ([7]) *Suppose that $u \in L^2(R^2) \cap W^{1,p}(R^2)$. For all $p > 1$, one has*

$$\|u\|_{L^\infty} \le C(1 + \|\nabla u\|_{L^2}\left(1 + \log^+\left(\|\nabla u\|_{L^p}\right)\right)^{\frac{1}{2}} + C\|u\|_{L^2}, \tag{12}$$

where $C > 0$ is a constant.

3 Proof of Theorem 1

The goal of this section is to prove Theorem 1. The proof is divided into two main parts showing global existence and uniqueness.

3.1 Global existence

The proof of global existence is based on several steps of careful energy estimates. First, we start with estimates of $\|u(t)\|$ and $\|\theta(t)\|$.

Lemma 4 *Under the assumptions of Theorem 1, one has*

$$\left\|u(t)\right\| \in C\left(0,+\infty;L^2(\Omega)\right) \cap L^2\left(0,+\infty;H^\alpha(\Omega)\right), \tag{13}$$

$$\left\|\theta(t)\right\| \in C\left(0,+\infty;L^2(\Omega)\right) \cap L^2\left(0,+\infty;H^\beta(\Omega)\right). \tag{14}$$

Moreover, there exist positive constant λ independents of t and such that

$$\left\|\theta(t)\right\|^2 \le \|\theta_0\|^2 e^{-2\kappa\lambda^{2\beta}t}, \tag{15}$$

as well as

$$\begin{aligned}&\left\|u(t)\right\|^2 \le e^{-\nu\lambda^{2\alpha}t}\|u_0\|^2 + \frac{1}{\nu\lambda^{2\alpha}}\left|\frac{e^{-\nu\lambda^{2\alpha}t} - e^{-\kappa\lambda^{2\beta}t}}{\nu\lambda^{2\alpha} - \kappa\lambda^{2\beta}}\right|\|\theta_0\|^2,\\ &\nu\lambda^{2\alpha} \ne \kappa\lambda^{2\beta},\end{aligned} \tag{16}$$

$$\begin{aligned}&\left\|u(t)\right\|^2 \le e^{-\nu\lambda^{2\alpha}t}\|u_0\|^2 + \frac{t}{\nu\lambda^{2\alpha}}e^{-\nu\lambda^{2\alpha}t}\|\theta_0\|^2,\\ &\nu\lambda^{2\alpha} = \kappa\lambda^{2\beta}.\end{aligned} \tag{17}$$

Proof Taking L^2-inner product of $(1)_3$ with θ, and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \kappa\|\Lambda^\beta\theta\|^2 = 0. \tag{18}$$

Since $\theta(x,t)|_{\partial\Omega} = 0$, using Poincaré inequality, we find

$$\frac{d}{dt}\|\theta\|^2 + 2\kappa\lambda^{2\beta}\|\theta\|^2 = 0, \tag{19}$$

where λ is the first eigenvalue of Λ. Then, we can obtain that, for all $t \in [0,+\infty)$,

$$\|\theta(t)\|^2 \leq \|\theta_0\|e^{-2\kappa\lambda^{2\beta}t}. \tag{20}$$

Integrating (18) in time gives

$$\|\theta(t)\|^2 + 2\kappa\int_0^t \|\Lambda^\beta\theta(\tau)\|^2\,d\tau \leq \|\theta_0\|^2. \tag{21}$$

Similarly, we can also deduce a uniform L^p estimate of θ, for all $p \in [2,+\infty)$,

$$\|\theta(t)\|_{L^p} \leq e^{-\frac{\kappa\lambda^{2\beta}t}{p}}\|\theta_0\|_{L^p}. \tag{22}$$

Multiplying $(1)_1$ by u and integrating the resulting equation by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|\Lambda^\alpha u\|^2 &\leq \int_\Omega \theta e_2\cdot u\,dx \\ &\leq \int_\Omega |\Lambda^{-\alpha}\theta|\cdot|\Lambda^\alpha u|\,dx \\ &\leq \frac{1}{2\nu}\|\Lambda^{-\alpha}\theta\|^2 + \frac{\nu}{2}\|\Lambda^\alpha u\|^2.\end{aligned} \tag{23}$$

Hence,

$$\frac{d}{dt}\|u\|^2 + \nu\|\Lambda^\alpha u\|^2 \leq \frac{1}{\nu}\|\Lambda^{-\alpha}\theta\|^2. \tag{24}$$

By Poincaré inequality, we have

$$\frac{d}{dt}\|u\|^2 + \nu\lambda^{2\alpha}\|u\|^2 \leq \frac{1}{\nu\lambda^{2\alpha}}\|\theta\|^2. \tag{25}$$

Integrating in time and using (20), we have, in the case when $\nu\lambda^{2\alpha} \neq \kappa\lambda^{2\beta}$,

$$\|u(t)\|^2 \leq e^{-\nu\lambda^{2\alpha}t}\|u_0\|^2 + \frac{1}{\nu\lambda^{2\alpha}}\left|\frac{e^{-\nu\lambda^{2\alpha}t} - e^{-\kappa\lambda^{2\beta}t}}{\nu\lambda^{2\alpha} - \kappa\lambda^{2\beta}}\right|\|\theta_0\|^2, \tag{26}$$

and in the case when $\nu\lambda^{2\alpha} = \kappa\lambda^{2\beta}$,

$$\|u(t)\|^2 \leq e^{-\nu\lambda^{2\alpha}t}\|u_0\|^2 + \frac{t}{\nu\lambda^{2\alpha}}e^{-\nu\lambda^{2\alpha}t}\|\theta_0\|^2. \tag{27}$$

After integration (24) in time and by (20), we obtain

$$\|u(t)\|^2 + \nu \int_0^t \|\Lambda^\alpha u(\tau)\|^2 \, d\tau \le \|u_0\|^2 + \frac{1}{2\nu\kappa\lambda^{2(\alpha+\beta)}} \|\theta_0\|^2, \tag{28}$$

completing the proof. □

In the next lemma, we shall obtain estimates of $\|\nabla u(t)\|$ and $\|\nabla\theta(t)\|$.

Lemma 5 *Under the assumptions of Theorem 1, one has*

$$\|u(t)\| \in C\big(0,+\infty;H^1(\Omega)\big) \cap L^2\big(0,+\infty;H^{1+\alpha}(\Omega)\big), \tag{29}$$

$$\|\theta(t)\| \in C\big(0,+\infty;H^1(\Omega)\big) \cap L^2\big(0,+\infty;H^{1+\beta}(\Omega)\big). \tag{30}$$

Moreover, there exist positive constants λ *and* C *independent of* t *and such that*

$$\|\nabla u(t)\| \le C, \qquad \|\nabla\theta(t)\| \le C. \tag{31}$$

Proof In order to complete the proof, we need to use vorticity formulation. Taking the curl of $(1)_1$, we have

$$\omega_t + \nu\Lambda^{2\alpha}\omega + u\cdot\nabla\omega = \theta_{x_1}, \tag{32}$$

where $\omega = \partial_{x_1}u_2 - \partial_{x_2}u_1$, with the Dirichlet boundary condition

$$\omega = 0, \quad \text{on } \partial\Omega.$$

Taking L^2-inner product of (32) with ω, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\omega\|^2 + \nu\|\Lambda^\alpha\omega\|^2 &= \int_\Omega \theta_{x_1}\cdot\omega \, dx \\ &\le \left| \int_\Omega \Lambda^{1-\alpha}\theta\cdot\Lambda^\alpha\omega \, dx \right| \\ &\le \frac{1}{2\nu}\|\Lambda^{1-\alpha}\theta\|^2 + \frac{\nu}{2}\|\Lambda^\alpha\omega\|^2, \end{aligned} \tag{33}$$

from which it follows that

$$\frac{d}{dt}\|\omega\|^2 + \nu\|\Lambda^\alpha\omega\|^2 \le \frac{1}{\nu}\|\Lambda^{1-\alpha}\theta\|^2. \tag{34}$$

Then Poincaré inequality implies

$$\frac{d}{dt}\|\omega\|^2 + \nu\lambda^{2\alpha}\|\omega\|^2 \le \frac{1}{\nu}\|\Lambda^{1-\alpha}\theta\|^2. \tag{35}$$

Since $\alpha,\beta \in (\frac{2}{3},1)$, we know that $1-\alpha < \beta$, and so, using the interpolation inequality and by (21), we have

$$\int_0^t \|\Lambda^{1-\alpha}\theta(\tau)\|^2 \, d\tau \le \int_0^t \|\Lambda^\beta\theta(\tau)\|^2 \, d\tau \le C. \tag{36}$$

Applying a variant of the uniform Gronwall lemma, and by the Biot–Savart law and (36), we have a uniform estimate $\|u(t)\|_{H^1}$ for all $t \in [0,+\infty)$. Furthermore, integrating (34) in time, we can get, for all $t \in [0,+\infty)$,

$$\|\omega(t)\|^2 + \nu\int_0^t \|\Lambda^\alpha \omega(\tau)\|^2\,d\tau \le \|\omega_0\|^2 + \frac{1}{\nu}\int_0^t \|\Lambda^{1-\alpha}\theta(\tau)\|^2\,d\tau. \tag{37}$$

As an immediate consequence, and by Sobolev embedding theorem, we have a uniform L^p estimate for u, that is, for all $1 < p < +\infty$,

$$\|u\|_{L^p} \le C(p) \tag{38}$$

and

$$\int_0^t \|\Lambda^{1+\alpha} u\|\,d\tau \le C(\|\nabla u_0\|, \|\theta_0\|), \tag{39}$$

where the constant $C(p) > 0$ only depends on p and $C(\|\nabla u_0\|, \|\theta_0\|)$ only depends the initial data $\|\nabla u_0\|$ and $\|\theta_0\|$.

Taking L^2-inner product of $(1)_3$ with $\Delta\theta$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 + \kappa\|\Lambda^{1+\beta}\theta\|^2 &= -\int_\Omega (u\cdot\nabla\theta)\cdot\Delta\theta\,dx \\ &\le \int_\Omega |\Lambda^{1-\beta}(u\cdot\nabla\theta)||\Lambda^{1+\beta}\theta|\,dx.\end{aligned} \tag{40}$$

Since u is divergence-free, $u\cdot\nabla\theta = \nabla\cdot(u\theta)$, and so, using Cauchy–Schwarz inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 + \kappa\|\Lambda^{1+\beta}\theta\|^2 &\le \int_\Omega |\Lambda^{2-\beta}(u\theta)||\Lambda^{1+\beta}\theta|\,dx \\ &\le \frac{1}{\kappa}\|\Lambda^{2-\beta}(u\theta)\|^2 + \frac{\kappa}{4}\|\Lambda^{1+\beta}\theta\|^2.\end{aligned} \tag{41}$$

Using Lemma 1, and by (22) and (38), we have

$$\begin{aligned}\|\Lambda^{2-\beta}(u\theta)\|^2 &\le C\|\Lambda^{2-\beta}u\|_{L^4}^2\|\theta\|_{L^4}^2 + C\|u\|_{L^6}^2\|\Lambda^{2-\beta}\theta\|_{L^3}^2 \\ &\le C\|\Lambda^{2-\beta}u\|_{L^4}^2 + C\|\Lambda^{2-\beta}\theta\|_{L^3}^2,\end{aligned} \tag{42}$$

so by Sobolev embedding theorem, and applying Gagliardo–Nirenberg and Young inequalities, we can obtain

$$\begin{aligned}&\|\Lambda^{2-\beta}(u\theta)\|^2 \\ &\quad\le C\|\Lambda^{1+\alpha}u\|^2 + C\|\Lambda^{-\alpha}\theta\|^{2(3\alpha+3\beta-4)/3(1+2\alpha)}\|\Lambda^{1+\alpha}\theta\|^{2(7+3\alpha-3\beta)/3(1+2\alpha)} \\ &\quad\le C\|\Lambda^{1+\alpha}u\|^2 + a_1C\|\Lambda^{-\alpha}\theta\|^2 + \frac{\kappa}{4}\|\Lambda^{1+\alpha}\theta\|^2,\end{aligned} \tag{43}$$

where $a_1 = \frac{3\alpha+3\beta-4}{3(1+2\alpha)}(\frac{3\kappa(1+2\alpha)}{4(7+3\alpha-3\beta)})^{(7+3\alpha-3\beta)/(4-3\alpha-3\beta)}$. Inserting (43) into (41), we can obtain that

$$\frac{d}{dt}\|\nabla\theta\|^2 + \kappa\|\Lambda^{1+\beta}\theta\|^2 \le C(\|\Lambda^{1+\alpha}u\|^2 + \|\Lambda^{-\alpha}\theta\|^2). \tag{44}$$

Then Poincaré inequality implies

$$\frac{d}{dt}\|\nabla\theta\|^2 + \kappa\lambda^{2\beta}\|\nabla\theta\|^2 \le C\left(\|\Lambda^{1+\alpha}u\|^2 + \frac{1}{\lambda^{2\alpha}}\|\theta\|^2\right). \tag{45}$$

By (20) and (39), we know that

$$\int_0^t (\|\Lambda^{1+\alpha}u\|^2 + \|\theta\|^2)\,d\tau \le C. \tag{46}$$

Applying a variant of the uniform Gronwall lemma again and (46), we have a uniform estimate of $\|\nabla\theta(t)\|$ for all $t \in [0,+\infty)$. Integrating over $[0,t]$, we obtain, for all $t \in [0,+\infty)$,

$$\|\nabla\theta(t)\|^2 + \kappa\int_0^t \|\Lambda^{1+\beta}\theta(\tau)\|^2\,d\tau \le \|\nabla\theta_0\|^2 + C, \tag{47}$$

where C only depends on p and the initial data. □

Now let us focus on the persistence in $H^{1+s}(R^2) \times H^{1+s}(R^2)$, $s \in (0,1)$.

Lemma 6 *Under the assumptions of Theorem 1, one has*

$$\|u(t)\| \in C(0,+\infty; H^{1+s}(\Omega)) \cap L^2(0,+\infty; H^{1+s+\alpha}(\Omega)), \tag{48}$$

$$\|\theta(t)\| \in C(0,+\infty; H^{1+s}(\Omega)) \cap L^2(0,+\infty; H^{1+s+\beta}(\Omega)). \tag{49}$$

Moreover, in the case when $\min\{\nu\lambda^{2\alpha}, \kappa\lambda^{2\beta}\} > \frac{1}{2}$, *there exist positive constants* λ *and* C *independent of* t, *and it holds that*

$$\|\Lambda^{1+s}\theta(t)\|^2 \le C, \qquad \|\Lambda^{1+s}u(t)\|^2 \le C. \tag{50}$$

Proof Taking L^2-inner product of $(1)_3$ with $\Lambda^{2+2s}\theta$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{1+s}\theta\|^2 + \kappa\|\Lambda^{1+s+\beta}\theta\|^2 = -\int_\Omega (u\cdot\nabla\theta)\cdot\Lambda^{2+2s}\theta\,dx. \tag{51}$$

Since u is divergence-free, $u\cdot\nabla\theta = \nabla\cdot(u\theta)$, using Lemma 1 and (22) with (38), we obtain

$$\begin{aligned}
&-\int_\Omega (u\cdot\nabla\theta)\cdot\Lambda^{2+2s}\theta\,dx \\
&\quad\le \left|\int_\Omega \Lambda^{2+s-\beta}(u\theta)\cdot\Lambda^{1+s+\beta}\theta\,dx\right| \\
&\quad\le \|\Lambda^{2+s-\beta}(u\theta)\|\,\|\Lambda^{1+s+\beta}\theta\| \\
&\quad\le C(\|\Lambda^{2+s-\beta}u\|_{L^3}\|\theta\|_{L^6} + \|u\|_{L^6}\|\Lambda^{2+s-\beta}\theta\|_{L^3})\|\Lambda^{1+s+\beta}\theta\|
\end{aligned}$$

$$\leq \frac{C}{\kappa}\left(\left\|\Lambda^{2+s-\beta}u\right\|_{L^3}^2 + \left\|\Lambda^{2+s-\beta}\theta\right\|_{L^3}^2\right) + \frac{\kappa}{4}\left\|\Lambda^{1+s+\beta}\theta\right\|^2. \tag{52}$$

Applying Gagliardo–Nirenberg and Young inequalities, we can get

$$\begin{aligned}\left\|\Lambda^{2+s-\beta}u\right\|_{L^3}^2 &\leq C\left\|\Lambda^{-\beta}u\right\|^{2(3\alpha+3\beta-4)/3(1+s+\alpha+\beta)}\left\|\Lambda^{1+s+\alpha}u\right\|^{2(7+3s)/3(1+s+\alpha+\beta)} \\ &\leq a_2C\left\|\Lambda^{-\beta}u\right\|^2 + \frac{\nu}{4}\left\|\Lambda^{1+s+\alpha}u\right\|^2\end{aligned} \tag{53}$$

and

$$\begin{aligned}\left\|\Lambda^{2+s-\beta}\theta\right\|_{L^3}^2 &\leq C\left\|\Lambda^{-\beta}\theta\right\|^{4(3\beta-2)/3(1+s+2\beta)}\left\|\Lambda^{1+s+\beta}\theta\right\|^{2(7+3s)/3(1+s+2\beta)} \\ &\leq a_3C\left\|\Lambda^{-\beta}\theta\right\|^2 + \frac{\kappa}{4}\left\|\Lambda^{1+s+\beta}\theta\right\|^2,\end{aligned} \tag{54}$$

where $a_2 = \frac{3\alpha+3\beta-4}{3(1+s+\alpha+\beta)}(\frac{3\nu(1+s+\alpha+\beta)}{4(7+3s)})^{(7+3s)/(4-3\alpha-3\beta)}$ and $a_3 = \frac{2(3\beta-2)}{3(1+s+2\beta)}(\frac{3\kappa(1+s+2\beta)}{4(7+3s)})^{(7+3s)/2(2-3\beta)}$. Inserting (52)–(54) into (51), we arrive at

$$\begin{aligned}&\frac{d}{dt}\left\|\Lambda^{1+s}\theta\right\|^2 + \kappa\left\|\Lambda^{1+s+\beta}\theta\right\|^2 \\ &\quad\leq \frac{\nu}{2}\left\|\Lambda^{1+s+\alpha}u\right\|^2 + C\left\|\Lambda^{-\beta}u\right\|^2 + C\left\|\Lambda^{-\beta}\theta\right\|^2.\end{aligned} \tag{55}$$

Applying the operator Λ^{1+s} to $(1)_1$, and taking the scalar product of both sides with $\Lambda^{1+s}u$, and then integrating the result by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\Lambda^{1+s}u\right\|^2 + \nu\left\|\Lambda^{1+s+\alpha}u\right\|^2 \\ &\quad= -\int_\Omega \Lambda^{1+s}(u_j\partial_j u_k)\Lambda^{1+s}u_k\,dx + \int_\Omega \Lambda^{1+s}(\theta e_2)\Lambda^{1+s}u\,dx.\end{aligned} \tag{56}$$

Using Lemma 1 and applying fractional embedding theorems together with Young inequality again, we obtain

$$\begin{aligned}&-\int_\Omega \Lambda^{1+s}(u_j\partial_j u_k)\Lambda^{1+s}u_k\,dx \\ &\quad\leq \left|\int_\Omega \Lambda^{1+s-\alpha}(u_j\partial_j u_k)\Lambda^{1+s+\alpha}u_k\,dx\right| \\ &\quad\leq C\left(\left\|\Lambda^{1+s-\alpha}u\right\|_{L^3}\|\nabla u\|_{L^6} + \|u\|_{L^6}\left\|\Lambda^{2+s-\alpha}u\right\|_{L^3}\right)\left\|\Lambda^{1+s+\alpha}u\right\| \\ &\quad\leq \frac{\nu}{4}\left\|\Lambda^{1+s+\alpha}u\right\|^2 + \frac{C}{\nu}\left(\left\|\Lambda^{1+s-\alpha}u\right\|_{L^4}^2\|\nabla u\|_{L^4}^2 + \left\|\Lambda^{2+s-\alpha}u\right\|_{L^3}^2\right).\end{aligned} \tag{57}$$

Applying Gagliardo–Nirenberg and Young inequalities, we can get

$$\begin{aligned}\left\|\Lambda^{1+s-\alpha}u\right\|_{L^3}^2\|\nabla u\|_{L^6}^2 &\leq \left\|\Lambda^{-\alpha}u\right\|^{4(3\alpha-2)/3(1+s+2\alpha)}\left\|\Lambda^{1+s+\alpha}u\right\|^{2(7+3s)/3(1+s+2\alpha)} \\ &\quad\times \left\|\Lambda^{-\alpha}u\right\|^{2(1+3\alpha)/3(1+2\alpha)}\left\|\Lambda^{1+\alpha}u\right\|^{2(2+3\alpha)/3(1+2\alpha)} \\ &\leq a_4C\left\|\Lambda^{-\alpha}u\right\|^{b_1}\left\|\Lambda^{1+\alpha}u\right\|^{b_2} + \frac{\nu}{8}\left\|\Lambda^{1+s+\alpha}u\right\|^2\end{aligned} \tag{58}$$

and

$$\begin{aligned}\|\Lambda^{2+s-\alpha}u\|_{L^3}^2 &\le C\|\Lambda^{-\alpha}u\|^{4(3\alpha-2)/3(1+s+2\alpha)}\|\Lambda^{1+s+\alpha}u\|^{2(7+3s)/3(1+s+2\alpha)} \\ &\le a_5C\|\Lambda^{-\alpha}u\|^2+\frac{\nu}{8}\|\Lambda^{1+s+\alpha}u\|^2,\end{aligned} \tag{59}$$

where $a_4=a_5=\frac{2(3\alpha-2)}{3(1+s+2\alpha)}(\frac{3\nu(1+s+2\alpha)}{8(7+3s)})^{(7+3s)/2(2-3\alpha)}$, $b_1=2+\frac{(1+3\alpha)(1+s+2\alpha)}{(1+2\alpha)(3\alpha-2)}$, and $b_2=\frac{(2+3\alpha)(1+s+2\alpha)}{(1+2\alpha)(3\alpha-2)}$. Using Hölder and Cauchy–Schwarz inequalities, we can get

$$\int_\Omega \Lambda^{1+s}(\theta e_2)\Lambda^{1+s}u\,dx\le\frac{1}{2}\|\Lambda^{1+s}\theta\|^2+\frac{1}{2}\|\Lambda^{1+s}u\|^2. \tag{60}$$

Inserting (58)–(60) into (56), we arrive at

$$\begin{aligned}&\frac{d}{dt}\|\Lambda^{1+s}u\|^2+\frac{3\nu}{2}\|\Lambda^{1+s+\alpha}u\|^2 \\ &\quad\le\frac{1}{2}(\|\Lambda^{1+s}\theta\|^2+\|\Lambda^{1+s}u\|^2)+C(\|\Lambda^{-\alpha}u\|^2+\|\Lambda^{-\alpha}u\|^{b_1}\|\Lambda^{1+\alpha}u\|^{b_2}).\end{aligned} \tag{61}$$

Summing up (55) and (61), we obtain that

$$\begin{aligned}&\frac{d}{dt}(\|\Lambda^{1+s}u\|^2+\|\Lambda^{1+s}\theta\|^2)+\nu\|\Lambda^{1+s+\alpha}u\|^2+\kappa\|\Lambda^{1+s+\beta}\theta\|^2 \\ &\quad\le\frac{1}{2}(\|\Lambda^{1+s}\theta\|^2+\|\Lambda^{1+s}u\|^2)+C(\|\Lambda^{-\alpha}u\|^2+\|\Lambda^{-\beta}u\|^2 \\ &\qquad+\|\Lambda^{-\beta}\theta\|^2+\|\Lambda^{-\alpha}u\|^{b_1}\|\Lambda^{1+\alpha}u\|^{b_2}).\end{aligned} \tag{62}$$

Then Poincaré inequality implies

$$\begin{aligned}&\frac{d}{dt}(\|\Lambda^{1+s}u\|^2+\|\Lambda^{1+s}\theta\|^2)+\nu\lambda^{2\alpha}\|\Lambda^{1+s}u\|^2+\kappa\lambda^{2\beta}\|\Lambda^{1+s}\theta\|^2 \\ &\quad\le\frac{1}{2}(\|\Lambda^{1+s}\theta\|^2+\|\Lambda^{1+s}u\|^2) \\ &\qquad+C\left(\frac{1}{\lambda^{2\alpha}}\|u\|^2+\frac{1}{\lambda^{2\beta}}\|u\|^2+\frac{1}{\lambda^{2\beta}}\|\theta\|^2+\frac{1}{\lambda^{b_1}}\|u\|^{b_1}\|\Lambda^{1+\alpha}u\|^{b_2}\right).\end{aligned} \tag{63}$$

Hence

$$\frac{d}{dt}X(t)+\left(c_1-\frac{1}{2}\right)X(t)\le C(\|u\|^2+\|\theta\|^2+\|u\|^{b_1}\|\Lambda^{1+\alpha}u\|^{b_2}), \tag{64}$$

where $X(t)=\|\Lambda^{1+s}u\|^2+\|\Lambda^{1+s}\theta\|^2$ and $c_1=\min\{\nu\lambda^{2\alpha},\kappa\lambda^{2\beta}\}$. By (20), (26), (27), and (39), we know that

$$\int_0^t(\|u\|^2+\|\theta\|^2+\|u\|^{b_1}\|\Lambda^{1+\alpha}u\|^{b_2})\,d\tau\le C. \tag{65}$$

Applying a variant of the uniform Gronwall lemma again and (65), in the case $(c_1-\frac{1}{2})>0$, we have uniform estimates of $\|\Lambda^{1+s}u\|^2$ and $\|\Lambda^{1+s}\theta\|^2$, for all $t\in[0,+\infty)$. Integrating (62) over $[0,t]$, we have

$$\begin{aligned}
&\|\Lambda^{1+s}u(t)\|^2+\|\Lambda^{1+s}\theta(t)\|^2+\nu\int_0^t\|\Lambda^{1+s+\alpha}u(\tau)\|^2\,d\tau\\
&\quad+\kappa\int_0^t\|\Lambda^{1+s+\beta}\theta(\tau)\|^2\,d\tau\\
&\quad\le\|\Lambda^{1+s}u_0\|^2+\|\Lambda^{1+s}\theta_0\|^2+C+\frac{1}{2}\int_0^t\bigl(\|\Lambda^{1+s}\theta\|^2+\|\Lambda^{1+s}u\|^2\bigr)\,d\tau.
\end{aligned}\tag{66}$$

Using Gronwall inequality, we find that, for all $t\in[0,T]$,

$$\begin{aligned}
&\|\Lambda^{1+s}u(t)\|^2+\|\Lambda^{1+s}\theta(t)\|^2+\nu\int_0^t\|\Lambda^{1+s+\alpha}u(\tau)\|^2\,d\tau\\
&\quad+\kappa\int_0^t\|\Lambda^{1+s+\beta}\theta(\tau)\|^2\,d\tau\\
&\quad\le e^{\frac{t}{2}}\bigl(\|\Lambda^{1+s}u_0\|^2+\|\Lambda^{1+s}\theta_0\|^2+C\bigr)\\
&\quad\le e^{\frac{T}{2}}\bigl(\|\Lambda^{1+s}u_0\|^2+\|\Lambda^{1+s}\theta_0\|^2+C\bigr)\\
&\quad\le C,
\end{aligned}\tag{67}$$

where $C=C(\|\Lambda^{1+s}u_0\|,\|\Lambda^{1+s}\theta_0\|,\nu,\kappa,s,T)$ is a positive constant. □

3.2 Uniqueness

With the global regularity established in Lemmas 4–6, we are able to prove the uniqueness of the solution.

Lemma 7 *Under the assumptions of Theorem 1, the solution of Boussinesq equations (1) is unique.*

Proof For any fixed $T>0$, suppose there are two solutions (u_1,θ_1,P_1) and (u_2,θ_2,P_2) to Boussinesq equations (1). Setting $\tilde{u}=u_1-u_2$, $\tilde{\theta}=\theta_1-\theta_2$ and $\tilde{P}=P_1-P_2$, we get that $(\tilde{u},\tilde{\theta},\tilde{P})$ satisfies

$$\tilde{u}_t+\nu\Lambda^{2\alpha}\tilde{u}+u_1\cdot\nabla\tilde{u}+\tilde{u}\cdot\nabla u_2+\nabla\tilde{P}=\tilde{\theta}e_2,\qquad e_2=(0,1),\tag{68}$$

$$\operatorname{div}\tilde{u}=0,\tag{69}$$

$$\tilde{\theta}_t+\kappa\Lambda^{2\beta}\tilde{\theta}+u_1\cdot\nabla\tilde{\theta}+\tilde{u}\cdot\nabla\theta_2=0,\tag{70}$$

$$\tilde{u}(x,0)=0,\qquad\tilde{\theta}(x,0)=0.\tag{71}$$

Taking the L^2-inner product of (68) with $\tilde{u}$ and (70) with $\tilde{\theta}$, respectively, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigl(\|\tilde{u}\|^2+\|\tilde{\theta}\|^2\bigr)+\nu\|\nabla\tilde{u}\|^2+\kappa\|\nabla\tilde{\theta}\|^2\\
&\quad\le\int_\Omega\tilde{\theta}e_2\cdot\tilde{u}\,dx-\int_\Omega\tilde{u}\cdot\nabla u_2\cdot\tilde{u}\,dx-\int_\Omega\tilde{u}\cdot\nabla\theta_2\tilde{\theta}\,dx.
\end{aligned}\tag{72}$$

Using Hölder and Cauchy–Schwarz inequalities, a standard calculation gives us the following:

$$\int_\Omega \tilde{\theta} e_2 \cdot \tilde{u}\,dx \le \frac{1}{2}\|\tilde{\theta}\|^2 + \frac{1}{2}\|\tilde{u}\|^2, \tag{73}$$

$$\begin{aligned} -\int_\Omega \tilde{u}\cdot\nabla u_2\cdot\tilde{u}\,dx &\le \left|-\int_\Omega \tilde{u}\cdot\nabla u_2\cdot\tilde{u}\,dx\right| \\ &\le \|\nabla u_2\|\|\tilde{u}\|_{L^4}^2 \\ &\le C\|\nabla u_2\|\|\tilde{u}\|\|\nabla\tilde{u}\| \\ &\le C\|\nabla u_2\|^2\|\tilde{u}\|^2 + \frac{\nu}{4}\|\nabla\tilde{u}\|, \end{aligned} \tag{74}$$

and

$$\begin{aligned} &-\int_\Omega \tilde{u}\cdot\nabla\theta_2\cdot\tilde{\theta}\,dx \\ &\quad\le \left|-\int_\Omega \tilde{u}\cdot\nabla\theta_2\cdot\tilde{\theta}\,dx\right| \\ &\quad\le \|\nabla\theta_2\|\|\tilde{\theta}\|_{L^4}\|\tilde{u}\|_{L^4} \\ &\quad\le C\|\nabla\theta_2\|\|\tilde{\theta}\|^{1/2}\|\nabla\tilde{\theta}\|^{1/2}\|\tilde{u}\|^{1/2}\|\nabla\tilde{u}\|^{1/2} \\ &\quad\le C\big(\|\nabla\theta_2\|\|\tilde{\theta}\|\|\nabla\tilde{\theta}\| + \|\nabla\theta_2\|\|\tilde{u}\|\|\nabla\tilde{u}\|\big) \\ &\quad\le C\|\nabla\theta_2\|^2\|\tilde{\theta}\|^2 + \frac{\nu}{4}\|\nabla\tilde{\theta}\| + C\|\nabla\theta_2\|^2\|\tilde{u}\|^2 + \frac{\kappa}{2}\|\nabla\tilde{u}\|^2. \end{aligned} \tag{75}$$

Inserting (73)–(75) into (72), we obtain

$$\begin{aligned} &\frac{d}{dt}\big(\|\tilde{u}\|^2 + \|\tilde{\theta}\|^2\big) + \nu\|\nabla\tilde{u}\|^2 + \kappa\|\nabla\tilde{\theta}\|^2 \\ &\quad\le C\big(\|\nabla\theta_2\|^2 + \|\nabla u_2\|^2 + 1\big)\big(\|\tilde{\theta}\|^2 + \|\tilde{u}\|^2\big). \end{aligned} \tag{76}$$

Using Gronwall inequality and the estimates for θ_2 and u_2, (76) implies that, for any $t \ge 0$,

$$e^{-CT}\big(\|\tilde{u}\|^2 + \|\tilde{\theta}\|^2\big) \le \big\|\tilde{u}(0)\big\|^2 + \big\|\tilde{\theta}(0)\big\|^2 = 0,$$

i.e., $\tilde{u} = 0, \tilde{\theta} = 0, \theta_1 = \theta_2, u_1 = u_2$. So the solution of Boussinesq equations (1) is unique. □

4 Proof of Theorem 2

The goal of this section is to prove Theorem 2. First of all, we multiply the first equation in (1) with $u|u|^{p-2}$ ($p > 2$) and, integrating it over R^2, have

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|u\|_{L^p}^p + \int_\Omega \Lambda^2 u\cdot u|u|^{p-2}\,dx \\ &\quad= -\int_\Omega (u\cdot\nabla u)\cdot u|u|^{p-2}\,dx - \int_\Omega \nabla P\cdot u|u|^{p-2}\,dx \end{aligned}$$

$$+\int_{\Omega}\theta e_2\cdot u|u|^{p-2}\,dx. \tag{77}$$

Since u is divergence-free, by Lemma 2 and using Hölder inequality, we can get

$$\frac{1}{p}\frac{d}{dt}\|u\|_{L^p}^p+\frac{4(p-2)}{p^2}\big\|\nabla|u|^{\frac{p}{2}}\big\|_{L^2}^2\le\|\nabla P\|_{L^p}\|u\|_{L^p}^{p-1}+\|\theta\|_{L^p}\|u\|_{L^p}^{p-1}. \tag{78}$$

Taking the divergence of the first equation in (1), we can obtain

$$-\Delta P=\operatorname{div}(u\cdot\nabla u)-\partial_{x_2}\theta. \tag{79}$$

Hence

$$\nabla P=\nabla(-\Delta)^{-1}\big(\operatorname{div}(u\cdot\nabla u)-\partial_{x_2}\theta\big). \tag{80}$$

Applying Calderón–Zygmund theorem, we get

$$\begin{aligned}\|\nabla P\|_{L^p}&\le C\big(\|u\cdot\nabla u\|_{L^p}+\|\theta\|_{L^p}\big)\\&\le C\big(\|u\|_{L^\infty}\|\nabla u\|_{L^p}+\|\theta\|_{L^p}\big)\\&\le C\big(\|u\|_{W^{1,p}}\|\nabla u\|_{L^p}+\|\theta\|_{L^p}\big)\\&\le C\big(\|u\|_{L^p}+\|\nabla u\|_{L^p}\big)\|\nabla u\|_{L^p}+C\|\theta\|_{L^p}\\&\le C\big(\|u\|_{L^p}+\|\omega\|_{L^p}\big)\|\omega\|_{L^p}+C\|\theta\|_{L^p}.\end{aligned} \tag{81}$$

Multiplying the third equation in (1) with $\theta|\theta|^{p-2}$ ($p>2$) and integrating it over R^2, we deduce that

$$\frac{1}{p}\frac{d}{dt}\|\theta\|_{L^p}^p+\int_{\Omega}\Lambda^{2\beta}\theta\cdot\theta|\theta|^{p-2}\,dx=0, \tag{82}$$

where we have used the divergence-free condition again. By Lemma 2 and integrating over $[0,t]$, we have, for all $t\in[0,T]$,

$$\|\theta\|_{L^p}^p+\frac{4(p-2)}{p}\int_0^t\big\|\Lambda^{\beta}|\theta|^{\frac{p}{2}}\big\|_{L^2}^2\,d\tau=\|\theta_0\|_{L^p}^p. \tag{83}$$

Combining (78) with (81) and (83) leads to

$$\frac{1}{p}\frac{d}{dt}\|u\|_{L^p}^p+\frac{4(p-2)}{p^2}\big\|\nabla|u|^{\frac{p}{2}}\big\|_{L^2}^2\le C\big(\|u\|_{L^p}\|\omega\|_{L^p}+\|\omega\|_{L^p}^2+1\big)\|u\|_{L^p}^{p-1}. \tag{84}$$

Taking the L^p-inner product of (27) with $\omega|\omega|^{p-2}$ ($p>2$) and integrating it over R^2, we arrive at

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p+\int_{\Omega}\Lambda^{2\alpha}\omega\cdot\omega|\omega|^{p-2}\,dx=\int_{\Omega}\partial_{x_1}\theta\cdot\omega|\omega|^{p-2}\,dx. \tag{85}$$

Using Lemma 2 again, we know that

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p+\frac{4(p-2)}{p^2}\big\|\nabla|\omega|^{\frac{p}{2}}\big\|^2\le\int_{\Omega}\partial_{x_1}\theta\cdot\omega|\omega|^{p-2}\,dx. \tag{86}$$

By Hölder and Young inequalities, and using Lemma 2 with $m = 2$, we have

$$\begin{aligned}
\int_\Omega \partial_{x_1}\theta \cdot \omega|\omega|^{p-2}\,dx &\le (p-1)\left|\int_\Omega \theta \cdot \partial_{x_1}\omega|\omega|^{p-2}\,dx\right| \\
&\le (p-1)\left|\int_\Omega \theta \cdot \nabla\omega|\omega|^{\frac{p-2}{2}}|\omega|^{\frac{p-2}{2}}\,dx\right| \\
&\le \frac{2(p-1)}{p}\|\theta\|_{L^p}\big\|\nabla|\omega|^{\frac{p}{2}}\big\|_{L^2}\|\omega\|_{L^p}^{\frac{p-2}{2}} \\
&\le \frac{2(p-1)}{p^2}\big\|\nabla|\omega|^{\frac{p}{2}}\big\|_{L^2}^2 + C\|\theta\|_{L^p}^2\|\omega\|_{L^p}^{p-2},
\end{aligned} \tag{87}$$

where constant C depends on p. Inserting (87) into (86), we can obtain

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p + \frac{2(p-2)}{p^2}\big\|\nabla|\omega|^{\frac{p}{2}}\big\|^2 \le C\|\theta\|_{L^p}^2\|\omega\|_{L^p}^{p-2}. \tag{88}$$

Hence, by Young inequality, we get

$$\frac{d}{dt}\|\omega\|_{L^p}^p + \frac{2(p-2)}{p}\big\|\nabla|\omega|^{\frac{p}{2}}\big\|^2 \le C\big(\|\theta\|_{L^p}^p + \|\omega\|_{L^p}^p\big). \tag{89}$$

Integrating over $[0,t]$, we have, for all $t \in [0,T]$,

$$\begin{aligned}
&\|\omega\|_{L^p}^p + \frac{2(p-2)}{p}\int_0^t \big\|\nabla|\omega|^{\frac{p}{2}}\big\|^2\,d\tau \\
&\quad\le \|\omega_0\|_{L^p}^p + C\int_0^t \big(\|\theta\|_{L^p}^p + \|\omega\|_{L^p}^p\big)\,d\tau \\
&\quad\le \|\omega_0\|_{L^p}^p + CT\|\theta_0\|_{L^p}^p + C\int_0^t \|\omega\|_{L^p}^p\,d\tau.
\end{aligned} \tag{90}$$

Using Gronwall inequality, we find from (90) that, for all $t \in [0,T]$,

$$\begin{aligned}
\|\omega\|_{L^p}^p + \frac{2(p-2)}{p}\int_0^t \big\|\nabla|\omega|^{\frac{p}{2}}\big\|^2\,d\tau &\le e^{Ct}\big(\|\omega_0\|_{L^p}^p + CT\|\theta_0\|_{L^p}^p\big) \\
&\le e^{CT}\big(\|\omega_0\|_{L^p}^p + CT\|\theta_0\|_{L^p}^p\big) \\
&\le C.
\end{aligned} \tag{91}$$

Then inequality (84), together with (91), implies that

$$\|u\|_{L^p}^p + \frac{4(p-2)}{p^2}\int_0^t \big\|\nabla|u|^{\frac{p}{2}}\big\|_{L^2}^2\,d\tau \le C. \tag{92}$$

Taking the derivative $D = (\partial_{x_1}, \partial_{x_2})$ of both sides of (27), and then multiplying the result equation array by $D\omega|D\omega|^{p-2}$, after integration by parts, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p + \frac{4(p-2)}{p^2}\big\|\nabla|D\omega|^{\frac{p}{2}}\big\|^2 \le &-\int_\Omega D(u\cdot\nabla\omega)\cdot D\omega|D\omega|^{p-2}\,dx \\
&-\int_\Omega D\theta_{x_1}\cdot D\omega|D\omega|^{p-2}\,dx.
\end{aligned} \tag{93}$$

Using Hölder and Cauchy–Schwarz inequalities, a standard calculation gives us the following:

$$\begin{aligned}
&-\int_\Omega D(u\cdot\nabla\omega)\cdot D\omega|D\omega|^{p-2}\,dx\\
&\quad\le C\left|\int_\Omega u\cdot\nabla\omega\cdot D^2\omega|D\omega|^{p-2}\,dx\right|\\
&\quad\le (p-1)\left|\int_\Omega u\cdot\nabla\omega\cdot D^2\omega|D\omega|^{\frac{p-2}{2}}|D\omega|^{\frac{p-2}{2}}\,dx\right|\\
&\quad\le \frac{2(p-1)}{p}\|u\|_{L^p}\bigl\|\nabla|D\omega|^{\frac{p}{2}}\bigr\|_{L^2}\|D\omega\|_{L^p}^{\frac{p-2}{2}}\\
&\quad\le \frac{p-1}{p^2}\bigl\|\nabla|D\omega|^{\frac{p}{2}}\bigr\|_{L^2}^2+C\|u\|_{L^p}^2\|D\omega\|_{L^p}^{p-2}
\end{aligned}\tag{94}$$

and

$$\begin{aligned}
&-\int_\Omega D\theta_{x_1}\cdot D\omega|D\omega|^{p-2}\,dx\\
&\quad\le \left|\int_\Omega D\theta_{x_1}\cdot D\omega|D\omega|^{p-2}\,dx\right|\\
&\quad\le (p-1)\left|\int_\Omega D\theta_{x_1}\cdot D\omega|D\omega|^{p-2}\,dx\right|\\
&\quad\le \frac{2(p-1)}{p}\|D\theta\|_{L^p}\bigl\|\partial_{x_1}|D\omega|^{\frac{p}{2}}\bigr\|_{L^2}\|D\omega\|_{L^p}^{\frac{p-2}{2}}\\
&\quad\le \frac{p-1}{p^2}\bigl\|\nabla|D\omega|^{\frac{p}{2}}\bigr\|_{L^2}^2+C\|D\theta\|_{L^p}^2\|D\omega\|_{L^p}^{p-2}\\
&\quad\le \frac{p-1}{p^2}\bigl\|\nabla|D\omega|^{\frac{p}{2}}\bigr\|_{L^2}^2+C\bigl(\|\nabla\theta\|_{L^p}^p+\|D\omega\|_{L^p}^p\bigr).
\end{aligned}\tag{95}$$

Inserting (94) and (95) into (93), we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|D\omega\|_{L^p}^p+\frac{2(p-2)}{p^2}\bigl\|\nabla|D\omega|^{\frac{p}{2}}\bigr\|^2\\
&\quad\le C\bigl(\|u\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p+\|D\omega\|_{L^p}^p\bigr).
\end{aligned}\tag{96}$$

Taking the derivative $\nabla^\perp=(-\partial_{x_2},\partial_{x_1})$ of both sides of $(1)_3$, we can show that

$$\nabla^\perp\theta_t+\kappa\Lambda^{2\beta}\nabla^\perp\theta+\nabla^\perp(u\cdot\nabla\theta)=0.\tag{97}$$

Multiplying (96) by $\nabla^\perp\theta|\nabla^\perp\theta|^{p-2}$, after integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\bigl\|\nabla^\perp\theta\bigr\|_{L^p}^p+\int_\Omega\Lambda^{2\beta}\nabla^\perp\theta\cdot\nabla^\perp\theta\bigl|\nabla^\perp\theta\bigr|^{p-2}\,dx\\
&\quad=-\int_\Omega\nabla^\perp(u\cdot\nabla\theta)\cdot\nabla^\perp\theta\bigl|\nabla^\perp\theta\bigr|^{p-2}\,dx.
\end{aligned}\tag{98}$$

Since u is divergence-free, using Lemma 2 again, we know that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\nabla^\perp\theta\|_{L^p}^p+\frac{4(p-2)}{p^2}\big\|\Lambda^\beta|\nabla^\perp\theta|^{\frac{p}{2}}\big\|_{L^2}^2\\
&\quad\le-\int_\Omega\nabla u\cdot\nabla^\perp\theta\cdot\nabla^\perp\theta\big|\nabla^\perp\theta\big|^{p-2}\,dx\\
&\quad\le\|\nabla u\|_{L^\infty}\big\|\nabla^\perp\theta\big\|_{L^p}^p.
\end{aligned}\tag{99}$$

By Lemma 3, we know that

$$\|\nabla u\|_{L^\infty}\le C\big(1+\|D\omega\|_{L^2}^2\big)\big(1+\log^+\big(\|D\omega\|_{L^p}^p\big)\big)+C\|\omega\|_{L^2}.\tag{100}$$

By (91) for $p=2$, and inserting (100) into (99), we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\nabla^\perp\theta\|_{L^p}^p+\frac{4(p-2)}{p^2}\big\|\Lambda^\beta|\nabla^\perp\theta|^{\frac{p}{2}}\big\|_{L^2}^2\\
&\quad\le C\big(1+\|D\omega\|_{L^2}^2\big)\big(1+\log^+\big(\|D\omega\|_{L^p}^p\big)\big)\big\|\nabla^\perp\theta\big\|_{L^p}^p.
\end{aligned}\tag{101}$$

Using the obvious identity $\|\nabla\theta\|_{L^p}=\|\nabla^\perp\theta\|_{L^p}$, and summing up (96) and (101), we obtain that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\big(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\big)+\frac{2(p-2)}{p^2}\big\|\nabla|D\omega|^{\frac{p}{2}}\big\|^2+\frac{4(p-2)}{p^2}\big\|\Lambda^\beta|\nabla\theta|^{\frac{p}{2}}\big\|_{L^2}^2\\
&\quad\le C\big(1+\|D\omega\|_{L^2}^2\big)\big(1+\log^+\big(\|D\omega\|_{L^p}^p\big)\big)\big(\|\nabla\theta\|_{L^p}^p+\|D\omega\|_{L^p}^p\big)\\
&\quad\le C\big(1+\|D\omega\|_{L^2}^2\big)\big(1+\log^+\big(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\big)\big)\big(\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p\big).
\end{aligned}\tag{102}$$

Setting $X(t)=\|D\omega\|_{L^p}^p+\|\nabla\theta\|_{L^p}^p$, we easily show that

$$\frac{d}{dt}X\le C\big(1+\|D\omega\|_{L^2}^2\big)\big(1+\log^+X\big)X.\tag{103}$$

Setting $Y=\log^+X$, we know that

$$\frac{d}{dt}X=X\frac{d}{dt}Y.\tag{104}$$

So, inequality (103), along with (104), implies that

$$\frac{d}{dt}Y\le C\big(1+\|D\omega\|_{L^2}^2\big)(1+Y).\tag{105}$$

By (91) for $p=2$, and integrating over $[0,t]$, we have, for all $t\in[0,T]$,

$$Y(t)\le Y(0)+C\int_0^t\big(1+\|D\omega\|_{L^2}^2\big)\,d\tau+C\int_0^t\big(1+\|D\omega\|_{L^2}^2\big)Y\,d\tau.\tag{106}$$

Using Gronwall inequality, from (105) we find that, for all $t\in[0,T]$,

$$Y(t)\le C\bigg(Y(0)+\int_0^t\big(1+\|D\omega\|_{L^2}^2\big)\,d\tau\bigg)e^{C\int_0^t(1+\|D\omega\|_{L^2}^2)\,d\tau}$$

$$\leq C\left(Y(0) + \int_0^T \left(1 + \|D\omega\|_{L^2}^2\right) d\tau\right) e^{C\int_0^T (1+\|D\omega\|_{L^2}^2)\,d\tau}$$
$$\leq C\left(Y(0) + CT\right)e^{CT}, \tag{107}$$

which implies

$$X(t) \leq e^{C(\log^+ X(0)+CT)e^{CT}}. \tag{108}$$

This thus completes the proof of Theorem 2.

5 Conclusions

In this paper, we study the well-posedness and related problem on Boussinesq equations with fractional dissipation which have recently attracted considerable interest. This paper proves the persistence of global well-posedness of strong solutions and their long-time decay, as well as investigates the existence of the solutions in Sobolev spaces. The obtained results will not only further improve the theory of fractional nonlinear evolution equations, but also provide support for the innovation on research methods and the related properties of fluid dynamics models.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally to this work. All authors read and approved the final manuscript.

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