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Oscillation and nonoscillation theorems of neutral dynamic equations on time scales

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Abstract

We present the oscillation criteria for the following neutral dynamic equation on time scales:

$$\big(y(t)-C(t)y(t-\zeta)\big)^{\Delta}+P(t)y(t-\eta)-Q(t)y(t-\delta)=0,\quad t\in\mathbb{T},$$

where $C,P,Q\in C_{rd}([t_0,\infty),\mathbb{R}^+)$, $\mathbb{R}^+=[0,\infty)$, $\gamma,\eta,\delta\in\mathbb{T}$ and $\gamma>0$, $\eta>\delta\geq 0$. New conditions for the existence of nonoscillatory solutions of the given equation are also obtained.

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1 Introduction

In the past two decades, there has been shown a growing interest in the study of oscillation and stability of delay dynamic equations on time scales. Several excellent monographs [1–5] on the topic indeed reflect its popularity. Some recent results on oscillation and existence of nonoscillatory solutions for dynamic equations can be found in the articles [6–23] and the references cited therein.

Motivated by aforementioned work, in this paper, we consider the following neutral dynamic equation on time scales:

$$\big(y(t)-C(t)y(t-\zeta)\big)^{\Delta}+P(t)y(t-\eta)-Q(t)y(t-\delta)=0,\quad t\in\mathbb{T},\tag{1}$$

where $C,P,Q\in C_{rd}([t_0,\infty),\mathbb{R}^+)$, $\mathbb{R}^+=[0,\infty)$, C_{rd} denotes the class of right-dense continuous functions, $\zeta,\eta,\delta\in\mathbb{T}$ and $\zeta>0$, $\eta>\delta\geq 0$. Some conditions for oscillation of Eq. (1) are obtained. We also discuss the existence of nonoscillatory solutions for Eq. (1).

A time scale is an arbitrary nonempty closed subset of the real numbers. We denote the time scale by the symbol $\mathbb{T}$. For $t\in\mathbb{T}$ we define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}$. Let $C_{rd}(\mathbb{T},\mathbb{R})$ denote the space of functions which are right-dense continuous on $\mathbb{T}$. In addition, we define the interval $[t_0,\infty)$ in $\mathbb{T}$ by $[t_0,\infty):=\{t\in\mathbb{T}:t_0\leq t<\infty\}$.

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Definition 1.1 For $h \geq 0$, we define the cylinder transformation ξ_h by

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Definition 1.2 A solution of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

Lemma 1.3 *If $f : \mathbb{T} \to \mathbb{R}$ is differentiable and $f^{\Delta} \geq 0$, then f is nondecreasing on $\mathbb{T}$.*

Lemma 1.4 *If $f : \mathbb{T} \to \mathbb{R}$ is differentiable at t, then f is continuous at t.*

2 Oscillation

In this section, we derive the main results for oscillation of Eq. (1). For that, we assume the following conditions:

(c_1) $0 \leq C(t) + \int_{t-\eta}^{t-\delta} Q(s+\delta)\Delta s \leq 1$;

(c_2) $\bar{R}(t) = P(t) - Q(t-\eta+\delta) \geq 0$ and $\liminf_{t\to\infty} \int_{t-\eta}^{t} \bar{R}(s)\Delta s > \gamma > 0$.

The following lemmas are useful in proving the main results of this section.

Lemma 2.1 *Assume that the conditions (c_1) and (c_2) are satisfied. Let $y(t)$ be an eventually positive solution of (1) such that*

$$u(t) = y(t) - C(t)y(t-\zeta) - \int_{t-\eta}^{t-\delta} Q(s+\delta)y(s)\Delta s. \tag{2}$$

Then eventually

$$u^{\Delta}(t) \leq 0, \qquad u(t) > 0.$$

Proof Since $y(t)$ is an eventually positive solution of (1), there exists $t_1 \geq t_0$ such that $y(t - m) > 0$ for $t \geq t_1$, where $m = \max\{\zeta, \eta, \delta\}$. In view of (1) and (2), we get

$$\begin{aligned} u^{\Delta}(t) &= \big(y(t) - C(t)y(t-\zeta)\big)^{\Delta} - \left(\int_{t-\eta}^{t-\delta} Q(s+\delta)y(s)\Delta s\right)^{\Delta} \\ &= -P(t)y(t-\eta) + Q(t)y(t-\delta) - Q(t)y(t-\delta) + Q(t-\eta+\delta)y(t-\eta) \\ &= -\big(P(t) - Q(t-\eta+\delta)\big)y(t-\eta) \\ &= -\bar{R}(t)y(t-\eta) \\ &\leq 0, \end{aligned}$$

which implies that $u(t)$ is decreasing. Next, we shall show that $u(t) > 0$. If $u(t) \to -\infty$ as $t \to \infty$, then $y(t)$ must be unbounded. Therefore there exists $\{t'_n\}$ with $t'_n \geq t_2$, $t_2 = t_1 + m$ such that

$$\lim_{n\to\infty} t'_n = \infty, \qquad \lim_{n\to\infty} y\big(t'_n\big) = \infty,$$

and $y(t'_n) = \max_{t_2 \le t \le t'_n} y(t)$. Hence, we have

$$\begin{aligned} u(t'_n) &= y(t'_n) - C(t'_n)y(t'_n - \zeta) - \int_{t'_n-\eta}^{t'_n-\delta} Q(s+\delta)y(s)\Delta s \\ &\ge y(t'_n)\left(1 - C(t'_n) - \int_{t'_n-\eta}^{t'_n-\delta} Q(s+\delta)\Delta s\right) \ge 0. \end{aligned}$$

In consequence, we get

$$\lim_{t\to\infty} u(t) = \lim_{n\to\infty} u(t'_n) \ge 0,$$

which is a contradiction. Hence $\lim_{t\to\infty} u(t) = l$ exists. As before, if $y(t)$ is unbounded, then $l \ge 0$. Now we consider the case when $y(t)$ is bounded. Let $\bar{l} = \limsup_{t\to\infty} y(t) = \lim_{t'\to\infty} y(t')$. Then

$$\begin{aligned} y(t') - u(t') &= C(t')y(t'-\zeta) + \int_{t'-\eta}^{t'-\delta} Q(s+\delta)y(s)\Delta s \\ &\le y(\xi_{t'})\left(C(t') + \int_{t'-\eta}^{t'-\delta} Q(s+\delta)\Delta s\right), \end{aligned}$$

where $y(\xi_{t'}) = \max\{\{y(s) : s \in (t'-\eta, t'-\delta)\}, y(t'-\zeta)\}$. Hence, it follows that $\xi_{t'} \to \infty$ as $t' \to \infty$ and $\limsup_{t'\to\infty} y(\xi_{t'}) \le \bar{l}$. Thus, we get

$$y(t') - u(t') \le y(\xi_{t'}), \tag{3}$$

which, on taking superior limit, leads to $\bar{l} - l \le \bar{l}$. Therefore $l \ge 0$. Hence $u(t) > 0$ eventually. The proof is complete. □

Lemma 2.2 *Suppose that the conditions* (c_1) *and* (c_2) *hold and that* $y(t)$ *is an eventually positive solution of* (1) *satisfying* (2). *Then the set* $\Lambda = \{\lambda > 0 : u^{\Delta}(t) + \lambda \bar{R}u(t) \le 0, \text{eventually}\}$ *is nonempty and there exists an upper bound of* Λ *which is independent of solution* $y(t)$.

Proof From the given assumptions, there exists a $t_1 \ge t_0$, such that $y(t-m) > 0$ for $t \ge t_1$, where $m = \{\zeta, \eta, \delta\}$. It follows from (2) that $u(t) \le y(t)$ for $t \ge t_1$. Then

$$u^{\Delta}(t) = -\bar{R}(t)y(t-\eta) \le -\bar{R}(t)u(t-\eta) \le -\bar{R}(t)u(t), \quad t \ge t_1 + m, \tag{4}$$

that is, $\lambda = 1 \in \Lambda$. Therefore Λ is nonempty.

Let

$$3k = \liminf_{t\to\infty} \int_{t-\eta}^{t} \bar{R}(s)\Delta s.$$

By (c_2), we have $k > 0$, and there exists a $t_2 > t_1 + m$ such that

$$\int_{t-\eta}^{t} \bar{R}(s)\Delta s > 2k := \gamma, \quad t \ge t_2.$$

Therefore, for any $t \geq t_2$, there exists $t^* > t > t^* - \eta$ such that

$$\int_t^{t^*} \bar{R}(s)\Delta s > k, \qquad \int_{t^*-\eta}^{t} \bar{R}(s)\Delta s > k.$$

Integrating (4) from t to t^* and noting that $u^\Delta(t) \leq 0$, $u(t) > 0$ for $t \geq t_2$, we find that

$$u(t) - u(t^*) \leq -\int_t^{t^*} \bar{R}(s)u(s-\eta)\Delta s,$$

which implies that

$$u(t) \geq \int_t^{t^*} \bar{R}(s)u(s-\eta)\Delta s \geq u(t^*-\eta)\int_t^{t^*} \bar{R}(s)\Delta s > ku(t^*-\eta).$$

Next, integrating (4) from $t^* - \eta$ to t, we get

$$u(t^*-\eta) > ku(t-\eta).$$

Hence

$$u(t) > k^2 u(t-\eta), \quad t \geq t_2. \tag{5}$$

Let us define

$$\liminf_{t\to\infty} y(t-\eta) = I. \tag{6}$$

Since $y(t-m) > 0$, (6) implies that $I \geq 0$. On the other hand, there exists a sequence $\{t'_n\}$ such that $t'_n \geq t_2$ and $t'_n \to \infty$ as $n \to \infty$ and

$$\liminf_{t\to\infty} \int_{t-\eta}^{t} \bar{R}(s)\Delta s = \lim_{n\to\infty} \int_{t'_n-\eta}^{t'_n} \bar{R}(s)\Delta s. \tag{7}$$

From (4), we have

$$y(\xi_n - \eta)\int_{t'_n-\eta}^{t'_n} \bar{R}(s)\Delta s = \int_{t'_n-\eta}^{t'_n} \bar{R}(s)y(s-\eta)\Delta s = -u(t'_n) + u(t'_n - \eta), \tag{8}$$

where $\xi_n \in [t'_n - \eta, t'_n]$, and $\xi_n \to \infty$ as $n \to \infty$. Hence, we can find an increasing subsequence in $\{\xi_n\}$ and so, without loss of generality, we may assume that the sequence numbers $\{\xi_n\}$ is also increasing. Let

$$F(t) = \inf\{y(s-\eta) : s \geq t\}, \quad t \geq t_2.$$

Then we have

$$\lim_{t\to\infty} F(t) = \liminf_{t\to\infty} y(t-\eta).$$

Since $\{\xi_n\}$ is an increasing sequence of numbers, we get

$$\{y(\xi'_n-\eta):n'\geq n\}\subset\{y(s-\eta):s\geq\xi_n\}.$$

Therefore

$$F(\xi_n)=\inf\{y(s-\eta):s\geq\xi_n\}\leq\inf\{y(\xi'_n-\eta):n'\geq n\},$$

which implies that

$$\liminf_{t\to\infty}y(t-\eta)=\lim_{n\to\infty}F(\xi_n)\leq\lim_{n\to\infty}\inf y(\xi_n-\eta),$$

that is,

$$\liminf_{t\to\infty}y(t-\eta)\leq\lim_{n\to\infty}\inf y(\xi_n-\eta). \tag{9}$$

On the other hand, $\lim_{t\to\infty}u(t)$ exists and is a finite number. Therefore, it follows from (7)–(9) that

$$\begin{aligned}
I\left(\liminf_{t\to\infty}\int_{t-\eta}^{t}\bar{R}(s)\Delta s\right) &= \left(\liminf_{t\to\infty}y(t-\eta)\right)\left(\liminf_{t\to\infty}\int_{t-\eta}^{t}\bar{R}(s)\Delta s\right)\\
&\leq\left(\lim_{n\to\infty}\inf y(\xi_n-\eta)\right)\left(\lim_{n\to\infty}\int_{t'_n-\eta}^{t'_n}\bar{R}(s)\Delta s\right)\\
&=\left(\lim_{n\to\infty}\inf y(\xi_n-\eta)\right)\left(\lim_{n\to\infty}\inf\int_{t'_n-\eta}^{t'_n}\bar{R}(s)\Delta s\right)\\
&\leq\lim_{n\to\infty}\inf\bigl(y(\xi_n-\eta)\bigr)\int_{t'_n-\eta}^{t'_n}\bar{R}(s)\Delta s\\
&=\lim_{n\to\infty}\inf\int_{t'_n-\eta}^{t'_n}\bar{R}(s)y(s-\eta)\Delta s)\\
&=-\lim_{n\to\infty}u(t'_n)+\lim_{n\to\infty}u(t'_n-\eta)\\
&=0,
\end{aligned}$$

that is,

$$I\left(\liminf_{t\to\infty}\int_{t-\eta}^{t}\bar{R}(s)\Delta s\right)\leq 0. \tag{10}$$

From condition (c_2), (10) and the fact that $I\geq 0$, we deduce that $I=0$. Thus, we obtain

$$\liminf_{t\to\infty}y(t-\eta)=0.$$

Hence there exists a sequence $\{s_n\}$ with $s_n\geq t_2+2m$, such that $y(s_n)\to 0$ as $n\to\infty$ and $y(s_n-\eta)=\min_{t_2\leq s\leq s_n-\eta}y(s)$ for $n=1,2,\ldots$. Then, from (4) for $n=1,2,\ldots$, we have

$$u(s_n)-u(s_n-\eta)=-\int_{s_n-\eta}^{s_n}\bar{R}(s)y(s-\eta)\Delta s$$

$$\begin{aligned}&\leq -y(s_n-\eta)\int_{s_n-\eta}^{s_n}\bar{R}(s)\Delta s\\&< -2ky(s_n-\eta).\end{aligned}$$

Hence

$$u(s_n-\eta) > 2ky(s_n-\eta), \quad n=1,2,\ldots. \tag{11}$$

Also, from (4), (5) and (11), for $n=1,2,\ldots$, we have

$$u^{\Delta}(s_n) = -\bar{R}(s_n)y(s_n-\eta) > -\frac{1}{2k}\bar{R}(s_n)u(s_n-\eta) \geq -\frac{1}{2k^3}\bar{R}(s_n)u(s_n),$$

which implies that

$$u^{\Delta}(s_n)+\frac{1}{2k^3}\bar{R}(s_n)u(s_n) > 0, \quad n=1,2,\ldots. \tag{12}$$

Now we may assert that $\frac{1}{2k^3}\bar{\in}\Lambda$. In fact, if $\frac{1}{2k^3}\in\Lambda$, then there exists some T' by the definition of Λ such that, for all $t\geq T'$, the following inequality holds true:

$$u^{\Delta}(t)+\frac{1}{2k^3}\bar{R}(t)u(t)\leq 0. \tag{13}$$

On the other hand, in view of the fact that $s_n\to 0$ as $n\to\infty$, from $\{s_n\}$ we find some s_n' such that $s_n'\geq T'$. Then it follows from (12) that

$$u^{\Delta}(s_n')+\frac{1}{2k^3}\bar{R}(s_n')u(s_n') > 0,$$

which contradicts (13). Therefore, $\frac{1}{2k^3}$ is an upper bound of Λ which is independent of solution $y(t)$. The proof is complete. □

Theorem 2.3 *Assume that the conditions* (c_1) *and* (c_2) *are satisfied. In addition it is assumed that there exist* $T\geq t_1+m$ *and* $\lambda>0$ *such that*

$$\begin{aligned}\inf_{t\geq T,\lambda>0}\biggl\{&\frac{1}{\lambda}\exp\biggl(-\int_{t-\eta}^{t}\xi_\mu\bigl(-\lambda\bar{R}(s)\bigr)\Delta s\biggr)+C(t-\eta)\exp\biggl(-\int_{t-\zeta}^{t}\xi_\mu\bigl(-\lambda\bar{R}(s)\bigr)\Delta s\biggr)\\&+\int_{t-\eta}^{t-\delta}Q(s+\delta-\eta)\exp\biggl(-\int_{s}^{t}\xi_\mu\bigl(-\lambda\bar{R}(u)\bigr)\Delta u\biggr)\Delta s\biggr\}>1.\end{aligned} \tag{14}$$

Then every solution of Eq. (1) *is oscillatory.*

Proof On the contrary, let $y(t)$ be a nonoscillatory solution of Eq. (1). Without loss of generality, it can be assumed that $y(t)$ is an eventually positive solution. Moreover, let $u(t)$ be the same as defined in (2) and the set Λ as given in Lemma 2.2. Then, by Lemma 2.2, we see that there exists a $t_2\geq t_0$ such that

$$u^{\Delta}(t)\leq 0, \qquad u(t)>0, \quad \text{for } t\geq t_2.$$

From condition (14), there exists a constant $\alpha > 1$ such that

$$\inf_{t\geq T,\lambda>0}\left\{\frac{1}{\lambda}\exp\left(-\int_{t-\eta}^{t}\xi_{\mu}\big(-\lambda\bar{R}(s)\big)\Delta s\right)+C(t-\eta)\exp\left(-\int_{t-\zeta}^{t}\xi_{\mu}\big(-\lambda\bar{R}(s)\big)\Delta s\right)\right.$$
$$\left.+\int_{t-\eta}^{t-\delta}Q(s+\delta-\eta)\exp\left(-\int_{s}^{t}\xi_{\mu}\big(-\lambda\bar{R}(u)\big)\Delta u\right)\Delta s\right\}\geq\alpha>1. \tag{15}$$

Let $\lambda_0 \in \Lambda$. Then we shall show that $\alpha\lambda_0 \in \Lambda$. In fact, $\lambda_0 \in \Lambda$ implies that

$$u^{\Delta}(t)+\lambda_0\bar{R}(t)u(t)\leq 0. \tag{16}$$

Define

$$w(t)=u(t)\exp\left(-\int_{t_0}^{t}\xi_{\mu}\big(-\lambda_0\bar{R}(s)\big)\Delta s\right) \tag{17}$$

and note that $w(t)$ is well defined. Let us introduce

$$v(t)=\exp\left(\int_{t_0}^{t}\xi_{\mu}\big(-\lambda_0\bar{R}(s)\big)\Delta s\right)$$

and note that

$$\begin{aligned}
w^{\Delta}(t) &= \left(\frac{u(t)}{v(t)}\right)^{\Delta}\\
&= \frac{u^{\Delta}(t)v(t)-u(t)v^{\Delta}(t)}{v(t)v(\sigma(t))}\\
&\leq \frac{-\lambda_0\bar{R}(t)u(t)v(t)-u(t)[-\lambda_0\bar{R}(t)v(t)]}{v(t)v(\sigma(t))}\\
&= \frac{-\lambda_0\bar{R}(t)u(t)v(t)+u(t)\lambda_0\bar{R}(t)v(t)}{v(t)v(\sigma(t))}\\
&= 0.
\end{aligned}$$

Hence, $w(t)$ is nonincreasing. From (2), we get $u^{\Delta}(t)=-\bar{R}(t)y(t-\eta)$, which together with (16) yields $y(t-\eta)\geq\lambda_0 u(t)$. Therefore

$$\begin{aligned}
u^{\Delta}(t) &= -\bar{R}(t)y(t-\eta)\\
&= -\bar{R}(t)\left[u(t-\eta)+C(t-\eta)y(t-\eta-\zeta)+\int_{t-2\eta}^{t-\eta-\delta}Q(s+\delta)y(s)\Delta s\right]\\
&\leq -\bar{R}(t)\left[u(t-\eta)+\lambda_0 C(t-\eta)u(t-\zeta)+\lambda_0\int_{t-2\eta}^{t-\eta-\delta}Q(s+\delta)u(s+\eta)\Delta s\right]\\
&= -\bar{R}(t)\left[u(t-\eta)+\lambda_0 C(t-\eta)u(t-\zeta)+\lambda_0\int_{t-\eta}^{t-\delta}Q(s+\delta-\eta)u(s)\Delta s\right]\\
&= -\bar{R}(t)\left[w(t-\eta)\exp\left(\int_{t_0}^{t-\eta}\xi_{\mu}\big(-\lambda_0\bar{R}(s)\big)\Delta s\right)\right.\\
&\quad +\lambda_0 C(t-\eta)w(t-\zeta)\exp\left(\int_{t_0}^{t-\zeta}\xi_{\mu}\big(-\lambda_0\bar{R}(s)\big)\Delta s\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \lambda_0 \int_{t-\eta}^{t-\delta} Q(s+\delta-\eta) w(s) \exp\left(\int_{t_0}^{s} \xi_\mu\left(-\lambda_0 \bar{R}(u)\right)\Delta u\right)\Delta s\Bigg] \\
&\le -\bar{R}(t)\Bigg[w(t)\exp\left(\int_{t_0}^{t-\eta} \xi_\mu\left(-\lambda_0 \bar{R}(s)\right)\Delta s\right) \\
&\quad + \lambda_0 C(t-\eta) w(t)\exp\left(\int_{t_0}^{t-\zeta} \xi_\mu\left(-\lambda_0 \bar{R}(s)\right)\Delta s\right) \\
&\quad + \lambda_0 \int_{t-\eta}^{t-\delta} Q(s+\delta-\eta) w(s) \exp\left(\int_{t_0}^{s} \xi_\mu\left(-\lambda_0 \bar{R}(u)\right)\Delta u\right)\Delta s\Bigg] \\
&= -\bar{R}(t)\Bigg[u(t)\exp\left(-\int_{t-\eta}^{t} \xi_\mu\left(-\lambda_0 \bar{R}(s)\right)\Delta s\right) \\
&\quad + \lambda_0 C(t-\eta) u(t)\exp\left(-\int_{t-\zeta}^{t} \xi_\mu\left(-\lambda_0 \bar{R}(s)\right)\Delta s\right) \\
&\quad + \lambda_0 \int_{t-\eta}^{t-\delta} Q(s+\delta-\eta) u(s) \exp\left(-\int_{s}^{t} \xi_\mu\left(-\lambda_0 \bar{R}(u)\right)\Delta u\right)\Delta s\Bigg] \\
&\le -\bar{R}(t)\Bigg[\exp\left(-\int_{t-\eta}^{t} \xi_\mu\left(-\lambda_0 \bar{R}(s)\right)\Delta s\right) + \lambda_0 C(t-\eta)\exp\left(-\int_{t-\zeta}^{t} \xi_\mu\left(-\lambda_0 \bar{R}(s)\right)\Delta s\right) \\
&\quad + \lambda_0 \int_{t-\eta}^{t-\delta} Q(s+\delta-\eta) \exp\left(-\int_{s}^{t} \xi_\mu\left(-\lambda_0 \bar{R}(u)\right)\Delta u\right)\Delta s\Bigg] u(t) \\
&\le -\inf_{t\ge T}\Bigg[\exp\left(-\int_{t-\eta}^{t} \xi_\mu\left(-\lambda_0 \bar{R}(s)\right)\Delta s\right) + \lambda_0 C(t-\eta)\exp\left(-\int_{t-\zeta}^{t} \xi_\mu\left(-\lambda_0 \bar{R}(s)\right)\Delta s\right) \\
&\quad + \lambda_0 \int_{t-\eta}^{t-\delta} Q(s+\delta-\eta) \exp\left(-\int_{s}^{t} \xi_\mu\left(-\lambda_0 \bar{R}(u)\right)\Delta u\right)\Delta s\Bigg] \bar{R}(t) u(t) \\
&\le -\alpha\lambda_0 \bar{R}(t) u(t).
\end{aligned}
$$

Thus, $\alpha\lambda_0 \in \Lambda$. Repeating this procedure, one finds that $\alpha^m\lambda_0 \in \Lambda$ for any integer m, which contradicts the boundedness of Λ. The proof is complete. □

Corollary 2.4 *Assume that* $P(t) \ge 0$, $\liminf_{t\to\infty} \int_{t-\eta}^{t} P(s)\Delta s > 0$ *and there exist* T *and* $\lambda > 0$ *such that*

$$
\inf_{t\ge T, \lambda>0} \left\{ \frac{1}{\lambda} \exp\left(-\int_{t-\eta}^{t} \xi_\mu\left(-\lambda \bar{R}(s)\right)\Delta s\right)\right\} > 1.
$$

Then every solution of the equation

$$
y^{\Delta}(t) + P(t) y(t-\eta) = 0
$$

is oscillatory.

3 Nonoscillation

Here we derive some results for the existence of a positive solution of (1).

Lemma 3.1 *Assume that*

(i) $\bar{R}(t) = P(t) - Q(t-\eta-\delta) \ge 0$;

(ii) *the inequality*

$$C(t)z(t-\zeta)+\int_{t-\eta}^{t-\delta}Q(s+\delta)z(s)\Delta s+\int_{t-\eta}^{\infty}\bar{R}(s+\eta)z(s)\Delta s\le z(t),\quad \textit{for } t\ge t_1, \tag{18}$$

has a continuous positive solution $Z(t)$: $[t_1-m,\infty)\to(0,\infty)$ *with* $\lim_{t\to\infty}Z(t)=0$.

Then the equation

$$C(t)y(t-\zeta)+\int_{t-\eta}^{t-\delta}Q(s+\delta)y(s)\Delta s+\int_{t-\eta}^{\infty}\bar{R}(s+\eta)y(s)\Delta s=y(t),\quad \textit{for } t\ge t_1, \tag{19}$$

has a continuous positive solution $y(t)$ *with* $0<y(t)\le Z(t)$ *for* $t\ge t_1$.

Proof Take $T>t_1$ large enough so that $z(t)>Z(t)$ for $t\in[t_1-m,T)$. Define a set

$$\Omega=\bigl\{\omega\in C_{rd}\bigl([t_1-m,\infty),\mathbb{R}^+\bigr):0\le\omega(t)\le Z(t),t\ge t_1-m\bigr\}$$

and introduce an operator S on Ω as follows:

$$(S\omega)(t)=\begin{cases}C(t)\omega(t-\zeta)+\int_{t-\eta}^{t-\delta}Q(s+\delta)\omega(s)\Delta s+\int_{t-\eta}^{\infty}\bar{R}(s+\eta)\omega(s)\Delta s, & t\in(T,\infty),\\ (S\omega)(T)+z(t)-Z(T), & t\in[t_1-m,T].\end{cases}$$

It is clear that $S\Omega\subset\Omega$, and $\omega_1,\omega_2\in\Omega$ with $\omega_1\le\omega_2$ implies $S\omega_1\le S\omega_2$.

Define a sequence on Ω as

$$z_0(t)=Z(t),\qquad z_k(t)=Sz_{k-1}(t),\quad k=1,2,\ldots.$$

It is not difficult to prove that

$$0\le z_k(t)\le z_{k-1}(t)\le\cdots\le z_1(t)\le z(t),\quad t\in[t_1-m,\infty).$$

Therefore, the sequence $\{z_k(t)\}$ has a limiting function $y(t)$ with $\lim_{t\to\infty}z_k(t)=y(t)$ for $t\in[t_1-m,\infty)$ and $y(t)$ satisfies (19) by Lebesgue's convergence theorem. It is easy to see that $y(t)>0$ for $t\in[t_1-m,T]$ and hence $y(t)>0$ for all $t\in[t_1-m,\infty)$ with $0<y(t)\le Z(t)$. The proof is complete. □

Theorem 3.2 *Assume that*

(i) $\bar{R}(t)=P(t)-Q(t-\eta-\delta)\ge 0$;

(ii) *there exist* $T\ge t_1+m$ *and* $\lambda^*>0$ *such that*

$$\sup_{t\ge T}\biggl\{\frac{1}{\lambda^*}\exp\biggl(-\int_{t-\eta}^{t}\xi_\mu\bigl(-\lambda^*\bar{R}(u)\bigr)\Delta u\biggr)+C(t-\eta)\exp\biggl(-\int_{t-\zeta}^{t}\xi_\mu\bigl(-\lambda^*\bar{R}(s)\bigr)\Delta s\biggr)\\+\int_{t-\eta}^{t-\delta}Q(s+\delta-\eta)\exp\biggl(-\int_{s}^{t}\xi_\mu\bigl(-\lambda^*\bar{R}(u)\bigr)\Delta u\biggr)\Delta s\biggr\}\le 1. \tag{20}$$

Then Eq. (1) *has a positive solution* $y(t)$ *with* $\lim_{t\to\infty}y(t)=0$.

Proof Set

$$z(t)=\exp\biggl(\int_{t_1}^{t+\eta}\xi_\mu\bigl(-\lambda^*\bar{R}(s)\bigr)\Delta s\biggr). \tag{21}$$

Obviously $z(t)$ is well defined, positive and continuous. From the condition (20), for $t\geq T\geq T-\eta$, we have

$$\begin{aligned}&\frac{1}{\lambda^*}\biggl\{\exp\biggl(-\int_t^{t+\eta}\xi_\mu\bigl(-\lambda^*\bar{R}(u)\bigr)\Delta u\biggr)+C(t)\exp\biggl(-\int_{t+\eta-\zeta}^{t+\eta}\xi_\mu\bigl(-\lambda^*\bar{R}(s)\bigr)\Delta s\biggr)\\&\quad+\int_t^{t-\delta+\eta}Q(s+\delta-\eta)\exp\biggl(-\int_s^{t+\eta}\xi_\mu\bigl(-\lambda^*\bar{R}(u)\bigr)\Delta u\biggr)\Delta s\biggr\}\leq 1.\end{aligned} \tag{22}$$

Substituting (21) into (22), we get

$$\frac{1}{\lambda^*}\frac{z(t-\eta)}{z(t)}+C(t)\frac{z(t-\zeta)}{z(t)}+\int_t^{t-\delta+\eta}Q(s+\delta-\eta)\frac{z(s-\eta)}{z(t)}\Delta s\leq 1. \tag{23}$$

From (21), it is easy to see that $z^\Delta(t)=-\lambda^*\bar{R}(t+\eta)z(t)$, and hence we have

$$\int_{t-\eta}^\infty\bar{R}(s+\eta)z(s)\Delta s=-\frac{1}{\lambda^*}\int_{t-\eta}^\infty z^\Delta(s)\Delta s=\frac{z(t-\eta)}{\lambda^*}. \tag{24}$$

Combining (23) and (24), we obtain

$$\int_{t-\eta}^\infty\bar{R}(s+\eta)z(s)\Delta s+C(t)z(t-\zeta)+\int_t^{t+\eta-\delta}Q(s+\delta-\eta)z(s-\eta)\leq z(t).$$

Thus the desired conclusion follows by Lemma 3.1. The proof is complete. □

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