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Unique solution for a new system of fractional differential equations

Chengbo Zhai[1*] and Xiaolin Zhu[1]

*Correspondence:
cbzhai@sxu.edu.cn
[1]School of Mathematical Sciences, Shanxi University, Taiyuan, P.R. China

Abstract

In this article, we discuss a new system of fractional differential equations

$$\begin{cases} D_{0^+}^{s_1}u(t)+f(t,u(t),v(t))=z_1(t), & 0<t<1, \\ D_{0^+}^{s_2}v(t)+g(t,u(t),v(t))=z_2(t), & 0<t<1, \\ u(0)=u(1)=u'(0)=u'(1)=0, & D_{0^+}^{\beta_1}u(0)=0, \quad D_{0^+}^{\beta_1}u(1)=b_1D_{0^+}^{\beta_1}u(\eta_1), \\ v(0)=v(1)=v'(0)=v'(1)=0, & D_{0^+}^{\beta_2}v(0)=0, \quad D_{0^+}^{\beta_2}v(1)=b_2D_{0^+}^{\beta_2}v(\eta_2), \end{cases}$$

where $s_i=\alpha_i+\beta_i$, $\alpha_i\in(1,2]$, $\beta_i\in(3,4]$, $z_i:[0,1]\to[0,+\infty)$ is continuous, $D_{0^+}^{\alpha_i}$ and $D_{0^+}^{\beta_i}$ are the standard Riemann–Liouville derivatives, $\eta_i\in(0,1)$, $b_i\in(0,{\eta_i}^{1-\alpha_i})$, $i=1,2$, and $f,g\in C([0,1]\times\mathbf{R}^2,\mathbf{R})$. We establish the existence and uniqueness of solutions for the problem by a recent fixed point theorem of increasing Ψ-(h,e)-concave operators defined on ordered sets. Furthermore, the results obtained are well proven by means of a specific example.

MSC: 26A33; 34B18; 34B15

Keywords: Existence and uniqueness; Fractional differential system; Ψ-(h,e)-concave operator

1 Introduction

In this article, we investigate the following system of fractional differential equations:

$$\begin{cases} D_{0^+}^{s_1}u(t)+f(t,u(t),v(t))=z_1(t), & 0<t<1, \\ D_{0^+}^{s_2}v(t)+g(t,u(t),v(t))=z_2(t), & 0<t<1, \\ u(0)=u(1)=u'(0)=u'(1)=0, & D_{0^+}^{\beta_1}u(0)=0, \quad D_{0^+}^{\beta_1}u(1)=b_1D_{0^+}^{\beta_1}u(\eta_1), \\ v(0)=v(1)=v'(0)=v'(1)=0, & D_{0^+}^{\beta_2}v(0)=0, \quad D_{0^+}^{\beta_2}v(1)=b_2D_{0^+}^{\beta_2}v(\eta_2), \end{cases} \tag{1.1}$$

where $s_i=\alpha_i+\beta_i$, $\alpha_i\in(1,2]$, $\beta_i\in(3,4]$, $z_i:[0,1]\to[0,+\infty)$ is continuous, $D_{0^+}^{\alpha_i}$ and $D_{0^+}^{\beta_i}$ are the standard Riemann–Liouville derivatives, $\eta_i\in(0,1)$, $b_i\in(0,{\eta_i}^{1-\alpha_i})$, $i=1,2$, and $f,g\in C([0,1]\times\mathbf{R}^2,\mathbf{R})$. A pair of functions $(u,v)\in C([0,1])\times C([0,1])$ is called a solution of system (1.1) if it satisfies (1.1). We seek a new method which is a recent fixed point theorem for Ψ-(h,e)-concave operators to discuss system (1.1).

In the last few decades, fractional problems have attracted wide attention by scholars because of their wide applications and important positions in biology, physics, chemistry, and engineering, see [1–28] and the references therein. In [16], Xu and Dong considered the following fractional equation:

$$\begin{cases} -D_{0^+}^{\alpha}(\varphi_p(D_{0^+}^{\beta}u(t))) = f(t,u(t)), & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \quad D_{0^+}^{\beta}u(0) = 0, & D_{0^+}^{\beta}u(1) = bD_{0^+}^{\beta}u(\eta), \end{cases} \tag{1.2}$$

where $\alpha \in (1,2]$, $\beta \in (3,4]$, $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Riemann–Liouville derivatives, $b \in (0,\eta^{\frac{1-\alpha}{p-1}})$, $f \in C([0,1] \times [0,+\infty),[0,+\infty))$. By using Schauder's fixed point theorem and the upper and lower solutions method, the existence and uniqueness of solutions were established. On the other hand, fractional system have been investigated extensively, see [29–38]. In [33], the authors considered the existence of positive solutions for the following fractional system with parameters:

$$\begin{cases} -D_{0^+}^{\alpha_1}(\varphi_{p_1}(D_{0^+}^{\beta_1}u(t))) = \lambda f(t,u(t),v(t)), & 0 < t < 1, \\ -D_{0^+}^{\alpha_2}(\varphi_{p_2}(D_{0^+}^{\beta_2}v(t))) = \mu g(t,u(t),v(t))), & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \quad D_{0^+}^{\beta_1}u(0) = 0, & D_{0^+}^{\beta_1}u(1) = b_1 D_{0^+}^{\beta_1}u(\eta_1), \\ v(0) = v(1) = v'(0) = v'(1) = 0, \quad D_{0^+}^{\beta_2}v(0) = 0, & D_{0^+}^{\beta_2}v(1) = b_2 D_{0^+}^{\beta_2}v(\eta_2), \end{cases} \tag{1.3}$$

where $\alpha_i \in (1,2]$, $\beta_i \in (3,4]$, $D_{0^+}^{\alpha_i}$ and $D_{0^+}^{\beta_i}$ are the Riemann–Liouville derivatives, $\eta_i \in (0,1)$, $b_i \in (0,\eta_i^{\frac{1-\alpha_i}{p_i-1}})$, $i = 1,2$. $f,g \in C([0,1] \times \mathbf{R}^2,\mathbf{R})$. The authors applied Guo–Krasnosel'skii's fixed point theorem to get various existence results for positive solutions in terms of different values of λ and μ.

In the existing literature, most of the scholars have studied the existence of solutions, but there is little discussion about the uniqueness. Moreover, the usual methods used are Guo–Krasnosel'skii's fixed point theorem, the upper and lower solutions method, monotone iterative method, and fixed index theory. In 2017, the authors [29] used a new method to study the existence and uniqueness of solutions for the following fractional system:

$$\begin{cases} D^{\alpha}u(t) + f(t,v(t)) = a, & 0 < t < 1, \\ D^{\beta}v(t) + g(t,u(t)) = b, & 0 < t < 1, \\ u(0) = 0, & u(1) = \int_0^1 \phi(t)u(t)\,dt, \\ v(0) = 0, & v(1) = \int_0^1 \psi(t)v(t)\,dt, \end{cases} \tag{1.4}$$

where $1 < \alpha,\beta \leq 2$, $f,g \in C([0,1] \times (-\infty,+\infty),(-\infty,+\infty))$, $\phi,\psi \in L^1[0,1]$, a, b are constants and D denotes the usual Riemann–Liouville derivatives. The authors gave the existence and uniqueness of solutions for the coupled system dependent on constants a and b by using a fixed point theorem of increasing Ψ-(h,e)-concave operators. We found that the method can resolve some new differential systems and can obtain some good unique results.

Inspired by the aforementioned works, in this article, based upon a fixed point theorem of increasing Ψ-(h,e)-concave operators, we aim to establish the existence and uniqueness of solutions for system (1.1). Our results show that the unique solution exists in a product set and can be approximated by making an iterative sequence for any initial point in the product set.

The paper is organized as follows. In Sect. 2, we propose not only some definitions and lemmas to be used to prove our main results, but also some useful properties of Green functions. In Sect. 3, we discuss the existence and uniqueness of solutions to system (1.1). Finally, in Sect. 4, a concrete example is given as the application of our main results.

2 Preliminaries and lemmas

We present here some definitions and related properties of Riemann–Liouville fractional derivatives and integrals. Some auxiliary results which will be used to prove our main results are also given.

Definition 2.1 ([39, 40]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to (-\infty, +\infty)$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([39, 40]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to (-\infty, +\infty)$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

From Lemmas 2.3, 2.5 in [33], we can easily get the following conclusions.

Lemma 2.1 *Let* $s_1 = \alpha_1 + \beta_1$, $\alpha_1 \in (1, 2]$, $\beta_1 \in (3, 4]$, $\eta_1 \in (0, 1)$, $b_1 \in (0, {\eta_1}^{1-\alpha_1})$. *If* $y_1 \in C[0, 1]$, *then the following fractional boundary value problem*

$$\begin{cases} D_{0^+}^{s_1} u(t) + y_1(t) = 0, & 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \\ D_{0^+}^{\beta_1} u(0) = 0, & D_{0^+}^{\beta_1} u(1) = b_1 D_{0^+}^{\beta_1} u(\eta_1), \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) y_1(\tau)\, d\tau\, ds,$$

where

$$
\begin{aligned}
&G_1(t,s) = \frac{1}{\Gamma(\beta_1)} \begin{cases} t^{\beta_1-2}(1-s)^{\beta_1-2}[(s-t)+(\beta_1-2)(1-t)s], \\ \quad 0 \le t \le s \le 1, \\ t^{\beta_1-2}(1-s)^{\beta_1-2}[(s-t)+(\beta_1-2)(1-t)s]+(t-s)^{\beta_1-1}, \\ \quad 0 \le s \le t \le 1, \end{cases} \\
&H_1(t,s) = h_1(t,s) + \frac{b_1 t^{\alpha_1-1}}{1-b_1\eta_1^{\alpha_1-1}} h_1(\eta_1,s), \\
&h_1(t,s) = \frac{1}{\Gamma(\alpha_1)} \begin{cases} [t(1-s)]^{\alpha_1-1}, & 0 \le t \le s \le 1, \\ [t(1-s)]^{\alpha_1-1} - (t-s)^{\alpha_1-1}, & 0 \le s \le t \le 1. \end{cases}
\end{aligned}
\tag{2.2}
$$

Lemma 2.2 *Let $s_2 = \alpha_2 + \beta_2$, $\alpha_2 \in (1,2]$, $\beta_2 \in (3,4]$, $\eta_2 \in (0,1)$, $b_2 \in (0, {\eta_2}^{1-\alpha_2})$. If $y_2 \in C[0,1]$, then the following fractional boundary value problem*

$$
\begin{cases} D_{0^+}^{s_2} v(t) + y_2(t) = 0, \quad 0 < t < 1, \\ v(0) = v(1) = v'(0) = v'(1) = 0, \\ D_{0^+}^{\beta_2} v(0) = 0, \qquad D_{0^+}^{\beta_2} v(1) = b_2 D_{0^+}^{\beta_2} v(\eta_2), \end{cases}
\tag{2.3}
$$

has a unique solution

$$
v(t) = \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) y_2(\tau)\, d\tau\, ds,
$$

where

$$
\begin{aligned}
&G_2(t,s) = \frac{1}{\Gamma(\beta_2)} \begin{cases} t^{\beta_2-2}(1-s)^{\beta_2-2}[(s-t)+(\beta_2-2)(1-t)s], \\ \quad 0 \le t \le s \le 1, \\ t^{\beta_2-2}(1-s)^{\beta_2-2}[(s-t)+(\beta_2-2)(1-t)s]+(t-s)^{\beta_2-1}, \\ \quad 0 \le s \le t \le 1, \end{cases} \\
&H_2(t,s) = h_2(t,s) + \frac{b_2 t^{\alpha_2-1}}{1-b_2\eta_2^{\alpha_2-1}} h_2(\eta_2,s), \\
&h_2(t,s) = \frac{1}{\Gamma(\alpha_2)} \begin{cases} [t(1-s)]^{\alpha_2-1}, & 0 \le t \le s \le 1, \\ [t(1-s)]^{\alpha_2-1} - (t-s)^{\alpha_2-1}, & 0 \le s \le t \le 1. \end{cases}
\end{aligned}
\tag{2.4}
$$

From Lemmas 2.4, 2.6 in [33], we can easily get the following lemma.

Lemma 2.3 *The functions $G_i(t,s)$, $i = 1,2$, defined by (2.2) and (2.4) have the following properties*:

(i) *$G_i(t,s)$ is continuous on $[0,1] \times [0,1]$ and $G_i(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$*;

(ii) *$(\beta_i - 2)k_i(t)r_i(s) \le \Gamma(\beta_i)G_i(t,s) \le M_i r_i(s)$, $(t,s) \in [0,1] \times [0,1]$*;

(iii) *$(\beta_i - 2)k_i(t)r_i(s) \le \Gamma(\beta_i)G_i(t,s) \le M_i k_i(s)$, $(t,s) \in [0,1] \times [0,1]$*;

where

$$k_i(t) = t^{\beta_i-2}(1-t)^2, \qquad r_i(s) = s^2(1-s)^{\beta_i-2},$$
$$M_i = \max\{\beta_i - 1, (\beta_i - 2)^2\}.$$

Let $(X, \|\cdot\|)$ be a real Banach space with a partial order induced by a cone $P \subset X$. For any $x, y \in X$, the notation $x \sim y$ denotes that there are $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), we consider a set $P_h = \{x \in X | x \sim h\}$. It is clear that $P_h \subset P$. Let $e \in P$ with $\theta \leq e \leq h$, we define $P_{h,e} = \{x \in X \mid x + e \in P_h\}$.

Now, we present the definition of Ψ-(h, e)-concave operator and a fixed point theorem which can be easily used to study some systems of differential equations.

Definition 2.3 ([4]) Suppose that $T : P_{h,e} \to X$ is a given operator which satisfies: for any $x \in P_{h,e}$ and $\xi \in (0, 1)$, there exists $\Psi(\xi) > \xi$ such that

$$T\big(\xi x + (\xi - 1)e\big) \geq \Psi(\xi)Tx + \big(\Psi(\xi) - 1\big)e.$$

Then T is called a Ψ-(h, e)-concave operator.

Lemma 2.4 ([4]) *Assume that T is an increasing Ψ-(h, e)-concave operator satisfying $Th \in P_{h,e}$ and P is normal. Then T has a unique fixed point x^* in $P_{h,e}$. For any $w_0 \in P_{h,e}$, constructing the sequence $w_n = Tw_{n-1}$, $n = 1, 2, \dots$, then $\|w_n - x^*\| \to 0$ as $n \to \infty$.*

For $h_1, h_2 \in P$ with $h_1, h_2 \neq \theta$. Suppose $h = (h_1, h_2)$, then $h \in \bar{P} := P \times P$. Take $\theta \leq e_1 \leq h_1$, $\theta \leq e_2 \leq h_2$, and write $\bar{\theta} = (\theta, \theta)$, $e = (e_1, e_2)$. Then $\bar{\theta} = (\theta, \theta) \leq (e_1, e_2) \leq (h_1, h_2) = h$. That is, $\bar{\theta} \leq e \leq h$. If P is normal, then $\bar{P} = (P, P)$ is normal.

Lemma 2.5 ([31]) $\bar{P}_h = P_{h_1} \times P_{h_2}$.

Lemma 2.6 ([31]) $\bar{P}_{h,e} = P_{h_1,e_1} \times P_{h_2,e_2}$.

3 Main results

In this section, let $X = \{u \mid u \in C[0, 1]\}$, a Banach space with the norm $\|u\| = \sup\{|u(t)| : t \in [0, 1]\}$. We will consider (1.1) in $X \times X$. For $(u, v) \in X \times X$, let $\|(u, v)\| = \max\{\|u\|, \|v\|\}$. Then $(X \times X, \|(u, v)\|)$ is a Banach space. Let $\bar{P} = \{(u, v) \in X \times X \mid u(t) \geq 0, v(t) \geq 0, t \in [0, 1]\}$, $P = \{u \in X \mid u(t) \geq 0, t \in [0, 1]\}$, then the cone $\bar{P} \subset X \times X$ and $\bar{P} = P \times P$ is normal, and the space $X \times X$ has a partial order: $(u_1, v_1) \leq (u_2, v_2)$ if and only if $u_1(t) \leq u_2(t)$, $v_1(t) \leq v_2(t)$, $t \in [0, 1]$.

Lemma 3.1 *Suppose that $f(t, u, v)$, $g(t, u, v)$ are continuous. By Lemmas 2.1, 2.2 and the result of [33], we can claim that $(u, v) \in X \times X$ is a solution of problem (1.1) if and only if $(u, v) \in X \times X$ is a solution of the following equations*:

$$\begin{cases} u(t) = \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau)f(\tau, u(\tau), v(\tau))\,d\tau\,ds - \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau)z_1(\tau)\,d\tau\,ds, \\ v(t) = \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau)g(\tau, u(\tau), v(\tau))\,d\tau\,ds - \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau)z_2(\tau)\,d\tau\,ds. \end{cases}$$

For $(u,v) \in X \times X$, we consider three operators $A_1, A_2 : X \times X \to X$ and $T : X \times X \to X \times X$ by

$$\begin{aligned}
A_1(u,v)(t) &= \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) f\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\, ds \\
&\quad - \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) z_1(\tau)\, d\tau\, ds, \\
A_2(u,v)(t) &= \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) g\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\, ds \\
&\quad - \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) z_2(\tau)\, d\tau\, ds, \\
T(u,v)(t) &= \bigl(A_1(u,v)(t), A_2(u,v)(t)\bigr).
\end{aligned}$$

According to the above results, we can easily get that $(u,v) \in X \times X$ is a solution of system (1.1) if and only if $(u,v) \in X \times X$ is a fixed point of operator T. Let

$$\begin{aligned}
e_1(t) &= \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) z_1(\tau)\, d\tau\, ds, \\
e_2(t) &= \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) z_2(\tau)\, d\tau\, ds, \\
h_1(t) &= N_1 t^{\beta_1 - 2}(1-t)^2, \qquad h_2(t) = N_2 t^{\beta_2 - 2}(1-t)^2, \quad t \in [0,1],
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
N_1 &\geq \frac{M_1}{\Gamma(\beta_1)} \int_0^1 \int_0^1 H_1(s,\tau) z_1(\tau)\, d\tau\, ds, \\
N_2 &\geq \frac{M_2}{\Gamma(\beta_2)} \int_0^1 \int_0^1 H_2(s,\tau) z_2(\tau)\, d\tau\, ds,
\end{aligned}$$

with M_i, $i = 1,2$, given as in Lemma 2.3.

Theorem 3.1 *Let $\alpha_i \in (1,2]$, $\beta_i \in (3,4]$, $z_i : [0,1] \to (0,+\infty)$ be continuous and e_1, e_2, h_1, h_2 be given as in (3.1). Assume that $f, g \in C([0,1] \times \mathbf{R}^2, \mathbf{R})$ and $z_1, z_2 \in C([0,1], [0,+\infty))$. Moreover,*

- (H_1) *$f : [0,1] \times [-e_1{}^*, +\infty) \times [-e_2{}^*, +\infty) \to (-\infty, +\infty)$ and $g : [0,1] \times [-e_1{}^*, +\infty) \times [-e_2{}^*, +\infty) \to (-\infty, +\infty)$ are both increasing with respect to the second and third variables, where $e_1{}^* = \max\{e_1(t) : t \in [0,1]\}$ and $e_2{}^* = \max\{e_2(t) : t \in [0,1]\}$;*
- (H_2) *for $\xi \in (0,1)$, there exists $\psi(\xi) > \xi$ such that*

$$\begin{aligned}
&f\bigl(t, \xi x_1 + (\xi - 1) y_1, \xi x_2 + (\xi - 1) y_2\bigr) \geq \psi(\xi) f(t, x_1, x_2), \\
&g\bigl(t, \xi x_1 + (\xi - 1) y_1, \xi x_2 + (\xi - 1) y_2\bigr) \geq \psi(\xi) g(t, x_1, x_2),
\end{aligned}$$

  where $t \in [0,1]$, $x_1, x_2 \in (-\infty, +\infty)$, $y_1 \in [0, e_1{}^*]$ and $y_2 \in [0, e_2{}^*]$;
- (H_3) *$f(t,0,0) \geq 0$, $g(t,0,0) \geq 0$, with $f(t,0,0) \not\equiv 0$, $g(t,0,0) \not\equiv 0$ for $t \in [0,1]$.*

Then:

(1) *system* (1.1) *has a unique solution* (u^*,v^*) *in* $\bar{P}_{h,e}$, *where*

$$e(t) = \bigl(e_1(t), e_2(t)\bigr), \qquad h(t) = \bigl(h_1(t), h_2(t)\bigr), \quad t \in [0,1];$$

(2) *for a given point* $(u_0,v_0) \in \bar{P}_{h,e}$, *construct the following sequences*:

$$\begin{aligned} u_{n+1}(t) &= \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) f\bigl(\tau, u_n(\tau), v_n(\tau)\bigr)\,d\tau\,ds \\ &\quad - \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) z_1(\tau)\,d\tau\,ds, \\ v_{n+1}(t) &= \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) g\bigl(\tau, u_n(\tau), v_n(\tau)\bigr)\,d\tau\,ds \\ &\quad - \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) z_2(\tau)\,d\tau\,ds, \end{aligned}$$

$n = 0,1,2,\ldots$, *we have* $u_{n+1}(t) \to u^*(t)$, $v_{n+1}(t) \to v^*(t)$ *as* $n \to \infty$.

Proof According to the properties of $G_i(t,s)$, since $H_i(t,s) \geq 0$, $i = 1,2$, we can get

$$\begin{aligned} e_1(t) &= \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) z_1(\tau)\,d\tau\,ds \geq 0, \quad t \in [0,1], \\ e_2(t) &= \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) z_2(\tau)\,d\tau\,ds \geq 0, \quad t \in [0,1]. \end{aligned}$$

From Lemma 2.3, one can obtain that, for $t \in [0,1]$,

$$\begin{aligned} e_1(t) &= \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) z_1(\tau)\,d\tau\,ds \\ &\leq \int_0^1 \frac{M_1 k_1(t)}{\Gamma(\beta_1)} \int_0^1 H_1(s,\tau) z_1(\tau)\,d\tau\,ds \\ &= \frac{M_1}{\Gamma(\beta_1)} t^{\beta_1-2}(1-t)^2 \int_0^1 \int_0^1 H_1(s,\tau) z_1(\tau)\,d\tau\,ds \\ &\leq N_1 t^{\beta_1-2}(1-t)^2 = h_1(t); \\ e_2(t) &= \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) z_1(\tau)\,d\tau\,ds \\ &\leq \int_0^1 \frac{M_2 k_2(t)}{\Gamma(\beta_2)} \int_0^1 H_2(s,\tau) z_2(\tau)\,d\tau\,ds \\ &= \frac{M_2}{\Gamma(\beta_2)} t^{\beta_2-2}(1-t)^2 \int_0^1 \int_0^1 H_2(s,\tau) z_2(\tau)\,d\tau\,ds \\ &\leq N_2 t^{\beta_2-2}(1-t)^2 = h_2(t). \end{aligned}$$

That is, $0 \leq e_1 \leq h_1$ and $0 \leq e_2 \leq h_2$.

In the following, we prove that $T : \bar{P}_{h,e} \to X \times X$ is a Ψ-(h,e)-concave operator. For $(u,v) \in \bar{P}_{h,e}$, $\xi \in (0,1)$, $t \in [0,1]$, one can obtain

$$\begin{aligned}
&T\bigl(\xi(u,v) + (\xi - 1)e\bigr)(t) \\
&\quad = T\bigl(\xi(u,v) + (\xi - 1)(e_1, e_2)\bigr)(t) \\
&\quad = T\bigl(\xi u + (\xi - 1)e_1, \xi v + (\xi - 1)e_2\bigr)(t) \\
&\quad = \bigl(A_1\bigl(\xi u + (\xi - 1)e_1, \xi v + (\xi - 1)e_2\bigr), A_2\bigl(\xi u + (\xi - 1)e_1, \xi v + (\xi - 1)e_2\bigr)\bigr)(t).
\end{aligned}$$

We consider $A_1(\xi u + (\xi - 1)e_1, \xi v + (\xi - 1)e_2)(t)$ and $A_2(\xi u + (\xi - 1)e_1, \xi v + (\xi - 1)e_2)(t)$, respectively. From (H_2),

$$\begin{aligned}
&A_1\bigl(\xi u + (\xi - 1)e_1, \xi v + (\xi - 1)e_2\bigr)(t) \\
&\quad = \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) f\bigl(\tau, \bigl(\xi u + (\xi - 1)e_1\bigr)(\tau), \bigl(\xi v + (\xi - 1)e_2\bigr)(\tau)\bigr)\, d\tau\, ds - e_1(t) \\
&\quad \geq \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) \psi(\xi) f\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\, ds - e_1(t) \\
&\quad = \psi(\xi) \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) f\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\, ds - e_1(t) \\
&\quad = \psi(\xi) \biggl[\int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) f\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\, ds - e_1(t)\biggr] + \bigl(\psi(\xi) - 1\bigr) e_1(t) \\
&\quad = \psi(\xi) A_1(u,v)(t) + \bigl(\psi(\xi) - 1\bigr) e_1(t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&A_2\bigl(\xi u + (\xi - 1)e_1, \xi v + (\xi - 1)e_2\bigr)(t) \\
&\quad = \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) f\bigl(\tau, \bigl(\xi u + (\xi - 1)e_1\bigr)(\tau), \bigl(\xi v + (\xi - 1)e_2\bigr)(\tau)\bigr)\, d\tau\, ds - e_2(t) \\
&\quad \geq \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) \psi(\xi) f\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\, ds - e_2(t) \\
&\quad = \psi(\xi) \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) f\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\, ds - e_2(t) \\
&\quad = \psi(\xi) \biggl[\int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) f\bigl(\tau, u(\tau), v(\tau)\bigr)\, d\tau\, ds - e_2(t)\biggr] + \bigl(\psi(\xi) - 1\bigr) e_2(t) \\
&\quad = \psi(\xi) A_2(u,v)(t) + \bigl(\psi(\xi) - 1\bigr) e_2(t).
\end{aligned}$$

So we have

$$\begin{aligned}
&T\bigl(\xi(u,v) + (\xi - 1)e\bigr)(t) \\
&\quad \geq \bigl(\psi(\xi) A_1(u,v)(t) + \bigl(\psi(\xi) - 1\bigr) e_1(t), \psi(\xi) A_2(u,v)(t) + \bigl(\psi(\xi) - 1\bigr) e_2(t)\bigr) \\
&\quad = \bigl(\psi(\xi) A_1(u,v)(t), \psi(\xi) A_2(u,v)(t)\bigr) + \bigl(\bigl(\psi(\xi) - 1\bigr) e_1(t), \bigl(\psi(\xi) - 1\bigr) e_2(t)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&= \psi(\xi)\bigl(A_1(u,v)(t), A_2(u,v)(t)\bigr) + \bigl(\psi(\xi)-1\bigr)\bigl(e_1(t), e_2(t)\bigr) \\
&= \psi(\xi)T(u,v)(t) + \bigl(\psi(\xi)-1\bigr)e(t).
\end{aligned}
$$

That is,

$$
T\bigl(\xi(u,v)+(\xi-1)e\bigr) \geq \psi(\xi)T(u,v) + \bigl(\psi(\xi)-1\bigr)e, \quad (u,v)\in \bar{P}_{h,e}, \xi\in(0,1).
$$

Therefore, T is a Ψ-(h,e)-concave operator.

Next we show that $T:\bar{P}_{h,e}\to X\times X$ is increasing. For $(u,v)\in\bar{P}_{h,e}$, we have $(u,v)+e\in\bar{P}_h$. From Lemma 2.5, $(u+e_1, v+e_2)\in P_{h_1}\times P_{h_2}$. So there are $\lambda_1,\lambda_2>0$ such that

$$
u(t)+e_1(t)\geq \lambda_1 h_1(t), \qquad v(t)+e_2(t)\geq\lambda_2 h_2(t), \quad t\in[0,1].
$$

Therefore, $u(t)\geq\lambda_1 h_1(t)-e_1(t)\geq -e_1(t)\geq -e_1{}^*$, $v(t)\geq \lambda_2 h_2(t)-e_2(t)\geq -e_2(t)\geq -e_2{}^*$. This fact and (H_1) imply that $T:\bar{P}_{h,e}\to X\times X$ is increasing.

Now we show $Th\in\bar{P}_{h,e}$, which needs to prove $Th+e\in\bar{P}_h$. For $t\in[0,1]$,

$$
\begin{aligned}
Th(t)+e(t) &= T(h_1,h_2)(t)+e(t) \\
&= \bigl(A_1(h_1,h_2)(t), A_2(h_1,h_2)(t)\bigr) + \bigl(e_1(t), e_2(t)\bigr) \\
&= \bigl(A_1(h_1,h_2)(t)+e_1(t), A_2(h_1,h_2)(t)+e_2(t)\bigr).
\end{aligned}
$$

We consider $A_1(h_1,h_2)(t)+e_1(t)$ and $A_2(h_1,h_2)(t)+e_2(t)$, respectively. By Lemmas 2.2, 2.3 and (H_1), (H_3), we can get

$$
\begin{aligned}
&A_1(h_1,h_2)(t)+e_1(t) \\
&\quad= \int_0^1 G_1(t,s)\int_0^1 H_1(s,\tau)f\bigl(\tau,h_1(\tau),h_2(\tau)\bigr)\,d\tau\,ds \\
&\quad\geq \int_0^1 \frac{(\beta_1-2)k_1(t)r_1(s)}{\Gamma(\beta_1)}\int_0^1 H_1(s,\tau)f\bigl(\tau,N_1\tau^{\beta_1-2}(1-\tau)^2, N_2\tau^{\beta_2-2}(1-\tau)^2\bigr)\,d\tau\,ds \\
&\quad\geq \frac{(\beta_1-2)k_1(t)}{\Gamma(\beta_1)}\int_0^1 r_1(s)\int_0^1 H_1(s,\tau)f(\tau,0,0)\,d\tau\,ds \\
&\quad= \frac{\beta_1-2}{N_1\Gamma(\beta_1)}h_1(t)\int_0^1 r_1(s)\int_0^1 H_1(s,\tau)f(\tau,0,0)\,d\tau\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
&A_1(h_1,h_2)(t)+e_1(t) \\
&\quad= \int_0^1 G_1(t,s)\int_0^1 H_1(s,\tau)f\bigl(\tau,h_1(\tau),h_2(\tau)\bigr)\,d\tau\,ds \\
&\quad\leq \int_0^1 \frac{M_1k_1(t)}{\Gamma(\beta_1)}\int_0^1 H_1(s,\tau)f(\tau,N_1,N_2)\,d\tau\,ds \\
&\quad= \frac{M_1}{N_1\Gamma(\beta_1)}h_1(t)\int_0^1\int_0^1 H_1(s,\tau)f(\tau,N_1,N_2)\,d\tau\,ds.
\end{aligned}
$$

According to (H_1), (H_3), since $r_1(s) = s^2(1-s)^{\beta_1-2} \leq 1$, $s \in [0,1]$, then

$$\int_0^1 \int_0^1 H_1(s,\tau)f(\tau,N_1,N_2)\,d\tau\,ds \geq \int_0^1 r_1(s)\int_0^1 H_1(s,\tau)f(\tau,0,0)\,d\tau\,ds \geq 0.$$

By the definition of M_1, it is clear that $M_1 \geq \beta_1 - 2$, then one can get

$$\begin{aligned} l_1 &:= \frac{\beta_1-2}{N_1\Gamma(\beta_1)}\int_0^1 r_1(s)\int_0^1 H_1(s,\tau)f(\tau,0,0)\,d\tau\,ds \\ &\leq l_2 := \frac{M_1}{N_1\Gamma(\beta_1)}\int_0^1\int_0^1 H_1(s,\tau)f(\tau,N_1,N_2)\,d\tau\,ds. \end{aligned}$$

Thus we obtain that $l_1h_1(t) \leq A_1(h_1,h_2)(t) + e_1(t) \leq l_2h_1(t)$. That is, $A_1(h_1,h_2) + e_1 \in P_{h_1}$. Similarly, by using Lemma 2.3 and (H_1), (H_3), we also can get $A_2(h_1,h_2) + e_2 \in P_{h_2}$. Consequently, according to Lemma 2.5,

$$Th + e = \bigl(A_1(h_1,h_2) + e_1, A_2(h_1,h_2) + e_2\bigr) \in P_{h_1} \times P_{h_2} = \bar{P}_h.$$

Finally, by using Lemma 2.4, T has a unique fixed point $(u^*,v^*) \in \bar{P}_{h,e}$. In addition, for any given $(u_0,v_0) \in \bar{P}_{h,e}$, the sequence

$$(u_n,v_n) = \bigl(A_1(u_{n-1},v_{n-1}), A_2(u_{n-1},v_{n-1})\bigr), \quad n = 1,2,\ldots,$$

converges to (u^*,v^*) as $n \to \infty$. Therefore, system (1.1) has a unique solution $(u^*,v^*) \in \bar{P}_{h,e}$; taking any point $(u_0,v_0) \in \bar{P}_{h,e}$, construct the following sequences:

$$\begin{aligned} u_{n+1}(t) &= \int_0^1 G_1(t,s)\int_0^1 H_1(s,\tau)f\bigl(\tau,u_n(\tau),v_n(\tau)\bigr)\,d\tau\,ds \\ &\quad - \int_0^1 G_1(t,s)\int_0^1 H_1(s,\tau)z_1(\tau)\,d\tau\,ds, \\ v_{n+1}(t) &= \int_0^1 G_2(t,s)\int_0^1 H_2(s,\tau)g\bigl(\tau,u_n(\tau),v_n(\tau)\bigr)\,d\tau\,ds \\ &\quad - \int_0^1 G_2(t,s)\int_0^1 H_2(s,\tau)z_2(\tau)\,d\tau\,ds, \end{aligned}$$

$n = 0,1,2,\ldots$, we have $u_{n+1}(t) \to u^*(t)$, $v_{n+1}(t) \to v^*(t)$ as $n \to \infty$. □

4 An example

We consider the following fractional system:

$$\begin{cases} D_{0^+}^{\frac{3}{2}+\frac{10}{3}}u(t) + (\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma(\frac{29}{6})u+1)^{\frac{1}{5}}[\frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}(A_1+B_1t+C_1t^{\frac{5}{2}}+Dt^{\frac{9}{2}})]^{\frac{1}{5}}t^{\frac{4}{15}} \\ \quad + (\frac{27-3\sqrt{3}}{88}\Gamma(\frac{29}{6})v+1)^{\frac{1}{5}}[\frac{3}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}(A_2+B_2t+C_2t^{\frac{5}{2}}+Dt^{\frac{9}{2}})]^{\frac{1}{5}}t^{\frac{4}{15}} = t, \quad 0<t<1, \\ D_{0^+}^{\frac{3}{2}+\frac{10}{3}}v(t) + (\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma(\frac{29}{6})u+1)^{\frac{1}{3}}[\frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}(A_1+B_1t+C_1t^{\frac{5}{2}}+Dt^{\frac{9}{2}})]^{\frac{1}{3}}t^{\frac{4}{9}} \\ \quad + (\frac{27-3\sqrt{3}}{88}\Gamma(\frac{29}{6})v+1)^{\frac{1}{3}}[\frac{3}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}(A_2+B_2t+C_2t^{\frac{5}{2}}+Dt^{\frac{9}{2}})]^{\frac{1}{3}}t^{\frac{4}{9}} = 3t, \quad 0<t<1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \qquad D_{0^+}^{\frac{10}{3}}u(0) = 0, \qquad D_{0^+}^{\frac{10}{3}}u(1) = \frac{1}{2}D_{0^+}^{\frac{10}{3}}u(\frac{1}{2}), \\ v(0) = v(1) = v'(0) = v'(1) = 0, \qquad D_{0^+}^{\frac{10}{3}}v(0) = 0, \qquad D_{0^+}^{\frac{10}{3}}v(1) = \frac{1}{3}D_{0^+}^{\frac{10}{3}}v(\frac{1}{3}), \end{cases} \tag{4.1}$$

where $\alpha_1 = \alpha_2 = \frac{3}{2}$, $\beta_1 = \beta_2 = \frac{10}{3}$, $z_1(t) = t$, $z_2(t) = 3t$, $\eta_1 = \frac{1}{2}$, $\eta_2 = \frac{1}{3}$, $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{3}$, and

$$\begin{aligned}
A_1 &= \frac{7}{3}\left(\frac{4}{15} + \frac{1}{10\sqrt{2}-5}\right)\frac{\Gamma(\frac{5}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{29}{6})} - \frac{7}{3}\frac{4}{15}\frac{\Gamma(\frac{9}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{41}{6})},\\
B_1 &= -\frac{4}{3}\left(\frac{4}{15} + \frac{1}{10\sqrt{2}-5}\right)\frac{\Gamma(\frac{5}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{29}{6})} + \frac{4}{3}\frac{4}{15}\frac{\Gamma(\frac{9}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{41}{6})}\\
&\quad - \left(\frac{4}{15} + \frac{1}{10\sqrt{2}-5}\right)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{23}{6})} + \frac{4}{15}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{35}{6})},\\
C_1 &= \left(\frac{4}{15} + \frac{1}{10\sqrt{2}-5}\right)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{29}{6})},\\
D &= -\frac{4}{15}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{41}{6})},\\
A_2 &= \frac{7}{3}\left(\frac{4}{15} + \frac{12+8\sqrt{3}}{135\sqrt{3}-45}\right)\frac{\Gamma(\frac{5}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{29}{6})} - \frac{7}{3}\frac{4}{15}\frac{\Gamma(\frac{9}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{41}{6})},\\
B_2 &= -\frac{4}{3}\left(\frac{4}{15} + \frac{12+8\sqrt{3}}{135\sqrt{3}-45}\right)\frac{\Gamma(\frac{5}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{26}{9})} + \frac{4}{3}\frac{4}{15}\frac{\Gamma(\frac{9}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{41}{6})}\\
&\quad - \left(\frac{4}{15} + \frac{12+8\sqrt{3}}{135\sqrt{3}-45}\right)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{23}{6})} + \frac{4}{15}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{35}{6})},\\
C_2 &= \left(\frac{4}{15} + \frac{12+8\sqrt{3}}{135\sqrt{3}-45}\right)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{29}{6})},\\
f(t,u,v) &= \left(\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma\left(\frac{29}{6}\right)u+1\right)^{\frac{1}{5}}\left[\frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}\left(A_1 + B_1 t + C_1 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\right)\right]^{\frac{1}{5}} t^{\frac{4}{15}}\\
&\quad + \left(\frac{27-3\sqrt{3}}{88}\Gamma\left(\frac{29}{6}\right)v+1\right)^{\frac{1}{5}}\left[\frac{3}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}\left(A_2 + B_2 t + C_2 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\right)\right]^{\frac{1}{5}} t^{\frac{4}{15}},\\
g(t,u,v) &= \left(\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma\left(\frac{29}{6}\right)u+1\right)^{\frac{1}{3}}\left[\frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}\left(A_1 + B_1 t + C_1 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\right)\right]^{\frac{1}{3}} t^{\frac{4}{9}}\\
&\quad + \left(\frac{27-3\sqrt{3}}{88}\Gamma\left(\frac{29}{6}\right)v+1\right)^{\frac{1}{3}}\left[\frac{3}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}\left(A_2 + B_2 t + C_2 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\right)\right]^{\frac{1}{3}} t^{\frac{4}{9}}.
\end{aligned}$$

So

$$\begin{aligned}
f(t,0,0) &= \left[\frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}\left(A_1 + B_1 t + C_1 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\right)\right]^{\frac{1}{5}} t^{\frac{4}{15}}\\
&\quad + \left[\frac{3}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}\left(A_2 + B_2 t + C_2 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\right)\right]^{\frac{1}{5}} t^{\frac{4}{15}},\\
g(t,0,0) &= \left[\frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}\left(A_1 + B_1 t + C_1 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\right)\right]^{\frac{1}{3}} t^{\frac{4}{9}}\\
&\quad + \left[\frac{3}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}\left(A_2 + B_2 t + C_2 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\right)\right]^{\frac{1}{3}} t^{\frac{4}{9}}.
\end{aligned}$$

Figure 1 Image of function $f(t,0,0)$ on $t \in [0,1]$

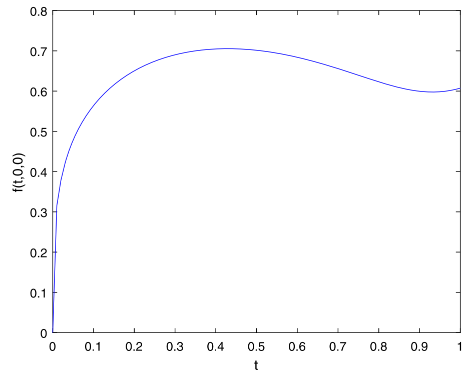
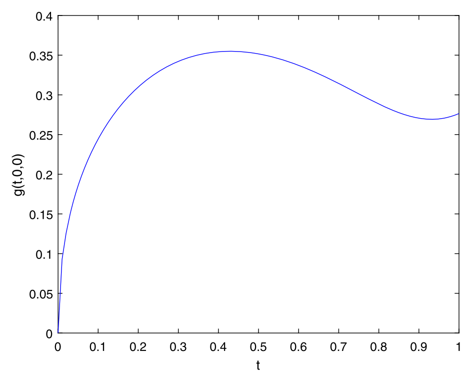


Figure 2 Image of function $g(t,0,0)$ on $t \in [0,1]$



We can obviously get $f(t,0,0) \geq 0$, $g(t,0,0) \geq 0$ from Figs. 1, 2. And $f(t,0,0) \not\equiv 0$, $g(t,0,0) \not\equiv 0$. Moreover,

$$G_1(t,s) = \frac{1}{\Gamma(\frac{10}{3})} \begin{cases} t^{\frac{4}{3}}(1-s)^{\frac{4}{3}}[(s-t) + \frac{4}{3}(1-t)s], & 0 \leq t \leq s \leq 1, \\ t^{\frac{4}{3}}(1-s)^{\frac{4}{3}}[(s-t) + \frac{4}{3}(1-t)s] + (t-s)^{\frac{7}{3}}, & 0 \leq s \leq t \leq 1, \end{cases} \tag{4.2}$$

$$H_1(t,s) = h_1(t,s) + \frac{\frac{1}{2}t^{\frac{1}{2}}}{1-\frac{1}{2}^{\frac{3}{2}}} h_1\left(\frac{1}{2}, s\right),$$

$$h_1(t,s) = \frac{1}{\Gamma(\frac{3}{2})} \begin{cases} [t(1-s)]^{\frac{1}{2}}, & 0 \leq t \leq s \leq 1, \\ [t(1-s)]^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}, & 0 \leq s \leq t \leq 1. \end{cases}$$

$$G_2(t,s) = \frac{1}{\Gamma(\frac{10}{3})} \begin{cases} t^{\frac{4}{3}}(1-s)^{\frac{4}{3}}[(s-t) + \frac{4}{3}(1-t)s], & 0 \leq t \leq s \leq 1, \\ t^{\frac{4}{3}}(1-s)^{\frac{4}{3}}[(s-t) + \frac{4}{3}(1-t)s] + (t-s)^{\frac{7}{3}}, & 0 \leq s \leq t \leq 1, \end{cases} \tag{4.3}$$

$$H_2(t,s) = h_2(t,s) + \frac{\frac{1}{3}t^{\frac{1}{2}}}{1-\frac{1}{3}^{\frac{3}{2}}} h_1\left(\frac{1}{3}, s\right),$$

$$h_2(t,s) = \frac{1}{\Gamma(\frac{3}{2})} \begin{cases} [t(1-s)]^{\frac{1}{2}}, & 0 \leq t \leq s \leq 1, \\ [t(1-s)]^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Further,

$$\begin{aligned}
e_1(t) &= \int_0^1 G_1(t,s)\int_0^1 H_1(s,\tau)z_1(\tau)\,d\tau\,ds \\
&= \int_0^1 G_1(t,s)\biggl[\int_0^1 \tau h_1(s,\tau)\,d\tau + \int_0^1 \tau\frac{\frac{1}{2}s^{\frac{1}{2}}}{1-\frac{1}{2}^{\frac{3}{2}}}h_1\biggl(\frac{1}{2},\tau\biggr)\,d\tau\biggr]\,ds \\
&= \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 G_1(t,s)\biggl[\int_0^1 \tau\bigl(s(1-\tau)\bigr)^{\frac{1}{2}}\,d\tau - \int_0^s \tau(s-\tau)^{\frac{1}{2}}\,d\tau \\
&\quad + \int_0^1 \tau\frac{\frac{1}{2}s^{\frac{1}{2}}}{1-\frac{1}{2}^{\frac{3}{2}}}\biggl(\frac{1}{2}(1-\tau)\biggr)^{\frac{1}{2}}\,d\tau - \int_0^{\frac{1}{2}} \tau\frac{\frac{1}{2}s^{\frac{1}{2}}}{1-\frac{1}{2}^{\frac{3}{2}}}\biggl(\frac{1}{2}-\tau\biggr)^{\frac{1}{2}}\,d\tau\biggr]\,ds \\
&= \frac{1}{\Gamma(\frac{3}{2})}\int_0^1 G_1(t,s)\biggl[\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)s^{\frac{1}{2}} - \frac{4}{15}s^{\frac{5}{2}}\biggr]\,ds \\
&= \frac{1}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}\biggl\{\int_0^1\biggl[t^{\frac{4}{3}}(1-s)^{\frac{4}{3}}\biggl((s-t)+\frac{4}{3}(1-t)s\biggr)\biggr] \\
&\quad \times\biggl[\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)s^{\frac{1}{2}} - \frac{4}{15}s^{\frac{5}{2}}\biggr]\,ds \\
&\quad + \int_0^t (t-s)^{\frac{7}{3}}\biggl[\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)s^{\frac{1}{2}} - \frac{4}{15}s^{\frac{5}{2}}\biggr]\,ds\biggr\} \\
&= \frac{1}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}\biggl\{t^{\frac{4}{3}}\biggl(\frac{7}{3}-\frac{4}{3}t\biggr)\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)\int_0^1 s^{\frac{3}{2}}(1-s)^{\frac{4}{3}}\,ds \\
&\quad - t^{\frac{4}{3}}\frac{4}{15}\biggl(\frac{7}{3}-\frac{4}{3}t\biggr)\int_0^1 s^{\frac{7}{2}}(1-s)^{\frac{4}{3}}\,ds \\
&\quad - t^{\frac{7}{3}}\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)\int_0^1 s^{\frac{1}{2}}(1-s)^{\frac{4}{3}}\,ds + t^{\frac{7}{3}}\frac{4}{15}\int_0^1 s^{\frac{5}{2}}(1-s)^{\frac{4}{3}}\,ds \\
&\quad + \int_0^t (t-s)^{\frac{7}{3}}\biggl[\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)^{\frac{1}{2}} - \frac{4}{15}s^{\frac{5}{2}}\biggr]\,ds\biggr\} \\
&= \frac{1}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}\biggl\{t^{\frac{4}{3}}\biggl(\frac{7}{3}-\frac{4}{3}t\biggr)\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)\frac{\Gamma(\frac{5}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{29}{6})} \\
&\quad - t^{\frac{4}{3}}\frac{4}{15}\biggl(\frac{7}{3}-\frac{4}{3}t\biggr)\frac{\Gamma(\frac{9}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{41}{6})} - t^{\frac{7}{3}}\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{23}{6})} \\
&\quad + t^{\frac{7}{3}}\frac{4}{15}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{35}{6})} + t^{\frac{23}{6}}\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{29}{6})} \\
&\quad - t^{\frac{35}{6}}\frac{4}{15}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{41}{6})}\biggr\} \\
&= \frac{1}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}t^{\frac{4}{3}}\bigl(A_1 + B_1 t + C_1 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\bigr), \\
e_2(t) &= \int_0^1 G_2(t,s)\int_0^1 H_2(s,\tau)z_2(\tau)\,d\tau\,ds \\
&= \int_0^1 G_2(t,s)\biggl[\int_0^1 3\tau h_2(s,\tau)\,d\tau + \int_0^1 3\tau\frac{\frac{1}{3}s^{\frac{1}{2}}}{1-\frac{1}{3}^{\frac{3}{2}}}h_2\biggl(\frac{1}{3},\tau\biggr)\,d\tau\biggr]\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{\Gamma(\frac{3}{2})}\int_0^1 G_2(t,s)\biggl[\int_0^1 \tau\bigl(s(1-\tau)\bigr)^{\frac{1}{2}}\,d\tau - \int_0^s \tau(s-\tau)^{\frac{1}{2}}\,d\tau \\
&\quad + \int_0^1 \tau\frac{\frac{1}{3}s^{\frac{1}{2}}}{1-\frac{1}{3}^{\frac{3}{2}}}\biggl(\frac{1}{3}(1-\tau)\biggr)^{\frac{1}{2}}\,d\tau - \int_0^{\frac{1}{3}} \tau\frac{\frac{1}{3}s^{\frac{1}{2}}}{1-\frac{1}{3}^{\frac{3}{2}}}\biggl(\frac{1}{3}-\tau\biggr)^{\frac{1}{2}}\,d\tau\biggr]\,ds \\
&= \frac{3}{\Gamma(\frac{3}{2})}\int_0^1 G_2(t,s)\biggl[\biggl(\frac{4}{15}+\frac{12+8\sqrt{3}}{135\sqrt{3}-45}\biggr)s^{\frac{1}{2}} - \frac{4}{15}s^{\frac{5}{2}}\biggr]\,ds \\
&= \frac{3}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}\biggl\{\int_0^1\biggl[t^{\frac{4}{3}}(1-s)^{\frac{4}{3}}\biggl((s-t)+\frac{4}{3}(1-t)s\biggr)\biggr] \\
&\quad \times\biggl[\biggl(\frac{4}{15}+\frac{12+8\sqrt{3}}{135\sqrt{3}-45}\biggr)s^{\frac{1}{2}} - \frac{4}{15}s^{\frac{5}{2}}\biggr]\,ds \\
&\quad + \int_0^t (t-s)^{\frac{7}{3}}\biggl[\biggl(\frac{4}{15}+\frac{12+8\sqrt{3}}{135\sqrt{3}-45}\biggr)s^{\frac{1}{2}} - \frac{4}{15}s^{\frac{5}{2}}\biggr]\,ds\biggr\} \\
&= \frac{3}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}\biggl\{t^{\frac{4}{3}}\biggl(\frac{7}{3}-\frac{4}{3}t\biggr)\biggl(\frac{4}{15}+\frac{12+8\sqrt{3}}{135\sqrt{3}-45}\biggr)\int_0^1 s^{\frac{3}{2}}(1-s)^{\frac{4}{3}}\,ds \\
&\quad - t^{\frac{4}{3}}\frac{4}{15}\biggl(\frac{7}{3}-\frac{4}{3}t\biggr)\int_0^1 s^{\frac{7}{2}}(1-s)^{\frac{4}{3}}\,ds \\
&\quad - t^{\frac{7}{3}}\biggl(\frac{4}{15}+\frac{12+8\sqrt{3}}{135\sqrt{3}-45}\biggr)\int_0^1 s^{\frac{1}{2}}(1-s)^{\frac{4}{3}}\,ds + t^{\frac{7}{3}}\frac{4}{15}\int_0^1 s^{\frac{5}{2}}(1-s)^{\frac{4}{3}}\,ds \\
&\quad + \int_0^t (t-s)^{\frac{7}{3}}\biggl[\biggl(\frac{4}{15}+\frac{12+8\sqrt{3}}{135\sqrt{3}-45}\biggr)^{\frac{1}{2}} - \frac{4}{15}s^{\frac{5}{2}}\biggr]\,ds\biggr\} \\
&= \frac{3}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}\biggl\{t^{\frac{4}{3}}\biggl(\frac{7}{3}-\frac{4}{3}t\biggr)\biggl(\frac{4}{15}+\frac{12+8\sqrt{3}}{135\sqrt{3}-45}\biggr)\frac{\Gamma(\frac{5}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{29}{6})} \\
&\quad - t^{\frac{4}{3}}\frac{4}{15}\biggl(\frac{7}{3}-\frac{4}{3}t\biggr)\frac{\Gamma(\frac{9}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{41}{6})} - t^{\frac{7}{3}}\biggl(\frac{4}{15}+\frac{12+8\sqrt{3}}{135\sqrt{3}-45}\biggr)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{23}{6})} \\
&\quad + t^{\frac{7}{3}}\frac{4}{15}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{35}{6})} + t^{\frac{23}{6}}\biggl(\frac{4}{15}+\frac{12+8\sqrt{3}}{135\sqrt{3}-45}\biggr)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{29}{6})} \\
&\quad - t^{\frac{35}{6}}\frac{4}{15}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{41}{6})}\biggr\} \\
&= \frac{1}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}t^{\frac{4}{3}}\bigl(A_2+B_2t+C_2t^{\frac{5}{2}}+Dt^{\frac{9}{2}}\bigr).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
e_1^* &= \max\bigl\{e_1(t): t\in[0,1]\bigr\} \\
&\le \frac{1}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}\biggl[\frac{7}{3}\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)\frac{\Gamma(\frac{5}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{29}{6})} + \biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{29}{6})}\biggr] \\
&\le \frac{1}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}\biggl[\frac{10}{3}\biggl(\frac{4}{15}+\frac{1}{10\sqrt{2}-5}\biggr)\frac{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{29}{6})}\biggr] \\
&= \frac{1}{\Gamma(\frac{29}{6})}\frac{2}{3}\biggl(\frac{4}{3}+\frac{1}{2\sqrt{2}-1}\biggr),
\end{aligned}
$$

$$\begin{aligned}
e_2^* &= \max\{e_2(t) : t \in [0,1]\} \\
&\le \frac{3}{\Gamma(\frac{3}{2})} \frac{1}{\Gamma(\frac{10}{3})} \Bigg[\frac{7}{3} \left(\frac{4}{15} + \frac{12 + 8\sqrt{3}}{135\sqrt{3} - 45} \right) \frac{\Gamma(\frac{5}{2})\Gamma(\frac{7}{3})}{\Gamma(\frac{29}{6})} \\
&\quad + \left(\frac{4}{15} + \frac{12 + 8\sqrt{3}}{135\sqrt{3} - 45} \right) \frac{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{29}{6})} \Bigg] \\
&\le \frac{3}{\Gamma(\frac{3}{2})} \frac{1}{\Gamma(\frac{10}{3})} \left[\frac{10}{3} \left(\frac{4}{15} + \frac{12 + 8\sqrt{3}}{135\sqrt{3} - 45} \right) \frac{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})}{\Gamma(\frac{29}{6})} \right] \\
&= \frac{2}{\Gamma(\frac{29}{6})} \frac{2}{3} \left(\frac{4}{3} + \frac{12 + 8\sqrt{3}}{27\sqrt{3} - 9} \right).
\end{aligned}$$

Take $h_1(t) = N_1 t^{\frac{4}{3}}(1-t)^2$, $h_2(t) = N_2 t^{\frac{4}{3}}(1-t)^2$, where

$$\begin{aligned}
N_1 &\ge \frac{M_1}{\Gamma(\beta_1)} \int_0^1 \int_0^1 H_1(s,\tau) z_1(\tau)\, d\tau\, ds \\
&= \frac{7}{3\Gamma(\frac{10}{3})} \int_0^1 \int_0^1 \tau H_1(s,\tau)\, d\tau\, ds \\
&= \frac{7}{3\Gamma(\frac{10}{3})\Gamma(\frac{3}{2})} \int_0^1 \left(\frac{4}{15} + \frac{1}{10\sqrt{2} - 5} \right) s^{\frac{1}{2}} - \frac{4}{15} s^{\frac{5}{2}}\, ds \\
&= \frac{7}{3\Gamma(\frac{10}{3})\Gamma(\frac{3}{2})} \left(\frac{32}{315} + \frac{2}{30\sqrt{2} - 15} \right), \\
N_2 &\ge \frac{M_2}{\Gamma(\beta_2)} \int_0^1 \int_0^1 H_2(s,\tau) z_2(\tau)\, d\tau\, ds \\
&= \frac{7}{3\Gamma(\frac{10}{3})} \int_0^1 \int_0^1 3\tau H_2(s,\tau)\, d\tau\, ds \\
&= \frac{7}{\Gamma(\frac{10}{3})\Gamma(\frac{3}{2})} \int_0^1 \left(\frac{4}{15} + \frac{20\sqrt{3} - 12}{135\sqrt{3} - 45} \right) s^{\frac{1}{2}} - \frac{4}{15} s^{\frac{5}{2}}\, ds \\
&= \frac{7}{\Gamma(\frac{10}{3})\Gamma(\frac{3}{2})} \left(\frac{32}{315} + \frac{24 + 16\sqrt{3}}{405\sqrt{3} - 135} \right).
\end{aligned}$$

Then

$$\begin{aligned}
e_1(t) &= \frac{1}{\Gamma(\frac{3}{2})} \frac{1}{\Gamma(\frac{10}{3})} t^{\frac{4}{3}} \left(A_1 + B_1 t + C_1 t^{\frac{5}{2}} + D t^{\frac{9}{2}} \right) \\
&\le \frac{7}{3} \frac{1}{\Gamma(\frac{10}{3})} t^{\frac{4}{3}} (1-t)^2 \int_0^1 \frac{1}{\Gamma(\frac{3}{2})} \left(\frac{4}{15} + \frac{1}{10\sqrt{2} - 5} \right) s^{\frac{1}{2}} - \frac{4}{15} s^{\frac{5}{2}}\, ds \\
&= \frac{7}{3} \frac{1}{\Gamma(\frac{10}{3})} t^{\frac{4}{3}} (1-t)^2 \frac{1}{\Gamma(\frac{3}{2})} \left(\frac{32}{315} + \frac{2}{30\sqrt{2} - 15} \right) \\
&\le N_1 t^{\frac{4}{3}} (1-t)^2 = h_1(t), \\
e_2(t) &= \frac{1}{\Gamma(\frac{3}{2})} \frac{1}{\Gamma(\frac{10}{3})} t^{\frac{4}{3}} \left(A_2 + B_2 t + C_2 t^{\frac{5}{2}} + D t^{\frac{9}{2}} \right) \\
&\le \frac{7}{3} \frac{1}{\Gamma(\frac{10}{3})} t^{\frac{4}{3}} (1-t)^2 \int_0^1 \frac{3}{\Gamma(\frac{3}{2})} \left(\frac{4}{15} + \frac{20\sqrt{3} - 12}{135\sqrt{3} - 45} \right) s^{\frac{1}{2}} - \frac{4}{15} s^{\frac{5}{2}}\, ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{7}{\Gamma(\frac{10}{3})} t^{\frac{4}{3}} (1-t)^2 \frac{1}{\Gamma(\frac{3}{2})} \left(\frac{32}{315} + \frac{24+16\sqrt{3}}{405\sqrt{3}-135} \right) \\
&\leq N_2 t^{\frac{4}{3}} (1-t)^2 = h_2(t).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
f(t,u,v) &= \left(\frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) u + 1 \right)^{\frac{1}{5}} \left[\frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})} \left(A_1 + B_1 t + C_1 t^{\frac{5}{2}} + D t^{\frac{9}{2}} \right) \right]^{\frac{1}{5}} t^{\frac{4}{15}} \\
&\quad + \left(\frac{27-3\sqrt{3}}{88} \Gamma\left(\frac{29}{6}\right) v + 1 \right)^{\frac{1}{5}} \left[\frac{3}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})} \left(A_2 + B_2 t + C_2 t^{\frac{5}{2}} + D t^{\frac{9}{2}} \right) \right]^{\frac{1}{5}} t^{\frac{4}{15}} \\
&= \left(\frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) u + 1 \right)^{\frac{1}{5}} \left(e_1(t)\right)^{\frac{1}{5}} + \left(\frac{27-3\sqrt{3}}{88} \Gamma\left(\frac{29}{6}\right) v + 1 \right)^{\frac{1}{5}} \left(e_2(t)\right)^{\frac{1}{5}} \\
&= \left(\frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) u e_1(t) + e_1(t) \right)^{\frac{1}{5}} + \left(\frac{27-3\sqrt{3}}{88} \Gamma\left(\frac{29}{6}\right) v e_2(t) + e_2(t) \right)^{\frac{1}{5}}, \\
g(t,u,v) &= \left(\frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) u + 1 \right)^{\frac{1}{3}} \left[\frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})} \left(A_1 + B_1 t + C_1 t^{\frac{5}{2}} + D t^{\frac{9}{2}} \right) \right]^{\frac{1}{3}} t^{\frac{4}{9}} \\
&\quad + \left(\frac{27-3\sqrt{3}}{88} \Gamma\left(\frac{29}{6}\right) v + 1 \right)^{\frac{1}{3}} \left[\frac{3}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})} \left(A_2 + B_2 t + C_2 t^{\frac{5}{2}} + D t^{\frac{9}{2}} \right) \right]^{\frac{1}{3}} t^{\frac{4}{9}} \\
&= \left(\frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) u + 1 \right)^{\frac{1}{3}} \left(e_1(t)\right)^{\frac{1}{3}} + \left(\frac{27-3\sqrt{3}}{88} \Gamma\left(\frac{29}{6}\right) v + 1 \right)^{\frac{1}{3}} \left(e_2(t)\right)^{\frac{1}{3}} \\
&= \left(\frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) u e_1(t) + e_1(t) \right)^{\frac{1}{3}} + \left(\frac{27-3\sqrt{3}}{88} \Gamma\left(\frac{29}{6}\right) v e_2(t) + e_2(t) \right)^{\frac{1}{3}}.
\end{aligned}
$$

For $\xi \in (0,1)$, $x_1, x_2 \in (-\infty, +\infty)$, $y_1 \in [0, e_1^*]$, $y_2 \in [0, e_2^*]$,

$$
\begin{aligned}
& f\left(t, \xi x_1 + (\xi-1) y_1, \xi x_2 + (\xi-1) y_2\right) \\
&\quad = \left[\frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) e_1(t) \left[\xi x_1 + (\xi-1) y_1 \right] + e_1(t) \right]^{\frac{1}{5}} \\
&\qquad + \left[\frac{27-3\sqrt{3}}{88} \Gamma\left(\frac{29}{6}\right) e_2(t) \left[\xi x_2 + (\xi-1) y_2 \right] + e_2(t) \right]^{\frac{1}{5}} \\
&\quad = \xi^{\frac{1}{5}} \left[\frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) e_1(t) \left[x_1 + \left(1 - \frac{1}{\xi}\right) y_1 \right] + \frac{1}{\xi} e_1(t) \right]^{\frac{1}{5}} \\
&\qquad + \xi^{\frac{1}{5}} \left[\frac{27-3\sqrt{3}}{88} \Gamma\left(\frac{29}{6}\right) e_2(t) \left[x_2 + \left(1 - \frac{1}{\xi}\right) y_2 \right] + \frac{1}{\xi} e_2(t) \right]^{\frac{1}{5}} \\
&\quad = \xi^{\frac{1}{5}} \left[\frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) e_1(t) x_1 + \left(1 - \frac{1}{\xi}\right) \frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) e_1(t) y_1 + \frac{1}{\xi} e_1(t) \right]^{\frac{1}{5}} \\
&\qquad + \xi^{\frac{1}{5}} \left[\frac{27-3\sqrt{3}}{88} \Gamma\left(\frac{29}{6}\right) e_2(t) x_2 + \left(1 - \frac{1}{\xi}\right) \frac{27-3\sqrt{3}}{88} \Gamma\left(\frac{29}{6}\right) e_2(t) y_2 + \frac{1}{\xi} e_2(t) \right]^{\frac{1}{5}} \\
&\quad \geq \xi^{\frac{1}{5}} \left[\frac{18\sqrt{2}-9}{16\sqrt{2}-2} \Gamma\left(\frac{29}{6}\right) e_1(t) x_1 + \left(1 - \frac{1}{\xi}\right) e_1(t) + \frac{1}{\xi} e_1(t) \right]^{\frac{1}{5}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \xi^{\frac{1}{5}}\left[\frac{27-3\sqrt{3}}{88}\Gamma\left(\frac{29}{6}\right)e_2(t)x_2 + \left(1-\frac{1}{\xi}\right)e_2(t) + \frac{1}{\xi}e_2(t)\right]^{\frac{1}{5}} \\
&= \xi^{\frac{1}{5}}\left[\left(\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma\left(\frac{29}{6}\right)e_1(t)x_1 + e_1(t)\right)^{\frac{1}{5}}\right. \\
&\quad \left. + \left(\frac{27-3\sqrt{3}}{88}\Gamma\left(\frac{29}{6}\right)e_2(t)x_2 + e_2(t)\right)^{\frac{1}{5}}\right] \\
&= \xi^{\frac{1}{5}} f(t,x_1,x_2) \geq \psi(\xi) f(t,x_1,x_2), \\
& g\bigl(t, \xi x_1 + (\xi-1)y_1, \xi x_2 + (\xi-1)y_2\bigr) \\
&= \left[\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma\left(\frac{29}{6}\right)e_1(t)\bigl[\xi x_1 + (\xi-1)y_1\bigr] + e_1(t)\right]^{\frac{1}{3}} \\
&\quad + \left[\frac{27-3\sqrt{3}}{88}\Gamma\left(\frac{29}{6}\right)e_2(t)\bigl[\xi x_2 + (\xi-1)y_2\bigr] + e_2(t)\right]^{\frac{1}{3}} \\
&= \xi^{\frac{1}{3}}\left[\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma\left(\frac{29}{6}\right)e_1(t)\left[x_1 + \left(1-\frac{1}{\xi}\right)y_1\right] + \frac{1}{\xi}e_1(t)\right]^{\frac{1}{3}} \\
&\quad + \xi^{\frac{1}{3}}\left[\frac{27-3\sqrt{3}}{88}\Gamma\left(\frac{29}{6}\right)e_2(t)\left[x_2 + \left(1-\frac{1}{\xi}\right)y_2\right] + \frac{1}{\xi}e_2(t)\right]^{\frac{1}{3}} \\
&= \xi^{\frac{1}{3}}\left[\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma\left(\frac{29}{6}\right)e_1(t)x_1 + \left(1-\frac{1}{\xi}\right)\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma\left(\frac{29}{6}\right)e_1(t)y_1 + \frac{1}{\xi}e_1(t)\right]^{\frac{1}{3}} \\
&\quad + \xi^{\frac{1}{3}}\left[\frac{27-3\sqrt{3}}{88}\Gamma\left(\frac{29}{6}\right)e_2(t)x_2 + \left(1-\frac{1}{\xi}\right)\frac{27-3\sqrt{3}}{88}\Gamma\left(\frac{29}{6}\right)e_2(t)y_2 + \frac{1}{\xi}e_2(t)\right]^{\frac{1}{3}} \\
&\geq \xi^{\frac{1}{3}}\left[\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma\left(\frac{29}{6}\right)e_1(t)x_1 + \left(1-\frac{1}{\xi}\right)e_1(t) + \frac{1}{\xi}e_1(t)\right]^{\frac{1}{3}} \\
&\quad + \xi^{\frac{1}{3}}\left[\frac{27-3\sqrt{3}}{88}\Gamma\left(\frac{29}{6}\right)e_2(t)x_2 + \left(1-\frac{1}{\xi}\right)e_2(t) + \frac{1}{\xi}e_2(t)\right]^{\frac{1}{3}} \\
&= \xi^{\frac{1}{3}}\left[\left(\frac{18\sqrt{2}-9}{16\sqrt{2}-2}\Gamma\left(\frac{29}{6}\right)e_1(t)x_1 + e_1(t)\right)^{\frac{1}{3}}\right. \\
&\quad \left. + \left(\frac{27-3\sqrt{3}}{88}\Gamma\left(\frac{29}{6}\right)e_2(t)x_2 + e_2(t)\right)^{\frac{1}{3}}\right] \\
&= \xi^{\frac{1}{3}} f(t,x_1,x_2) = \psi(\xi) f(t,x_1,x_2),
\end{aligned}
$$

here $\psi(\xi) = \xi^{\frac{1}{3}} > \xi$. By Theorem 3.1, system (4.1) has a unique solution (u^*, v^*) in $\bar{P}_{h,e}$, where

$$
\begin{aligned}
e(t) &= \bigl(e_1(t), e_2(t)\bigr) \\
&= \left(\frac{1}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}t^{\frac{4}{3}}\bigl(A_1 + B_1 t + C_1 t^{\frac{5}{2}} + D t^{\frac{9}{2}}\bigr),\right. \\
&\quad \left.\frac{1}{\Gamma(\frac{3}{2})}\frac{1}{\Gamma(\frac{10}{3})}t^{\frac{4}{3}}\bigl(A_2 + B_2 t + C_2 t^{\frac{5}{2}} + D t^{\frac{9}{2}}\bigr)\right), \\
h(t) &= \bigl(h_1(t), h_2(t)\bigr) = \bigl(N_1 t^{\frac{4}{3}}(1-t)^2, N_2 t^{\frac{4}{3}}(1-t)^2\bigr), \quad t \in [0,1].
\end{aligned}
$$

Taking any point $(u_0, v_0) \in \bar{P}_{h,e}$, we construct the following sequences:

$$\begin{aligned} u_{n+1}(t) &= \int_0^1 G_1(t,s) \int_0^1 H_1(s,\tau) f\bigl(\tau, u_n(\tau), v_n(\tau)\bigr)\, d\tau\, ds \\ &\quad - \frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})} t^{\frac{4}{3}} \bigl(A_1 + B_1 t + C_1 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\bigr), \\ v_{n+1}(t) &= \int_0^1 G_2(t,s) \int_0^1 H_2(s,\tau) g\bigl(\tau, u_n(\tau), v_n(\tau)\bigr)\, d\tau\, ds \\ &\quad - \frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{10}{3})} t^{\frac{4}{3}} \bigl(A_2 + B_2 t + C_2 t^{\frac{5}{2}} + Dt^{\frac{9}{2}}\bigr), \end{aligned}$$

$n = 0, 1, 2, \ldots$, then we have $u_{n+1}(t) \to u^*(t)$, $v_{n+1}(t) \to v^*(t)$ as $n \to \infty$.

5 Conclusion

Recently, fractional differential systems have been increasingly used to describe problems in optical and thermal systems, rheology and materials and mechanics systems, signal processing and system identification, control, robotics, and other applications. Because of their deep realistic background and important role, people are paying more and more attention. For nonlinear fractional differential systems subject to different boundary conditions, there are many articles studying the existence or multiplicity of solutions or positive solutions. But the unique results are very rare. In this paper, we study a system of fractional differential Eqs. (1.1). By constructing two functions e and h and using fixed point theorem of increasing Ψ-(h,e)-concave operators defined on ordered set $P_{h,e}$, we establish some new existence and uniqueness criteria for system (1.1). Our result shows that the unique solution exists in a product set $\bar{P}_{h,e} = P_{h_1,e_1} \times P_{h_2,e_2}$ and can be approximated by making an iterative sequence for any initial point in $\bar{P}_{h,e}$. Finally, an interesting example is given to illustrate the application of our main results.

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