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Existence and iteration of positive solution for fractional integral boundary value problems with p-Laplacian operator

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Abstract

This paper is concerned with an integral boundary value problem of fractional differential equations with p-Laplacian operator. Sufficient conditions ensuring the existence of extremal solutions for the given problem are obtained. Our results are based on the method of upper and lower solutions and monotone iterative technique.

MSC: 34B15

Keywords: Monotone iterative; p-Laplacian operator; Integral boundary value problem; Upper and lower solutions

1 Introduction

This paper studies the existence of extremal solutions for the boundary value problem of a fractional p-Laplacian equation with the following form:

$$\begin{cases} -D^{\sigma}(\phi_p(-D^{\tau}u(t))) = h(t,u(t),D^{\tau}u(t)), \quad 0<t<1, \\ D^{\tau}u(0)=0, \\ D^{\sigma-1}(\phi_p(-D^{\tau}u(1))) \\ \quad = I^{\gamma}k(\theta,\phi_p(-D^{\tau}u(\theta)))+d = \frac{1}{\Gamma(\gamma)}\int_0^{\theta}(\theta-s)^{\gamma-1}k(s,\phi_p(-D^{\tau}u(s)))\,ds+d, \\ u(0)=0, \qquad D^{\tau-1}u(1)=I^{\epsilon}u(\zeta)+e=\frac{1}{\Gamma(\epsilon)}\int_0^{\zeta}(\zeta-s)^{\epsilon-1}u(s)\,ds+e, \end{cases} \tag{1.1}$$

where D^{τ} and D^{σ} are the standard Riemann–Liouville fractional derivatives, I^{γ}, I^{ϵ} are the Riemann–Liouville fractional integral, and $1<\sigma,\tau<2$, $\gamma,\epsilon>1$, $0<\theta,\zeta<1$, $d,e\in\mathbb{R}$, $h\in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $k\in C([0,1]\times\mathbb{R},\mathbb{R})$. The p-Laplacian operator is defined as $\phi_p(t)=|t|^{p-2}t$, $p>1$, and $(\phi_p)^{-1}=\phi_q$, $\frac{1}{p}+\frac{1}{q}=1$.

Recently, much attention has been paid to the study of the existence of extremal solutions, for fractional differential equations with corresponding initial or boundary conditions; see [1–9]. The monotone iterative technique, combined with the method of upper and lower solutions, provides an effective mechanism to prove constructive existence results for nonlinear differential equations, the advantage and importance of the technique needs no special emphasis [10, 11]. By using the monotone iterative technique, Ahmad

[12] and Alsaedi [13] successfully investigated initial value problems for nonlinear fractional differential equations with fractional derivatives. Han [14] considered the existence of positive solutions for the following problem:

$$\begin{cases} D^{\beta}(\phi_p(D^{\alpha}u(t))) = \lambda f(u(t)), & 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, & \phi_p(D^{\alpha}u(0)) = (\phi_p(D^{\alpha}u(1)))' = 0, \end{cases}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$ are real numbers, $\lambda > 0$ is a parameter, and $f : (0, +\infty) \to (0, +\infty)$ is continuous. By using the properties of Green function and the Guo–Krasnosel'skii fixed-point theorem on cones, several existence results of at least one or two positive solutions in terms of different eigenvalue interval are obtained. By means of the monotone iterative method, Wang [15] investigated the fractional integral boundary problem

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t), u(\theta(t))), & n < \alpha \leq n+1, n \geq 2, t \in [0,1], \\ u'(0) = u''(0) = u'''(0) = \cdots = u^{n}(0) = 0, \\ u(0) = \int_0^1 g(s, u(s))\,ds + \lambda, \end{cases}$$

where $\lambda > 0$, and f, g are continuous functions. However, the existence results in [10] mainly depend upon a restrictive condition, i.e.,

$$f(t, u, v) \geq f(t, \overline{u}, \overline{v}).$$

It is a critical condition in order to discuss the monotone iterative sequences. Therefore, it is natural to ask whether similar results can be obtained if

$$f\bigl(t, u(t), D^{\tau}u(t)\bigr) - f\bigl(t, v(t), D^{\tau}v(t)\bigr) \leq L\bigl[\phi_p\bigl(-D^{\tau}v(t)\bigr) - \phi_p\bigl(-D^{\tau}u(t)\bigr)\bigr].$$

Being directly inspired by Wang [15], the purpose of this paper is to study the nonlinear integral boundary value problem for p-Laplacian differential equations. The nonlinear terms h, k are not required to satisfy monotonicity conditions on the unknown function u or their derivatives. The monotone iterative technique combined with the method of upper and lower solutions is applied. In particular, we construct two well-defined monotone iterative sequences of upper and lower solutions and prove that they converge uniformly to the actual solution of the problem.

2 Preliminaries

In this section, we deduce some preliminary results which will be used in the next section.

Denote $C_{\tau}[0,1] = \{u : u \in C[0,1], D^{\tau}u(t) \in C[0,1]\}$ and endow it with the norm $\|u\|_{\tau} = \|u\| + \|D^{\tau}u\|$, where $\|u\| = \max_{0 \leq t \leq 1}|u(t)|$ and $\|D^{\tau}u\| = \max_{0 \leq t \leq 1}|D^{\tau}u(t)|$. Then $(C_{\tau}[0,1], \|\cdot\|_{\tau})$ is a Banach space (see [16]).

Definition 2.1 A function $u(t) \in C_{\tau}[0,1]$ satisfying $D^{\sigma}(\phi_p(-D^{\tau}u(t))) \in C[0,1]$ is called a lower solution of problem (1.1) if

$$\begin{cases} -D^{\sigma}(\phi_p(-D^{\tau}u(t))) \leq h(t, u(t), D^{\tau}u(t)), & 0 < t < 1, \\ D^{\tau}u(0) = 0, & D^{\sigma-1}(\phi_p(-D^{\tau}u(1))) \leq I^{\gamma}k(\theta, \phi_p(-D^{\tau}u(\theta))) + d, \\ u(0) = 0, & D^{\tau-1}u(1) \leq I^{\epsilon}u(\zeta) + e. \end{cases}$$

A function $v(t) \in C_\tau[0,1]$ satisfying $D^\sigma(\phi_p(-D^\tau v(t))) \in C[0,1]$ is called an upper solution of problem (1.1) if the above inequalities are reversed.

For the sake of convenience, we now present some assumptions as follows.

(H_1) Assume that $u_0, v_0 \in C_\tau[0,1]$ satisfying $D^\sigma(\phi_p(-D^\tau u_0(t)))$, $D^\sigma(\phi_p(-D^\tau v_0(t))) \in C[0,1]$ are lower and upper solutions of problem (1.1), respectively, and $u_0(t) \le v_0(t)$, $D^\tau v_0(t) \le D^\tau u_0(t)$, $t \in [0,1]$.

(H_2) There exists a constant $L \in \mathbb{R}$ such that

$$h\bigl(t,u(t),D^\tau u(t)\bigr) - h\bigl(t,v(t),D^\tau v(t)\bigr) \le L\bigl[\phi_p\bigl(-D^\tau v(t)\bigr) - \phi_p\bigl(-D^\tau u(t)\bigr)\bigr],$$

for $u_0(t) \le u(t) \le v(t) \le v_0(t)$, $D^\tau v_0(t) \le D^\tau v(t) \le D^\tau u(t) \le D^\tau u_0(t)$, $t \in [0,1]$.

(H_3) There exists a constant $\mu \ge 0$, such that

$$k\bigl(t,\phi_p\bigl(-D^\tau v(t)\bigr)\bigr) - k\bigl(t,\phi_p\bigl(-D^\tau u(t)\bigr)\bigr) \ge \mu\bigl[\phi_p\bigl(-D^\tau v(t)\bigr) - \phi_p\bigl(-D^\tau u(t)\bigr)\bigr],$$

for $u_0(t) \le u(t) \le v(t) \le v_0(t)$, $D^\tau v_0(t) \le D^\tau v(t) \le D^\tau u(t) \le D^\tau u_0(t)$, $t \in [0,1]$.

(H_4) $\Gamma(\sigma+\gamma) > \mu\theta^{\sigma+\gamma-1}$.

(H_5) $2\Gamma(\sigma+\gamma)|L| < \Gamma(\sigma)[\Gamma(\sigma+\gamma) - \mu\theta^{\sigma+\gamma-1}]$.

(H_6) For any $t \in (0,1)$, we have

$$\Gamma(2-\sigma)t^\sigma L > 1-\sigma$$

and

$$\Gamma(2-\sigma)\mu\theta^\gamma < \Gamma(\gamma).$$

Lemma 2.1 ([17]) *Let $f(t) \in C[0,1]$, $a \in \mathbb{R}$, and $\Gamma(\sigma+\gamma) \neq \mu\theta^{\sigma+\gamma-1}$, then the fractional boundary value problem*

$$\begin{cases} -D^\sigma w(t) = f(t), \quad t \in [0,1], \\ w(0) = 0, \\ D^{\sigma-1}w(1) = \mu I^\gamma w(\theta) + a = \frac{\mu}{\Gamma(\gamma)}\int_0^\theta (\theta-s)^{\gamma-1}w(s)\,ds + a, \end{cases} \tag{2.1}$$

is equivalent to

$$w(t) = \int_0^1 J(t,s)f(s)\,ds + \frac{a\Gamma(\sigma+\gamma)t^{\sigma-1}}{\Gamma(\sigma)[\Gamma(\sigma+\gamma) - \mu\theta^{\sigma+\gamma-1}]},$$

where

$$J(t,s) = \frac{1}{\Delta}\begin{cases} [\Gamma(\sigma+\gamma) - \mu(\theta-s)^{\sigma+\gamma-1}]t^{\sigma-1} \\ \quad - [\Gamma(\sigma+\gamma) - \mu\theta^{\sigma+\gamma-1}](t-s)^{\sigma-1}, & s \le t, s \le \theta; \\ \Gamma(\sigma+\gamma)t^{\sigma-1} - \mu(\theta-s)^{\sigma+\gamma-1}t^{\sigma-1}, & t \le s \le \theta; \\ \Gamma(\sigma+\gamma)[t^{\sigma-1} - (t-s)^{\sigma-1}] + \mu\theta^{\sigma+\gamma-1}(t-s)^{\sigma-1}, & \theta \le s \le t; \\ \Gamma(\sigma+\gamma)t^{\sigma-1}, & s \ge t, s \ge \theta, \end{cases}$$

and $\Delta = \Gamma(\sigma)[\Gamma(\sigma+\gamma) - \mu\theta^{\sigma+\gamma-1}]$.

Lemma 2.2 ([17]) *If* (H_4) *holds, then the Green's function* $J(t,s)$ *satisfies*

$$0 \le J(t,s) \le \frac{\Gamma(\sigma+\gamma)}{\Gamma(\sigma)[\Gamma(\sigma+\gamma)-\mu\theta^{\sigma+\gamma-1}]}\bigl(1+t^{\sigma-1}\bigr).$$

Lemma 2.3 *Let* $L, a \in \mathbb{R}$, $w(t), f(t) \in C[0,1]$ *and* (H_4), (H_5) *hold, then the boundary value problem*

$$\begin{cases} -D^{\sigma} w(t) + Lw(t) = f(t), \quad t \in [0,1], \\ w(0) = 0, \\ D^{\sigma-1} w(1) = \mu I^{\gamma} w(\theta) + a, \end{cases} \tag{2.2}$$

has a unique solution $w(t) \in C[0,1]$.

Proof It follows from Lemma 2.1 that problem (2.2) is equivalent to the following integral equation:

$$w(t) = \int_0^1 J(t,s)\bigl[f(s) - Lw(s)\bigr]\,ds + \frac{a\Gamma(\sigma+\gamma)t^{\sigma-1}}{\Gamma(\sigma)[\Gamma(\sigma+\gamma)-\mu\theta^{\sigma+\gamma-1}]}, \quad \forall t \in [0,1].$$

Let

$$Aw(t) = \int_0^1 J(t,s)\bigl[f(s) - Lw(s)\bigr]\,ds + \frac{a\Gamma(\sigma+\gamma)t^{\sigma-1}}{\Gamma(\sigma)[\Gamma(\sigma+\gamma)-\mu\theta^{\sigma+\gamma-1}]}, \quad \forall t \in [0,1].$$

For any $u, w \in C[0,1]$, by (H_4) and Lemma 2.2, we have

$$\begin{aligned} \|Au - Aw\| &= \max_{0\le t\le 1}\bigl|Au(t) - Aw(t)\bigr| \\ &\le \max_{0\le t\le 1}\left(\int_0^1 J(t,s)|L|\cdot|u-w|\,ds\right) \\ &\le \frac{|L|\Gamma(\sigma+\gamma)\|u-w\|}{\Gamma(\sigma)[\Gamma(\sigma+\gamma)-\mu\theta^{\sigma+\gamma-1}]}\max_{0\le t\le 1}\bigl(1+t^{\sigma-1}\bigr) \\ &\le \frac{2\Gamma(\sigma+\gamma)\|u-w\||L|}{\Gamma(\sigma)[\Gamma(\sigma+\gamma)-\mu\theta^{\sigma+\gamma-1}]}. \end{aligned}$$

Noting that (H_5) holds, which implies $\frac{2\Gamma(\sigma+\gamma)|L|}{\Gamma(\sigma)[\Gamma(\sigma+\gamma)-\mu\theta^{\sigma+\gamma-1}]} < 1$, we have

$$\|Au - Aw\| < \|u - w\|.$$

By the Banach fixed point theorem, the operator A has a unique fixed point. That is, (2.2) has a unique solution. □

Lemma 2.4 *Assume that* $z(t) \in C[0,1]$, $k \in \mathbb{R}$. *Then the fractional boundary value problem*

$$\begin{cases} -D^{\tau} u(t) = z(t), \quad 0 < t < 1, \\ u(0) = 0, \qquad D^{\tau-1}u(1) = k, \end{cases} \tag{2.3}$$

is equivalent to

$$u(t) = \int_0^1 H(t,s)z(s)\,ds + \frac{kt^{\tau-1}}{\Gamma(\tau)},$$

where

$$H(t,s) = \begin{cases} t^{\tau-1} - (t-s)^{\tau-1}, & 0 \le s \le t \le 1, \\ t^{\tau-1}, & 0 \le t \le s \le 1. \end{cases}$$

Proof We can transform the equation $-D^{\tau} v(t) = z(t)$ to an equivalent integral equation

$$u(t) = -I^{\tau} z(t) + C_1 t^{\tau-1} + C_2 t^{\tau-2}.$$

Note that $u(0) = 0$, we have $C_2 = 0$. Consequently, we have the following form:

$$u(t) = -I^{\tau} z(t) + C_1 t^{\tau-1}$$

and

$$\begin{aligned} D^{\tau-1} u(t) &= -D^{\tau-1} I^{\tau} z(t) + C_1 D^{\tau-1} t^{\tau-1} \\ &= -I^{\tau-(\tau-1)} z(t) + C_1 D^{\tau-1} t^{\tau-1} \\ &= -\int_0^t z(s)\,ds + C_1 \Gamma(\tau). \end{aligned}$$

On the other hand $D^{\tau-1} u(1) = k$, and we obtain

$$C_1 = \frac{1}{\Gamma(\tau)} \int_0^1 z(s)\,ds + \frac{k}{\Gamma(\tau)}. \tag{2.4}$$

Therefore, the solution of problem (2.3) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\tau)} \int_0^t (t-s)^{\tau-1} z(s)\,ds + \frac{t^{\tau-1}}{\Gamma(\tau)} \int_0^1 z(s)\,ds + \frac{kt^{\tau-1}}{\Gamma(\tau)} \\ &= \int_0^1 H(t,s) z(s)\,ds + \frac{kt^{\tau-1}}{\Gamma(\tau)}. \end{aligned}$$

□

Lemma 2.5 *Assume that* $k, a \in \mathbb{R}$, $u(t) \in C_{\tau}[0,1], f(t) \in C[0,1]$ *and* (H_4), (H_5) *hold. Then the boundary value problem*

$$\begin{cases} -D^{\sigma}(\phi_p(-D^{\tau} u(t))) + L\phi_p(-D^{\tau} u(t)) = f(t), & 0 < t < 1, \\ D^{\tau} u(0) = 0, \quad D^{\sigma-1}(\phi_p(-D^{\tau} u(1))) = \mu I^{\gamma} \phi_p(-D^{\tau} u(\theta)) + a, \\ u(0) = 0, \quad D^{\tau-1} u(1) = k, \end{cases} \tag{2.5}$$

has a unique solution $u(t)$.

Proof Let $\phi_p(-D^\tau u(t)) = w(t)$, and consider the boundary value problem:

$$\begin{cases} -D^\sigma w(t) + Lw(t) = f(t), & 0 < t < 1, \\ w(0) = 0, & D^{\sigma-1}w(1) = \mu I^\gamma w(\theta) + a. \end{cases} \tag{2.6}$$

From Lemma 2.3, we know that (2.6) has a unique solution $w(t) \in C[0,1]$. Note that $\phi_p(-D^\tau u(t)) = w(t) \in C[0,1]$, and $-D^\tau u(t) = \phi_q(w(t)) \in C[0,1]$. Then the problem (2.5) is transformed into to the following problem:

$$\begin{cases} -D^\tau u(t) = \phi_q(w(t)), & 0 < t < 1, \\ u(0) = 0, & D^{\tau-1}u(1) = k. \end{cases} \tag{2.7}$$

By Lemma 2.4, the solution of (2.7) can be written

$$u(t) = \int_0^1 H(t,s)\phi_q\bigl(w(s)\bigr)\,ds + \frac{kt^{\tau-1}}{\Gamma(\tau)}.$$

Combining with (2.6) and (2.7), we assert that the boundary problem (2.5) has a unique solution $u(t)$. □

Lemma 2.6 ([18, Lemma 2.6]) *Assume that* (H_6) *holds,* $w(t) \in C[0,1]$, *satisfying* $D^\sigma w(t) \in C[0,1]$ *and*

$$\begin{cases} -D^\sigma w(t) \geq -Lw(t), & t \in [0,1], \\ w(0) = 0, \\ D^{\sigma-1}w(1) \geq \mu I^\gamma w(\theta), \end{cases} \tag{2.8}$$

then $w(t) \geq 0$, $\forall t \in [0,1]$.

Lemma 2.7 *If* $u(t) \in C[0,1]$ *satisfies*

$$\begin{cases} -D^\tau u(t) \geq 0, & 0 < t < 1, \\ u(0) = 0, \\ D^{\tau-1}u(1) \geq 0, \end{cases} \tag{2.9}$$

then $u(t) \geq 0$, $\forall t \in [0,1]$.

Proof By Lemma 2.4, we know that (2.3) has a unique solution

$$u(t) = \int_0^1 H(t,s)z(s)\,ds + \frac{kt^{\tau-1}}{\Gamma(\tau)}.$$

It is easy to verify that the Green's function $H(t,s) \geq 0$, $t, s \in [0,1]$. Let $z(t) \geq 0$ and $k \geq 0$. Then we obtain (2.9) and $u(t) \geq 0$, $\forall t \in [0,1]$. □

3 Main results

Theorem 3.1 *Suppose that* (H_1)–(H_6) *hold. Then problem* (1.1) *has extremal solution* $u^*, v^* \in [u_0, v_0]$. *Moreover,*

$$u_0(t) \le u^*(t) \le v^*(t) \le v_0(t)$$

and

$$D^\tau v_0(t) \le D^\tau v^*(t) \le D^\tau u^*(t) \le D^\tau u_0(t), \quad \forall t \in [0,1].$$

Proof For $n = 0, 1, 2, \ldots$, we define

$$\begin{cases} -D^\sigma(\phi_p(-D^\tau u_{n+1}(t))) \\ \quad = h(t, u_n(t), D^\tau u_n(t)) - L[\phi_p(-D^\tau u_{n+1}(t)) - \phi_p(-D^\tau u_n(t))], \\ D^\tau u_{n+1}(0) = 0, \\ D^{\sigma-1}(\phi_p(-D^\tau u_{n+1}(1))) \\ \quad = I^\gamma\{k(\theta, \phi_p(-D^\tau u_n(\theta))) + \mu[\phi_p(-D^\tau u_{n+1}(\theta)) - \phi_p(-D^\tau u_n(\theta))]\} + d, \\ u_{n+1}(0) = 0, \qquad D^{\tau-1} u_{n+1}(1) = I^\epsilon u_n(\zeta) + e, \end{cases} \tag{3.1}$$

and

$$\begin{cases} -D^\sigma(\phi_p(-D^\tau v_{n+1}(t))) \\ \quad = h(t, v_n(t), D^\tau v_n(t)) - L[\phi_p(-D^\tau v_{n+1}(t)) - \phi_p(-D^\tau v_n(t))], \\ D^\tau v_{n+1}(0) = 0, \\ D^{\sigma-1}(\phi_p(-D^\tau v_{n+1}(1))) \\ \quad = I^\gamma\{k(\theta, \phi_p(-D^\tau v_n(\theta))) + \mu[\phi_p(-D^\tau v_{n+1}(\theta)) - \phi_p(-D^\tau v_n(\theta))]\} + d, \\ v_{n+1}(0) = 0, \qquad D^{\tau-1} v_{n+1}(1) = I^\epsilon v_n(\zeta) + e. \end{cases} \tag{3.2}$$

In view of Lemma 2.5, the functions u_1 and v_1 are well defined. First, we show that $u_0(t) \le u_1(t) \le v_1(t) \le v_0(t)$, and $D^\tau v_0(t) \le D^\tau v_1(t) \le D^\tau u_1(t) \le D^\tau u_0(t)$, $t \in [0,1]$. Let $\delta(t) = \phi_p(-D^\tau u_1(t)) - \phi_p(-D^\tau u_0(t))$. From (3.1) and (H_1), we obtain

$$\begin{aligned} -D^\sigma \delta(t) &\ge h\bigl(t, u_0(t), D^\tau u_0(t)\bigr) - L\bigl[\phi_p\bigl(-D^\tau u_1(t)\bigr) - \phi_p\bigl(-D^\tau u_0(t)\bigr)\bigr] \\ &\quad - h\bigl(t, u_0(t), D^\tau u_0(t)\bigr) \\ &\ge -L\delta(t). \end{aligned}$$

Also $\delta(0) = 0$ and

$$\begin{aligned} D^{\sigma-1}\delta(1) &= D^{\sigma-1}\bigl(\phi_p\bigl(-D^\tau u_1(1)\bigr)\bigr) - D^{\sigma-1}\bigl(\phi_p\bigl(-D^\tau u_0(1)\bigr)\bigr) \\ &\ge I^\gamma\bigl\{k\bigl(\theta, \phi_p\bigl(-D^\tau u_0(\theta)\bigr)\bigr) + \mu\bigl[\phi_p\bigl(-D^\tau u_1(\theta)\bigr) - \phi_p\bigl(-D^\tau u_0(\theta)\bigr)\bigr]\bigr\} \\ &\quad - I^\gamma k\bigl(\theta, \phi_p\bigl(-D^\tau u_0(\theta)\bigr)\bigr) \\ &\ge \mu I^\gamma \delta(\theta). \end{aligned}$$

In view of Lemma 2.6, we have $\phi_p(-D^\tau u_1(t)) \geq \phi_p(-D^\tau u_0(t))$, $t \in [0,1]$, since $\phi_p(x)$ is nondecreasing, thus

$$D^\tau u_1(t) \leq D^\tau u_0(t). \tag{3.3}$$

Let $\alpha(t) = u_1(t) - u_0(t)$, it follows from (3.1) and (3.3) that

$$\begin{cases} -D^\tau \alpha(t) = -D^\tau u_1(t) + D^\tau u_0(t) \geq 0, \quad t \in [0,1], \\ \alpha(0) = 0, \\ D^{\tau-1}\alpha(1) = D^{\tau-1}u_1(1) - D^{\tau-1}u_0(1) \geq I^\epsilon u_0(\zeta) - I^\epsilon u_0(\zeta) = 0. \end{cases}$$

According to Lemma 2.7, we have $u_1(t) \geq u_0(t)$, $\forall t \in [0,1]$.

By a similar way, we can show that $v_0(t) \geq v_1(t)$, and $D^\tau v_0(t) \leq D^\tau v_1(t)$. Now, we put $p(t) = \phi_p(-D^\tau v_1(t)) - \phi_p(-D^\tau u_1(t))$. From ($H_2$) and ($H_3$), we have

$$\begin{aligned} -D^\sigma p(t) &= h\bigl(t, v_0(t), D^\tau v_0(t)\bigr) - L\bigl[\phi_p\bigl(-D^\tau v_1(t)\bigr) - \phi_p\bigl(-D^\tau v_0(t)\bigr)\bigr] \\ &\quad - h\bigl(t, u_0(t), D^\tau u_0(t)\bigr) + L\bigl[\phi_p\bigl(-D^\tau u_1(t)\bigr) - \phi_p\bigl(-D^\tau u_0(t)\bigr)\bigr] \\ &\geq -L\bigl[(\phi_p\bigl(-D^\tau v_0(t)\bigr) - \phi_p\bigl(-D^\tau u_0(t)\bigr)\bigr] - L\bigl[\phi_p\bigl(-D^\tau v_1(t)\bigr) - \phi_p\bigl(-D^\tau v_0(t)\bigr)\bigr] \\ &\quad + L\bigl[\phi_p\bigl(-D^\tau u_1(t)\bigr) - \phi_p\bigl(-D^\tau u_0(t)\bigr)\bigr] \\ &= -Lp(t), \end{aligned}$$

also $p(0) = 0$, and

$$\begin{aligned} D^{\sigma-1}p(1) &= I^\gamma\bigl\{k\bigl(\theta, \phi_p\bigl(-D^\tau v_0(\theta)\bigr)\bigr) + \mu\bigl[\phi_p\bigl(-D^\tau v_1(\theta)\bigr) - \phi_p\bigl(-D^\tau v_0(\theta)\bigr)\bigr]\bigr\} \\ &\quad - I^\gamma\bigl\{k\bigl(\theta, \phi_p\bigl(-D^\tau u_0(\theta)\bigr)\bigr) + \mu\bigl[\phi_p\bigl(-D^\tau u_1(\theta)\bigr) - \phi_p\bigl(-D^\tau u_0(\theta)\bigr)\bigr]\bigr\} \\ &\geq I^\gamma\bigl\{\mu\bigl[\phi_p\bigl(-D^\tau v_0(\theta)\bigr) - \phi_p\bigl(-D^\tau u_0(\theta)\bigr)\bigr] \\ &\quad + \mu\bigl[\phi_p\bigl(-D^\tau v_1(\theta)\bigr) - \phi_p\bigl(-D^\tau v_0(\theta)\bigr)\bigr] \\ &\quad - \mu\bigl[\phi_p\bigl(-D^\tau u_1(\theta)\bigr) - \phi_p\bigl(-D^\tau u_0(\theta)\bigr)\bigr]\bigr\} \\ &= \mu I^\gamma p(\theta). \end{aligned}$$

In view of Lemma 2.6, we have $p(t) \geq 0$, $\forall t \in [0,1]$. Thus we have $\phi_p(-D^\tau v_1(t)) \geq \phi_p(-D^\tau u_1(t))$, that is, $D^\tau v_1(t) \leq D^\tau u_1(t)$, since ϕ_p is nondecreasing. Therefore $D^\tau v_0(t) \leq D^\tau v_1(t) \leq D^\tau u_1(t) \leq D^\tau u_0(t)$ $\forall t \in [0,1]$ holds.

Let $\theta(t) = v_1(t) - u_1(t)$. From ($H_1$), we have

$$\begin{cases} -D^\tau \theta(t) = -D^\tau v_1(t) + D^\tau u_1(t) \geq 0, \\ \theta(0) = 0, \qquad D^{\tau-1}\theta(1) = D^{\tau-1}v_1(1) - D^{\tau-1}u_1(1) = I^\epsilon v_0(\zeta) - I^\epsilon u_0(\zeta) \geq 0. \end{cases}$$

Moreover, we get $v_1(t) \geq u_1(t)$, from Lemma 2.7. Hence, we have the relation $u_0(t) \leq u_1(t) \leq v_1(t) \leq v_0(t)$, $\forall t \in [0,1]$.

In the following, we show that $u_1(t)$, $v_1(t)$ are lower and upper solutions of problem (1.1), respectively. From (3.1)–(3.2) and (H_2), (H_3), one gets

$$\begin{aligned} -D^{\sigma}\big(\phi_p\big(-D^{\tau}u_1(t)\big)\big) &= h\big(t,u_0(t),D^{\tau}u_0(t)\big) - h\big(t,u_1(t),D^{\tau}u_1(t)\big) + h\big(t,u_1(t),D^{\tau}u_1(t)\big) \\ &\quad - L\big[\phi_p\big(-D^{\tau}u_1(t)\big) - \phi_p\big(-D^{\tau}u_0(t)\big)\big] \\ &\le L\big[\phi_p\big(-D^{\tau}u_1(t)\big) - \phi_p\big(-D^{\tau}u_0(t)\big)\big] \\ &\quad - L\big[\phi_p\big(-D^{\tau}u_1(t)\big) - \phi_p\big(-D^{\tau}u_0(t)\big)\big] \\ &\quad + h\big(t,u_1(t),D^{\tau}u_1(t)\big) \\ &= h\big(t,u_1(t),D^{\tau}u_1(t)\big). \end{aligned}$$

Also $D^{\tau}u_1(0) = 0$, $u_1(0) = 0$, and

$$\begin{aligned} D^{\sigma-1}\big(\phi_p\big(-D^{\tau}u_1(1)\big)\big) &= I^{\gamma}\big\{k\big(\theta,\phi_p\big(-D^{\tau}u_0(\theta)\big)\big) - k\big(\theta,\phi_p\big(-D^{\tau}u_1(\theta)\big)\big) \\ &\quad + k\big(\theta,\phi_p\big(-D^{\tau}u_1(\theta)\big)\big) \\ &\quad + \mu\big[\phi_p\big(-D^{\tau}u_1(\theta)\big) - \phi_p\big(-D^{\tau}u_0(\theta)\big)\big]\big\} + d \\ &\le I^{\gamma}\big\{\mu\big[\phi_p\big(-D^{\tau}u_0(\theta)\big) - \phi_p\big(-D^{\tau}u_1(\theta)\big)\big] \\ &\quad + \mu\big[\phi_p\big(-D^{\tau}u_1(\theta)\big) - \phi_p\big(-D^{\tau}u_0(\theta)\big)\big] \\ &\quad + k\big(\theta,\phi_p\big(-D^{\tau}u_1(\theta)\big)\big)\big\} + d \\ &= I^{\gamma}k\big(\theta,\phi_p\big(-D^{\tau}u_1(\theta)\big)\big) + d, \\ D^{\tau-1}u_1(1) &= I^{\epsilon}u_0(\zeta) + e \le I^{\epsilon}u_1(\zeta) + e. \end{aligned}$$

This proves that $u_1(t)$ is a lower solution of the problem (1.1). Similarly, we find that $v_1(t)$ is an upper solution of (1.1).

Using mathematical induction, we see that

$$u_0(t) \le u_1(t) \le \cdots \le u_n(t) \le \cdots \le v_n(t) \le \cdots \le v_1(t) \le v_0(t)$$

and

$$D^{\tau}v_0(t) \le D^{\tau}v_1(t) \le \cdots \le D^{\tau}v_n(t) \le \cdots \le D^{\tau}u_n(t) \le \cdots \le D^{\tau}u_1(t) \le D^{\tau}u_0(t),$$

for $t \in [0,1]$ and $n = 1,2,3,\ldots$.

Since the space of solutions is $C_{\tau}[0,1]$, the sequences $\{u_n\}$ and $\{v_n\}$ are uniformly bounded and equi-continuous. The Arzela–Ascoli theorem guarantees that they are relatively compact sets in the space $C_{\tau}[0,1]$. Therefore, $\{u_n\}$ and $\{v_n\}$ converge, say to $u^*(t)$ and $v^*(t)$, uniformly on $[0,1]$, respectively. That is,

$$\lim_{n\to\infty} u_n(t) = u^*(t), \qquad \lim_{n\to\infty} v_n(t) = v^*(t),$$

and

$$\lim_{n\to\infty} D^{\tau}u_n(t) = D^{\tau}u^*(t), \qquad \lim_{n\to\infty} D^{\tau}v_n(t) = D^{\tau}v^*(t),$$

uniformly on $t \in [0,1]$. Moreover, from (3.1) and (3.2), we find that $u^*(t)$ and $v^*(t)$ are solutions of problem of (1.1).

Finally, we prove that $u^*(t)$, $v^*(t)$ are the minimal and maximal solutions of problem (1.1), respectively. Let $u(t) \in [u_0, v_0]$ be any solution of the problem (1.1). We suppose that $u_n(t) \le u(t) \le v_n(t)$, $D^\tau v_n(t) \le D^\tau u(t) \le D^\tau u_n(t)$ $\forall t \in [0,1]$ for some n. Let $y(t) = \phi_p(-D^\tau u(t)) - \phi_p(-D^\tau u_{n+1}(t))$, $x(t) = \phi_p(-D^\tau v_{n+1}(t)) - \phi_p(-D^\tau u(t))$. Then, by assumptions (H_2) and (H_3), we see that

$$\begin{aligned} -D^\sigma y(t) &= h\bigl(t, u(t), D^\tau u(t)\bigr) - h\bigl(t, u_n(t), D^\tau u_n(t)\bigr) \\ &\quad + L\bigl[\phi_p\bigl(-D^\tau u_{n+1}(t)\bigr) - \phi_p\bigl(-D^\tau u_n(t)\bigr)\bigr] \\ &\ge -L\bigl[\phi_p\bigl(-D^\tau u(t)\bigr) - \phi_p\bigl(-D^\tau u_n(t)\bigr)\bigr] \\ &\quad + L\bigl[\phi_p\bigl(-D^\tau u_{n+1}(t)\bigr) - \phi_p\bigl(-D^\tau u_n(t)\bigr)\bigr] \\ &= -Ly(t), \end{aligned}$$

also $y(0) = 0$, and

$$\begin{aligned} D^{\sigma-1} y(1) &= I^\gamma k\bigl(\theta, \phi_p\bigl(-D^\tau u(\theta)\bigr)\bigr) - I^\gamma \bigl\{k\bigl(\theta, \phi_p\bigl(-D^\tau u_n(\theta)\bigr)\bigr) \\ &\quad + \mu\bigl[\phi_p\bigl(-D^\tau u_{n+1}(\theta)\bigr) - \phi_p\bigl(-D^\tau u_n(\theta)\bigr)\bigr]\bigr\} \\ &= I^\gamma \bigl\{k\bigl(\theta, \phi_p\bigl(-D^\tau u(\theta)\bigr)\bigr) - k\bigl(\theta, \phi_p\bigl(-D^\tau u_n(\theta)\bigr)\bigr) \\ &\quad - \mu\bigl[\phi_p\bigl(-D^\tau u_{n+1}(\theta)\bigr) - \phi_p\bigl(-D^\tau u_n(\theta)\bigr)\bigr]\bigr\} \\ &\ge I^\gamma \bigl\{\mu\bigl[\phi_p\bigl(-D^\tau u(\theta)\bigr) - \phi_p\bigl(-D^\tau u_n(\theta)\bigr)\bigr] \\ &\quad - \mu\bigl[\phi_p\bigl(-D^\tau u_{n+1}(\theta)\bigr) - \phi_p\bigl(-D^\tau u_n(\theta)\bigr)\bigr]\bigr\} \\ &= \mu I^\gamma y(\theta). \end{aligned}$$

On the other hand

$$\begin{cases} -D^\sigma x(t) \ge -Lx(t), \\ x(0) = 0, \\ D^{\sigma-1} x(1) \ge \mu I^\gamma x(\theta). \end{cases}$$

Using Lemma 2.6, we have

$$D^\tau v_{n+1}(t) \le D^\tau u(t) \le D^\tau u_{n+1}(t). \tag{3.4}$$

Let $m(t) = u(t) - u_{n+1}(t)$, $\xi(t) = v_{n+1}(t) - u(t)$, by (3.4), we get

$$\begin{cases} -D^\tau m(t) = -D^\tau u(t) + D^\tau u_{n+1}(t) \ge 0, \quad t \in [0,1], \\ m(0) = 0, \\ D^{\tau-1} m(1) = D^{\tau-1} u(1) - D^{\tau-1} u_{n+1}(1) = I^\epsilon u(\zeta) - I^\epsilon u_n(\zeta) \ge 0, \end{cases}$$

and

$$\begin{cases} -D^{\tau}\xi(t) \geq 0, & t \in [0,1], \\ \xi(0) = 0, \\ D^{\tau-1}\xi(1) \geq 0. \end{cases}$$

These results and Lemma 2.7 imply that $u_{n+1}(t) \leq u(t) \leq v_{n+1}(t)$, $t \in [0,1]$, so by induction $u^*(t) \leq u(t) \leq v^*(t)$, $D^{\tau}v^*(t) \leq D^{\tau}u(t) \leq D^{\tau}u^*(t)$, $t \in [0,1]$ by taking as $n \longrightarrow \infty$. The proof is complete. □

4 Iteration procedure and a numerical example

In this section, a numerical procedure is introduced to obtain an appropriate solution of (1.1). For a given accuracy δ, we take u_n and v_n as δ-accurate approximations of x and y, respectively, according to the stopping criterion $E(N) < \delta$, where for each n, $E(n)$ is defined by

$$E(n) = \left\| u_n(t) - v_n(t) \right\|_1 = \int_0^1 \left| u_n(t) - v_n(t) \right| dt.$$

For the iteration equation (3.1), let $\phi_p(-D^{\tau}u_{n+1}(t)) = x_{n+1}$, and with the boundary conditions $u_{n+1}(0) = 0$, $D^{\tau-1}u_{n+1}(1) = k$, and by Lemma 2.4,

$$u(t) = \frac{k}{\Gamma(\tau)}t^{\tau-1} + \int_0^1 H(t,s)\phi_q\big(x_{n+1}(s)\big)\,ds := Bx_{n+1}(t), \tag{4.1}$$

where $k = I^{\epsilon}u_n(\zeta) + e = \frac{1}{\Gamma(\epsilon)}\int_0^{\zeta}(\zeta - s)^{\epsilon-1}u_n(s)\,ds + e$ and

$$H(t,s) = \begin{cases} t^{\tau-1} - (t-s)^{\tau-1}, & 0 \leq s \leq t \leq 1, \\ t^{\tau-1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Thus the iteration equation (3.1) can be rewritten as

$$\begin{cases} -D^{\sigma}x_{n+1} = -Lx_{n+1} + h(t, Bx_n, -\phi_q x_n) + Lx_n, \\ x_{n+1}(0) = 0, \\ D^{\sigma-1}x_{n+1}(1))) = \mu I^{\gamma}x_{n+1}(\theta) + a. \end{cases} \tag{4.2}$$

Applying Lemma 2.1 to (4.2), we obtain

$$\begin{aligned} x_{n+1}(t) = {} & \frac{a\Gamma(\sigma+\gamma)}{\Gamma(\sigma)[\Gamma(\sigma+\gamma) - \mu\theta^{\sigma+\gamma-1}]}t^{\sigma-1} \\ & + \int_0^1 J(t,s)\big[-Lx_{n+1}(s) + h\big((s), Bx_n(s), -\phi_q\big(x_n(s)\big)\big) + Lx_n(s)\big]\,ds, \end{aligned} \tag{4.3}$$

where $a = I^{\gamma}k(\theta, x_n(\theta)) - \mu I^{\gamma}x_n(\theta) + d$, and

$$J(t,s) = \frac{1}{\Delta}\begin{cases} [\Gamma(\sigma+\gamma) - \mu(\theta-s)^{\sigma+\gamma-1}]t^{\sigma-1} \\ \quad - [\Gamma(\sigma+\gamma) - \mu\theta^{\sigma+\gamma-1}](t-s)^{\sigma-1}, & s \le t, s \le \theta; \\ \Gamma(\sigma+\gamma)t^{\sigma-1} - \mu(\theta-s)^{\sigma+\gamma-1}t^{\sigma-1}, & t \le s \le \theta; \\ \Gamma(\sigma+\gamma)[t^{\sigma-1} - (t-s)^{\sigma-1}] + \mu\theta^{\sigma+\gamma-1}(t-s)^{\sigma-1}, & \theta \le s \le t; \\ \Gamma(\sigma+\gamma)t^{\sigma-1}, & s \ge t, s \ge \theta, \end{cases}$$

$$\Delta = \Gamma(\sigma)\big[\Gamma(\sigma+\gamma) - \mu\theta^{\sigma+\gamma-1}\big].$$

Discretize the interval $[0,1]$ with the nodes $t_i = ih$, $b = \frac{1}{K}$, $K = \mathbb{N}$. Let $u_{n+1}^{(i)} \approx u_{n+1}(t_i)$, $x_{n+1}^{(i)} \approx x_{n+1}(t_i)$, $H(i,j) = H(t_i,s_j)$, $J(i,j) = J(t_i,s_j)$ and $h_n^{(j)} = h(s_j, Bx_n(s_j), -\phi_q(x_n(s_j))) + Lx_n(s_j)$. Using the trapezoidal quadrature rule to approximate the integral in the right hand sides of (4.3) and (4.1), we obtain the following linear systems of equations:

$$\begin{aligned} x_{n+1}^{(i)} &= \frac{a\Gamma(\sigma+\gamma)}{\Gamma(\sigma)[\Gamma(\sigma+\gamma) - \mu\theta^{\sigma+\gamma-1}]}t_i^{\sigma-1} - \frac{b}{2}\sum_{j=0}^{K} LJ(i,j)d_j x_{n+1}^{(j)} \\ &\quad + \frac{b}{2}\sum_{j=0}^{K} J(i,j)d_j h_n^{(j)}, \quad 0 \le i \le K, \end{aligned} \tag{4.4}$$

and

$$u_{n+1}^{(i)} = Bx_{n+1}^{(i)} = \frac{k}{\Gamma(\tau)}t_i^{\tau-1} + \frac{b}{2}\sum_{j=0}^{K} H(i,j)d_j\phi_q\big(x_{n+1}^{(j)}\big), \quad 0 \le i \le K, \tag{4.5}$$

for the unknown $x_{n+1}^{(i)}$, $u_{n+1}^{(i)}$, $0 \le i \le K$, where $\{d_j\}$ are the coefficients in the rule, $d_0 = d_K = 1$, $d_j = 2$ for $0 \le i \le K-1$.

Setting $J_{ij} = \frac{b}{2}J(i,j)d_j$, $H_{ij} = \frac{b}{2}H(i,j)d_j$, the matrix $\Phi = (J_{ij})$, $A = \mathbb{I} + L\Phi$, and $G = (H_{(ij)})$ with $\mathbb{I}$ the identity matrix. The systems (4.4) and (4.5) can be written as a system of matrix-vector equations

$$\begin{cases} A\overrightarrow{x}_{n+1} = \frac{a\Gamma(\sigma+\gamma)}{\Gamma(\sigma)[\Gamma(\sigma+\gamma)-\mu\theta^{\sigma+\gamma-1}]}S^{\sigma-1} + \overrightarrow{F}_n, \\ \overrightarrow{U}_{n+1} = \frac{k}{\Gamma(\tau)}S^{\sigma-1} + G\phi_q(\overrightarrow{x}_{n+1}), \end{cases}$$

where $\overrightarrow{x}_{n+1} = [x_{n+1}^{(0)}, x_{n+1}^{(1)}, \ldots, x_{n+1}^{(K)}]$, $\overrightarrow{U}_{n+1} = [u_{n+1}^{(0)}, u_{n+1}^{(1)}, \ldots, u_{n+1}^{(K)}]$, $S = [t_0, t_1, \ldots, t_K]^T$ and $\overrightarrow{F}_n$ is a column vector of its component $F_n^i = \frac{b}{2}\sum_{j=0}^{K} J(i,j)d_j h_n^{(j)}$.

Example 4.1 Consider the following fractional boundary value problem:

$$\begin{cases} -D^{\frac{5}{4}}(\phi_4(-D^{\frac{3}{2}}u(t))) = -\frac{1}{2(1+t)^3}(-D^{\frac{3}{2}}u)^3 + tu, \quad 0 < t < 1, \\ D^{\frac{3}{2}}u(0) = 0, \\ D^{\frac{1}{4}}(\phi_4(-D^{\frac{3}{2}}u(1))) \\ \quad = I^{\frac{5}{4}}k(\frac{1}{4}, \phi_4(-D^{\frac{3}{2}}u(\frac{1}{4}))) + 0.1 \\ \quad = \frac{1}{\Gamma(\frac{5}{4})}\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^{\frac{1}{4}}(s+1)(\phi_4(-D^{\frac{3}{2}}u(s)))\,ds + 0.1, \\ u(0) = 0, \qquad D^{\frac{1}{2}}u(1) = \frac{1}{\Gamma(\frac{3}{2})}\int_0^{\frac{1}{2}}(\frac{1}{2}-s)^{\frac{1}{2}}u(s)\,ds + 0.2, \end{cases} \tag{4.6}$$

where $\sigma = \frac{5}{4}$, $\tau = \frac{3}{2}$, $\gamma = \frac{5}{4}$, $\epsilon = \frac{3}{2}$, $\theta = \frac{1}{4}$, $\zeta = \frac{1}{2}$, $d = 0.1$, $e = 0.2$, $p = 4$, and

$$\begin{cases} h(t,u,D^{\frac{3}{2}}u) = -\frac{1}{2(1+t)^3}(-D^{\frac{3}{2}}u)^3 + tu, \\ k(t,\phi_4(-D^{\frac{3}{2}}u)) = (t+1)\phi_4(-D^{\frac{3}{2}}u). \end{cases}$$

Take $u_0(t) = \frac{1}{2}t^{\frac{1}{2}}$, $v_0(t) = 2t^{\frac{1}{2}} - \frac{\sqrt{\Pi}}{4}t^2 + \frac{8}{15\sqrt{\Pi}}t^{\frac{5}{2}}$, then $-1 \le -t^{\frac{1}{2}} + t = D^{\frac{3}{2}}v_0(t) \le D^{\frac{3}{2}}u_0(t) = 0$. It is not difficult to verify that u_0, v_0 are lower and upper solutions of problem (4.6), respectively. So (H_1) holds. In addition, we have

$$\begin{aligned} &h\big(t,u(t),D^{\frac{3}{2}}u(t)\big) - h\big(t,v(t),D^{\frac{3}{2}}v(t)\big) \\ &\quad = -\frac{1}{2(1+t)^3}\big[-D^{\frac{3}{2}}u\big]^3 + \frac{1}{2(1+t)^3}\big[-D^{\frac{3}{2}}v\big]^3 + t(u-v) \\ &\quad \le \frac{1}{2(1+t)^3}\big[\big(-D^{\frac{3}{2}}v\big)^3 - \big(-D^{\frac{3}{2}}u\big)^3\big] \\ &\quad \le \frac{1}{16}\big[\phi_4\big(-D^{\frac{3}{2}}v\big) - \phi_4\big(-D^{\frac{3}{2}}u\big)\big] \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &k\big(t,\phi_4\big(-D^{\frac{3}{2}}v\big)\big) - k\big(t,\phi_4\big(-D^{\frac{3}{2}}u\big)\big) \\ &\quad = (t+1)\big[\phi_4\big(-D^{\frac{3}{2}}v\big) - \phi_4\big(-D^{\frac{3}{2}}u\big)\big] \ge \phi_4\big(-D^{\frac{3}{2}}v\big) - \phi_4\big(-D^{\frac{3}{2}}u\big), \end{aligned} \tag{4.8}$$

where $u_0(t) \le u(t) \le v(t) \le v_0(t)$. Therefore (H_2) and (H_3) hold.

From (4.7) and (4.8), we have $L = \frac{1}{16}$, $\mu = 1$. Then

$$\begin{aligned} &\Gamma(\sigma+\gamma) = \Gamma\left(\frac{5}{4}+\frac{5}{4}\right) = \Gamma\left(\frac{5}{2}\right) \approx 1.3293 > \mu\theta^{\sigma+\gamma-1} = 1\cdot\left(\frac{1}{4}\right)^{\frac{3}{2}} = 0.125, \\ &2\Gamma(\sigma+\gamma)|L| = 2\cdot\Gamma\left(\frac{5}{2}\right)\cdot\frac{1}{16} \approx 0.1662 < \Gamma(\sigma)\big[\Gamma(\sigma+\gamma) - \mu\theta^{\sigma+\gamma-1}\big] \\ &\qquad = \Gamma\left(\frac{5}{4}\right)\left[\Gamma\left(\frac{5}{2}\right) - 1\cdot\left(\frac{1}{4}\right)^{\frac{3}{2}}\right] \approx 1.1609, \\ &\Gamma(2-\sigma)\cdot t^{\sigma}\cdot L = \Gamma\left(\frac{3}{4}\right)\cdot t^{\sigma}\cdot\frac{1}{16} > 1-\sigma = -\frac{1}{4}, \\ &\Gamma(2-\sigma)\mu\theta^{\gamma} = \Gamma\left(\frac{3}{4}\right)\cdot 1\cdot\left(\frac{1}{4}\right)^{\frac{5}{4}} \approx 0.2332 < \Gamma(\gamma) = \Gamma\left(\frac{5}{4}\right) \approx 0.9064. \end{aligned}$$

It appears that (H_4), (H_5) and (H_6) hold. By Theorem 3.1, the boundary value problem (4.6) has extremal solutions in $[u_0(t), v_0(t)]$.

Applying the numerical scheme to Example 4.1, we obtain approximate solutions to u_n and v_n for $1 \le n \le N$, with N some integer. The graphs of to u_n and v_n for $n = 0,1,2,3,4,34$ are shown in Fig. 1. Computed error values $E(n)$ are displayed in Table 1. We found that, for $\delta = 10^{-10}$, it took $N = 34$ iterations for $E(N) < \delta$.

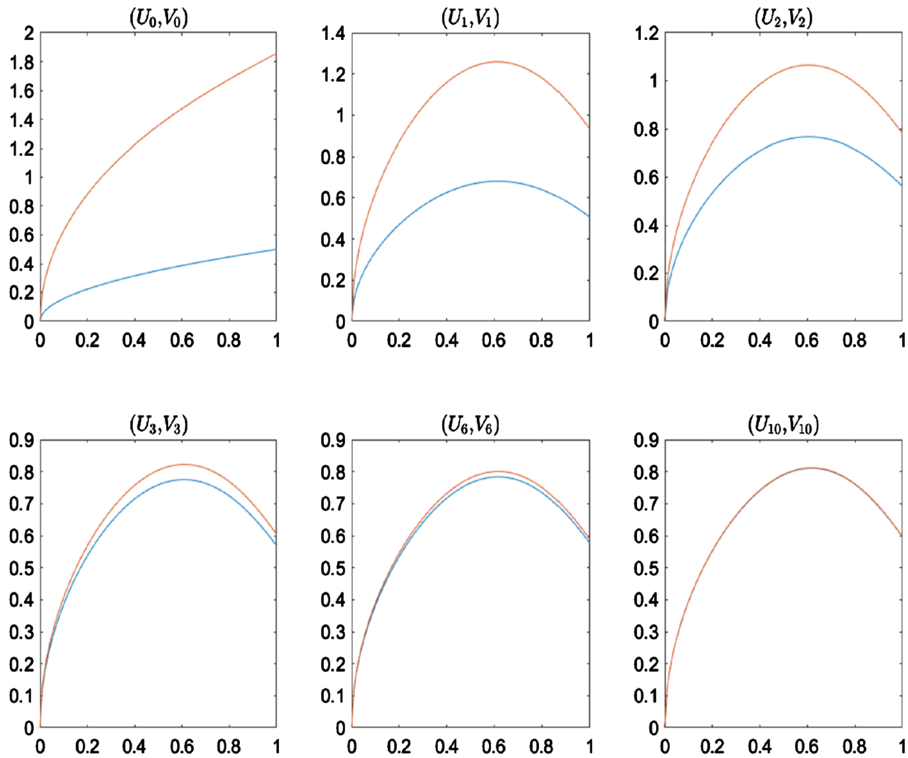

Figure 1 Graph U_n and V_n

Table 1 $E(n) = 6n + 4, n = 0, 1, 2, 3, 4, 5$

n	4	10	16	22	28	34
$E(n)$	0.2448	3.2744e–05	2.3971e–07	1.9264e–09	1.5492e–10	1.2416e–11

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