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Positive radial solutions of n -dimensional elliptic systems with indefinite weight functions and n parameters

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Abstract

Under simple conditions on \mathbf{f} and \mathbf{a} , we show the existence of positive radial solutions for the n -dimensional elliptic differential system

$$\begin{cases} \Delta \mathbf{u}(\mathbf{x}) + \Lambda \mathbf{a}(|\mathbf{x}|)\mathbf{f}(\mathbf{u}(\mathbf{x})) = \mathbf{0}, & R_1 < |\mathbf{x}| < R_2, \\ \mathbf{u}|_{|\mathbf{x}|=R_1} = \mathbf{u}|_{|\mathbf{x}|=R_2} = \mathbf{0}. \end{cases}$$

Here Δ denotes the Laplace operator, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_i(\mathbf{x}), \dots, f_n(\mathbf{x}))^T$, $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$, $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), \dots, u_i(\mathbf{x}), \dots, u_n(\mathbf{x}))^T$, $\Delta \mathbf{u}(\mathbf{x}) = (\Delta u_1(\mathbf{x}), \dots, \Delta u_i(\mathbf{x}), \dots, \Delta u_n(\mathbf{x}))^T$, $\Lambda = \text{diag}[\lambda_1, \dots, \lambda_i, \dots, \lambda_n]$, $\mathbf{a}(|\mathbf{x}|) = \text{diag}[a_1(|\mathbf{x}|), \dots, a_i(|\mathbf{x}|), \dots, a_n(|\mathbf{x}|)]$, $R_2 > R_1 > 0$, $\mathbf{x} \in \mathcal{R}^n$, $n \geq 2$. The interest is that $\mathbf{a}(|\mathbf{x}|)$ is allowed to change sign on $[R_1, R_2]$, which needs some new ingredients in the arguments. An example is also given to illustrate the main results.

Keywords: Positive radial solutions; n -dimensional elliptic systems; Indefinite weight function; n parameters

1 Introduction

In this paper, we analyze the existence and multiplicity of positive radial solutions for the following n -dimensional elliptic differential system:

$$\begin{cases} \Delta \mathbf{u}(\mathbf{x}) + \Lambda \mathbf{a}(|\mathbf{x}|)\mathbf{f}(\mathbf{u}(\mathbf{x})) = \mathbf{0}, & R_1 < |\mathbf{x}| < R_2, \\ \mathbf{u}|_{|\mathbf{x}|=R_1} = \mathbf{u}|_{|\mathbf{x}|=R_2} = \mathbf{0}, \end{cases} \quad (1.1)$$

where Δ denotes the Laplace operator, $R_2 > R_1 > 0$, $\mathbf{x} \in \mathcal{R}^n$, $n \geq 2$, $\mathbf{a}(|\mathbf{x}|)$ is allowed to change sign on $[R_1, R_2]$, and

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, \dots, x_n)^T, \\ \mathbf{u}(\mathbf{x}) &= (u_1(\mathbf{x}), \dots, u_i(\mathbf{x}), \dots, u_n(\mathbf{x}))^T, \\ \Delta \mathbf{u}(\mathbf{x}) &= (\Delta u_1(\mathbf{x}), \dots, \Delta u_i(\mathbf{x}), \dots, \Delta u_n(\mathbf{x}))^T, \\ \Lambda &= \text{diag}[\lambda_1, \dots, \lambda_i, \dots, \lambda_n], \end{aligned}$$

$$\mathbf{a}(|\mathbf{x}|) = \text{diag}[a_1(|\mathbf{x}|), \dots, a_i(|\mathbf{x}|), \dots, a_n(|\mathbf{x}|)],$$

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_i(\mathbf{x}), \dots, f_n(\mathbf{x}))^T,$$

where we understand $f_i(\mathbf{x})$ to mean $f_i(x_1, x_2, \dots, x_n), i = 1, 2, \dots, n$.

Therefore, system (1.1) means that ($i = 1, 2, \dots, n$)

$$\begin{cases} \Delta u_i(\mathbf{x}) + \lambda_i a_i(|\mathbf{x}|) f_i(\mathbf{u}(\mathbf{x})) = 0, & R_1 < |\mathbf{x}| < R_2, \\ u_i|_{|\mathbf{x}|=R_1} = u_i|_{|\mathbf{x}|=R_2} = 0. \end{cases} \tag{1.2}$$

Let $J = [0, 1], \mathcal{R}_+ = [0, +\infty), \mathcal{R} = (-\infty, +\infty), \mathcal{R}_+^n = \underbrace{\mathcal{R}_+ \times \mathcal{R}_+ \times \dots \times \mathcal{R}_+}_n$. By a positive radial solution \mathbf{u}^* of system (1.1) we understand a solution \mathbf{u}^* with $u_i^* \geq 0 (i = 1, 2, \dots, n)$ and either $u_i^* \neq 0 (i = 1, 2, \dots, n)$. By the maximum principle, each nontrivial component of \mathbf{u}^* is thus positive in $\Omega = \{\mathbf{x} \in \mathcal{R}^n : R_1 < |\mathbf{x}| < R_2, R_1, R_2 > 0\}$. For $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, we define $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i, i = 1, 2, \dots, n$.

The study of boundary value problems with positive solutions has attracted recently the attention of different researchers and it is a topic of current interest; see [1–16], and the references therein.

At the same time, we note that the existence and multiplicity of solutions to the elliptic differential systems:

$$\begin{cases} \Delta u + k_1(|x|)f(u, v) = 0, \\ \Delta v + k_2(|x|)g(u, v) = 0, \end{cases} \tag{1.3}$$

under different boundary conditions have been studied extensively in the past decades (see [17–36]). Kawano and Kusano [17] gave sufficient conditions which guarantee the existence of entire solutions of (1.3) by means of the method of sub- and super-solutions. By applying the linking theorem and with the assistance of the Nehari manifold, Benrhouma [18] showed the existence of at least two solutions of (1.3) in the whole space \mathcal{R}^n . Serrin and Zou [37] gave an excellent survey on the existence results for problem (1.3).

However, there is almost no paper dealing with the n -dimensional elliptic differential system with indefinite weight functions and parameters; for instance, see [38–41] and the references therein. Dalmasso [38] investigated the existence and uniqueness of positive solutions for the following elliptic system:

$$\begin{cases} \Delta u + g(v) = 0 & \text{in } \Omega, \\ \Delta v + f(u) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \Omega, \end{cases}$$

where $\Omega \subset \mathcal{R}^n (n \geq 1)$ denotes a bounded domain of class $C^{2,\alpha}, \alpha \in (0, 1]$. Precup [39] considered the existence, localization and multiplicity of positive radial solutions of the elliptic differential system:

$$\begin{cases} \Delta u_1 + f_1(|x|)g_1(u_1, u_2) = 0, \\ \Delta u_2 + f_2(|x|)g_2(u_1, u_2) = 0, \end{cases}$$

in $\Omega := \{x \in \mathcal{R}^n : |x| > r_0\}$ ($n \geq 3$), under the conditions

$$u_1 = u_2 = 0 \quad \text{for } |x| = r_0 \quad \text{and} \quad u_1, u_2 \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Recently, in [40], Maniwa studied the uniqueness and existence of positive solutions for the following elliptic differential system:

$$\begin{cases} -\Delta u_i = \prod_{j=1}^N u_j^{p_{ij}} & \text{in } \Omega, \\ u_i = 0 & \text{on } \Omega, i = 1, 2, \dots, N, \end{cases}$$

where p_{ij} ($1 \leq i, j \leq N$) are nonnegative constants and $\Omega \subset \mathcal{R}^n$ ($n \geq 1$) denotes a bounded domain of class $C^{2,\alpha}$, $\alpha \in (0, 1)$.

To the best of our knowledge, in the literature there are no articles on multiple radial positive solutions for the analogous of n -dimensional elliptic differential system with indefinite weights and n parameters. More precisely, the study of $\Lambda \neq 1$, and \mathbf{a} changing sign on $[R_1, R_2]$ is still open for the elliptic systems. Specially, comparing with [17–39, 41, 42], the main features of this paper are as follows.

- (i) $\lambda_i > 0$ is available, not only $\lambda_i \equiv 1, i = 1, 2, \dots, n$.
- (ii) $\mathbf{a}(|\mathbf{x}|)$ being allowed to change sign on $[R_1, R_2]$ is considered.
- (iii) n -dimensional elliptic system is investigated.

In [40], the author considered n -dimensional elliptic system and obtained several excellent results of uniqueness and existence of positive solutions, but Maniwa only studied the case $\lambda_i \equiv 1$ and $a_i(|\mathbf{x}|) \equiv 1$ ($i = 1, 2, \dots, n$). On the other hand, in [43], Yao pointed out that it is of particular mathematical interest when the weight function $\mathbf{a}(|\mathbf{x}|)$ is allowed to change sign on $[R_1, R_2]$. Therefore, the present work is new and significant.

In this paper, we always suppose that the following conditions hold:

1. $f_i : \mathcal{R}_+^n \rightarrow \mathcal{R}_+$ is continuous and there exists $0 < c_i \leq 1$ such that

$$f_i(\mathbf{x}) \geq c_i \varphi_i(\mathbf{x}), \quad \mathbf{x} \in \mathcal{R}_+^n,$$

where $\varphi_i(\mathbf{x}) = \max\{f_i(\mathbf{y}) : 0 \leq \mathbf{y} \leq \mathbf{x}\}, i = 1, 2, \dots, n$.

2. $a_i : [R_1, R_2] \rightarrow \mathcal{R}$ is continuous and there exists $R_1 < \zeta < R_2$ such that

$$a_i(r) \geq 0, \quad r \in [R_1, \zeta]; \quad a_i(r) \leq 0, \quad r \in [\zeta, R_2], i = 1, 2, \dots, n.$$

Moreover, $a_i(r)$ ($i = 1, 2, \dots, n$) does not vanish identically on any subintervals of $[R_1, R_2]$.

In Sect. 2 we list several preliminary results that will be used in the subsequent sections. Section 3 is devoted to stating and proving the main results. Several special cases and an example are also given in Sect. 4.

2 Preliminaries

Looking for radial solutions, let us first introduce the radial coordinates form of system (1.2); for details to see Precup [39] and Lee [44]. By the radial variable $r = |x|$, we can write

(1.2) as

$$\begin{cases} u_i''(r) + \frac{n-1}{r}u_i'(r) + \lambda_i a_i(r)f_i(\mathbf{u}(r)) = 0, & R_1 < r < R_2, \\ u_i(R_1) = u_i(R_2) = 0, \end{cases} \tag{2.1}$$

where $i = 1, 2, \dots, n$.

Let

$$s(r) = - \int_r^{R_2} \frac{dt}{t^{n-1}}. \tag{2.2}$$

Then

$$s(R_1) = - \int_{R_1}^{R_2} \frac{dt}{t^{n-1}} =: m$$

and $s(R_2) = 0$.

Set $v_i(s) = u_i(r(s))$. Then taking the derivative of this equation with respect to r , and together with (2.2), we get

$$\begin{aligned} u'_{ir} &= v'_{is} \times s'(r) = v'_{is} \times \frac{1}{r^{n-1}}, \\ u''_{irr} &= v''_{iss} \times \frac{1}{r^{2n-2}} + v'_{is} \times (1-n)r^{2-n}. \end{aligned}$$

For convenience, we write v''_{iss} as $v''_i(s)$. Thus submitting u'_{irr} and u''_{irr} into (2.1), we get

$$\begin{cases} v''_i(s) + \lambda_i r^{2(n-1)}(s)a_i(r(s))f_i(\mathbf{v}(s)) = 0, & m < s < 0, \\ v_i(m) = v_i(0) = 0, \end{cases} \tag{2.3}$$

where $i = 1, 2, \dots, n, \mathbf{v} = (v_1, v_2, \dots, v_n)^T$.

Let $t = \frac{m-s}{m}$. Then $s = m(1-t)$. Set $w_i(t) = v_i(s)$. Then similarly system (2.3) can be written as

$$\begin{cases} w''_i(t) + \lambda_i m^2 r^{2(n-1)}(m(1-t))a_i(r(m(1-t)))f_i(\mathbf{w}(t)) = 0, & 0 < t < 1, \\ w_i(1) = w_i(0) = 0, \end{cases} \tag{2.4}$$

where $i = 1, 2, \dots, n, \mathbf{w} = (w_1, w_2, \dots, w_n)^T$.

Letting $h_i(t) = m^2 r^{2(n-1)}(m(1-t))a_i(r(m(1-t)))$, then system (2.4) is equal to

$$\begin{cases} w''_i(t) + \lambda_i h_i(t)f_i(\mathbf{w}) = 0, & 0 < t < 1, \\ w_i(1) = w_i(0) = 0, \end{cases} \tag{2.5}$$

where $i = 1, 2, \dots, n$.

The following conditions will be assumed throughout this paper:

(H₁) $h_i : J \rightarrow \mathcal{R}$ is continuous and there exists $0 < \xi < 1$ such that

$$h_i(t) \geq 0, \quad t \in [0, \xi]; \quad h_i(t) \leq 0, \quad t \in [\xi, 1], i = 1, 2, \dots, n.$$

Moreover, $h(t)$ does not vanish identically on any subintervals of J .

(H₂) $f_i : \mathcal{R}_+^n \rightarrow \mathcal{R}_+$ is continuous and there exists $0 < c_i \leq 1$ such that

$$f_i(\mathbf{w}) \geq c_i \varphi_i(\mathbf{w}), \quad i = 1, 2, \dots, n,$$

where $\varphi_i(\mathbf{w}) = \max\{f_i(\mathbf{y}), 0 \leq y_j \leq w_j, j = 1, 2, \dots, n\}$.

(H₃) There exist $0 < \theta_i < +\infty, \theta_i \neq 1$ and $k_i, l_i > 0$ such that

$$k_i \left(\sum_{i=1}^n w_i \right)^{\theta_i} \leq f_i(\mathbf{w}) \leq l_i \left(\sum_{i=1}^n w_i \right)^{\theta_i}, \quad \mathbf{w} \in \mathcal{R}_+^n, i = 1, 2, \dots, n.$$

(H₄) There exists $0 < \sigma < \xi$ such that

$$c_i^2 k_i \sigma^{\theta_2} \int_{\sigma}^{\xi} G(t, s) h_i^+(s) ds \geq l_i \xi^{\theta_2} \int_{\xi}^1 G(t, s) h_i^-(s) ds, \quad i = 1, 2, \dots, n.$$

We define

$$h_i^+(t) = \max\{h_i(t), 0\}, \quad h_i^-(t) = -\min\{h_i(t), 0\}, \quad i = 1, 2, \dots, n.$$

Then

$$h_i(t) = h_i^+(t) - h_i^-(t), \quad i = 1, 2, \dots, n.$$

Next we give some lemmas which we will need later.

Lemma 2.1 (See [39]) *By (2.1)–(2.4), system (1.1) admitting positive radial solutions is equal to system (2.4) having positive solutions.*

Lemma 2.2 (Lemma 1 of [45]) *Assume that (H₁)–(H₂) hold. Then system (2.5) has a solution $\mathbf{w} = (w_1, \dots, w_i, \dots, w_n)$ given by*

$$w_i(t) = \lambda_i \int_0^1 G(t, s) h_i(s) f_i(\mathbf{w}(s)) ds, \quad t \in J, \tag{2.6}$$

where

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.7}$$

It is well known that $C[0, 1]$ is a real Banach space with the norm given by $\|x\|_{\infty} = \max_{t \in J} |x(t)|$. Let $X = \prod_{i=1}^n C[0, 1]$, and for any $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in X$,

$$\|\mathbf{x}\| = \sum_{i=1}^n \|x_i\|_{\infty}.$$

Then $(X, \|\cdot\|)$ is a real Banach space.

Define a cone K in X by

$$K = \prod_{i=1}^n K_i,$$

where

$$K_i = \{x_i \in C^+[0, 1] : x_i(t) \text{ is concave on } [0, \xi], \text{ and convex on } [\xi, 1]\}, \quad i = 1, 2, \dots, n,$$

$$C^+[0, 1] = \{x \in C[0, 1] : x(t) \geq 0, \forall t \in J \text{ and } x(0) = x(1) = 0\}.$$

We define some sets as follows:

$$K_{i,r} = \{x_i \in K_i \mid \|x_i\| < r\},$$

$$K_{i,r,R} = \{x_i \in K_i \mid r < \|x_i\| < R\},$$

$$\partial K_{i,r} = \{x_i \in K_i \mid \|x_i\| = r\},$$

where $R > r > 0, i = 1, 2, \dots, n$.

Lemma 2.3 (Proposition 2.1–2.2 of [16]) *From (2.7), it is easy to verify that $G(t, s)$ has following properties:*

- (i) $G(t, s) > 0, \forall t, s \in (0, 1)$;
- (ii) $G(t, s) \leq G(s, s), \forall t, s \in J$;
- (iii) $G(t, s) \geq \sigma G(s, s), \forall t \in [\sigma, 1 - \sigma], s \in J$, where σ is defined in (H_4) .

We define the map $\mathbf{T}^\lambda : \mathcal{R}_+^n \rightarrow \mathcal{R}_+^n$ with components $(T_1^{\lambda_1}, \dots, T_i^{\lambda_i}, \dots, T_n^{\lambda_n})$. Here, we understand $\mathbf{T}^\lambda \mathbf{w} = (T_1^{\lambda_1} \mathbf{w}, \dots, T_i^{\lambda_i} \mathbf{w}, \dots, T_n^{\lambda_n} \mathbf{w})$, which

$$(T_i^{\lambda_i} \mathbf{w})(t) = \lambda_i \int_0^1 G(t, s) h_i(s) f_i(\mathbf{w}(s)) ds, \quad t \in J, i = 1, 2, \dots, n. \tag{2.8}$$

Cheng and Zhang [46] pointed out that the existence of a positive solutions of system (2.5) is equivalent to the existence of nontrivial fixed points of \mathbf{T}^λ in K .

Lemma 2.4 *Assume (H_1) – (H_4) hold. Then $\mathbf{T}^\lambda : K \rightarrow K$ is completely continuous.*

Proof We show that $\mathbf{T}^\lambda(K) \subset K$, that is, for $\mathbf{w} \in K$, we show that $T_i^{\lambda_i} \in K_i, i = 1, 2, \dots, n$. By (2.8), it is clear that $(T_i^{\lambda_i} \mathbf{w})(0) = (T_i^{\lambda_i} \mathbf{w})(1) = 0 (i = 1, 2, \dots, n)$.

Define a function $q : J \rightarrow J$ as follows:

$$q(t) = \min \left\{ \frac{t}{\xi}, \frac{1-t}{1-\xi} \right\}.$$

Then $\min_{\sigma \leq t \leq \xi} q(t) = \frac{\sigma}{\xi}, \max_{\xi \leq t \leq 1} q(t) = 1$.

For any $\mathbf{w} \in K$, we prove that

$$\int_0^1 G(t, s) h_i(s) f_i(\mathbf{w}(s)) ds \geq \int_0^\sigma G(t, s) h_i^+(s) f_i(\mathbf{w}(s)) ds.$$

Since $\mathbf{w} \in K$ and $w_i(0) = w_i(1) = 0, i = 1, 2, \dots, n$, then we have

$$\begin{aligned} \frac{w_i(t) - w_i(0)}{t - 0} &\geq \frac{w_i(\xi) - w_i(0)}{\xi - 0}, \quad t \in [0, \xi] \Rightarrow w_i(t) \geq q(t)w_i(\xi), \quad t \in [0, \xi], \\ \frac{w_i(t) - w_i(1)}{t - 1} &\geq \frac{w_i(\xi) - w_i(1)}{\xi - 1}, \quad t \in [\xi, 1] \Rightarrow w_i(t) \leq q(t)w_i(\xi), \quad t \in [\xi, 1], \end{aligned}$$

where $i = 1, 2, \dots, n$. Accordingly, we know that

$$\mathbf{w}(t) \geq q(t)\mathbf{w}(\xi), \quad t \in [0, \xi]; \quad \mathbf{w}(t) \leq q(t)\mathbf{w}(\xi), \quad t \in [\xi, 1].$$

From the definition of $\varphi_i (i = 1, 2, \dots, n)$, we also have

$$\begin{aligned} \varphi_i(\mathbf{w}(t)) &\geq \varphi_i(q(t)\mathbf{w}(\xi)), \quad t \in [0, \xi], \\ \varphi_i(\mathbf{w}(t)) &\leq \varphi_i(q(t)\mathbf{w}(\xi)), \quad t \in [\xi, 1]. \end{aligned}$$

Then, for $0 \leq t \leq 1$,

$$\begin{aligned} &\int_0^1 G(t, s)h_i(s)f_i(\mathbf{w}(s)) \, ds - \int_0^\sigma G(t, s)h_i^+(s)f_i(\mathbf{w}(s)) \, ds \\ &= \int_\sigma^\xi G(t, s)h_i^+(s)f_i(\mathbf{w}(s)) \, ds - \int_\xi^1 G(t, s)h_i^-(s)f_i(\mathbf{w}(s)) \, ds \\ &\geq c_i \int_\sigma^\xi G(t, s)h_i^+(s)\varphi_i(\mathbf{w}(s)) \, ds - \int_\xi^1 G(t, s)h_i^-(s)\varphi_i(\mathbf{w}(s)) \, ds \\ &\geq c_i \int_\sigma^\xi G(t, s)h_i^+(s)\varphi_i(q(s)\mathbf{w}(\xi)) \, ds - \int_\xi^1 G(t, s)h_i^-(s)\varphi_i(q(s)\mathbf{w}(\xi)) \, ds \\ &\geq c_i \int_\sigma^\xi G(t, s)h_i^+(s)f_i(q(s)\mathbf{w}(\xi)) \, ds - \frac{1}{c_i} \int_\xi^1 G(t, s)h_i^-(s)f_i(q(s)\mathbf{w}(\xi)) \, ds \\ &\geq c_i \int_\sigma^\xi G(t, s)h_i^+(s)k_i \left(q(s) \sum_{i=1}^n w_i(\xi) \right)^{\theta_i} \, ds \\ &\quad - \frac{1}{c_i} \int_\xi^1 G(t, s)h_i^-(s)l_i \left(q(s) \sum_{i=1}^n w_i(\xi) \right)^{\theta_i} \, ds \\ &\geq c_i k_i \left(\frac{\sigma}{\xi} \right)^{\theta_i} \left(\sum_{i=1}^n w_i(\xi) \right)^{\theta_i} \int_\sigma^\xi G(t, s)h_i^+(s) \, ds \\ &\quad - \frac{1}{c_i} l_i \left(\sum_{i=1}^n w_i(\xi) \right)^{\theta_i} \int_\xi^1 G(t, s)h_i^-(s) \, ds \\ &\geq \left(\sum_{i=1}^n w_i(\xi) \right)^{\theta_i} \left(c_i k_i \left(\frac{\sigma}{\xi} \right)^{\theta_i} \int_\sigma^\xi G(t, s)h_i^+(s) \, ds - \frac{1}{c_i} l_i \int_\xi^1 G(t, s)h_i^-(s) \, ds \right) \\ &\geq 0. \end{aligned}$$

Thus $T_i^{\lambda_i} \mathbf{w} \geq 0, i = 1, 2, \dots, n$.

By the above definitions and the properties of the Green’s function $G(t, s)$, we have

$$\begin{aligned} (T_i^{\lambda_i} \mathbf{w})''(t) &= \lambda_i h_i^+(t) f_i(\mathbf{w}(t)) \geq 0, \quad t \in [0, \xi], \\ (T_i^{\lambda_i} \mathbf{w})''(t) &= -\lambda_i h_i^-(t) f_i(\mathbf{w}(t)) \leq 0, \quad t \in [0, \xi], \end{aligned}$$

where $i = 1, 2, \dots, n$. Thus, $T_i^{\lambda_i} \mathbf{w} \in K_i$ and $T_i^{\lambda_i}(K) \subset K_i, i = 1, 2, \dots, n$, that is, $\mathbf{T}^A(K) \subset K$.

Similar to the proof of Lemma 2.4 in [47], one can prove $\mathbf{T}^A : K \rightarrow K$ is completely continuous. The proof of Lemma 2.5 is complete. \square

The main proof is based on the well-known fixed point theorem of a cone expansion and a compression of norm type.

Lemma 2.5 (Theorem 2.3.4 of [48]) *(Fixed point theorem of cone expansion and compression of norm type). Let Ω_1 and Ω_2 be two bounded open sets in a real Banach space E such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let the operator $T : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$ be completely continuous, where P is a cone in E . Suppose that one of the two conditions*

- (i) $\|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$ and
 - (ii) $\|Tx\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$ and $\|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$
- is satisfied. Then T has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.*

3 Main results

Based on the lemmas mentioned above, we give the following theorems and their proofs.

Theorem 3.1 *Assume that (H_1) – (H_4) hold. If $\theta_i > 1, i = 1, 2, \dots, n$, then there exists $\lambda_{i,0} > 0$ such that system (1.1) has at least two positive radial solutions for $\lambda_i \in [\lambda_{i,0}, +\infty)$.*

Proof On one hand, since $\theta_i > 1$, by (H_3) , we get

$$\lim_{\max_{1 \leq j \leq n} w_j \rightarrow 0} \frac{f_i(\mathbf{w})}{\sum_{i=1}^n w_i} \leq \lim_{\max_{1 \leq j \leq n} w_j \rightarrow 0} \frac{l_i (\sum_{i=1}^n w_i)^{\theta_i}}{\sum_{i=1}^n w_i} = 0,$$

where $i = 1, 2, \dots, n$.

Furthermore, there exists a $r_1 > 0$ such that

$$f_i(\mathbf{w}) \leq \varepsilon_i \sum_{i=1}^n w_i, \quad 0 < w_i < r_1,$$

where ε_i satisfies $\sum_{i=1}^n \max_{t \in J} (\lambda_i \varepsilon_i \int_0^\xi G(t, s) h_i^+(s) ds) < 1, i = 1, 2, \dots, n$.

Then, for any $w_i \in \partial K_{i,r_1}$, we have

$$\begin{aligned} \|T_i^{\lambda_i}(\mathbf{w})\|_\infty &= \max_{t \in J} \left| \lambda_i \int_0^1 G(t, s) h_i(s) f_i(\mathbf{w}(s)) ds \right| \\ &= \max_{t \in J} \left| \lambda_i \int_0^\xi G(t, s) h_i^+(s) f_i(\mathbf{w}(s)) ds - \lambda_i \int_\xi^1 G(t, s) h_i^-(s) f_i(\mathbf{w}(s)) ds \right| \\ &\leq \max_{t \in J} \left| \lambda_i \int_0^\xi G(t, s) h_i^+(s) f_i(\mathbf{w}(s)) ds \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \max_{t \in J} \left| \lambda_i \int_0^\xi G(t,s) h_i^+(s) \varepsilon_i \sum_{i=1}^n w_i(s) ds \right| \\
 &\leq \max_{t \in J} \lambda_i \int_0^\xi G(t,s) h_i^+(s) \varepsilon_i \left| \sum_{i=1}^n w_i(s) \right| ds \\
 &\leq \max_{t \in J} \lambda_i \varepsilon_i \int_0^\xi G(t,s) h_i^+(s) ds \left(\sum_{i=1}^n \|w_i\|_\infty \right) \\
 &= \max_{t \in J} \lambda_i \varepsilon_i \int_0^\xi G(t,s) h_i^+(s) ds \|\mathbf{w}\|,
 \end{aligned}$$

where $i = 1, 2, \dots, n$.

Thus, for $w_i \in \partial K_{i,r_1}, i = 1, 2, \dots, n$, we have

$$\begin{aligned}
 \|\mathbf{T}^A \mathbf{w}\| &= \sum_{i=1}^n \|T_i^{\lambda_i} \mathbf{w}\|_\infty \\
 &\leq \sum_{i=1}^n \max_{t \in J} \left(\lambda_i \varepsilon_i \int_0^\xi G(t,s) h_i^+(s) ds \right) \|\mathbf{w}\| \\
 &< \|\mathbf{w}\|.
 \end{aligned} \tag{3.1}$$

We denote $\delta(t) = \min\{\frac{t}{\xi}, \frac{\xi-t}{\xi}\}, t \in [0, \xi]$. If $w_i \in K_i, i = 1, 2, \dots, n$, then from the concave on $[0, \xi]$. So

$$w_i(t) \geq \delta(t) \|w_i\|_\infty, \quad t \in [0, \xi], i = 1, 2, \dots, n.$$

It follows that $w_i(t) \geq \alpha \|w_i\|_\infty, t \in [\frac{\sigma}{2}, \sigma]$, where $\alpha = \min_{\frac{\sigma}{2} \leq t \leq \sigma} \delta(t), i = 1, 2, \dots, n$. Thus we have

$$w_i(t) \geq \alpha \|w_i\|, \quad t \in \left[\frac{\sigma}{2}, \sigma \right], i = 1, 2, \dots, n.$$

Since $\theta_i > 1, i = 1, 2, \dots, n$, by (H_3) , we have

$$\lim_{\min_{1 \leq j \leq n} w_j \rightarrow \infty} \frac{f_i(\mathbf{w})}{\sum_{i=1}^n w_i} \geq \lim_{\min_{1 \leq j \leq n} w_j \rightarrow \infty} \frac{k_i (\sum_{i=1}^n w_i)^{\theta_i}}{\sum_{i=1}^n w_i} = \infty,$$

where $i = 1, 2, \dots, n$.

Furthermore, there exists $0 < r_1 < R'_1$ such that

$$f_i(\mathbf{w}) \geq \eta_i \sum_{i=1}^n w_i, \quad w_i \geq R'_1,$$

where η_i satisfies $\sum_{i=1}^n \max_{t \in J} (\lambda_i \eta_i \alpha \int_{\frac{\sigma}{2}}^\sigma G(t,s) h_i^+(s) ds) \geq 1, i = 1, 2, \dots, n$.

Choose $R_1 \geq \frac{R'_1}{\alpha}$. Then, for any $w_i \in \partial K_{i,R_1}, i = 1, 2, \dots, n$, we have $\min_{\frac{\sigma}{2} \leq t \leq \sigma} w_i(t) \geq \min_{\frac{\sigma}{2} \leq t \leq \sigma} \delta(t) \|w_i\|_\infty = \alpha R_1 \geq R'_1$ and $f_i(\mathbf{w}(t)) \geq \eta_i (\sum_{i=1}^n w_i(t)), t \in [\frac{\sigma}{2}, \sigma], i = 1, 2, \dots, n$.

Then, for $w_i \in \partial K_{i,R_1}$, we have

$$\|T_i^{\lambda_i} \mathbf{w}\|_\infty = \max_{t \in J} \left| \lambda_i \int_0^1 G(t,s) h_i(s) f_i(\mathbf{w}(s)) ds \right|$$

$$\begin{aligned}
 &\geq \max_{t \in J} \left| \lambda_i \int_0^\sigma G(t,s) h_i^+(s) f_i(\mathbf{w}(s)) \, ds \right| \\
 &\geq \max_{t \in J} \left| \lambda_i \int_{\frac{\sigma}{2}}^\sigma G(t,s) h_i^+(s) \eta_i \sum_{i=1}^n w_i(s) \, ds \right| \\
 &\geq \max_{t \in J} \lambda_i \int_{\frac{\sigma}{2}}^\sigma G(t,s) h_i^+(s) \eta_i \alpha \sum_{i=1}^n \|w_i\|_\infty \, ds \\
 &= \max_{t \in J} \lambda_i \eta_i \alpha \int_{\frac{\sigma}{2}}^\sigma G(t,s) h_i^+(s) \, ds \|\mathbf{w}\|,
 \end{aligned}$$

where $i = 1, 2, \dots, n$.

Thus, for $w_i \in \partial K_{i,R_1}, i = 1, 2, \dots, n$, we have

$$\begin{aligned}
 \|\mathbf{T}^A \mathbf{w}\| &= \sum_{i=1}^n \|T_i^{\lambda_i} \mathbf{w}\|_\infty \\
 &\geq \sum_{i=1}^n \max_{t \in J} \left(\lambda_i \eta_i \alpha \int_{\frac{\sigma}{2}}^\sigma G(t,s) h_i^+(s) \, ds \right) \|\mathbf{w}\| \\
 &\geq \|\mathbf{w}\|.
 \end{aligned} \tag{3.2}$$

In addition, choose a number $0 < r < r_1$. Noticing that $f_i(\mathbf{w}) > 0$ for all $\mathbf{w} > \mathbf{0}$, we can define

$$f_{i,r} = \min\{f_i(\mathbf{w}) : \alpha r \leq w_i \leq r\}, \quad i = 1, 2, \dots, n.$$

$$\text{Let } \lambda_{i,0} = \frac{r}{\max_{t \in J} \int_{\frac{\sigma}{2}}^\sigma G(t,s) h_i^+(s) f_{i,r} \, ds}.$$

If $w_i \in \partial K_{i,r}$, then $\|w_i\|_\infty = r$ and $\alpha r = \min_{\frac{\sigma}{2} \leq t \leq \sigma} \delta(t) \|w_i\|_\infty \leq w_i(t) \leq \|w_i\|_\infty = r, t \in [\frac{\sigma}{2}, \sigma], i = 1, 2, \dots, n$. It is clear that $f_i(\mathbf{w}(t)) \geq f_{i,r}, t \in [\frac{\sigma}{2}, \sigma], i = 1, 2, \dots, n$. Then, for $w_i \in \partial K_{i,r}$, we have

$$\begin{aligned}
 \|T_i^{\lambda_{i,0}} \mathbf{w}\|_\infty &= \max_{t \in J} \left| \lambda_{i,0} \int_0^1 G(t,s) h_i(s) f_i(\mathbf{w}(s)) \, ds \right| \\
 &\geq \max_{t \in J} \left| \lambda_{i,0} \int_0^\sigma G(t,s) h_i^+(s) f_i(\mathbf{w}(s)) \, ds \right| \\
 &\geq \max_{t \in J} \lambda_{i,0} \int_{\frac{\sigma}{2}}^\sigma G(t,s) h_i^+(s) f_{i,r} \, ds,
 \end{aligned}$$

where $i = 1, 2, \dots, n$.

Thus, for $w_i \in \partial K_{i,r}, i = 1, 2, \dots, n$, we have

$$\begin{aligned}
 \|\mathbf{T}^A \mathbf{w}\| &= \sum_{i=1}^n \|T_i^{\lambda_{i,0}} \mathbf{w}\|_\infty \\
 &\geq \sum_{i=1}^n \max_{t \in J} \left(\lambda_{i,0} \int_{\frac{\sigma}{2}}^\sigma G(t,s) h_i^+(s) f_{i,r} \, ds \right) \\
 &\geq \sum_{i=1}^n \max_{t \in J} \lambda_{i,0} \left(\int_{\frac{\sigma}{2}}^\sigma G(t,s) h_i^+(s) f_{i,r} \, ds \right)
 \end{aligned}$$

$$= nr = \sum_{i=1}^n \|w_i\|_\infty = \|\mathbf{w}\|. \tag{3.3}$$

Applying Lemma 2.5 to (3.1), (3.2) and (3.3) shows that \mathbf{T}^A admits at least two fixed points $\mathbf{w}_1, \mathbf{w}_2$, where $\mathbf{w}_1 \in \overline{K}_{R_1} \setminus \overline{K}_{r_1}$ and $\mathbf{w}_2 \in K_{r_1} \setminus K_r$. Thus it follows from Lemma 2.1 that, if $\theta_i > 1, i = 1, 2, \dots, n$, there exists $\lambda_{i,0} > 0$ such that system (1.1) has at least two positive radial solutions for $\lambda_i \in [\lambda_{i,0}, +\infty)$. This finishes the proof of Theorem 3.1. \square

Theorem 3.2 *Assume that (H_1) – (H_4) hold. If $0 < \theta_i < 1, i = 1, 2, \dots, n$, then there exists $\lambda_i^0 > 0$ such that system (1.1) admits at least two positive radial solutions for $\lambda_i \in (0, \lambda_i^0], i = 1, 2, \dots, n$.*

Proof On one hand, since $0 < \theta_i < 1$, by (H_3) , we get

$$\lim_{\max_{1 \leq j \leq n} w_j \rightarrow 0} \frac{f_i(\mathbf{w})}{\sum_{i=1}^n w_i} \geq \lim_{\max_{1 \leq j \leq n} w_j \rightarrow 0} \frac{k_i (\sum_{i=1}^n w_i)^{\theta_i}}{\sum_{i=1}^n w_i} = \infty,$$

where $i = 1, 2, \dots, n$.

Furthermore, there exists a $r_2 > 0$ such that

$$f_i(\mathbf{w}) \geq \eta_i^1 \sum_{i=1}^n w_i, \quad 0 < w_i < r_2,$$

where η_i^1 satisfies $\sum_{i=1}^n \max_{t \in J} \lambda_i \eta_i^1 \alpha \int_{\frac{\sigma}{2}}^\sigma G(t, s) h_i^+(s) ds > 1, i = 1, 2, \dots, n$. Thus $\min\{f_i(\mathbf{w}) : \alpha r_2 \leq w_i \leq r_2, i = 1, 2, \dots, n\} \geq \eta_i^1 \sum_{i=1}^n w_i, i = 1, 2, \dots, n$.

Then, for any $w_i \in \partial K_{i,r_2}, i = 1, 2, \dots, n$, we have

$$\begin{aligned} \|T_i^{\lambda_i} \mathbf{w}\|_\infty &= \max_{t \in J} \left| \lambda_i \int_0^1 G(t, s) h_i(s) f_i(\mathbf{w}(s)) ds \right| \\ &\geq \max_{t \in J} \left| \lambda_i \int_0^\sigma G(t, s) h_i(s) f_i(\mathbf{w}(s)) ds \right| \\ &\geq \max_{t \in J} \left| \lambda_i \int_{\frac{\sigma}{2}}^\sigma G(t, s) h_i^+(s) \eta_i^1 \sum_{i=1}^n w_i(s) ds \right| \\ &\geq \max_{t \in J} \lambda_i \int_{\frac{\sigma}{2}}^\sigma G(t, s) h_i^+(s) \eta_i^1 \alpha \sum_{i=1}^n \|w_i\|_\infty ds \\ &\geq \max_{t \in J} \lambda_i \eta_i^1 \alpha \int_{\frac{\sigma}{2}}^\sigma G(t, s) h_i^+(s) ds \|\mathbf{w}\|, \end{aligned}$$

where $i = 1, 2, \dots, n$.

Thus, for $w_i \in \partial K_{i,r_2}, i = 1, 2, \dots, n$, we have

$$\begin{aligned} \|\mathbf{T}^A \mathbf{w}\| &= \sum_{i=1}^n \|T_i^{\lambda_i} \mathbf{w}\|_\infty \\ &\geq \sum_{i=1}^n \max_{t \in J} \lambda_i \eta_i^1 \alpha \int_{\frac{\sigma}{2}}^\sigma G(t, s) h_i^+(s) ds \|\mathbf{w}\| > \|\mathbf{w}\|. \end{aligned} \tag{3.4}$$

On the other hand, since $0 < \theta_i < 1$, by (H_3) , we have

$$\lim_{\min_{1 \leq j \leq n} w_j \rightarrow \infty} \frac{f_i(\mathbf{w})}{\sum_{i=1}^n w_i} \leq \lim_{\min_{1 \leq j \leq n} w_j \rightarrow \infty} \frac{l_i (\sum_{i=1}^n w_i)^{\theta_i}}{\sum_{i=1}^n w_i} = 0,$$

where $i = 1, 2, \dots, n$.

Furthermore, there exists $0 < r_2 < R'_2$ such that

$$f_i(\mathbf{w}) \leq \varepsilon_i^1 \left(\sum_{i=1}^n w_i \right), \quad w_i \geq R'_2,$$

where ε_i^1 satisfies $\sum_{i=1}^n \max_{t \in J} \lambda_i \int_0^\xi G(t, s) h_i^+(s) \varepsilon_i^1 ds \leq \frac{1}{2}, i = 1, 2, \dots, n$.

Let $M_i = \max\{f_i(\mathbf{w}) : 0 \leq w_i \leq R'_2, i = 1, 2, \dots, n\}$. It implies that

$$f_i(\mathbf{w}) \leq \varepsilon_i^1 \left(\sum_{i=1}^n w_i \right) + M_i, \quad x, y \geq 0, i = 1, 2, \dots, n.$$

Choose $R_2 \geq \{R'_2, \frac{2}{n} \sum_{i=1}^n \max_{t \in J} \lambda_i \int_0^\xi G(t, s) h_i^+(s) M_i ds\}$. If $w_i \in \partial K_{i, R_2}$, then $\|w_i\|_\infty = R_2$ and $0 \leq w_i(t) \leq R_2, t \in J, i = 1, 2, \dots, n$. It is easy to see that $f_i(\mathbf{w}(t)) \leq \varepsilon_i^1 (\sum_{i=1}^n w_i(t)) + M_i, t \in J, i = 1, 2, \dots, n$. Then, for any $w_i \in \partial K_{i, R_2}$, we have

$$\begin{aligned} \|T_i^{\lambda_i} \mathbf{w}\|_\infty &= \max_{t \in J} \left| \lambda_i \int_0^1 G(t, s) h_i(s) f_i(\mathbf{w}(s)) ds \right| \\ &= \max_{t \in J} \left| \lambda_i \int_0^\xi G(t, s) h_i^+(s) f_i(\mathbf{w}(s)) ds - \lambda_i \int_\xi^1 G(t, s) h_i^-(s) f_i(\mathbf{w}(s)) ds \right| \\ &\leq \max_{t \in J} \left| \lambda_i \int_0^\xi G(t, s) h_i^+(s) f_i(\mathbf{w}(s)) ds \right| \\ &\leq \max_{t \in J} \left| \lambda_i \int_0^\xi G(t, s) h_i^+(s) \left[\varepsilon_i^1 \sum_{i=1}^n w_i(s) + M_i \right] ds \right| \\ &\leq \max_{t \in J} \lambda_i \int_0^\xi G(t, s) h_i^+(s) \varepsilon_i^1 \sum_{i=1}^n \|w_i\|_\infty ds + \max_{t \in J} \lambda_i \int_0^\xi G(t, s) h_i^+(s) M_i ds, \end{aligned}$$

where $i = 1, 2, \dots, n$.

Thus, for $w_i \in \partial K_{i, R_2}, i = 1, 2, \dots, n$, we have

$$\begin{aligned} \|\mathbf{T}^A \mathbf{w}\| &= \sum_{i=1}^n \|T_i^{\lambda_i} \mathbf{w}\|_\infty \\ &\leq \sum_{i=1}^n \left(\max_{t \in J} \lambda_i \int_0^\xi G(t, s) h_i^+(s) \varepsilon_i^1 \sum_{i=1}^n \|w_i\|_\infty ds \right. \\ &\quad \left. + \max_{t \in J} \lambda_i \int_0^\xi G(t, s) h_i^+(s) M_i ds \right) \\ &\leq \sum_{i=1}^n \max_{t \in J} \lambda_i \int_0^\xi G(t, s) h_i^+(s) \varepsilon_i^1 ds \|\mathbf{w}\| + \sum_{i=1}^n \max_{t \in J} \lambda_i \int_0^\xi G(t, s) h_i^+(s) M_i ds \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{2} \|\mathbf{w}\| + \frac{n}{2} R_2 \\
 &= \frac{1}{2} \|\mathbf{w}\| + \frac{1}{2} \|\mathbf{w}\| \\
 &= \|\mathbf{w}\|.
 \end{aligned} \tag{3.5}$$

In addition, choose a number $0 < r' < r_1$. Noticing that $f_i(\mathbf{w}) > 0$ for all $w_i > 0$, we can define

$$f_i^{r'} = \max\{f_i(\mathbf{w}) : 0 < w_j \leq r', j = 1, 2, \dots, n\},$$

where $i = 1, 2, \dots, n$.

$$\text{Let } \lambda_0 = \frac{r'}{\max_{t \in J} \int_0^\xi G(t,s) h_i^+(s) f_i^{r'} ds}.$$

If $w_i \in \partial K_{r'}$, then $\|w_i\|_\infty = r'$ and $0 \leq w_i(t) \leq \|w_i\|_\infty = r', t \in J, i = 1, 2, \dots, n$. It is clear that $f_i(\mathbf{w}(t)) \leq f_i^{r'}, t \in J, i = 1, 2, \dots, n$. Then, for $w_i \in \partial K_{i,r'}$, we have

$$\begin{aligned}
 \|T_i^{\lambda_i} \mathbf{w}\|_\infty &= \max_{t \in J} \left| \lambda_i \int_0^1 G(t,s) h_i(s) f_i(\mathbf{w}(s)) ds \right| \\
 &= \max_{t \in J} \left| \lambda_i \int_0^\xi G(t,s) h_i^+(s) f_i(\mathbf{w}(s)) ds - \lambda_i \int_\xi^1 G(t,s) h_i^-(s) f_i(\mathbf{w}(s)) ds \right| \\
 &\leq \max_{t \in J} \left| \lambda_i \int_0^\xi G(t,s) h_i^+(s) f_i(\mathbf{w}(s)) ds \right| \\
 &\leq \max_{t \in J} \left| \lambda_i \int_0^\xi G(t,s) h_i^+(s) f_i^{r'} ds \right| \\
 &\leq \max_{t \in J} \lambda_i^0 \int_0^\xi G(t,s) h_i^+(s) f_i^{r'} ds,
 \end{aligned}$$

where $i = 1, 2, \dots, n$.

Thus, for $w_i \in \partial K_{i,r'}, i = 1, 2, \dots, n$, we have

$$\begin{aligned}
 \|\mathbf{T}^\lambda \mathbf{w}\| &= \sum_{i=1}^n \|T_i^{\lambda_i} \mathbf{w}\|_\infty \\
 &\leq \sum_{i=1}^n \max_{t \in J} \lambda_i^0 \int_0^\xi G(t,s) h_i^+(s) f_i^{r'} ds \\
 &= nr' = \sum_{i=1}^n \|w_i\|_\infty = \|\mathbf{w}\|.
 \end{aligned} \tag{3.6}$$

Applying Lemma 2.5 to (3.4), (3.5) and (3.6) shows that \mathbf{T}^A admits at least two fixed points $\mathbf{w}^1, \mathbf{w}^2$, where $\mathbf{w}^1 \in \overline{K}_{R_2} \setminus \overline{K}_{r_2}, \mathbf{w}^2 \in K_{r_2} \setminus K_{r'}$. Thus it follows from Lemma 2.1 that, if $0 < \theta_i < 1, i = 1, 2, \dots, n$, there exists $\lambda_i^0 > 0$ such that system (1.1) has at least two positive radial solutions for $\lambda_i \in (0, \lambda_i^0]$. The proof of Theorem 3.2 is completed. \square

4 Some special cases and an example

In this part, we consider two special cases: $\Lambda \equiv \mathbf{1}$ of system (1.1) and the weight function $\mathbf{a}(|\mathbf{x}|)$ is positive on (R_1, R_2) .

4.1 Case of $\Lambda \equiv 1$

We consider $\Lambda \equiv 1$, that is, $\lambda_i \equiv 1$ ($i = 1, 2, \dots, n$). If $\Lambda \equiv 1$, system (1.1) translates into the system (4.1):

$$\begin{cases} \Delta \mathbf{u}(\mathbf{x}) + \mathbf{a}(|\mathbf{x}|)\mathbf{f}(\mathbf{u}(\mathbf{x})) = \mathbf{0}, & R_1 < |\mathbf{x}| < R_2, \\ \mathbf{u}|_{|\mathbf{x}|=R_1} = \mathbf{u}|_{|\mathbf{x}|=R_2} = \mathbf{0}. \end{cases} \tag{4.1}$$

Similar to system (2.5), we transform system (4.1) into the system (4.2):

$$\begin{cases} w_i''(t) + h_i(t)f_i(\mathbf{w}) = 0, & 0 < t < 1, \\ w_i(1) = w_i(0) = 0. \end{cases} \tag{4.2}$$

We define the map $\mathbf{T} : R_+^n \rightarrow R_+^n$ with components $(T_1, \dots, T_i, \dots, T_n)$. Here, we understand $\mathbf{T}\mathbf{w} = (T_1\mathbf{w}, \dots, T_i\mathbf{w}, \dots, T_n\mathbf{w})$, where

$$(T_i\mathbf{w})(t) = \int_0^1 G(t,s)h_i(s)f_i(\mathbf{w}(s)) ds, \quad i = 1, 2, \dots, n. \tag{4.3}$$

As Cheng and Zhang [46] pointed out, the existence of a positive solution of system (4.1) is equivalent to the existence of a nontrivial fixed point of \mathbf{T} in K .

Lemma 4.1 *Assume (H_1) – (H_4) hold. $\mathbf{T} : K \rightarrow K$ is completely continuous.*

Theorem 4.1 *Assume (H_1) – (H_4) hold. System (4.1) has at least one positive radial solution.*

Proof We denote

$$M = \left[\sum_{i=1}^n \int_0^\sigma \sigma G(s,s)h_i^+(s) ds \right]^{-1}, \quad N = \left[\sum_{i=1}^n \int_0^\xi G(s,s)h_i^+(s) ds \right]^{-1}.$$

Let $\theta_i > 1$ ($i = 1, 2, \dots, n$). On the one hand, since $\theta_i > 1$ ($i = 1, 2, \dots, n$), by (H_3) , we have

$$\lim_{\min_{1 \leq j \leq n} w_j \rightarrow 0} \frac{f_i(\mathbf{w})}{\sum_{i=1}^n w_i} \leq \lim_{\min_{1 \leq j \leq n} w_j \rightarrow 0} \frac{l_i(\sum_{i=1}^n w_i)^{\theta_i}}{\sum_{i=1}^n w_i} = 0.$$

Furthermore, there exists a $r_1 > 0$ such that

$$f_i(\mathbf{w}) \leq N \sum_{i=1}^n w_i, \quad 0 < w_i < r_1, i = 1, 2, \dots, n.$$

If $w_i \in \partial K_{i,r}$, then $\|w_i\|_\infty = r_1$ and $0 \leq w_i(t) \leq \|w_i\|_\infty = r_1, t \in J, i = 1, 2, \dots, n$. This implies that $f_i(\mathbf{w}(t)) \leq N \sum_{i=1}^n w_i(t) \leq Nnr_1, t \in J, i = 1, 2, \dots, n$. Then, for any $w_i \in \partial K_{i,r_1}$ ($i = 1, 2, \dots, n$), we have

$$\|\mathbf{T}\mathbf{w}\| = \sum_{i=1}^n \|T_i\mathbf{w}\|_\infty = \sum_{i=1}^n \max_{t \in J} \left| \int_0^1 G(t,s)h_i(s)f_i(\mathbf{w}(s)) ds \right|$$

$$\begin{aligned}
 &= \sum_{i=1}^n \max_{t \in J} \left| \int_0^\xi G(t,s)h_i^+(s)f_i(\mathbf{w}(s)) ds - \int_\xi^1 G(t,s)h_i^-(s)f_i(\mathbf{w}(s)) ds \right| \\
 &\leq \sum_{i=1}^n \max_{t \in J} \left| \int_0^\xi G(t,s)h_i^+(s)f_i(\mathbf{w}(s)) ds \right| \\
 &\leq \sum_{i=1}^n \left| \int_0^\xi G(s,s)h_i^+(s)f_i(\mathbf{w}(s)) ds \right| \\
 &\leq \sum_{i=1}^n \int_0^\xi G(s,s)h_i^+(s)Nnr_1 ds \\
 &= nr_1 = \|\mathbf{w}\|.
 \end{aligned}$$

Consequently,

$$\|\mathbf{Tw}\| \leq \|\mathbf{w}\|, \quad \forall \mathbf{w} \in \partial K_{r_1}. \tag{4.4}$$

On the other hand, if $w_i \in K_i, i = 1, 2, \dots, n$, then from the concavity on $[0, \xi]$,

$$w_i(t) \geq \delta(t)\|w_i\|_\infty, \quad t \in [0, \xi], i = 1, 2, \dots, n.$$

It follows that $w_i(t) \geq \alpha\|w_i\|_\infty, t \in [\frac{\sigma}{2}, \sigma]$, where $\alpha = \min_{\frac{\sigma}{2} \leq t \leq \sigma} \delta(t), i = 1, 2, \dots, n$. Thus we have

$$w_i(t) \geq \alpha\|w_i\|, \quad t \in \left[\frac{\sigma}{2}, \sigma\right], i = 1, 2, \dots, n.$$

Since $\theta_i > 1, i = 1, 2, \dots, n$, by (H_3) , we have

$$\lim_{\min_{1 \leq j \leq n} w_j \rightarrow \infty} \frac{f_i(\mathbf{w})}{\sum_{i=1}^n w_i} \geq \lim_{\min_{1 \leq j \leq n} w_j \rightarrow \infty} \frac{k_i(\sum_{i=1}^n w_i)^{\theta_i}}{\sum_{i=1}^n w_i} = \infty.$$

Furthermore, there exists $0 < r_1 < R'_1 < +\infty$ such that

$$f_i(\mathbf{w}) \geq \frac{M \sum_{i=1}^n w_i}{\alpha}, \quad w_i \geq R'_1, i = 1, 2, \dots, n.$$

Choose $R_1 \geq \frac{R'_1}{\alpha}$. Then, for any $w_i \in \partial K_{i,R_1}, i = 1, 2, \dots, n$, we have $\min_{\frac{\sigma}{2} \leq t \leq \sigma} w_i(t) \geq \min_{\frac{\sigma}{2} \leq t \leq \sigma} \delta(t)\|w_i\|_\infty = \alpha R_1 \geq R'_1$ and $f_i(\mathbf{w}(t)) \geq MnR_1, t \in [\frac{\sigma}{2}, \sigma], i = 1, 2, \dots, n$. Then, for $w_i \in \partial K_{i,R_1}$, we have

$$\begin{aligned}
 \|\mathbf{Tw}\| &= \sum_{i=1}^n \|T_i \mathbf{w}\|_\infty = \sum_{i=1}^n \max_{t \in J} \left| \int_0^1 G(t,s)h_i(s)f_i(\mathbf{w}(s)) ds \right| \\
 &\geq \sum_{i=1}^n \max_{t \in [\sigma, 1-\sigma]} \left| \int_0^\sigma G(t,s)h_i^+(s)f_i(\mathbf{w}(s)) ds \right| \\
 &\geq \sum_{i=1}^n \int_0^\sigma \sigma G(s,s)h_i^+(s)f_i(\mathbf{w}(s)) ds
 \end{aligned}$$

$$\geq \sum_{i=1}^n \int_{\frac{\sigma}{2}}^{\sigma} \sigma G(s, s) h_i^+(s) M n R_1 ds = n R_1 = \|\mathbf{w}\|.$$

Consequently,

$$\|\mathbf{T}\mathbf{w}\| \geq \|\mathbf{w}\|, \quad \forall \mathbf{w} \in \partial K_{R_1}. \tag{4.5}$$

Summing up we can show that \mathbf{T} has at least one fixed point \mathbf{w}_1 , where $\mathbf{w}_1 \in \overline{K}_{R_1} \setminus \overline{K}_{r_1}$ by applying Lemma 2.5 to (4.4) and (4.5). According to Lemma 2.1, if $\theta_i > 1, i = 1, 2, \dots, n$, system (4.1) has at least one positive solution.

If $0 < \theta_i < 1 (i = 1, 2, \dots, n)$, the proof is similar. We omit it.

The proof of Theorem 4.1 is completed. □

If $\min_{1 \leq i \leq n} \frac{l_i}{k_i}$ is sufficiency large, we have the following theorem.

Theorem 4.2 *Assume that (H_1) – (H_4) hold and there exist two positive numbers A_1, B_1 such that one of the following conditions is satisfied:*

- (i) $0 < \theta_i < 1, A_1 < B_1$ and

$$\begin{aligned} \max\{f_i(\mathbf{w}) : 0 \leq w_j \leq A_1, j = 1, 2, \dots, n\} &< A_1 M, \\ \min\{f_i(\mathbf{w}) : \alpha B_1 \leq w_j \leq B_1, j = 1, 2, \dots, n\} &> B_1 N. \end{aligned}$$

- (ii) $1 < \theta_i < +\infty, A_1 > B_1$ and

$$\begin{aligned} \min\{f_i(\mathbf{w}) : \alpha B_1 \leq w_j \leq B_1, j = 1, 2, \dots, n\} &> B_1 N, \\ \max\{f_i(\mathbf{w}) : 0 \leq w_j \leq A_1, j = 1, 2, \dots, n\} &< A_1 M. \end{aligned}$$

Then system (4.1) has at least three positive radial solutions.

Proof It is enough to prove the case (i).

We have the following claim.

Claim 4.1 *If there exist two different positive numbers A, B such that*

$$\begin{aligned} \max\{f_i(\mathbf{w}) : 0 \leq w_j \leq A, j = 1, 2, \dots, n\} &\leq AM, \\ \min\{f_i(\mathbf{w}) : \alpha B \leq w_j \leq B, j = 1, 2, \dots, n\} &\geq BN, \end{aligned}$$

then the operator \mathbf{T} has one fixed point $\mathbf{w}^ \in K$ and $\min\{nA, nB\} \leq \|\mathbf{w}^*\| \leq \max\{nA, nB\}$.*

The proof of Claim 4.1 is similar to the proof of Theorem 4.1.

Now, $\lim_{\max_{1 \leq j \leq n} w_j \rightarrow 0} \frac{f_i(\mathbf{w})}{\sum_{j=1}^n w_j} = +\infty$ and $\lim_{\min_{1 \leq j \leq n} w_j \rightarrow +\infty} \frac{f_i(\mathbf{w})}{\sum_{j=1}^n w_j} = 0, i = 1, 2, \dots, n$. By the proof of Theorem 4.1, we assert that there exist positive numbers A_2, B_2 such that $B_2 < A_1 < B_1 < A_2$ and

$$\begin{aligned} \min\{f_i(\mathbf{w}) : \alpha B_2 \leq w_j \leq B_2, j = 1, 2, \dots, n\} &> B_2 N, \\ \max\{f_i(\mathbf{w}) : 0 \leq w_j \leq A_2, j = 1, 2, \dots, n\} &< A_2 M, \end{aligned}$$

where $i = 1, 2, \dots, n$.

On the other hand, letting $\psi_i(\mathbf{w}) = \min\{f_i(\mathbf{y}) : \alpha \mathbf{w} \leq \mathbf{y} \leq \mathbf{w}\}$ ($i = 1, 2, \dots, n$), then $\varphi_i, \psi_i : \mathcal{R}_+^n \rightarrow \mathcal{R}_+$ ($i = 1, 2, \dots, n$) are continuous. Since $M < N, \psi_i(\mathbf{B}_2) > B_2N, \varphi_i(\mathbf{A}_2) < A_2M$, we assert that there exist

$$B_2 < A'_1 < A_1 < A''_1 < B'_1 < B_1 < B''_1 < A_2$$

such that

$$\begin{aligned} \max\{f_i(\mathbf{w}) : 0 \leq w_j \leq A'_1, j = 1, 2, \dots, n\} &= \varphi_i(\mathbf{A}'_1) < A'_1M, \\ \max\{f_i(\mathbf{w}) : 0 \leq w_j \leq A''_1, j = 1, 2, \dots, n\} &= \varphi_i(\mathbf{A}''_1) < A''_1M, \\ \min\{f_i(\mathbf{w}) : \alpha B'_1 \leq w_j \leq B'_1, j = 1, 2, \dots, n\} &= \psi_i(\mathbf{B}'_1) > B'_1N, \\ \min\{f_i(\mathbf{w}) : \alpha B''_1 \leq w_j \leq B''_1, j = 1, 2, \dots, n\} &= \psi_i(\mathbf{B}''_1) > B''_1N, \end{aligned}$$

where

$$\begin{aligned} \mathbf{A}_2 &= (\underbrace{A_2, \dots, A_2}_{n})^T, & \mathbf{B}_2 &= (\underbrace{B_2, \dots, B_2}_{n})^T, \\ \mathbf{A}'_1 &= (\underbrace{A'_1, \dots, A'_1}_{n})^T, & \mathbf{A}''_1 &= (\underbrace{A''_1, \dots, A''_1}_{n})^T, \\ \mathbf{B}'_1 &= (\underbrace{B'_1, \dots, B'_1}_{n})^T, & \mathbf{B}''_1 &= (\underbrace{B''_1, \dots, B''_1}_{n})^T. \end{aligned}$$

By using Claim 4.1 for $(B_2, A'_1), (A''_1, B'_1), (B''_1, A_2)$, respectively, we see that the operator \mathbf{T} has three fixed points $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in K$ satisfying

$$nB_2 \leq \|\mathbf{w}_1\| \leq nA'_1 < nA''_1 \leq \|\mathbf{w}_2\| \leq nB'_1 < nB''_1 \leq \|\mathbf{w}_3\| \leq nA_2.$$

By Lemma 2.1 and Lemma 2.5 we know that system (4.1) has at least three positive radial solutions. The proof of Theorem 4.1 is completed. \square

4.2 Case of definite weight function

We consider the multiplicity of elliptic system (1.1) with definite function. By a series of transformations, (1.1) is transformed to (2.4). Assume the following conditions throughout:

- (A₁) $\mathbf{\lambda} = (\lambda_1, \dots, \lambda_i, \dots, \lambda_n) > \mathbf{0}$ is a parameter vector;
- (A₂) $h_i \in L^1[0, 1]$ and there exists $\eta_i > 0$ such that $h_i(t) \geq \eta_i$ a.e. on J .
- (A₃) $f_i : \mathcal{R}_+^n \rightarrow \mathcal{R}_+$ is continuous with $f_i(\mathbf{w}) > 0$ for $\mathbf{w} > \mathbf{0}$;

where $i = 1, 2, \dots, n$.

Lemma 4.2 *Let $G(t, s)$ be given as (2.6) and $0 < \rho < \frac{1}{2}$. Then we have*

$$G(t, s) \geq \rho G(s, s), \quad t \in J_\rho, s \in J,$$

where $J_\rho = [\rho, 1 - \rho]$.

Proof For $t \in J_\rho$ and $s \in J$, we have

$$\frac{G(t,s)}{G(s,s)} = \frac{t(1-s)}{s(1-s)} = \frac{t}{s} \geq \rho, \quad t \leq s,$$

$$\frac{G(t,s)}{G(s,s)} = \frac{s(1-t)}{s(1-s)} = \frac{1-t}{1-s} \geq \rho, \quad s \leq t.$$

Then the proof is complete. □

We define a cone K in X by

$$K = \prod_{i=1}^n K_i,$$

where

$$K_i = \left\{ x_i \in C^+[0, 1] : x_i \geq 0 \text{ for } J, \min_{t \in J_\rho} x_i(t) \geq \rho \|x_i\|_\infty \right\}, \quad i = 1, 2, \dots, n. \tag{4.6}$$

Definition The map β is said to be a nonnegative continuous concave function on a cone K of a real Banach space E if $\beta : K \rightarrow K$ is continuous and

$$\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in K$ and $t \in J$.

Let K be a cone in a Banach space X . For positive numbers $0 < c < d$, we define the convex set $K(\beta, c, d)$ by

$$K(\beta, c, d) = \{x : x \in K, c \leq \beta(x), \|x\| \leq d\}.$$

In this part, $\beta(\mathbf{x}) = \sum_{i=1}^n \beta_i(x_i)$, and we understand $K(\beta, c, d) = (K_1(\beta_1, c, d), K_2(\beta_2, c, d), \dots, K_n(\beta_n, c, d))$, where $K_i(\beta_i, c, d) = \{x_i : x_i \in K_i, c \leq \beta_i(x_i), \|x_i\|_\infty \leq d\}, i = 1, 2, \dots, n$.

We define the map $\mathbf{T}^A : K \rightarrow X$ with components $(T_1^{\lambda_1}, T_2^{\lambda_2}, \dots, T_n^{\lambda_n})^T$. Hence, we understand $\mathbf{T}^A \mathbf{w} = (T_1^{\lambda_1} \mathbf{w}, T_2^{\lambda_2} \mathbf{w}, \dots, T_n^{\lambda_n} \mathbf{w})^T$, which

$$(T_i^{\lambda_i} \mathbf{w})(t) = \lambda_i \int_0^1 G(t,s)h_i(s)f_i(\mathbf{w}(s)) ds, \quad t \in J, i = 1, 2, \dots, n. \tag{4.7}$$

As Cheng and Zhang [46] pointed out, \mathbf{w} is a positive radial solution of system (1.1) if and only if $\mathbf{w} \in K$ is a positive point of \mathbf{T}^A .

Lemma 4.3 *Suppose that (A₁)–(A₃) hold. Then $\mathbf{T}^A : K \rightarrow K$ is completely continuous.*

Proof We just prove $T_i^{\lambda_i} : K \rightarrow K_i$ is completely continuous. For all $\mathbf{w} \in K, T_i^{\lambda_i} \mathbf{w} \geq 0$ on J and

$$\|T_i^{\lambda_i} \mathbf{w}\|_\infty = \max_{t \in J} \left| \lambda_i \int_0^1 G(t,s)h_i(s)f_i(\mathbf{x}(s)) ds \right|$$

$$\leq \max_{t \in J} \left| \lambda_i \int_0^1 G(s, s) h_i(s) f_i(\mathbf{x}(s)) \, ds \right|.$$

From Lemma 4.2, we have

$$\begin{aligned} \min_{t \in J_\rho} T_i^{\lambda_i} \mathbf{w}(t) &= \min_{t \in J_\rho} \lambda_i \int_0^1 G(t, s) h_i(s) f_i(\mathbf{w}(s)) \, ds \\ &\geq \min_{t \in J_\rho} \lambda_i \int_0^1 \rho G(s, s) h_i(s) f_i(\mathbf{w}(s)) \, ds \\ &\geq \min_{t \in J} \lambda_i \int_0^1 \rho G(s, s) h_i(s) f_i(\mathbf{w}(s)) \, ds \\ &\geq \|T_i^{\lambda_i} \mathbf{w}\|_\infty. \end{aligned}$$

Thus, we have $T_i^{\lambda_i}(K) \subset K_i$, therefore $\mathbf{T}^A(K) \subset K$.

Finally, from the standard process, it follows that $T_i^{\lambda_i} : K \rightarrow K_i$ is completely continuous, that is, $\mathbf{T}^A : K \rightarrow K$ is completely continuous. \square

Lemma 4.4 (Leggett–Williams fixed point theorem) *Let K be a cone in a real Banach space E , $A : \bar{K}_a \rightarrow \bar{K}_a$ be completely continuous and β be a nonnegative continuous concave functional on K with $\beta(x) \leq \|x\|$ for all $x \in K_a$. Suppose there exist $0 < d < a < b \leq c$ such that*

- (i) $\{x \in K(\beta, a, b) : \beta(x) > a\} \neq \emptyset$ and $\beta(Ax) > a$ for $x \in K(\beta, a, b)$;
- (ii) $\|Ax\| < d$ for $\|x\| \leq d$;
- (iii) $\beta(Ax) > a$ for $x \in K(\beta, a, c)$ with $\|Ax\| > b$.

Then A has at least three positive solutions x_1, x_2, x_3 satisfying

$$\|x_1\| < d, \quad a < \beta(x_2), \quad \|x_3\| > d, \quad \beta(x_3) < a.$$

Next, we begin by introducing the notation

$$\begin{aligned} f_i^0 &= \limsup_{\|\mathbf{w}\| \rightarrow 0^+} \frac{f_i(\mathbf{w})}{\|\mathbf{x}\|}, & f_i^\infty &= \limsup_{\|\mathbf{w}\| \rightarrow +\infty} \frac{f_i(\mathbf{w})}{\|\mathbf{x}\|}, \\ f_{i,0} &= \liminf_{\|\mathbf{w}\| \rightarrow 0^+} \frac{f_i(\mathbf{w})}{\|\mathbf{x}\|}, & f_{i,\infty} &= \liminf_{\|\mathbf{w}\| \rightarrow +\infty} \frac{f_i(\mathbf{w})}{\|\mathbf{x}\|}, \\ f_i^*(l) &= \min\{f_i(\mathbf{w}), 0 \leq \|\mathbf{w}\| \leq l\}, & f_i^{**}(l) &= \max\{f_i(\mathbf{w}), 0 \leq \|\mathbf{w}\| \leq l\}, \\ N_i &= \frac{1}{6} \lambda_i \rho \eta_i, & D_i &= \lambda_i \int_0^1 G(s, s) h_i(s) \, ds, \end{aligned}$$

where $i = 1, 2, \dots, n$.

Theorem 4.3 *Assume (A_1) – (A_3) hold. For $\lambda_i > 0$ there exist constants $0 < M < C < \frac{C}{\rho} < L$ such that*

- (A_4) $f_i^\infty < \frac{1}{N_i}$,
- (A_5) $f_i(\mathbf{w}) < \frac{M}{D_i}, t \in J_\rho, \rho M \leq \|w_i\|_\infty \leq M$, and $f_i(\mathbf{w}) > \frac{C}{N_i}, t \in J_\rho, C \leq \|w_i\|_\infty \leq \frac{C}{\rho}$,

where $i = 1, 2, \dots, n$.

Then system (1.1) has at least three positive solutions $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ satisfying

$$\|\mathbf{w}_1\| < nM, \quad nC < \beta(\mathbf{w}_2), \quad \|\mathbf{w}_3\| > nM, \quad \beta(\mathbf{w}_3) < nC.$$

Proof Let $\beta(\mathbf{w}) = \sum_{i=1}^n \beta_i(w_i), \beta_i(w_i) = \min_{t \in J_\rho} w_i(t)$. It is clear that $\beta(\mathbf{w})$ is a nonnegative continuous concave functional on the cone K satisfying $\beta(\mathbf{w}) \leq \|\mathbf{w}\|$ for all $\mathbf{w} \in K$.

By (A_4) there exist $0 < \varepsilon < \frac{1}{N_i}$ such that

$$f_i(\mathbf{w}) \leq \varepsilon \|w_i\|_\infty, \quad \|w_i\|_\infty > r.$$

By the definition of f_i^{**} , we have

$$0 \leq f_i(\mathbf{w}) \leq \varepsilon \|w_i\|_\infty + f_i^{**}, \quad \|w_i\|_\infty > 0.$$

Let $L > \max\left\{\frac{f_i^{**}}{\max_{t \in J} \lambda_i \int_0^1 G(t,s)h_i^+(s)\varepsilon_i ds}\right\}$. Then, for $\mathbf{w} \in \overline{K}_{nL}$,

$$\begin{aligned} \|\mathbf{T}^A \mathbf{w}\| &= \sum_{i=1}^n \|T_i^\lambda \mathbf{w}\|_\infty = \sum_{i=1}^n \max_{t \in J} \left| \lambda_i \int_0^1 G(t,s)h_i(s)f_i(\mathbf{w}(s)) ds \right| \\ &\leq \sum_{i=1}^n \max_{t \in J} \left| \lambda_i \int_0^1 G(t,s)h_i^+(s)f_i(\mathbf{w}(s)) ds \right| \\ &\leq \sum_{i=1}^n \max_{t \in J} \lambda_i \int_0^1 G(t,s)h_i^+(s)(\varepsilon_i \mathbf{w}(s) + f_i^{**}) ds \\ &\leq \sum_{i=1}^n \max_{t \in J} \lambda_i \int_0^1 G(t,s)h_i^+(s)(\varepsilon_i \|\mathbf{w}\| + f_i^{**}) ds \\ &= \sum_{i=1}^n \max_{t \in J} \lambda_i \int_0^1 G(t,s)h_i^+(s)(\varepsilon_i L + f_i^{**}) ds \\ &< nL, \end{aligned}$$

which implies that $\mathbf{T}^A \mathbf{w} \in K_{nL}$. Hence, we have shown that the map $\mathbf{T}^A : \overline{K}_{nL} \rightarrow \overline{K}_{nL}$ is completely continuous.

Next, we verify that $\{\mathbf{x} : \mathbf{x} \in K(\beta, nC, \frac{nC}{\rho}), \beta(\mathbf{x}) > nC\} \neq \emptyset$ and $\beta(\mathbf{T}^A \mathbf{x}) > nC$ for all $\mathbf{x} \in K(\beta, nC, \frac{nC}{\rho})$.

Take $\mathbf{w}_0 = (w_1^0(t), w_2^0(t), \dots, w_n^0(t)), w_i^0 = \frac{c}{\rho}, i = 1, 2, \dots, n$, for $t \in J$. Then

$$\mathbf{w}_0 = \left\{ \mathbf{w} : \mathbf{w} \in K\left(\beta, nC, \frac{nC}{\rho}\right), \beta(\mathbf{w}) > nC \right\},$$

which shows that

$$\left\{ \mathbf{w} : \mathbf{w} \in K\left(\beta, nC, \frac{nC}{\rho}\right), \beta(\mathbf{w}) > nC \right\} \neq \emptyset.$$

For all $\mathbf{w} \in K(\beta, nC, \frac{nC}{\rho})$, that is, $w_i \in K_i(\beta_i, C, \frac{C}{\rho})$, we have $\|w_i\|_\infty \leq \frac{C}{\rho}$, and from the definition of K_i , we know that $\min_{t \in J_\rho} w_i(t) \geq \rho \|w_i\|_\infty$. Thus we have

$$\begin{aligned} \beta(\mathbf{T}^A \mathbf{w}) &= \sum_{i=1}^n \beta_i(T_i^{\lambda_i} \mathbf{w}) = \sum_{i=1}^n \min_{t \in J_\rho} T_i^{\lambda_i} \mathbf{w}(t) \\ &= \sum_{i=1}^n \min_{t \in J_\rho} \lambda_i \int_0^1 G(t, s) h_i(s) f_i(\mathbf{w}(s)) \, ds \\ &\geq \sum_{i=1}^n \min_{t \in J_\rho} \lambda_i \left(\rho \int_0^1 G(s, s) \eta_i f_i(\mathbf{w}(s)) \, ds \right) \\ &> \sum_{i=1}^n \frac{1}{6} \lambda_i \rho \eta_i \frac{C}{N_i} \\ &= nC. \end{aligned}$$

This implies that condition (i) of Lemma 4.4 holds.

For $w_i \in \bar{K}_{i,M}$, we have

$$\begin{aligned} \|\mathbf{T}^A \mathbf{w}\| &= \sum_{i=1}^n \|T_i^{\lambda_i} \mathbf{w}\|_\infty \\ &= \sum_{i=1}^n \max_{t \in J} \lambda_i \int_0^1 G(t, s) h_i(s) f_i(\mathbf{w}(s)) \, ds \\ &\leq \sum_{i=1}^n \max_{t \in J} \lambda_i \int_0^1 G(s, s) h_i(s) f_i(\mathbf{w}(s)) \, ds \\ &< \sum_{i=1}^n \lambda_i \int_0^1 G(s, s) h_i(s) \frac{M}{D_i} \, ds \\ &= nM. \end{aligned}$$

This implies that condition (ii) of Lemma 4.4 holds.

Finally, we assert that if $w_i \in K_i(\beta_i, C, L)$ and $\|T_i^{\lambda_i} \mathbf{w}\|_\infty > \frac{C}{\rho}$, then $\beta(\mathbf{T}^A \mathbf{w}) > nC$.

Suppose that $w_i \in K_i(\beta_i, C, L)$ and $\|T_i^{\lambda_i} \mathbf{w}\|_\infty > \frac{C}{\rho}$. Then

$$\begin{aligned} \beta(\mathbf{T}^A \mathbf{w}) &= \sum_{i=1}^n \beta_i(T_i^{\lambda_i} \mathbf{w}) = \sum_{i=1}^n \min_{t \in J_\rho} T_i^{\lambda_i} \mathbf{w}(t) \\ &= \sum_{i=1}^n \min_{t \in J_\rho} \lambda_i \int_0^1 G(t, s) h_i(s) f_i(\mathbf{w}(s)) \, ds \\ &\geq \sum_{i=1}^n \rho \|T_i^{\lambda_i} \mathbf{w}\|_\infty > nC. \end{aligned}$$

This implies that condition (iii) of Lemma 4.4 holds.

To sum up, the hypotheses of Lemma 4.4 hold. Therefore, an application of Lemma 4.4 implies that system (1.1) has at least three positive radial solutions w_1, w_2, w_3 satisfying

$$\|w_1\| < nM, \quad nC < \beta(w_2), \quad \|w_3\| > nM, \quad \beta(w_3) < nC.$$

The proof is finished. □

4.3 An example

Example 4.1 We consider the example ($n = 2$)

$$\begin{cases} \Delta u(x) + \Lambda a(|x|)f(u(x)) = 0, & R_1 < |x| < R_2, \\ u|_{|x|=R_1} = u|_{|x|=R_2} = 0. \end{cases} \tag{4.8}$$

By appropriate transformations, (4.8) can be written

$$\begin{cases} w_1''(t) + \lambda_1 h_1(t) f_1(w) = 0, & 0 < t < 1, \\ w_2''(t) + \lambda_2 h_2(t) f_2(w) = 0, & 0 < t < 1, \\ w_1(1) = w_1(0) = w_2(1) = w_2(0) = 0. \end{cases}$$

Let

$$h_1(t) = \begin{cases} \frac{1}{2} - t, & t \in [0, \frac{1}{2}), \\ t - \frac{1}{2}, & t \in [\frac{1}{2}, 1], \end{cases} \quad h_2(t) = \begin{cases} \frac{1}{4} - t^2, & t \in [0, \frac{1}{2}), \\ t^2 - \frac{1}{4}, & t \in [\frac{1}{2}, 1], \end{cases}$$

$$f_1(w) = w_1^2, \quad f_2(w) = w_2^3, \quad w = (w_1, w_2)^T.$$

It is clear that $\xi = \frac{1}{2}, \sigma = \frac{1}{3}, c_1 = c_2 = 1, k_1 = k_2 = 1, l_1 = l_2 = 2, \delta(t) = \min\{\frac{t}{\xi}, \frac{\xi-t}{\xi}\} = \min\{2t, 1 - 2t\}, t \in [0, \xi], \alpha = \min_{\frac{\sigma}{2} \leq t \leq \sigma} \delta(t) = \frac{1}{3}$. Let $r = \frac{1}{5}$. Then

$$f_{1,r} = \min\{f_1(w) : \alpha r \leq w_j \leq r, j = 1, 2, \dots, n\} = \frac{1}{225},$$

$$f_{2,r} = \min\{f_2(w) : \alpha r \leq w_j \leq r, j = 1, 2, \dots, n\} = \frac{1}{3375}.$$

From the above, let $\lambda_{1,0} = \frac{311,040}{13}, \lambda_{2,0} = \frac{209,952,000}{437}$. By Theorem 3.1, system (4.8) has at least two positive radial solutions for $\lambda_i \in [\lambda_{i,0}, +\infty), i = 1, 2$.

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