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Some new weakly singular integral inequalities with discontinuous functions for two variables and their applications

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Abstract

This paper investigates some new retarded weakly singular integral inequalities with discontinuous functions for two independent variables. The inequalities given here can be used in the qualitative analysis of various problems for integral equations and differential equations. Some examples are also given to illustrate the application of the conclusion.

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1 Introduction

Integral inequalities are used as handy tools in the study of the qualitative properties of solutions to differential and integral equations, such as existence, uniqueness, boundedness, stability, and other properties. The literature on such inequalities and their applications is vast (see [1–12] and the references therein). With the development of theory for fractional differential equations, integral inequalities with weakly singular kernels have attracted great interest [13–18]. In 1981, Henry [8] proved global existence and exponential decay results for a parabolic Cauchy problem by using the following singular integral inequality:

$$u(t) \le a + b\int_0^t (t-s)^{\beta-1}u(s)\,\mathrm{d}s.$$

Sano and Kunimatsu [9] gave a sufficient condition for stabilization of semilinear parabolic distributed systems by using a modification of Henry-type inequalities

$$0 \le u(t) \le c_1 + c_2 t^{\alpha-1} + c_3\int_0^t u(s)\,\mathrm{d}s + c_4\int_0^t (t-s)^{\beta-1}u(s)\,\mathrm{d}s. \tag{1}$$

Ye et al. [11] provided a generalization of inequality (1)

$$u(t) \le a(t) + b(t)\int_0^t (t-s)^{\beta-1}u(s)\,\mathrm{d}s,$$

and used it to study the dependence of the solution and the initial condition to a certain fractional differential equation. All such inequalities are studied by an iteration argument, and the estimation formulas are expressed by a complicated power series which are sometimes not very convenient for applications. To avoid this, Medveď [12] presented a new method for studying Henry-type inequalities and established explicit bounds with relatively simple formulas which are similar to the classic Gronwall–Bellman inequalities. Recently, by using a modification of Medveď's method, Ma and Pečarić [14] studied a certain class of nonlinear inequalities of Henry-type

$$u^p(t) \le a(t) + b(t)\int_0^t \left(t^\alpha - s^\alpha\right)^{\beta-1} s^{\gamma-1} f(s) u^q(s)\,\mathrm{d}s, \quad t \in \mathbb{R}^+.$$

The results were further generalized by Cheung et al. [15] to the following form:

$$u^p(x,y) \le a(x,y) + b(x,y)\int_0^x \int_0^y \left(x^\alpha - s^\alpha\right)^{\beta-1} s^{\gamma-1} \left(y^\alpha - t^\alpha\right)^{\beta-1} t^{\gamma-1} f(s,t) u^q(s,t)\,\mathrm{d}t\,\mathrm{d}s,$$
$$(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

In 2017, Xu [18] studied the following new generalization of weakly singular integral inequalities in two variables:

$$u^p(x,y) \le a(x,y) + b(x,y)\int_0^x \int_0^y \left(x^\alpha - s^\alpha\right)^{\beta-1} s^{\gamma-1} \left(y^\alpha - t^\alpha\right)^{\beta-1} t^{\gamma-1}$$
$$\times \left[f(s,t) u^q(s,t) + h(s,t) u^r\left(\sigma(s), \sigma(t)\right)\right]\mathrm{d}t\,\mathrm{d}s,$$

with the initial condition $u(x,y) = \phi(x,y), (x,y) \in D' = [\mu,0] \times [\mu,0], \phi(\sigma(x),\sigma(y)) \le (a(x,y))^{1/p}$ for $(x,y) \in D$ with $\sigma(x) \le 0, \sigma(y) \le 0$.

In recent ten years, a series of achievements have been made in the research of integral inequalities for discontinuous functions (see [19–22]). In 2007, Iovane [19] studied the following discontinuous function integral inequality:

$$u(t) \le a(t) + \int_{t_0}^t f(s) u\left(\tau(s)\right)\mathrm{d}s + \sum_{t_0<t_i<t} \beta_i u^m(t_i - 0), \quad t \ge t_0.$$

In 2009, Gllo et al. [20] studied the impulsive integral inequality

$$u(t) \le a(t) + g(t)\int_{t_0}^t q(s) u^n\left(\tau(s)\right)\mathrm{d}s + p(t)\sum_{t_0<t_i<t} \beta_i u^m(t_i - 0), \quad t \ge t_0.$$

In 2015, Mi et al. [21] studied the integral inequality of complex functions with unknown function

$$u(t) \le a(t) + \int_{t_0}^t f(t,s)\int_{t_0}^s g(s,\tau) w\left(u(\tau)\right)\mathrm{d}\tau\,\mathrm{d}s + q(t)\sum_{t_0<t_i<t} \beta_i u^m(t_i - 0), \quad t \ge t_0.$$

Very recently, Li et al. [22] studied the following weakly singular retarded integral inequality for discontinuous function:

$$u^p(t) \le a(t) + b(t)\int_{t_0}^{\alpha(t)} \left(\alpha^\beta(t) - s^\beta\right)^{\gamma-1} s^{\xi-1} f(s)\left[u^m(s) + \int_{t_0}^{s} g(\tau)u^n(\tau)\,\mathrm{d}\tau\right]^q \mathrm{d}s + \sum_{t_0<t_i<t} \beta_i u^p(t_i - 0).$$

In this paper, we establish some new weakly singular retarded integral inequalities with discontinuous functions in two variables

$$\begin{aligned} u(x,y) \le{}& a(x,y) \\ &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} \left(\alpha^\zeta(x) - t^\zeta\right)^{(\gamma-1)} t^{(\xi-1)} \left(\beta^\zeta(y) - s^\zeta\right)^{(\gamma-1)} s^{(\xi-1)} f_1(t,s)u(t,s)\,\mathrm{d}s\,\mathrm{d}t \\ &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} \left(\alpha^\zeta(x) - t^\zeta\right)^{(\gamma-1)} t^{(\xi-1)} \left(\beta^\zeta(y) - s^\zeta\right)^{(\gamma-1)} s^{(\xi-1)} \\ &\times f_2(t,s)\int_0^t\int_0^s f_3(\tau,\eta)u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}t, \end{aligned} \tag{2}$$

$$\begin{aligned} u(x,y) \le{}& a(x,y) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \left(x^\zeta - t^\zeta\right)^{\gamma-1}\left(y^\zeta - s^\zeta\right)^{\gamma-1} f(t,s)u(t,s) \\ &\times \left[u^2(t,s) + \int_{x_0}^t\int_{y_0}^s g(\tau,\eta)u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^p \mathrm{d}s\,\mathrm{d}t \\ &+ \sum_{(x_0,y_0)<(x_i,y_i)<(x,y)} \zeta_i u(x_i - 0, y_i - 0), \end{aligned} \tag{3}$$

$$\begin{aligned} u^p(x,y) \le{}& a(x,y) + b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \left(\alpha^\zeta(x) - t^\zeta\right)^{\gamma-1} t^{\xi-1}\left(\beta^\zeta(y) - s^\zeta\right)^{\gamma-1} s^{\xi-1} f(t,s) \\ &\times \left[u^m(t,s) + \int_{x_0}^t\int_{y_0}^s g(\tau,\eta)u^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^q \mathrm{d}s\,\mathrm{d}t \\ &+ \sum_{(x_0,y_0)<(x_i,y_i)<(x,y)} \zeta_i u^p(x_i - 0, y_i - 0). \end{aligned} \tag{4}$$

Finally, two examples are included to illustrate the usefulness of our results.

2 Preliminaries

In this paper, let $\Omega = \bigcup_{i,j\ge 1} \Omega_{ij}$, $\Omega_{ij} = \{(x,y) : x_{i-1} \le x < x_i, y_{j-1} \le y < y_j, i,j = 1,2,\ldots, x_0 > 0, y_0 > 0\}$. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^+ = [0,\infty)$, $C(M,S)$ denote the class of all continuous functions defined on the set M with range in the set S.

Lemma 1 ([23]) *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers, $m > 1$ is a real number, and n is a natural number. Then*

$$(a_1 + a_2 + \cdots + a_n)^m \le n^{m-1}\left(a_1^m + a_2^m + \cdots + a_n^m\right).$$

Lemma 2 ([13]) *Let β,γ,ξ, and p be positive constants. Then*

$$\int_0^t \left(t^\beta - s^\beta\right)^{p(\gamma-1)} s^{p(\xi-1)}\,\mathrm{d}s = \frac{t^\theta}{\beta} B\left[\frac{p(\xi-1)+1}{\beta}, p(\gamma-1)+1\right], \quad t \in \mathbb{R}^+, \tag{5}$$

where $B[x,y] = \int_0^1 s^{x-1}(1-s)^{y-1}\,\mathrm{d}s$ $(x>0, y>0)$ is the well-known beta-function and $\theta = p[\beta(\gamma-1)+\xi-1]+1$.

In addition, Li et al. [22] gave a generalization of equality (5), that is,

$$\int_{\alpha(t_0)}^{\alpha(t)} \left(\alpha^\beta(t) - s^\beta\right)^{p(\gamma-1)} s^{p(\xi-1)}\,\mathrm{d}s \le \frac{\alpha^\theta(t)}{\beta} B\left[\frac{p(\xi-1)+1}{\beta}, p(\gamma-1)+1\right], \quad t \in \mathbb{R}^+,$$

where $\alpha(t)$ is a continuous, differentiable, and increasing function on $[t_0,+\infty)$ with $\alpha(t) \le t$, $\alpha(t_0) = t_0 \ge 0$.

Lemma 3 ([13]) *Suppose that the positive constants β,γ,ξ,p_1, and p_2 satisfy conditions:*

(1) *if $\beta \in (0,1]$, $\gamma \in (\frac{1}{2},1)$ and $\xi \ge \frac{3}{2}-\gamma$, $p_1 = \frac{1}{\gamma}$;*
(2) *if $\beta \in (0,1]$, $\gamma \in (0,\frac{1}{2})$ and $\xi > \frac{1-2\gamma^2}{1-\gamma^2}$, $p_2 = \frac{1+4\gamma}{1+3\gamma}$, then*

$$B\left[\frac{p_i(\xi-1)+1}{\beta}, p_i(\gamma-1)+1\right] \in \mathbb{R}^+, \qquad \theta_i = p_i\left[\beta(\gamma-1)+\xi-1\right]+1 \ge 0$$

are valid for $i = 1,2$.

Lemma 4 *Let $u(x,y), a(x,y), b(x,y), h(x,y) \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\alpha(x), \beta(y)$ be continuous, differentiable, and increasing functions on $\mathbb{R}^+$ with $\alpha(x) \le x$, $\beta(y) \le y$, $\alpha(0) = 0$, $\beta(0) = 0$. If $u(x,y)$ satisfied the following inequality*

$$u(x,y) \le a(x,y) + b(x,y)\int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)u(t,s)\,\mathrm{d}s\,\mathrm{d}t, \tag{6}$$

then

$$u(x,y) \le a(x,y) + \frac{b(x,y)}{e(\alpha(x),\beta(y))}\int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)a(t,s)e(t,s)\,\mathrm{d}s\,\mathrm{d}t, \tag{7}$$

where

$$e(x,y) = \exp\left(-\int_0^x\int_0^y h(t,s)b(t,s)\,\mathrm{d}s\,\mathrm{d}t\right).$$

Proof Define a function $v(x,y)$ on $\mathbb{R}^+$ by

$$v(x,y) = e\left(\alpha(x),\beta(y)\right)\int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)u(t,s)\,\mathrm{d}s\,\mathrm{d}t, \tag{8}$$

we have $v(x,0) = 0$, $v(0,y) = 0$. Differentiating $v(x,y)$ with respect to x,y, we have

$$v_{xy} = e\left(\alpha(x),\beta(y)\right)\alpha'(x)\beta'(y)h\left(\alpha(x),\beta(y)\right)u\left(\alpha(x),\beta(y)\right)$$

$$
\begin{aligned}
&- e\big(\alpha(x),\beta(y)\big)\alpha'(x)\beta'(y)h\big(\alpha(x),\beta(y)\big)b\big(\alpha(x),\beta(y)\big)\\
&\times \int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)u(t,s)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
$$

Using (6) and $\alpha(x) \leq x, \beta(y) \leq y$, we have

$$
\begin{aligned}
v_{xy} &\leq e\big(\alpha(x),\beta(y)\big)\alpha'(x)\beta'(y)h\big(\alpha(x),\beta(y)\big)\\
&\quad\times \left[a\big(\alpha(x),\beta(y)\big) + b\big(\alpha(x),\beta(y)\big)\int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)u(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]\\
&\quad- e\big(\alpha(x),\beta(y)\big)\alpha'(x)\beta'(y)h\big(\alpha(x),\beta(y)\big)b\big(\alpha(x),\beta(y)\big)\int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)u(t,s)\,\mathrm{d}s\,\mathrm{d}t\\
&= e\big(\alpha(x),\beta(y)\big)\alpha'(x)\beta'(y)h\big(\alpha(x),\beta(y)\big)a\big(\alpha(x),\beta(y)\big).
\end{aligned}
\tag{9}
$$

Integrating both sides of inequality (9), because $v(x,0) = v(0,y) = 0$, we have

$$
\begin{aligned}
v(x,y) &\leq \int_0^x\int_0^y e\big(\alpha(x),\beta(y)\big)\alpha'(x)\beta'(y)h\big(\alpha(x),\beta(y)\big)a\big(\alpha(x),\beta(y)\big)\,\mathrm{d}s\,\mathrm{d}t\\
&= \int_0^{\alpha(x)}\int_0^{\beta(y)} e(t,s)h(t,s)a(t,s)\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
\tag{10}
$$

From (8) and (10), we have

$$
\int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)u(t,s)\,\mathrm{d}s\,\mathrm{d}t \leq \frac{1}{e(\alpha(x),\beta(y))}\int_0^{\alpha(x)}\int_0^{\beta(y)} h(t,s)a(t,s)e(t,s)\,\mathrm{d}s\,\mathrm{d}t. \tag{11}
$$

Substituting inequality (11) into (6), we can get the required estimation (7). This completes the proof. □

Lemma 5 ([24]) *Let $a \geq 0, p \geq q \geq 0$, and $p \neq 0$, then*

$$
a^{\frac{q}{p}} \leq \frac{q}{p}K^{\frac{q-p}{p}}a + \frac{p-q}{p}K^{\frac{q}{p}}, \quad K > 0.
$$

We give two special cases of the above result:

(a) *If $K = 1$, we have*

$$
a^{\frac{q}{p}} \leq \frac{q}{p}a + \frac{p-q}{p}, \quad a \geq 0, p \geq q \geq 0, p \neq 0.
$$

(b) *If $K = 1, p = 1$, we have*

$$
a^q \leq qa + (1-q), \quad a \geq 0, q \geq 0.
$$

3 Main results

Firstly, we study inequality (2) and assume that the following conditions hold:

(H_1) $a(x,y) \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, and $a(x,y)$ is a nondecreasing function;

(H_2) $f_i(x,y)$ $(i = 1,2,3)$ are continuous and nonnegative on Ω;

(H_3) $\alpha(x), \beta(y)$ are continuous, differentiable, and increasing functions on $\mathbb{R}^+$ with $\alpha(x) \leq x, \beta(y) \leq y, \alpha(0) = 0, \beta(0) = 0$;

(H_4) ζ, γ, ξ are positive constants.

Theorem 1 *Suppose that* (H_1)–(H_4) *hold and* $u(x,y)$ *satisfies inequality* (2), *then we have the following results*:

(i) *If* $\zeta \in (0,1], \gamma \in (\frac{1}{2}, 1)$, *and* $\xi \geq \frac{3}{2} - \gamma$, *we have*

$$u(x,y) \leq \left(\tilde{a}_1(x,y) + \frac{\tilde{b}_1(x,y)}{\tilde{e}_1(\alpha(x),\beta(y))} \int_0^{\alpha(x)} \int_0^{\beta(y)} \tilde{h}_1(t,s)\tilde{a}_1(t,s)\tilde{e}_1(t,s)\,\mathrm{d}s\,\mathrm{d}t\right)^{1-\gamma}, \quad (x,y) \in \Omega, \tag{12}$$

where

$$\tilde{a}_1(x,y) = 3^{\frac{\gamma}{1-\gamma}} a^{\frac{1}{1-\gamma}}(x,y),$$

$$\tilde{b}_1(x,y) = \left(3M_1^2 \times \left(\alpha(x)\beta(y)\right)^{\theta_1}\right)^{\frac{\gamma}{1-\gamma}},$$

$$\tilde{h}_1(x,y) = f_1^{\frac{1}{1-\gamma}}(x,y) + \left(f_2(x,y)\int_0^x \int_0^y f_3(t,s)\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1}{1-\gamma}},$$

$$\tilde{e}_1(x,y) = \exp\left(-\int_0^x \int_0^y \tilde{h}_1(t,s)\tilde{b}_1(t,s)\,\mathrm{d}s\,\mathrm{d}t\right),$$

$$M_1 = \frac{1}{\zeta}B\left[\frac{\gamma+\xi-1}{\zeta\gamma}, \frac{2\gamma-1}{\gamma}\right],$$

$$\theta_1 = \frac{1}{\gamma}\left[\zeta(\gamma-1)+\xi-1\right]+1.$$

(ii) *If* $\zeta \in (0,1], \gamma \in (0, \frac{1}{2}]$, *and* $\xi > \frac{1-2\gamma^2}{1-\gamma^2}$, *we have*

$$u(x,y) \leq \left(\tilde{a}_2(x,y) + \frac{\tilde{b}_2(x,y)}{\tilde{e}_2(\alpha(x),\beta(y))} \int_0^{\alpha(x)} \int_0^{\beta(y)} \tilde{h}_2(t,s)\tilde{a}_2(t,s)\tilde{e}_2(t,s)\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{\gamma}{1+4\gamma}}, \quad (x,y) \in \Omega, \tag{13}$$

where

$$\tilde{a}_2(x,y) = 3^{\frac{1+3\gamma}{\gamma}} a^{\frac{1+4\gamma}{\gamma}}(x,y),$$

$$\tilde{b}_2(x,y) = \left(3M_2^2 \times \left(\alpha(x)\beta(y)\right)^{\theta_2}\right)^{\frac{1+3\gamma}{\gamma}},$$

$$\tilde{h}_2(x,y) = f_1^{\frac{1+4\gamma}{\gamma}}(x,y) + \left(f_2(x,y)\int_0^x \int_0^y f_3(t,s)\,\mathrm{d}s\,\mathrm{d}t\right)^{\frac{1+4\gamma}{\gamma}},$$

$$\tilde{e}_2(x,y) = \exp\left(-\int_0^x \int_0^y \tilde{h}_1(t,s)\tilde{b}_1(t,s)\,\mathrm{d}s\,\mathrm{d}t\right),$$

$$M_2 = \frac{1}{\zeta}B\left[\frac{\xi(1+4\gamma)-\gamma}{\zeta(1+3\gamma)}, \frac{4\gamma^2}{1+3\gamma}\right],$$

$$\theta_2 = \frac{1+4\gamma}{1+3\gamma}\left[\zeta(\gamma-1)+\xi-1\right]+1.$$

Proof If $\zeta \in (0,1]$, $\gamma \in (\frac{1}{2},1)$, and $\xi \geq \frac{3}{2}-\gamma$, let

$$p_1 = \frac{1}{\gamma}, \qquad q_1 = \frac{1}{1-\gamma}.$$

If $\zeta \in (0,1]$, $\gamma \in (0,\frac{1}{2}]$, and $\xi > \frac{1-2\gamma^2}{1-\gamma^2}$, let

$$p_2 = \frac{1+4\gamma}{1+3\gamma}, \qquad q_2 = \frac{1+4\gamma}{\gamma},$$

then

$$\frac{1}{p_i}+\frac{1}{q_i} = 1, \quad i = 1,2.$$

Using Hölder's inequality in (2), we have

$$\begin{aligned} u(x,y) &\leq a(x,y) \\ &\quad + \left[\int_0^{\alpha(x)}\int_0^{\beta(y)} \left(\alpha^\zeta(x)-t^\zeta\right)^{p_i(\gamma-1)} t^{p_i(\xi-1)} \left(\beta^\zeta(y)-s^\zeta\right)^{p_i(\gamma-1)} s^{p_i(\xi-1)}\,\mathrm{d}s\,\mathrm{d}t\right]^{1/p_i} \\ &\quad \times \left[\int_0^{\alpha(x)}\int_0^{\beta(y)} f_1^{q_i}(t,s)u^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]^{1/q_i} \\ &\quad + \left[\int_0^{\alpha(x)}\int_0^{\beta(y)} \left(\alpha^\zeta(x)-t^\zeta\right)^{p_i(\gamma-1)} t^{p_i(\xi-1)} \left(\beta^\zeta(y)-s^\zeta\right)^{p_i(\gamma-1)} s^{p_i(\xi-1)}\,\mathrm{d}s\,\mathrm{d}t\right]^{1/p_i} \\ &\quad \times \left[\int_0^{\alpha(x)}\int_0^{\beta(y)} \left(f_2(t,s)\int_0^t\int_0^s f_3(\tau,\eta)u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{q_i}\mathrm{d}s\,\mathrm{d}t\right]^{1/q_i}. \end{aligned}$$

Set $z(x,y)$ as the right-hand side of the above inequality, and

$$A(x,y) = \left[\int_0^{\alpha(x)}\int_0^{\beta(y)} \left(\alpha^\zeta(x)-t^\zeta\right)^{p_i(\gamma-1)} t^{p_i(\xi-1)} \left(\beta^\zeta(y)-s^\zeta\right)^{p_i(\gamma-1)} s^{p_i(\xi-1)}\,\mathrm{d}s\,\mathrm{d}t\right]^{1/p_i},$$

that is,

$$\begin{aligned} z(x,y) &= a(x,y) + A(x,y)\left[\int_0^{\alpha(x)}\int_0^{\beta(y)} f_1^{q_i}(t,s)u^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]^{1/q_i} \\ &\quad + A(x,y)\left[\int_0^{\alpha(x)}\int_0^{\beta(y)} \left(f_2(t,s)\int_0^t\int_0^s f_3(\tau,\eta)u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{q_i}\mathrm{d}s\,\mathrm{d}t\right]^{1/q_i}. \end{aligned}$$

Then $z(x,y)$ is a nondecreasing function, and $u(x,y) \leq z(x,y)$, we have

$$\begin{aligned} z(x,y) &\leq a(x,y) + A(x,y)\left[\int_0^{\alpha(x)}\int_0^{\beta(y)} f_1^{q_i}(t,s)z^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]^{1/q_i} \\ &\quad + A(x,y)\left[\int_0^{\alpha(x)}\int_0^{\beta(y)} \left(f_2(t,s)\int_0^t\int_0^s f_3(\tau,\eta)z(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{q_i}\mathrm{d}s\,\mathrm{d}t\right]^{1/q_i} \\ &\leq a(x,y) + A(x,y)\left[\int_0^{\alpha(x)}\int_0^{\beta(y)} f_1^{q_i}(t,s)z^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]^{1/q_i} \end{aligned}$$

$$+A(x,y)\left[\int_0^{\alpha(x)}\int_0^{\beta(y)}\left(f_2(t,s)\int_0^t\int_0^s f_3(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{q_i} z^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]^{1/q_i}. \qquad (14)$$

Using the discrete Jensen inequality in Lemma 1 with $n=3, m=q_i$, we get

$$\begin{aligned} & z^{q_i}(x,y) \\ &\le 3^{q_i-1}a^{q_i}(x,y)+3^{q_i-1}A^{q_i}(x,y)\int_0^{\alpha(x)}\int_0^{\beta(y)} f_1^{q_i}(t,s)z^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t \\ &\quad +3^{q_i-1}A^{q_i}(x,y)\int_0^{\alpha(x)}\int_0^{\beta(y)}\left(f_2(t,s)\int_0^t\int_0^s f_3(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{q_i} z^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \qquad (15)$$

Using Lemma 2, we obtain

$$\begin{aligned} A(x,y) &= \left[\int_0^{\alpha(x)}\int_0^{\beta(y)}\left(\alpha^\zeta(x)-t^\zeta\right)^{p_i(\gamma-1)}t^{p_i(\xi-1)}\left(\beta^\zeta(y)-s^\zeta\right)^{p_i(\gamma-1)}s^{p_i(\xi-1)}\,\mathrm{d}s\,\mathrm{d}t\right]^{1/p_i} \\ &\le \left(M_i^2\times\left(\alpha(x)\beta(y)\right)^{\theta_i}\right)^{1/p_i}, \end{aligned} \qquad (16)$$

for $(x,y)\in\Omega$, where

$$\begin{aligned} & M_i=\frac{1}{\zeta}B\left[\frac{p_i(\xi-1)+1}{\zeta}, p_i(\gamma-1)+1\right], \\ & \theta_i=p_i\left[\zeta(\gamma-1)+\xi-1\right]+1\ge 0, \quad i=1,2. \end{aligned}$$

From (15) and (16), we get

$$\begin{aligned} z^{q_i}(x,y) &\le 3^{q_i-1}a^{q_i}(x,y)+3^{q_i-1}\left[M_i^2\times\left(\alpha(x)\beta(y)\right)^{\theta_i}\right]^{q_i/p_i} \\ &\quad \times\int_0^{\alpha(x)}\int_0^{\beta(y)}\left[f_1^{q_i}(t,s)+\left(f_2(t,s)\int_0^t\int_0^s f_3(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{q_i}\right]z^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \qquad (17)$$

Set

$$\begin{aligned} & \tilde{a}_i(x,y)=3^{q_i-1}a^{q_i}(x,y), \\ & \tilde{b}_i(x,y)=3^{q_i-1}\left[M_i^2\times\left(\alpha(x)\beta(y)\right)^{\theta_i}\right]^{q_i/p_i}, \\ & \tilde{h}_i(t,s)=f_1^{q_i}(t,s)+\left(f_2(t,s)\int_0^t\int_0^s f_3(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{q_i}, \\ & \tilde{e}_i(x,y)=\exp\left(-\int_0^x\int_0^y\tilde{h}_i(t,s)\tilde{b}_i(t,s)\,\mathrm{d}s\,\mathrm{d}t\right), \quad i=1,2, \end{aligned}$$

we have

$$z^{q_i}(x,y)\le\tilde{a}_i(x,y)+\tilde{b}_i(x,y)\int_0^{\alpha(x)}\int_0^{\beta(y)}\tilde{h}_i(t,s)z^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t, \quad i=1,2,(x,y)\in\Omega. \qquad (18)$$

Applying Lemma 4 to (18), we obtain

$$u^{q_i}(x,y)\le z^{q_i}(x,y)\le\tilde{a}_i(x,y)+\frac{\tilde{b}_i(x,y)}{\tilde{e}_i(\alpha(x),\beta(y))}\int_0^{\alpha(x)}\int_0^{\beta(y)}\tilde{h}_i(t,s)\tilde{a}_i(t,s)\tilde{e}_i(t,s)\,\mathrm{d}s\,\mathrm{d}t,$$

$$i = 1, 2, (x, y) \in \Omega. \tag{19}$$

Substituting $p_1 = \frac{1}{\gamma}, q_1 = \frac{1}{1-\gamma}$, and $p_2 = \frac{1+4\gamma}{1+3\gamma}, q_2 = \frac{1+4\gamma}{\gamma}$ to (19), respectively, we can get the desired estimations (12) and (13). This completes the proof. □

Secondly, we study inequality (3) and assume that the following conditions hold:

(H_5) $a(x, y) \geq 1$;

(H_6) $f(x, y)$ is continuous and nonnegative on Ω;

(H_7) $\alpha(x), \beta(y)$ are continuous, differentiable, and increasing functions on $[x_0, +\infty)$, $[y_0, +\infty)$, respectively, and $\alpha(x) \leq x, \beta(y) \leq y, \alpha(x_i) = x_i, \beta(y_i) = y_i, i = 0, 1, 2, \ldots$;

(H_8) $u(x, y)$ is nonnegative and continuous on Ω with the exception of the points (x_i, y_i), where there is a finite jump: $u(x_i - 0, y_i - 0) \neq u(x_i + 0, y_i + 0), i = 1, 2, \ldots$;

(H_9) p, ζ, γ are positive constants;

(H_{10}) ζ_i are nonnegative constants for any positive integer i.

Theorem 2 *Suppose that* (H_1), (H_5)–(H_{10}) *hold and* $u(x, y)$ *satisfies inequality* (3), *then we have*

$$u(x, y) \leq \tilde{a}_i(x, y) + \frac{1}{\tilde{e}_i(\alpha(x), \beta(y))} \int_{x_i}^{\alpha(x)} \int_{y_i}^{\beta(y)} \tilde{h}(t, s)\tilde{a}_i(t, s)\tilde{e}_i(t, s)\, ds\, dt, \quad (x, y) \in \Omega, \tag{20}$$

where

$$\tilde{a}_i(x, y) = A_i^{\frac{1}{1-\gamma}}(x, y), \quad i = 0, 1, 2, \ldots,$$

$$A_i(x, y) = a(x, y) + \sum_{j=1}^{i} \int_{x_{j-1}}^{\alpha(x_j)} \int_{y_{j-1}}^{\beta(y_j)} \left(x^\zeta - t^\zeta\right)^{\gamma-1} \left(y^\zeta - s^\zeta\right)^{\gamma-1} f(t, s)\tilde{u}_j(t, s)$$
$$\times \left[\tilde{u}_j^2(t, s) + \int_{x_{j-1}}^{t} \int_{y_{j-1}}^{s} g(\tau, \eta)\tilde{u}_j(\tau, \eta)\, d\eta\, d\tau\right]^p ds\, dt + \sum_{j=1}^{i} \zeta_j \tilde{u}_j(x_j - 0, y_j - 0),$$
$$i = 0, 1, 2, \ldots,$$

$$\tilde{u}_j(x, y) = \tilde{a}_{j-1}(x, y) + \frac{1}{\tilde{e}_{j-1}(\alpha(x), \beta(y))} \int_{x_{j-1}}^{\alpha(x)} \int_{y_{j-1}}^{\beta(y)} \tilde{h}(t, s)\tilde{a}_{j-1}(t, s)\tilde{e}_{j-1}(t, s)\, ds\, dt,$$
$$j = 1, 2, 3, \ldots,$$

$$\tilde{h}(t, s) = \left(x^\zeta - t^\zeta\right)^{\gamma-1} \left(y^\zeta - s^\zeta\right)^{\gamma-1} f(t, s)\Phi\left(\alpha^{-1}(t), \beta^{-1}(s)\right),$$

$$\tilde{e}_i(x, y) = \exp\left(-\int_{x_i}^{x} \int_{y_i}^{y} \tilde{h}(t, s)\, ds\, dt\right), \quad i = 0, 1, 2, \ldots.$$

Proof Firstly, we consider the case $(x, y) \in \Omega_{11}$. Denoting

$$v(x, y) = a(x, y) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \left(x^\zeta - t^\zeta\right)^{\gamma-1} \left(y^\zeta - s^\zeta\right)^{\gamma-1} f(t, s)u(t, s)$$
$$\times \left[u^2(t, s) + \int_{x_0}^{t} \int_{y_0}^{s} g(\tau, \eta)u(\tau, \eta)\, d\eta\, d\tau\right]^p ds\, dt, \tag{21}$$

then $v(x,y)$ is a nonnegative and nondecreasing continuous function, and $u(x,y) \le v(x,y)$, $v(x_0,y_0) = a(x_0,y_0)$.

Differentiating (21), we have

$$\begin{aligned} v_x(x,y) &= a_x(x,y) + \alpha'(x) \int_{y_0}^{\beta(y)} \left(x^\zeta - \alpha^\zeta(x)\right)^{\gamma-1} \left(y^\zeta - s^\zeta\right)^{\gamma-1} f\left(\alpha(x),s\right) u\left(\alpha(x),s\right) \\ &\quad \times \left[u^2\left(\alpha(x),s\right) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{s} g(\tau,\eta) u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^p \mathrm{d}s \\ &\le a_x(x,y) + \alpha'(x) \int_{y_0}^{\beta(y)} \left(x^\zeta - \alpha^\zeta(x)\right)^{\gamma-1} \left(y^\zeta - s^\zeta\right)^{\gamma-1} f\left(\alpha(x),s\right) v\left(\alpha(x),s\right) \\ &\quad \times \left[v^2\left(\alpha(x),s\right) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{s} g(\tau,\eta) v(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^p \mathrm{d}s. \end{aligned} \tag{22}$$

$$\begin{aligned} v_{xy}(x,y) &\le a_{xy}(x,y) + \alpha'(x)\beta'(y)\left(x^\zeta - \alpha^\zeta(x)\right)^{\gamma-1} \left(y^\zeta - \beta^\zeta(y)\right)^{\gamma-1} \\ &\quad \times f\left(\alpha(x),\beta(y)\right) v\left(\alpha(x),\beta(y)\right) \\ &\quad \times \left[v^2\left(\alpha(x),\beta(y)\right) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} g(t,s) v(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]^p. \end{aligned} \tag{23}$$

Set

$$\begin{aligned} F(x,y) &= v^2\left(\alpha(x),\beta(y)\right) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} g(t,s) v(t,s)\,\mathrm{d}s\,\mathrm{d}t, \\ G(x,y) &= v^2(x,y) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} g(t,s) v(t,s)\,\mathrm{d}s\,\mathrm{d}t, \end{aligned} \tag{24}$$

then $F(x,y) \le G(x,y)$, $G(x,y)$ is a nonnegative and nondecreasing continuous function, and $G(x_0,y_0) = a^2(x_0,y_0)$. Since $a(x,y) \ge 1$, we have $v(x,y) \ge 1$, then $v(x,y) \le v^2(x,y) \le G(x,y)$, that is, $v(x,y) \le G(x,y)$. Differentiating (24) with respect to x, from (22), we have

$$\begin{aligned} G_x(x,y) &= 2v(x,y)v_x(x,y) + \alpha'(x) \int_{y_0}^{\beta(y)} g\left(\alpha(x),s\right) v\left(\alpha(x),s\right) \mathrm{d}s \\ &\le 2G(x,y)\left[a_x(x,y) + \alpha'(x) \int_{y_0}^{\beta(y)} \left(x^\zeta - \alpha^\zeta(x)\right)^{\gamma-1} \left(y^\zeta - s^\zeta\right)^{\gamma-1} f\left(\alpha(x),s\right) v\left(\alpha(x),s\right) \right. \\ &\quad \left. \times \left[v^2\left(\alpha(x),s\right) + \int_{x_0}^{\alpha(x)} \int_{y_0}^{s} g(\tau,\eta) v(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^p \mathrm{d}s\right] \\ &\quad + \alpha'(x) \int_{y_0}^{\beta(y)} g\left(\alpha(x),s\right) v\left(\alpha(x),s\right) \mathrm{d}s \\ &\le 2G(x,y)\left[a_x(x,y) + G^{p+1}(x,y)\alpha'(x) \right. \\ &\quad \left. \times \int_{y_0}^{\beta(y)} \left(x^\zeta - \alpha^\zeta(x)\right)^{\gamma-1} \left(y^\zeta - s^\zeta\right)^{\gamma-1} f\left(\alpha(x),s\right) \mathrm{d}s\right] \\ &\quad + \alpha'(x)G(x,y) \int_{y_0}^{\beta(y)} g\left(\alpha(x),s\right) \mathrm{d}s \end{aligned}$$

$$= \left[2\alpha'(x)\int_{y_0}^{\beta(y)}\left(x^\zeta-\alpha^\zeta(x)\right)^{\gamma-1}\left(y^\zeta-s^\zeta\right)^{\gamma-1}f\left(\alpha(x),s\right)\mathrm{d}s\right]G^{p+2}(x,y)$$
$$+\left[2a_x(x,y)+\alpha'(x)\int_{y_0}^{\beta(y)}g\left(\alpha(x),s\right)\mathrm{d}s\right]G(x,y). \tag{25}$$

Set

$$A(x,y) = 2a_x(x,y)+\alpha'(x)\int_{y_0}^{\beta(y)}g\left(\alpha(x),s\right)\mathrm{d}s,$$
$$B(x,y) = 2\alpha'(x)\int_{y_0}^{\beta(y)}\left(x^\zeta-\alpha^\zeta(x)\right)^{\gamma-1}\left(y^\zeta-s^\zeta\right)^{\gamma-1}f\left(\alpha(x),s\right)\mathrm{d}s,$$

then

$$G_x(x,y)\le B(x,y)G^{p+2}(x,y)+A(x,y)G(x,y). \tag{26}$$

From (26), we have

$$G^{-(p+2)}(x,y)G_x(x,y)\le B(x,y)+A(x,y)G^{-(p+1)}(x,y). \tag{27}$$

Let $\eta(x,y)=G^{-(p+1)}(x,y)$, then $\eta_x(x,y)=-(p+1)G^{-(p+2)}(x,y)G_x(x,y)$, (27) can be restated as

$$\eta_x(x,y)+(p+1)A(x,y)\eta(x,y)\ge(p+1)B(x,y). \tag{28}$$

Multiplying by $\exp((p+1)\int_{x_0}^x A(t,y)\,\mathrm{d}t)$ on both sides of (28), we have

$$\frac{\partial}{\partial x}\left[\eta(x,y)\times\exp\left((p+1)\int_{x_0}^x A(t,y)\,\mathrm{d}t\right)\right]$$
$$\ge -(p+1)B(x,y)\times\exp\left((p+1)\int_{x_0}^x A(t,y)\,\mathrm{d}t\right). \tag{29}$$

Integrating both sides of (29) from x_0 to x, we get

$$\eta(x,y)\times\exp\left((p+1)\int_{x_0}^x A(t,y)\,\mathrm{d}t\right)-\eta(x_0,y)$$
$$\ge\int_{x_0}^x -(p+1)B(t,y)\times\exp\left((p+1)\int_{x_0}^t A(\tau,y)\,\mathrm{d}\tau\,\mathrm{d}t\right),$$

set

$$\Delta(x,y)=\exp\left((p+1)\int_{x_0}^x A(t,y)\,\mathrm{d}t\right),$$

then

$$\eta(x,y)\ge\frac{\eta(x_0,y)-\int_{x_0}^x(p+1)B(t,y)\Delta(t,y)\,\mathrm{d}t}{\Delta(x,y)}. \tag{30}$$

Since $\eta(x_0,y) = G^{-(p+1)}(x_0,y) = a^{-2(p+1)}(x_0,y)$, from (30) we have

$$\eta(x,y) \geq \frac{1-(p+1)a^{2(p+1)}(x_0,y)\int_{x_0}^{x} B(t,y)\Delta(t,y)\,\mathrm{d}t}{a^{2(p+1)}(x_0,y)\Delta(x,y)}. \tag{31}$$

By the relation $\eta(x,y) = G^{-(p+1)}(x,y)$, from (31) we get

$$G^{p}(x,y) \leq \left[\frac{a^{2(p+1)}(x_0,y)\Delta(x,y)}{1-(p+1)a^{2(p+1)}(x_0,y)\int_{x_0}^{x} B(t,y)\Delta(t,y)\,\mathrm{d}t}\right]^{\frac{p}{p+1}}, \tag{32}$$

where $1-(p+1)a^{2(p+1)}(x_0,y)\int_{x_0}^{x} B(t,y)\Delta(t,y)\,\mathrm{d}t > 0$. Setting

$$\Phi(x,y) = \left[\frac{a^{2(p+1)}(x_0,y)\Delta(x,y)}{1-(p+1)a^{2(p+1)}(x_0,y)\int_{x_0}^{x} B(t,y)\Delta(t,y)\,\mathrm{d}t}\right]^{\frac{p}{p+1}}, \tag{33}$$

from (23), (24), (32), and (33), we obtain

$$\begin{aligned} v_{xy}(x,y) &\leq a_{xy}(x,y) + \alpha'(x)\beta'(y)\bigl(x^{\zeta}-\alpha^{\zeta}(x)\bigr)^{\gamma-1}\bigl(y^{\zeta}-\beta^{\zeta}(y)\bigr)^{\gamma-1} \\ &\quad \times f\bigl(\alpha(x),\beta(y)\bigr)v\bigl(\alpha(x),\beta(y)\bigr)\Phi(x,y). \end{aligned} \tag{34}$$

Integrating both sides of (34), we get

$$\begin{aligned} v(x,y) &\leq a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} \alpha'(t)\beta'(s)\bigl(x^{\zeta}-\alpha^{\zeta}(t)\bigr)^{\gamma-1}\bigl(y^{\zeta}-\beta^{\zeta}(s)\bigr)^{\gamma-1} \\ &\quad \times f\bigl(\alpha(t),\beta(s)\bigr)v\bigl(\alpha(t),\beta(s)\bigr)\Phi(t,s)\,\mathrm{d}s\,\mathrm{d}t \\ &= a(x,y) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \bigl(x^{\zeta}-t^{\zeta}\bigr)^{\gamma-1}\bigl(y^{\zeta}-s^{\zeta}\bigr)^{\gamma-1} f(t,s)\Phi\bigl(\alpha^{-1}(t),\beta^{-1}(s)\bigr)v(t,s)\,\mathrm{d}s\,\mathrm{d}t \\ &= a(x,y) + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \tilde{h}(t,s)v(t,s)\,\mathrm{d}s\,\mathrm{d}t, \end{aligned} \tag{35}$$

where $\tilde{h}(t,s) = (x^{\zeta}-t^{\zeta})^{\gamma-1}(y^{\zeta}-s^{\zeta})^{\gamma-1}f(t,s)\Phi(\alpha^{-1}(t),\beta^{-1}(s))$. Inequality (35) has the same form as inequality (6) of Lemma 4. By using Lemma 4, we can obtain the estimate of $u(x,y)$ as follows:

$$u(x,y) \leq \tilde{a}_0(x,y) + \frac{1}{\tilde{e}_0(\alpha(x),\beta(y))}\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \tilde{h}(t,s)\tilde{a}_0(t,s)\tilde{e}_0(t,s)\,\mathrm{d}s\,\mathrm{d}t, \quad (x,y)\in\Omega_{11}.$$

Set

$$\tilde{u}_1(x,y) = \tilde{a}_0(x,y) + \frac{1}{\tilde{e}_0(\alpha(x),\beta(y))}\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \tilde{h}(t,s)\tilde{a}_0(t,s)\tilde{e}_0(t,s)\,\mathrm{d}s\,\mathrm{d}t, \quad (x,y)\in\Omega_{11},$$

then $u(x,y) \leq \tilde{u}_1(x,y), (x,y)\in\Omega_{11}$.

Next, if $(x,y)\in\Omega_{22}$, (3) can be restated as

$$u(x,y) \leq a(x,y) + \int_{x_0}^{\alpha(x_1)}\int_{y_0}^{\beta(y_1)} \bigl(x^{\zeta}-t^{\zeta}\bigr)^{\gamma-1}\bigl(y^{\zeta}-s^{\zeta}\bigr)^{\gamma-1} f(t,s)u(t,s)$$

$$
\begin{aligned}
&\times\left[u^2(t,s)+\int_{x_0}^{t}\int_{y_0}^{s}g(\tau,\eta)u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^p\mathrm{d}s\,\mathrm{d}t\\
&+\int_{x_1}^{\alpha(x)}\int_{y_1}^{\beta(y)}\bigl(x^\zeta-t^\zeta\bigr)^{\gamma-1}\bigl(y^\zeta-s^\zeta\bigr)^{\gamma-1}f(t,s)u(t,s)\\
&\times\left[u^2(t,s)+\int_{x_0}^{t}\int_{y_0}^{s}g(\tau,\eta)u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^p\mathrm{d}s\,\mathrm{d}t+\zeta_1u(x_1-0,y_1-0)\\
\le{}& a(x,y)+\int_{x_0}^{\alpha(x_1)}\int_{y_0}^{\beta(y_1)}\bigl(x^\zeta-t^\zeta\bigr)^{\gamma-1}\bigl(y^\zeta-s^\zeta\bigr)^{\gamma-1}f(t,s)\tilde{u}_1(t,s)\\
&\times\left[\tilde{u}_1^2(t,s)+\int_{x_0}^{t}\int_{y_0}^{s}g(\tau,\eta)\tilde{u}_1(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^p\mathrm{d}s\,\mathrm{d}t\\
&+\int_{x_1}^{\alpha(x)}\int_{y_1}^{\beta(y)}\bigl(x^\zeta-t^\zeta\bigr)^{\gamma-1}\bigl(y^\zeta-s^\zeta\bigr)^{\gamma-1}f(t,s)u(t,s)\\
&\times\left[u^2(t,s)+\int_{x_0}^{t}\int_{y_0}^{s}g(\tau,\eta)u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^p\mathrm{d}s\,\mathrm{d}t+\zeta_1\tilde{u}_1(x_1-0,y_1-0).
\end{aligned}
\tag{36}
$$

Setting

$$
\begin{aligned}
A_1(x,y)={}&a(x,y)+\int_{x_0}^{\alpha(x_1)}\int_{y_0}^{\beta(y_1)}\bigl(x^\zeta-t^\zeta\bigr)^{\gamma-1}\bigl(y^\zeta-s^\zeta\bigr)^{\gamma-1}f(t,s)\tilde{u}_1(t,s)\\
&\times\left[\tilde{u}_1^2(t,s)+\int_{x_0}^{t}\int_{y_0}^{s}g(\tau,\eta)u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^p\mathrm{d}s\,\mathrm{d}t+\zeta_1\tilde{u}_1(x_1-0,y_1-0),\\
&(x,y)\in\Omega_{22},
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
\Psi(x,y)={}&A_1(x,y)+\int_{x_1}^{\alpha(x)}\int_{y_1}^{\beta(y)}\bigl(x^\zeta-t^\zeta\bigr)^{\gamma-1}\bigl(y^\zeta-s^\zeta\bigr)^{\gamma-1}f(t,s)u(t,s)\\
&\times\left[u^2(t,s)+\int_{x_0}^{t}\int_{y_0}^{s}g(\tau,\eta)u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^p\mathrm{d}s\,\mathrm{d}t,\quad (x,y)\in\Omega_{22},
\end{aligned}
$$

then $\Psi(x,y)$ is a nonnegative and nondecreasing function, and

$$
u(x,y)\le\Psi(x,y),\qquad u(x_1,y_1)\le\Psi(x_1,y_1)=A_1(x_1,y_1).
$$

Differentiating both sides of (37), we obtain

$$
\begin{aligned}
\Psi_{xy}(x,y)={}&\bigl(A_1(x,y)\bigr)_{xy}+\alpha'(x)\beta'(y)\bigl(x^\zeta-\alpha^\zeta(x)\bigr)^{\gamma-1}\bigl(y^\zeta-\beta^\zeta(y)\bigr)^{\gamma-1}\\
&\times f\bigl(\alpha(x),\beta(y)\bigr)u\bigl(\alpha(x),\beta(y)\bigr)\\
&\times\left[u^2\bigl(\alpha(x),\beta(y)\bigr)+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}g(t,s)u(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]^p\\
\le{}&\bigl(A_1(x,y)\bigr)_{xy}+\alpha'(x)\beta'(y)\bigl(x^\zeta-\alpha^\zeta(x)\bigr)^{\gamma-1}\bigl(y^\zeta-\beta^\zeta(y)\bigr)^{\gamma-1}\\
&\times f\bigl(\alpha(x),\beta(y)\bigr)\Psi\bigl(\alpha(x),\beta(y)\bigr)\\
&\times\left[\Psi^2\bigl(\alpha(x),\beta(y)\bigr)+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}g(t,s)\Psi(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]^p.
\end{aligned}
\tag{38}
$$

(38) has the same form as (23), and using the same procedure, we can get the desired estimations (20) for $(x,y) \in \Omega_{22}$.

Consequently, by using a similar procedure, we can get the desired estimations (20) for $(x,y) \in \Omega_{ii}$ $(i = 3,4,5,\ldots)$. Thus we complete the proof of Theorem 2. □

Finally, we study inequality (4) and assume that the following conditions hold:

(H_{11}) $g(x,y)$ is continuous and nonnegative on Ω;

(H_{12}) p,q,m,n,ξ,ζ,γ are positive constants with $p \geq m, p \geq n, q \in [0,1]$.

Theorem 3 *Suppose* (H_1), (H_5)–(H_8), (H_{10})–(H_{12}) *hold and* $u(x,y)$ *satisfies inequality* (4). *Then we have the following results*:

(i) *If* $\zeta \in (0,1], \gamma \in (\frac{1}{2},1)$, *and* $\xi \geq \frac{3}{2} - \gamma$, *we have*

$$\begin{aligned} u(x,y) \leq {}& \Bigg[E_i(x,y) + \bigg(\tilde{a}_i(x,y) + \frac{\tilde{b}_1(x,y)}{\tilde{e}_i(\alpha(x),\beta(y))} \\ & \times \int_{x_i}^{\alpha(x)} \int_{y_i}^{\beta(y)} \tilde{h}(t,s)\tilde{a}_i(t,s)\tilde{e}_i(t,s)\,\mathrm{d}s\,\mathrm{d}t\bigg)^{1-\gamma}\Bigg]^{1/p}, \\ & (x,y) \in \Omega, \end{aligned} \tag{39}$$

where M_1, θ_1 *are the same as in Theorem* 1, *and*

$$\begin{aligned} & E_0(x,y) = a(x,y), \quad (x,y) \in \Omega_{11}, \\ & E_i(x,y) \\ & \quad = a(x,y) + b(x,y) \sum_{j=1}^{i} \int_{x_{j-1}}^{\alpha(x_j)} \int_{y_{j-1}}^{\beta(y_j)} \big(\alpha^\zeta(x) - t^\zeta\big)^{\gamma-1} t^{\xi-1} \big(\beta^\zeta(y) - s^\zeta\big)^{\gamma-1} s^{\xi-1} f(t,s) \\ & \qquad \times \bigg[\tilde{u}_j^m(t,s) + \int_{x_{j-1}}^{t} \int_{y_{j-1}}^{s} g(\tau,\eta)\tilde{u}_j^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\bigg]^p \,\mathrm{d}s\,\mathrm{d}t \\ & \qquad + \sum_{j=1}^{i} \zeta_j \tilde{u}_j^p(x_j - 0, y_j - 0), \quad (x,y) \in \Omega_{ii}, i = 1,2,3,\ldots, \\ & \tilde{u}_j(x,y) = \Bigg[\bigg(\tilde{a}_{j-1}(x,y) + \frac{\tilde{b}_1(x,y)}{\tilde{e}_{j-1}(\alpha(x),\beta(y))} \\ & \qquad \times \int_{x_{j-1}}^{\alpha(x)} \int_{y_{j-1}}^{\beta(y)} \tilde{h}_{j-1}(t,s)\tilde{a}_{j-1}(t,s)\tilde{e}_{j-1}(t,s)\,\mathrm{d}s\,\mathrm{d}t\bigg)^{1-\gamma}\Bigg]^{1/p}, \quad j = 1,2,3,\ldots, \\ & \tilde{a}_i(x,y) = 3^{\frac{\gamma}{1-\gamma}} A_i^{\frac{1}{1-\gamma}}(x,y), \quad i = 0,1,2,\ldots, \\ & A_i(x,y) = b(x,y)\big(M_1^2 \times \big(\alpha(x)\beta(y)\big)^{\theta_1}\big)^\gamma \bigg[\int_{x_i}^{\alpha(x)} \int_{y_i}^{\beta(y)} B_i^{\frac{1}{1-\gamma}}(t,s)\,\mathrm{d}s\,\mathrm{d}t\bigg]^{1-\gamma}, \\ & \quad i = 0,1,2,\ldots, \\ & B_i(x,y) = f(x,y)\bigg[(1-q) + q\bigg(\frac{m}{p}E_i(x,y) + \frac{p-m}{p}\bigg)\bigg] \\ & \qquad + qf(x,y) \int_{x_i}^{x} \int_{y_i}^{y} g(\tau,\eta)\bigg[\frac{n}{p}E_i(\tau,\eta) + \frac{p-n}{p}\bigg]\,\mathrm{d}\eta\,\mathrm{d}\tau, \quad i = 0,1,2,\ldots, \end{aligned}$$

$$\tilde{b}_1(x,y) = \left(3M_1^2 \times \left(\alpha(x)\beta(y)\right)^{\theta_1}\right)^{\frac{\gamma}{1-\gamma}} b^{\frac{1}{1-\gamma}}(x,y),$$

$$\tilde{e}_i(x,y) = \exp\left(-\int_{x_i}^{x}\int_{y_i}^{y} \tilde{h}_i(t,s)\tilde{b}_1(t,s)\,\mathrm{d}s\,\mathrm{d}t\right), \quad i = 0,1,2,\ldots,$$

$$\tilde{h}_i(x,y) = g_1^{\frac{1}{1-\gamma}}(x,y) + \left(g_2(x,y)\int_{x_i}^{x}\int_{y_i}^{y} g_3(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{\frac{1}{1-\gamma}}, \quad i = 0,1,2,\ldots,$$

$$g_1(x,y) = \frac{mq}{p} f(x,y), \qquad g_2(x,y) = qf(x,y), \qquad g_3(x,y) = \frac{n}{p} g(x,y).$$

(ii) *If $\zeta \in (0,1]$, $\gamma \in (0,\frac{1}{2})$, and $\xi > \frac{1-2\gamma^2}{1-\gamma^2}$, we have*

$$\begin{aligned} u(x,y) \le {} & \Bigg[E_i(x,y) + \Bigg(\tilde{a}_i(x,y) + \frac{\tilde{b}_1(x,y)}{\tilde{e}_i(\alpha(x),\beta(y))} \\ & \times \int_{x_i}^{\alpha(x)}\int_{y_i}^{\beta(y)} \tilde{h}(t,s)\tilde{a}_i(t,s)\tilde{e}_i(t,s)\,\mathrm{d}s\,\mathrm{d}t\Bigg)^{\frac{\gamma}{1+4\gamma}}\Bigg]^{1/p}, \\ & (x,y) \in \Omega, \end{aligned} \tag{40}$$

where M_2, θ_2 are the same as in Theorem 1, E_i, B_i, h_i ($i = 0,1,2,\ldots$) are the same as in (2) of Theorem 3, and

$$\tilde{a}_i(x,y) = 3^{\frac{1+3\gamma}{\gamma}} A_i^{\frac{1+4\gamma}{\gamma}}(x,y), \quad i = 0,1,2,\ldots,$$

$$\tilde{b}_2(x,y) = \left(3M_2^2 \times \left(\alpha(x)\beta(y)\right)^{\theta_2}\right)^{\frac{1+3\gamma}{\gamma}} b^{\frac{1+4\gamma}{\gamma}}(x,y),$$

$$\tilde{e}_i(x,y) = \exp\left(-\int_{x_i}^{x}\int_{y_i}^{y} \tilde{h}_i(t,s)\tilde{b}_2(t,s)\,\mathrm{d}s\,\mathrm{d}t\right), \quad i = 0,1,2,\ldots,$$

$$\begin{aligned} A_i(x,y) = {} & b(x,y)\left(M_2^2 \cdot \left(\alpha(x)\beta(y)\right)^{\theta_2}\right)^{\frac{1+3\gamma}{1+4\gamma}} \left[\int_{x_i}^{\alpha(x)}\int_{y_i}^{\beta(y)} B_i^{\frac{1+4\gamma}{\gamma}}(t,s)\,\mathrm{d}s\,\mathrm{d}t\right]^{\frac{\gamma}{1+4\gamma}}, \\ & i = 0,1,2,\ldots, \end{aligned}$$

$$\begin{aligned} \tilde{u}_j(x,y) = {} & \Bigg[\Bigg(\tilde{a}_{j-1}(x,y) + \frac{\tilde{b}_2(x,y)}{\tilde{e}_{j-1}(\alpha(x),\beta(y))} \\ & \times \int_{x_{j-1}}^{\alpha(x)}\int_{y_{j-1}}^{\beta(y)} \tilde{h}_{j-1}(t,s)\tilde{a}_{j-1}(t,s)\tilde{e}_{j-1}(t,s)\,\mathrm{d}s\,\mathrm{d}t\Bigg)^{\frac{\gamma}{1+4\gamma}}\Bigg]^{1/p}, \\ & j = 1,2,3,\ldots. \end{aligned}$$

Proof If $(x,y) \in \Omega_{11}$, (4) can be restated as

$$\begin{aligned} u^p(x,y) \le {} & a(x,y) + b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \left(\alpha^\zeta(x) - t^\zeta\right)^{\gamma-1} t^{\xi-1}\left(\beta^\zeta(y) - s^\zeta\right)^{\gamma-1} s^{\xi-1} f(t,s) \\ & \times \left[u^m(t,s) + \int_{x_0}^{t}\int_{y_0}^{s} g(\tau,\eta)u^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^q \mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{41}$$

By Lemma 5, we obtain

$$\begin{aligned}&\left[u^m(t,s)+\int_{x_0}^{t}\int_{y_0}^{s}g(\tau,\eta)u^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^q\\&\quad\le q\left[u^m(t,s)+\int_{x_0}^{t}\int_{y_0}^{s}g(\tau,\eta)u^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]+(1-q).\end{aligned}\tag{42}$$

Substituting (42) into (41), we get

$$\begin{aligned}u^p(x,y)&\le a(x,y)+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1}t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1}s^{\xi-1}f(t,s)\\&\quad\times\left[q\left(u^m(t,s)+\int_{x_0}^{t}\int_{y_0}^{s}g(\tau,\eta)u^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right)+(1-q)\right]\mathrm{d}s\,\mathrm{d}t.\end{aligned}\tag{43}$$

Define a function $w(x,y)$ as the second items of the right-hand side of (43), i.e.,

$$\begin{aligned}w(x,y)&=b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1}t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1}s^{\xi-1}(1-q)f(t,s)\,\mathrm{d}s\,\mathrm{d}t\\&\quad+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1}t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1}s^{\xi-1}qf(t,s)u^m(t,s)\,\mathrm{d}s\,\mathrm{d}t\\&\quad+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1}t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1}s^{\xi-1}qf(t,s)\\&\quad\times\int_{x_0}^{t}\int_{y_0}^{s}g(\tau,\eta)u^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}\tag{44}$$

From (43) and (44), we have

$$u^p(x,y)\le a(x,y)+w(x,y)\quad\text{or}\quad u(x,y)\le\left(a(x,y)+w(x,y)\right)^{1/p}.\tag{45}$$

By Lemma 5 and (45), we obtain

$$u^m(x,y)\le\left(a(x,y)+w(x,y)\right)^{m/p}\le\frac{m}{p}\left(a(x,y)+w(x,y)\right)+\frac{p-m}{p},\tag{46}$$

$$u^n(x,y)\le\left(a(x,y)+w(x,y)\right)^{n/p}\le\frac{n}{p}\left(a(x,y)+w(x,y)\right)+\frac{p-n}{p}.\tag{47}$$

Substituting inequalities (46) and (47) into (44), we have

$$\begin{aligned}w(x,y)&\le b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1}t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1}s^{\xi-1}(1-q)f(t,s)\,\mathrm{d}s\,\mathrm{d}t\\&\quad+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1}t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1}s^{\xi-1}qf(t,s)\\&\quad\times\left[\frac{m}{p}\left(a(t,s)+w(t,s)\right)+\frac{p-m}{p}\right]\mathrm{d}s\,\mathrm{d}t\\&\quad+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1}t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1}s^{\xi-1}qf(t,s)\end{aligned}$$

$$
\begin{aligned}
&\times \int_{x_0}^{t}\int_{y_0}^{s} g(\tau,\eta)\left[\frac{n}{p}\bigl(a(\tau,\eta)+w(\tau,\eta)\bigr)+\frac{p-n}{p}\right]\mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}t\\
&= b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{\gamma-1}t^{\xi-1}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{\gamma-1}s^{\xi-1}f(t,s)\\
&\quad\times\left[(1-q)+q\left(\frac{m}{p}a(x,y)+\frac{p-m}{p}\right)\right]\mathrm{d}s\,\mathrm{d}t\\
&\quad+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{\gamma-1}t^{\xi-1}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{\gamma-1}s^{\xi-1}qf(t,s)\\
&\quad\times\int_{x_0}^{t}\int_{y_0}^{s} g(\tau,\eta)\left[\frac{n}{p}a(\tau,\eta)+\frac{p-n}{p}\right]\mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}t\\
&\quad+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{\gamma-1}t^{\xi-1}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{\gamma-1}s^{\xi-1}\frac{mq}{p}f(t,s)w(t,s)\\
&\quad+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{\gamma-1}t^{\xi-1}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{\gamma-1}s^{\xi-1}qf(t,s)\\
&\quad\times\int_{x_0}^{t}\int_{y_0}^{s}\frac{n}{p}g(\tau,\eta)w(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}t\\
&= b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(x^{\zeta}-t^{\zeta}\bigr)^{\gamma-1}t^{\xi-1}\bigl(y^{\zeta}-s^{\zeta}\bigr)^{\gamma-1}s^{\xi-1}B_0(t,s)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{\gamma-1}t^{\xi-1}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{\gamma-1}s^{\xi-1}g_1(t,s)w(t,s)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{\gamma-1}t^{\xi-1}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{\gamma-1}s^{\xi-1}g_2(t,s)\\
&\quad\times\int_{x_0}^{t}\int_{y_0}^{s} g_3(\tau,\eta)w(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned}
$$

that is,

$$
\begin{aligned}
w(x,y)\le{}& b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(x^{\zeta}-t^{\zeta}\bigr)^{\gamma-1}t^{\xi-1}\bigl(y^{\zeta}-s^{\zeta}\bigr)^{\gamma-1}s^{\xi-1}B_0(t,s)\,\mathrm{d}s\,\mathrm{d}t\\
&+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{\gamma-1}t^{\xi-1}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{\gamma-1}s^{\xi-1}g_1(t,s)w(t,s)\,\mathrm{d}s\,\mathrm{d}t\\
&+b(x,y)\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{\gamma-1}t^{\xi-1}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{\gamma-1}s^{\xi-1}g_2(t,s)\\
&\times\int_{x_0}^{t}\int_{y_0}^{s} g_3(\tau,\eta)w(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned}
\tag{48}
$$

where

$$
\begin{aligned}
B_0(x,y)={}&f(x,y)\left[(1-q)+q\left(\frac{m}{p}a(x,y)+\frac{p-m}{p}\right)\right]\\
&+qf(x,y)\int_{x_i}^{x}\int_{y_i}^{y} g(\tau,\eta)\left[\frac{n}{p}a(\tau,\eta)+\frac{p-n}{p}\right]\mathrm{d}\eta\,\mathrm{d}\tau,\quad i=0,1,2,\ldots,\\
g_1(x,y)={}&\frac{mq}{p}f(x,y),\qquad g_2(x,y)=qf(x,y),\qquad g_3(x,y)=\frac{n}{p}g(x,y).
\end{aligned}
$$

Using Hölder's inequality in (48), we have

$$\begin{aligned}
&w(x,y)\\
&\quad\le b(x,y)\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{p_i(\gamma-1)}t^{p_i(\xi-1)}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{p_i(\gamma-1)}s^{p_i(\xi-1)}\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/p_i}\\
&\qquad\times\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}B_0^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/q_i}\\
&\qquad+b(x,y)\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{p_i(\gamma-1)}t^{p_i(\xi-1)}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{p_i(\gamma-1)}s^{p_i(\xi-1)}\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/p_i}\\
&\qquad\times\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}g_1^{q_i}(t,s)w^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/q_i}\\
&\qquad+b(x,y)\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{p_i(\gamma-1)}t^{p_i(\xi-1)}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{p_i(\gamma-1)}s^{p_i(\xi-1)}\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/p_i}\\
&\qquad\times\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\biggl(g_2(t,s)\int_{x_0}^{t}\int_{y_0}^{s}g_3(\tau,\eta)w(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\biggr)^{q_i}\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/q_i}.
\end{aligned}$$

Using Lemma 2 to the first items of the right-hand side above, we have

$$\begin{aligned}
&w(x,y)\\
&\quad\le b(x,y)\bigl(M_i^2\times\bigl(\alpha(x)\beta(y)\bigr)^{\theta_i}\bigr)^{1/p_i}\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}B_0^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/q_i}\\
&\qquad+b(x,y)\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{p_i(\gamma-1)}t^{p_i(\xi-1)}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{p_i(\gamma-1)}s^{p_i(\xi-1)}\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/p_i}\\
&\qquad\times\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}g_1^{q_i}(t,s)w^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/q_i}\\
&\qquad+b(x,y)\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{p_i(\gamma-1)}t^{p_i(\xi-1)}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{p_i(\gamma-1)}s^{p_i(\xi-1)}\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/p_i}\\
&\qquad\times\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\biggl(g_2(t,s)\int_{x_0}^{t}\int_{y_0}^{s}g_3(\tau,\eta)w(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\biggr)^{q_i}\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/q_i}\\
&\quad=A_0(x,y)+\Gamma(x,y)\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}g_1^{q_i}(t,s)w^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/q_i}\\
&\qquad+\Gamma(x,y)\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\biggl(g_2(t,s)\int_{x_0}^{t}\int_{y_0}^{s}g_3(\tau,\eta)w(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\biggr)^{q_i}\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/q_i},
\end{aligned}\tag{49}$$

where

$$\begin{aligned}
&A_0(x,y)=b(x,y)\bigl(M_i^2\times\bigl(\alpha(x)\beta(y)\bigr)^{\theta_i}\bigr)^{\frac{1}{p_i}}\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}B_0^{q_i}(t,s)\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/q_i},\\
&\Gamma(x,y)=b(x,y)\biggl[\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}\bigl(\alpha^{\zeta}(x)-t^{\zeta}\bigr)^{p_i(\gamma-1)}t^{p_i(\xi-1)}\bigl(\beta^{\zeta}(y)-s^{\zeta}\bigr)^{p_i(\gamma-1)}s^{p_i(\xi-1)}\,\mathrm{d}s\,\mathrm{d}t\biggr]^{1/p_i}.
\end{aligned}$$

(49) has the same form as (14) of Theorem 1. Using the same procedure as that in Theorem 1, considering inequality (45), we can get the desired estimations (39) and (40) for $(x,y) \in \Omega_{11}$.

If $(x,y) \in \Omega_{22}$, (4) can be restated as

$$\begin{aligned} u^p(x,y) \le{}& a(x,y) + b(x,y)\int_{x_0}^{\alpha(x_1)}\int_{y_0}^{\beta(y_1)} \left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1} t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1} s^{\xi-1} f(t,s) \\ &\times\left[\tilde{u}_1^m(t,s) + \int_{x_0}^{t}\int_{y_0}^{s} g(\tau,\eta)\tilde{u}_1^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^q \mathrm{d}s\,\mathrm{d}t + \zeta_1\tilde{u}_1^p(x_1-0,y_1-0) \\ &+ b(x,y)\int_{x_1}^{\alpha(x)}\int_{y_1}^{\beta(y)} \left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1} t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1} s^{\xi-1} f(t,s) \\ &\times\left[u^m(t,s) + \int_{x_1}^{t}\int_{y_1}^{s} g(\tau,\eta)u^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^q \mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

Let

$$\begin{aligned} E_1(x,y) ={}& a(x,y) + b(x,y)\int_{x_0}^{\alpha(x_1)}\int_{y_0}^{\beta(y_1)} \left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1} t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1} s^{\xi-1} f(t,s) \\ &\times\left[\tilde{u}_1^m(t,s) + \int_{x_0}^{t}\int_{y_0}^{s} g(\tau,\eta)\tilde{u}_1^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^q \mathrm{d}s\,\mathrm{d}t \\ &+ \zeta_1\tilde{u}_1^p(x_1-0,y_1-0), \quad (x,y)\in\Omega_{22}, \end{aligned}$$

then we get

$$\begin{aligned} u^p(x,y) \le{}& E_1(x,y) + b(x,y)\int_{x_1}^{\alpha(x)}\int_{y_1}^{\beta(y)} \left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1} t^{\xi-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1} s^{\xi-1} f(t,s) \\ &\times\left[u^m(t,s) + \int_{x_1}^{t}\int_{y_1}^{s} g(\tau,\eta)u^n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^q \mathrm{d}s\,\mathrm{d}t, \quad (x,y)\in\Omega_{22}. \end{aligned} \tag{50}$$

Since (50) has the same form as (41), we can conclude that estimates (39) and (40) are valid for $(x,y) \in \Omega_{22}$. Consequently, by using a similar procedure for $(x,y) \in \Omega_{ii}$ $(i = 3,4,5,\dots)$, we complete the proof. □

4 Applications

In this section, let Ω, Ω_{ij} $(i,j = 1,2,3,\dots)$ be the as in the previous section.

- Consider the following Volterra-type retarded weakly singular integral equations:

$$\begin{aligned} u^p(x,y) -{}& \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \left(\alpha^\beta(x)-t^\beta\right)^{\gamma-1} t^{\beta(1+\delta)-1}\left(\beta^\beta(y)-s^\beta\right)^{\gamma-1} s^{\beta(1+\delta)-1} \\ &\times\left[u(t,s) + \int_{x_0}^{t}\int_{y_0}^{s} g(\tau,\eta)u(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right]^q \mathrm{d}s\,\mathrm{d}t = h(x,y), \quad (x,y)\in\Omega, \end{aligned} \tag{51}$$

which arise very often in various problems, especially in describing physical processes with after effect.

Example 1 Let $u(x,y), g(x,y)$, and $h(x,y)$ be continuous functions on Ω, and let $\alpha(x), \beta(y)$ be continuous, differentiable, and increasing functions on $\mathbb{R}^+$ with $\alpha(x) \le x, \beta(y) \le y, \alpha(x_0) = x_0, \beta(y_0) = y_0$. Let $p, q, \zeta, \gamma, \delta$ be positive constants with $p \ge q$. Suppose that $u(x,y)$ satisfies equation (51). Then we have the estimate for $u(x,y)$.

(¡) If $\zeta \in (0,1], \gamma \in (\frac{1}{2}, 1)$, and $\beta(1+\delta) \ge \frac{3}{2} - \gamma$, we have

$$\begin{aligned} \left|u(x,y)\right| \le {} & \Biggl[\left|h(x,y)\right| + \Biggl(\tilde{a}_1(x,y) + \frac{\tilde{b}_1(x,y)}{\tilde{e}_1(\alpha(x),\beta(y))} \\ & \times \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \tilde{h}_1(t,s)\tilde{a}_1(t,s)\tilde{e}_1(t,s)\,\mathrm{d}s\,\mathrm{d}t\Biggr)^{1-\gamma}\Biggr]^{1/p}, \\ & (x,y) \in \Omega, \end{aligned} \tag{52}$$

where M_1, θ_1 are the same as in Theorem 1, and

$$\begin{aligned} \tilde{a}_1(x,y) &= 3^{\frac{\gamma}{1-\gamma}} \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} A_1^{\frac{1}{1-\gamma}}(t,s)\,\mathrm{d}s\,\mathrm{d}t, \\ \tilde{b}_1(x,y) &= \bigl(3M_1^2 \times \bigl(\alpha(x)\beta(y)\bigr)^{\theta_1}\bigr)^{\frac{\gamma}{1-\gamma}}, \\ \tilde{h}_1(x,y) &= A_2^{\frac{1}{1-\gamma}}(x,y) + \biggl(A_3(x,y)\int_{x_0}^{x}\int_{y_0}^{y} A_4(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\biggr)^{\frac{1}{1-\gamma}}, \\ \tilde{e}_1(x,y) &= \exp\biggl(-\int_0^x\int_0^y \tilde{h}_1(t,s)\tilde{b}_1(t,s)\,\mathrm{d}s\,\mathrm{d}t\biggr), \\ A_1(x,y) &= (1-q) + q\biggl(\frac{1}{p}\left|h(x,y)\right| + \frac{p-1}{p}\biggr) \\ &\quad + q\int_0^x\int_0^y \left|g(\tau,\eta)\right|\biggl(\frac{1}{p}\left|h(\tau,\eta)\right| + \frac{p-1}{p}\biggr)\,\mathrm{d}\eta\,\mathrm{d}\tau, \\ A_2(x,y) &= \frac{q}{p}, \qquad A_3(x,y) = q, \qquad A_4(x,y) = \frac{1}{p}\left|g(x,y)\right|. \end{aligned}$$

(¡¡) If $\zeta \in (0,1], \gamma \in (0, \frac{1}{2})$, and $\xi > \frac{1-2\gamma^2}{1-\gamma^2}$, we have

$$\begin{aligned} \left|u(x,y)\right| \le {} & \Biggl[\left|h(x,y)\right| + \Biggl(\tilde{a}_2(x,y) + \frac{\tilde{b}_2(x,y)}{\tilde{e}_2(\alpha(x),\beta(y))} \\ & \times \int_{x_0}^{\alpha(x)} \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} \tilde{h}_2(t,s)\tilde{a}_2(t,s)\tilde{e}_2(t,s)\,\mathrm{d}s\,\mathrm{d}t\Biggr)^{\frac{\gamma}{1+4\gamma}}\Biggr]^{1/p}, \\ & (x,y) \in \Omega, \end{aligned} \tag{53}$$

where M_2, θ_2 are the same as in Theorem 1 and A_1, A_2, A_3, A_4 are the same as in (¡) of Example 1

$$\begin{aligned} \tilde{a}_2(x,y) &= 3^{\frac{1+3\gamma}{\gamma}} \int_{x_0}^{\alpha(x)} \int_{y_0}^{\beta(y)} A_1^{\frac{1+4\gamma}{\gamma}}(t,s)\,\mathrm{d}s\,\mathrm{d}t, \\ \tilde{b}_2(x,y) &= \bigl(3M_2^2 \times \bigl(\alpha(x)\beta(y)\bigr)^{\theta_2}\bigr)^{\frac{1+3\gamma}{\gamma}}, \end{aligned}$$

$$\tilde{h}_2(x,y) = A_2^{\frac{1+4\gamma}{\gamma}}(x,y) + \left(A_3(x,y)\int_{x_0}^{x}\int_{y_0}^{y} A_4(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{\frac{1+4\gamma}{\gamma}},$$

$$\tilde{e}_2(x,y) = \exp\left(-\int_0^x\int_0^y \tilde{h}_2(t,s)\tilde{b}_2(t,s)\,\mathrm{d}s\,\mathrm{d}t\right).$$

Proof From (51), we have

$$\begin{aligned}\left|u(x,y)\right|^p \le \left|h(x,y)\right| + \int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)} \left(\alpha^\zeta(x)-t^\zeta\right)^{\gamma-1} t^{\zeta(1+\delta)-1}\left(\beta^\zeta(y)-s^\zeta\right)^{\gamma-1} s^{\beta(1+\delta)-1} \\ \times \left[\left|u(t,s)\right| + \int_{x_0}^{t}\int_{y_0}^{s}\left|g(\tau,\eta)\right|\left|u(\tau,\eta)\right|\mathrm{d}\eta\,\mathrm{d}\tau\right]^q \mathrm{d}s\,\mathrm{d}t. \qquad (54)\end{aligned}$$

Applying Theorem 3 for $(x,y)\in\Omega_{11}$ (with $m=n=1, \xi=\zeta(1+\delta), a(x,y)=|h(x,y)|, b(x,y)=1$) to (54), we get the desired estimations (52) and (53). □

- Consider the following impulsive differential system:

$$\frac{\partial^2 v(x,y)}{\partial x\,\partial y} = H\left(x,y,v(x,y)\right), \quad (x,y)\in\Omega_{ii}, x\neq x_i, y\neq y_i, \qquad (55)$$

$$\Delta v|_{x=x_i,y=y_i} = \beta_i v(x_i-0,y_i-0), \qquad (56)$$

$$v(x_0,y_0) = v_0,$$

where $(x_i,y_i)<(x_{i+1},y_{i+1})$, $\lim_{i\to\infty} x_i=\infty$, $\lim_{i\to\infty} y_i=\infty$, $v_0>0$ is a constant, $H(x,y,v)$ is nonnegative and continuous on Ω.

Example 2 Suppose that $H(x,y,v)$ satisfies

$$H(x,y,v) \le \left(x^\zeta - t^\zeta\right)^{\gamma-1}\left(y^\zeta - s^\zeta\right)^{\gamma-1} f(x,y)\sqrt{|v|}, \qquad (57)$$

and $f(x,y)\in C(\Omega,\mathbb{R}^+)$, $\zeta\in(0,1]$, $\gamma\in(\frac{1}{2},1)$, then we have

$$\left|v(x,y)\right| \le E_i(x,y) + \left(\tilde{a}_i(x,y) + \frac{\tilde{b}(x,y)}{\tilde{e}_i(x,y)}\int_{x_i}^{x}\int_{y_i}^{y}\tilde{h}(t,s)\tilde{a}_i(t,s)\tilde{e}_i(t,s)\,\mathrm{d}s\,\mathrm{d}t\right)^{1-\gamma},$$
$$(x,y)\in\Omega,$$

where

$$E_0(x,y) = a(x,y), \quad (x,y)\in\Omega_{11},$$

$$E_i(x,y) = a(x,y) + \sum_{j=1}^{i}\int_{x_{j-1}}^{x_j}\int_{y_{j-1}}^{y_j}\left(x^\zeta - t^\zeta\right)^{\gamma-1}\left(y^\zeta - s^\zeta\right)^{\gamma-1} f(t,s)\sqrt{\tilde{u}_j(t,s)}$$
$$+ \sum_{j=1}^{i}\zeta_j u_j(x_j-0,y_j-0), \quad (x,y)\in\Omega_{ii}, i=1,2,3,\ldots,$$

$$\tilde{u}_j(x,y) = \left(\tilde{a}_{j-1}(x,y) + \frac{\tilde{b}(x,y)}{\tilde{e}_{j-1}(x,y)}\int_{x_{j-1}}^{x}\int_{y_{j-1}}^{y}\tilde{h}(t,s)\tilde{a}_{j-1}(t,s)\tilde{e}_{j-1}(t,s)\,\mathrm{d}s\,\mathrm{d}t\right)^{1-\gamma},$$

$$
\begin{aligned}
&j = 1, 2, 3, \ldots, \\
&\tilde{a}_i(x, y) = 2^{\frac{\gamma}{1-\gamma}} A_i^{\frac{1}{1-\gamma}}(x, y), \quad i = 0, 1, 2, \ldots, \\
&A_i(x, y) = b(x, y)\big(M_1^2 \times (xy)^{\theta_1}\big)^{\gamma} \left[\int_{x_i}^{x}\int_{y_i}^{y} B_i^{\frac{1}{1-\gamma}}(t, s)\, \mathrm{d}s\, \mathrm{d}t\right]^{1-\gamma}, \quad i = 0, 1, 2, \ldots, \\
&B_i(x, y) = f(x, y)\left(\frac{1}{2} + \frac{1}{2}E_i(x, y)\right), \quad i = 0, 1, 2, \ldots, \\
&\tilde{b}(x, y) = \big(2M_1^2 \times (xy)^{\theta_1}\big)^{\frac{\gamma}{1-\gamma}}, \\
&\tilde{e}_i(x, y) = \exp\left(-\int_{x_i}^{x}\int_{y_i}^{y} \tilde{h}(t, s)\tilde{b}(t, s)\, \mathrm{d}s\, \mathrm{d}t\right), \quad i = 0, 1, 2, \ldots, \\
&\tilde{h}(x, y) = g_1^{\frac{1}{1-\gamma}}(x, y), \qquad g_1(x, y) = \frac{1}{2}f(x, y), \\
&M_1 = \frac{1}{\zeta}B\left[\frac{1}{\zeta}, \frac{2\gamma - 1}{\gamma}\right], \qquad \theta_1 = \frac{1}{\gamma}\big[\zeta(\gamma - 1)\big] + 1.
\end{aligned}
$$

Proof The impulsive differential system (55) and (56) is equivalent to the integral equation

$$
v(x, y) = v_0 + \int_{x_0}^{x}\int_{y_0}^{y} H\big(t, s, v(t, s)\big)\, \mathrm{d}s\, \mathrm{d}t + \sum_{x_0 < x_i < x, y_0 < y_i < y} \zeta_i v(x_i - 0, y_i - 0). \tag{58}
$$

By using condition (57), from (58) we have

$$
\begin{aligned}
\big|v(x, y)\big| \le{}& v_0 + \int_{x_0}^{x}\int_{y_0}^{y} \big(x^{\zeta} - t^{\zeta}\big)^{\gamma-1}\big(y^{\zeta} - s^{\zeta}\big)^{\gamma-1} f(t, s)\sqrt{\big|v(t, s)\big|}\, \mathrm{d}s\, \mathrm{d}t \\
&+ \sum_{x_0 < x_i < x, y_0 < y_i < y} \zeta_i \big|v(x_i - 0, y_i - 0)\big|.
\end{aligned} \tag{59}
$$

Let $u(x, y) = |v(x, y)|$, from (59) we get

$$
\begin{aligned}
u(x, y) \le{}& v_0 + \int_{x_0}^{x}\int_{y_0}^{y} \big(x^{\beta} - t^{\beta}\big)^{\gamma-1}\big(y^{\beta} - s^{\beta}\big)^{\gamma-1} f(t, s)\sqrt{u(t, s)}\, \mathrm{d}s\, \mathrm{d}t \\
&+ \sum_{x_0 < x_i < x, y_0 < y_i < y} \zeta_i u(x_i - 0, y_i - 0).
\end{aligned} \tag{60}
$$

By Lemma 5, we have

$$
u^{\frac{1}{2}}(x, y) \le \frac{1}{2}u(x, y) + \frac{1}{2}. \tag{61}
$$

Substituting (61) to (60), we get

$$
\begin{aligned}
u(x, y) \le{}& v_0 + \int_{x_0}^{x}\int_{y_0}^{y} \big(x^{\zeta} - t^{\zeta}\big)^{\gamma-1}\big(y^{\zeta} - s^{\zeta}\big)^{\gamma-1} f(t, s)\left(\frac{1}{2}u(t, s) + \frac{1}{2}\right) \mathrm{d}s\, \mathrm{d}t \\
&+ \sum_{x_0 < x_i < x, y_0 < y_i < y} \zeta_i u(x_i - 0, y_i - 0) \\
\le{}& v_0 + \int_{x_0}^{x}\int_{y_0}^{y} \big(x^{\zeta} - t^{\zeta}\big)^{\gamma-1}\big(y^{\zeta} - s^{\zeta}\big)^{\gamma-1} \frac{f(t, s)}{2} u(t, s)\, \mathrm{d}s\, \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{x_0}^{x}\int_{y_0}^{y}\left(x^\zeta - t^\zeta\right)^{\gamma-1}\left(y^\zeta - s^\zeta\right)^{\gamma-1}\frac{f(t,s)}{2}\,\mathrm{d}s\,\mathrm{d}t + \sum_{x_0<x_i<x, y_0<y_i<y} \zeta_i u(x_i - 0, y_i - 0) \\
\le{}& a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y}\left(x^\zeta - t^\zeta\right)^{\gamma-1}\left(y^\zeta - s^\zeta\right)^{\gamma-1}\frac{f(t,s)}{2}u(t,s)\,\mathrm{d}s\,\mathrm{d}t \\
&+ \sum_{x_0<x_i<x, y_0<y_i<y} \zeta_i u(x_i - 0, y_i - 0),
\end{aligned}
$$

that is,

$$
\begin{aligned}
u(x,y) \le{}& a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y}\left(x^\zeta - t^\zeta\right)^{\gamma-1}\left(y^\zeta - s^\zeta\right)^{\gamma-1}\frac{f(t,s)}{2}u(t,s)\,\mathrm{d}s\,\mathrm{d}t \\
&+ \sum_{x_0<x_i<x, y_0<y_i<y} \zeta_i u(x_i - 0, y_i - 0), \qquad (62)
\end{aligned}
$$

where $a(x,y) = v_0 + \int_{x_0}^{x}\int_{y_0}^{y}(x^\zeta - t^\zeta)^{\gamma-1}(y^\zeta - s^\zeta)^{\gamma-1}\frac{f(t,s)}{2}\,\mathrm{d}s\,\mathrm{d}t$. We see that (62) is the particular form of (4), and the functions of (55) satisfy the conditions of Theorem 3. Using the result of Theorem 3, we complete the proof. □

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Availability of data and materials
We declare that the data in the paper can be used publicly.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
YL carried out the main results. RX participated in the revision of Sects. 1, 2, and 4-Applications. All authors read and approved the final manuscript.

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