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A fourth-order accurate difference Dirichlet problem for the approximate solution of Laplace's equation with integral boundary condition

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Abstract

A new constructive method for the finite-difference solution of the Laplace equation with the integral boundary condition is proposed and justified. In this method, the approximate solution of the given problem is defined as a sequence of 9-point solutions of the local Dirichlet problems. It is proved that when the exact solution $u(x,y)$ belongs to the Hölder calsses $C^{4,\lambda}$, $0<\lambda<1$, on the closed solution domain, the uniform estimate of the error of the approximate solution is of order $O(h^4)$, where h is the mesh step. Numerical experiments are given to support analysis made.

Keywords: Finite difference method; Nonlocal integral boundary condition; Laplace's equation; Uniform error

1 Introduction

Different finite-difference problems as approximations of the nonlocal problems with integral boundary condition have been studied by many authors (see [1–5] and references given therein). They all were basically focusing on the following difficulties related to the existence of a quadrature approximation of the integral condition on the side of the domain where nonlocal condition was given: (i) finding an approximate solution by solving the obtained system of equations which are non-band matrices, (ii) determining the rate of convergence of the approximate solution by appropriate smoothness conditions on the given data. In [1], a system of finite difference equations for the Poisson problem has been studied for the spectrum of the matrix to apply an iterative method. Moreover, the author obtained some conditions, under which this system has a unique solution. In [2] and [3], for the error of approximate solution, the order of estimation of $O(h^2)$ in the difference W_2^1 metric is obtained, where h is the mesh step. In [4], the radial basis function collocation technique is used to find an approximate solution of an elliptic equation with nonlocal integral boundary condition. In [5], a finite-difference approximation for the problem with integral boundary conditions is constructed by reducing the given problem to the problem with nonlocal conditions containing derivatives. The authors proved that when the fourth-order partial derivatives of the exact solution are continuous on the closed solu-

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tion domain, the uniform estimate of order $O(h^2|\ln h|)$ is obtained for the error of the approximate solution.

In this paper, we propose and justify a new constructive method to solve a system of nonlocal 9-point finite-difference problem for the Laplace equation with the integral boundary condition. The solution of this nonlocal difference problem is defined as a solution of the 9-point Dirichlet problem by constructing approximate values of the solution on the side where the integral condition was given. Therefore, the approximate solution is obtained by solving a system with 9 diagonal matrices, for the realization of which many fast algorithms have been proposed (see [6, 7]). Moreover, the uniform estimate of the error of approximate solution is of order $O(h^4)$ when the given boundary functions on the sides belong to the Hölder classes $C^{4,\lambda}$, $0 < \lambda < 1$. Finally, numerical experiments are demonstrated to support the theoretical results.

The proposed method with the 5-point scheme was announced in [8].

Other nonlocal boundary value problems are stated and developed in numerous papers (see [9–20] and references therein).

2 Nonlocal boundary value problem

Let

$$R = \{(x,y) : 0 < x < a, 0 < y < b\}$$

be an open rectangle, γ^m, $m = 1,2,3,4$, be its sides including the endpoints, numbered in the clockwise direction, beginning with the side lying on the y-axis, and let $\gamma = \bigcup_{m=1}^{4} \gamma^m$ be the boundary of R and $\overline{R} = R \cup \gamma$. Let C^0 denote the linear space of continuous functions of one variable x on the interval $[0,a]$ of the x-axis, and vanishing at the points $x = 0$ and $x = a$. For a function $f \in C^0$, we define the norm

$$\|f\|_{C^0} = \max_{0\le x\le a} |f(x)|.$$

It is clear that the space C^0 with this norm is complete.

Consider the following nonlocal boundary value problem:

$$\triangle u = 0 \quad \text{on } R, \qquad u = 0 \quad \text{on } \gamma^1 \cup \gamma^3, \qquad u = \tau \quad \text{on } \gamma^2, \tag{1}$$

$$u(x,0) = \alpha \int_{\xi}^{b} u(x,y)\,dy + \mu(x), \quad 0 < x < a, 0 < \xi < b, \tag{2}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial x^2$ is the Laplacian, $\tau = \tau(x)$ and $\mu = \mu(x)$ are given functions which belong to C^0, and α is a given constant which satisfies the following inequality:

$$|\alpha| < \frac{1}{b - \xi}. \tag{3}$$

3 Nonlocal finite-difference problem and its reduction to the Dirichlet problem

We define a square mesh with size $h = \frac{a}{N} = \frac{b}{M^*}$, where $N, M^* > 2$ are integers, constructed with the lines $x, y = h, 2h, \ldots$. Let D_h be the set of nodes of this square grid and let $R_h = R \cap D_h$, $\overline{R}_h = \overline{R} \cap D_h$. We put $\gamma_h^m = \gamma^m \cap D_h$, $m = 1,2,3,4$, and $\gamma_h = \bigcup_{m=1}^{4} \gamma_h^m$.

Let

$$[0,a]_h = \left\{x = x_i, x_i = ih, i = 0,1,\dots,N, h = \frac{a}{N}\right\}$$

be the set of points divided by the step size h on $[0,a]$.

Let C_h^0 be the linear space of grid functions defined on $[0,a]_h$ that vanish at $x = 0$ and $x = a$. The norm of a function $f_h \in C_h^0$ is defined as

$$\|f_h\|_{C_h^0} = \max_{x\in[0,a]_h} |f_h|.$$

We introduce the operator B_h by

$$\begin{aligned} Bu_h(x,y) \equiv{} & \bigl(u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)\bigr)/5 \\ & + \bigl(u(x+h,y+h) + u(x+h,y-h) + \\ & + u(x-h,y+h) + u(x-h,y-h)\bigr)/20. \end{aligned}$$

For the approximate solution of the nonlocal problem (1)–(2), we consider a solution of the following system of difference equations (see [1]):

$$u_h = Bu_h \quad \text{on } R_h, \qquad u_h = 0 \quad \text{on } \gamma_h^1 \cup \gamma_h^3, \qquad u_h = \tau_h \quad \text{on } \gamma_h^2, \tag{4}$$

$$u_h(x,0) = \alpha \sum_{k=1}^{M} \rho_k u_h(x,\eta_k) + \mu_h \quad \text{on } \gamma_h^4, \tag{5}$$

where equation (5) is obtained by approximating the integral in (2) and using Simpson's rule with $\rho_1 = \rho_M = \frac{h}{3}$, $\rho_j = \frac{h}{3}(3 + (-1)^j)$ for $j = 2,3,\dots,M-1$, $\eta_j = \xi + (j-1)h$, $j = 1,2,\dots,M$, $h = \frac{a}{N}$, $(M-1)h + \xi = b$, μ_h is the trace of μ on γ_h^4, and $\frac{\xi}{h}$ is an integer.

We reduce a solution of the nonlocal differential problem to the solution of the local Dirichlet problem.

Let v_h be the solution of the finite-difference Dirichlet problem

$$v_h = Bv_h \quad \text{on } R_h, \qquad v_h = \tau_h \quad \text{on } \gamma_h^2, \qquad v_h = 0 \quad \text{on } \gamma_h/\gamma_h^2, \tag{6}$$

and we put

$$\widetilde{\varphi}_{i,h}(x) = v_h(x,\eta_i), \quad i = 1,2,\dots,M, \tag{7}$$

where τ_h is the trace of τ on γ_h^2.

Let w_h be a solution of the following finite difference Dirichlet problem:

$$w_h = Bw_h \quad \text{on } R_h, \qquad w_h = 0 \quad \text{on } \gamma_h/\gamma_h^4, \qquad w_h = \widetilde{f}_h \quad \text{on } \gamma_h^4, \tag{8}$$

where $\widetilde{f}_h \in C_h^0$, is an arbitrary function.

We define a linear operator B_i^h from C_h^0 to C_h^0 as follows:

$$B_i^h\widetilde{f}_h(x) = w_h(x,\eta_i), \quad i = 1,2,\dots,M, \tag{9}$$

where w_h is the solution of problem (8).

Let

$$w_h^*(x,y) = \frac{1}{b}\|\widetilde{f}_h\|_{C_h^0}(b-y) \quad \text{on } \overline{R}_h.$$

We have

$$\left|w_h(x,y)\right| \le w_h^*(x,y), \quad (x,y) \in \gamma_h. \tag{10}$$

Since $w_h^* = Bw_h^*$ on R_h, from (9)–(10) and by a comparison theorem (see [21, Chap. 4]), we have

$$\left\|B_i^h\widetilde{f}_h\right\|_{C_h^0} \le \|\widetilde{f}_h\|_{C_h^0}\left(1 - \frac{\xi + (i-1)h}{b}\right), \quad i = 1,2,\dots,M, \tag{11}$$

and then for the norm of operator B_i^h, we get

$$\left|B_i^h\right| < 1, \quad i = 1,2,\dots,M. \tag{12}$$

Let

$$\widetilde{\varphi}_h = \alpha\sum_{k=1}^{M}\rho_k\widetilde{\varphi}_{k,h}(x), \quad x \in [0,a]_h, \tag{13}$$

where $\widetilde{\varphi}_{k,h}(x)$ is the function from (7).

In the view of inequality (3), we have

$$|\alpha|\sum_{k=1}^{M}\rho_k = q_0 < 1. \tag{14}$$

Inequalities (12) and (14) yield

$$q_0\left|B_1^h\right| = q < 1. \tag{15}$$

Lemma 1 *A solution of the finite difference problem* (4)–(5) *can be represented as*

$$u_h = v_h + w_h, \tag{16}$$

where v_h is the solution of problem (6) *and w_h is the solution of problem* (8) *with $\widetilde{f}_h$ being a solution of the following nonlinear equation*:

$$\widetilde{f}_h = \widetilde{\varphi}_h + \mu_h + \alpha\sum_{k=1}^{M}\rho_k B_k^h\widetilde{f}_h \quad \text{on } \gamma_h^4. \tag{17}$$

Proof According to (4), (6), and (8), relation (16) holds on R_h and the boundary sides γ_h^m, $m = 1,2,3$.

From (13) and (17), it follows that

$$\widetilde{f}_h = \mu_h + \alpha \sum_{k=1}^{M} \rho_k \big[\widetilde{\varphi}_{k,h}(x) + B_k^h \widetilde{f}_h\big] \quad \text{on } \gamma_h^4.$$

Relying on (7) and (9), we have

$$\widetilde{f}_h = \mu_h + \alpha \sum_{k=1}^{M} \rho_k \big[v_h(x,\eta_i) + w_h(x,\eta_i)\big] \quad \text{on } \gamma_h^4.$$

By virtue of (6) and (8), we obtain

$$v_h(x,0) + w_h(x,0) = \mu_h + \alpha \sum_{k=1}^{M} \rho_k \big[v_h(x,\eta_i) + w_h(x,\eta_i)\big] \quad \text{on } \gamma_h^4.$$

Due to (5), this shows that relation (16) is also satisfied on γ_h^4. □

Thus, the unknown function on γ_h^4 in problem (8) is a solution of the nonlinear equation (17).

Theorem 2 *There exists a unique solution $\widetilde{f}_h$ of the nonlinear equation* (17).

Proof Consider the following sequences in C_h^0:

$$\widetilde{\psi}_{i,h}^0 = 0, \qquad \widetilde{\psi}_{i,h}^n = B_i^h \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}^{n-1}\right),$$
$$i = 1,2,\ldots,M; n = 1,2,\ldots. \tag{18}$$

From this, for the positive integers m and n with $m > n$, we get

$$\widetilde{\psi}_{i,h}^m - \widetilde{\psi}_{i,h}^n = B_i^h \left(\alpha \sum_{k=1}^{M} \rho_k \big(\widetilde{\psi}_{k,h}^{m-1} - \widetilde{\psi}_{k,h}^{n-1}\big)\right), \quad i = 1,2,\ldots,M.$$

Applying inequality (11), we reach

$$\big\|\widetilde{\psi}_{i,h}^m - \widetilde{\psi}_{i,h}^n\big\|_{C_h^0} \le q \big\|\widetilde{\psi}_{i,h}^{m-1} - \widetilde{\psi}_{i,h}^{n-1}\big\|_{C_h^0}, \tag{19}$$

where q is defined by (15). In a similar way, from (19) we obtain

$$\big\|\widetilde{\psi}_{i,h}^m - \widetilde{\psi}_{i,h}^n\big\|_{C_h^0} \le q^{n+1} \frac{1-q^{m-n}}{1-q} \big(\|\widetilde{\varphi}_h\|_{C_h^0} + \|\mu_h\|_{C_h^0}\big),$$

which shows that sequences (18) are Cauchy. Since C_h^0 is complete, there are limits

$$\lim_{n\to\infty} \widetilde{\psi}_{i,h}^n = \widetilde{\psi}_{i,h} \in C_h^0, \quad i = 1,2,\ldots,M.$$

By using (11) and (15),

$$\lim_{n\to\infty} B_k^h \widetilde{\psi}_{i,h}^n = B_k^h \widetilde{\psi}_{i,h} \in C_h^0, \quad i,k = 1,2,\ldots,M. \tag{20}$$

Using (20) and taking the limit of (18) as $n \to \infty$, we have

$$\widetilde{\psi}_{i,h} = B_i^h \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h} \right), \quad i = 1,2,\ldots,M. \tag{21}$$

We multiply both sides of equation (21) by $\alpha\rho_i$ and sum over $i = 1,2,\ldots,M$ to get

$$\widetilde{\varphi}_h + \mu_h + \alpha \sum_{i=1}^{M} \rho_i \widetilde{\psi}_{i,h} = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{i=1}^{M} \rho_i B_i^h \left(\widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h} \right). \tag{22}$$

In view of relations (17) and (22), we obtain a solution of the nonlinear equation (17) as

$$\widetilde{f}_h = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}.$$

To show the uniqueness, let $\widetilde{f}_{h,p} \in C_h^0$, $p = 1,2$, be two functions satisfying relation (17). Then, we obtain the following inequality:

$$\|\widetilde{f}_{h,1} - \widetilde{f}_{h,2}\|_{C_h^0} = \left\| \alpha \sum_{k=1}^{m} \rho_k B_k^h (\widetilde{f}_{h,1} - \widetilde{f}_{h,2}) \right\|_{C_h^0} \le q \|\widetilde{f}_{h,1} - \widetilde{f}_{h,2}\|_{C_h^0},$$

where $0 < q < 1$ is defined by (15). Hence $\widetilde{f}_{h,1} = \widetilde{f}_{h,2}$. □

4 Convergence of the finite-difference problem

We say that $F \in C^{k,\lambda}(E)$, if F has kth derivatives on E satisfying Hölder condition with exponent λ. We assume that $\tau(x)$ and $\mu(x)$ in (1) and (2) are from $C^{4,\lambda}$, $0 < \lambda < 1$, on γ^2 and γ^4, respectively, and $\tau^{(2m)}(0) = \tau^{(2m)}(a) = 0$, $\mu^{(2m)}(0) = \mu^{(2m)}(a) = 0$, $m = 0,1,2$. By using the nth iteration $\widetilde{\psi}_{i,h}^n$, $n \ge 1$ of (18), we define the function

$$\widetilde{f}_h^n = \widetilde{\varphi}_h + \mu_h + \alpha \sum_{k=1}^{M} \rho_k \widetilde{\psi}_{k,h}^n. \tag{23}$$

Hence, for the approximate solution of the nonlocal problem (1)–(2), we define the following difference problem:

$$\widetilde{u}_h^n = B_h \widetilde{u}_h^n \quad \text{on } R_h, \qquad \widetilde{u}_h^n = \tau_h \quad \text{on } \gamma_h^2, \qquad \widetilde{u}_h^n = 0 \quad \text{on } \gamma_h^1 \cup \gamma_h^3, \tag{24}$$

$$\widetilde{u}_h^n = \widetilde{f}_h^n \quad \text{on } \gamma_h^4. \tag{25}$$

Theorem 3 *The following estimate holds*:

$$\max_{(x,y)\in \overline{R}_h} \left| \widetilde{u}_h^n - u \right| \le c_1 h^4 + q_0 \frac{q_1^{n+1}}{1 - q_1} c^*, \tag{26}$$

where $\widetilde{u}_h^n$ is a solution of problem (24)–(25), u is the exact solution of nonlocal boundary value problem (1)–(2), c_1 and c^* are constants independent of h, q_0 is defined by (14), and $q_1 = 1 - \frac{\xi}{b}$.

Proof Let U be the exact solution of the system of the following problem:

$$\triangle U = 0 \quad \text{on } R, \qquad U = \tau \quad \text{on } \gamma^2, \qquad U = 0 \quad \text{on } \gamma^1 \cup \gamma^3, \tag{27}$$

$$U(x,0) = \alpha \sum_{k=1}^{M} \rho_k U(x,\eta_k) + \mu(x), \quad 0 \le x \le a. \tag{28}$$

Let V be a solution of the Dirichlet problem

$$\triangle V = 0 \quad \text{on } R, \qquad V = \tau \quad \text{on } \gamma^2, \qquad V = 0 \quad \text{on } \gamma/\gamma^2, \tag{29}$$

and denote by

$$\varphi_k(x) = V(x,\eta_k) \quad \text{for } k = 1,2,\ldots,M, \tag{30}$$

where $\eta_k = \xi + (k-1)h$, $k = 1,2,\ldots,M$. We define the function

$$\varphi = \alpha \sum_{k=1}^{M} \rho_k \varphi_k. \tag{31}$$

Consider the Dirichlet problem

$$\triangle W = 0 \quad \text{on } R, \qquad W = 0 \quad \text{on } \gamma/\gamma^4, \qquad W = f \quad \text{on } \gamma^4, \tag{32}$$

where f is an unknown function from C^0. The linear operator $B_i : C^0 \to C^0$ is defined as

$$B_i f(x) = W(x,\eta_i) \in C^0, \quad i = 1,2,\ldots,M.$$

Then following inequality holds for the norm $|B_i|$:

$$|B_i| < \left(1 - \frac{\xi + (i-1)h}{b}\right), \quad i = 1,2,\ldots,M.$$

By analogy with the results in [18], it is shown that a solution U of problem (27)–(28) can be represented as $U = V + W$ where V and W are the solutions of problem (29) and (32), respectively, when f is defined by

$$f = \varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k. \tag{33}$$

Here the functions $\psi_1, \psi_2, \ldots, \psi_M$ are from C^0, and are defined as the solutions of the nonlinear equations

$$\psi_i = B_i\left(\varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k\right), \quad i = 1,2,\ldots,M. \tag{34}$$

Therefore, the nonlocal problem (27)–(28) is reduced to the following Dirichlet problem:

$$\triangle U = 0 \quad \text{on } R, \qquad U = \tau \quad \text{on } \gamma^2, \qquad U = 0 \quad \text{on } \gamma^1 \cup \gamma^3, \tag{35}$$

$$U(x,0) = f, \quad 0 \le x \le a, \tag{36}$$

where f is defined by (33). The solution ψ_i, $i = 1,2,\ldots,M$, of system (34) is found as a limit of the infinite sequence of functions $\{\psi_i^n\}_{n=0}^{\infty}$ in C^0 defined by

$$\psi_i^0 = 0, \qquad \psi_i^n = B_i\left(\varphi + \mu + \alpha \sum_{k=1}^{M} \rho_k \psi_k^{n-1}\right),$$
$$i = 1,2,\ldots,M; n = 1,2,\ldots. \tag{37}$$

Since $\tau(x)$ in (29) belongs to $C^{4,\lambda}(\gamma^2)$ and $\tau^{(2m)}(0) = \tau^{(2m)}(a) = 0$, $m = 0,1,2$, it follows from [22] that

$$\max_{(x,y)\in \overline{R}_h} |v_h - V_h| \le c_2 h^4, \tag{38}$$

where v_h is a solution of problem (6), V_h is the trace of the solution of (29) on $\overline{R}_h$ and c_2 is a constant independent of h. Let φ_h, $\psi_{i,h}$, and $\psi_{i,h}^n$ be the trace of φ, ψ_i, and ψ_i^n on $[0,a]_h$, respectively, and let $(B_i(F))_h$ be the trace of $B_i(F)$ on $[0,a]_h$ for any function $F \in C^{4,\lambda}[0,a]$. By (7), (13), (30), (31), and (38), we obtain

$$\|\widetilde{\varphi}_h - \varphi_h\|_{C_h^0} \le c_3 h^4, \tag{39}$$

where c_3 is a constant independent of h. By using (18) and (37), we have, for all $i = 1,2,\ldots,M$,

$$\begin{aligned}\left\|\widetilde{\psi}_{i,h}^1 - \psi_{i,h}^1\right\|_{C_h^0} \le{}& \left\|B_i^h(\widetilde{\varphi}_h - \varphi_h)\right\|_{C_h^0} \\ &+ \left\|B_i^h(\varphi_h + \mu_h) - \left(B_i(\varphi + \mu)\right)_h\right\|_{C_h^0}.\end{aligned} \tag{40}$$

Applying (11) and (39), it follows that

$$\left\|B_i^h(\widetilde{\varphi}_h - \varphi_h)\right\|_{C_h^0} \le c_4 h^4, \quad i = 1,2,\ldots,M, \tag{41}$$

where c_4 is a constant independent of h. Similar to inequality (38), we have

$$\left\|B_i^h(\varphi_h + \mu_h) - \left(B_i(\varphi + \mu)\right)_h\right\|_{C_h^0} \le c_5 h^4, \tag{42}$$

where c_5 is a constant independent of h. From the relations (40)–(42), we have

$$\left\|\widetilde{\psi}_{i,h}^1 - \psi_{i,h}^1\right\|_{C_h^0} \le c_6 h^4, \tag{43}$$

where c_6 is a constant independent of h. For $n \geq 2$, we have

$$\begin{aligned}\|\widetilde{\psi}_{i,h}^n - \psi_{i,h}^n\|_{C_h^0} = \Bigg\| & B_i^h\Bigg(\widetilde{\varphi}_h + \mu_h + \alpha\sum_{k=1}^M \rho_k \widetilde{\psi}_{k,h}^{n-1}\Bigg) \\ & - \Bigg(B_i\Bigg(\varphi + \mu + \alpha\sum_{k=1}^M \rho_k \psi_k^{n-1}\Bigg)\Bigg)_h\Bigg\|_{C_h^0}.\end{aligned}$$

Then,

$$\begin{aligned}\|\widetilde{\psi}_{i,h}^n - \psi_{i,h}^n\|_{C_h^0} \leq{} & \|B_i^h(\widetilde{\varphi}_h + \mu_h) - (B_i(\varphi+\mu))_h\|_{C_h^0} \\ & + \Bigg\|B_i^h\Bigg(\alpha\sum_{k=1}^M \rho_k\widetilde{\psi}_{k,h}^{n-1} - \alpha\sum_{k=1}^M \rho_k \psi_k^{n-1}\Bigg)\Bigg\|_{C_h^0} \\ & + \Bigg\|B_i^h\Bigg(\alpha\sum_{k=1}^M \rho_k \psi_k^{n-1}\Bigg) - \Bigg(B_i\Bigg(\alpha\sum_{k=1}^M \rho_k\psi_k^{n-1}\Bigg)\Bigg)_h\Bigg\|_{C_h^0}, \\ i = 1,2,\ldots,M. &\end{aligned} \tag{44}$$

By analogy with (54) in [20], it follows that

$$\max_{1\leq k\leq M}\|B_i^h\psi_k^{n-1} - (B_i\psi_k^{n-1})_h\|_{C_h^0} \leq c_7 h^4, \tag{45}$$

where c_7 is a constant independent of h. From (45), we find that

$$\begin{aligned}&\Bigg\|B_i^h\Bigg(\alpha\sum_{k=1}^M \rho_k\psi_k^{n-1}\Bigg) - \Bigg(B_i\Bigg(\alpha\sum_{k=1}^M \rho_k\psi_k^{n-1}\Bigg)\Bigg)_h\Bigg\|_{C_h^0} \\ &\quad\leq \sum_{k=1}^M |\alpha\rho_k|\,\|B_i^h\psi_k^{n-1} - (B_i\psi_k^{n-1})_h\|_{C_h^0} \\ &\quad\leq c_8 h^4,\end{aligned} \tag{46}$$

where $c_8 = |\alpha|(b-\xi)c_7$. In the view of (11), (14), (42), (44), and (46), we have

$$\|\widetilde{\psi}_{i,h}^n - \psi_{i,h}^n\|_{C_h^0} \leq c_9 h^4 + q_0\|\widetilde{\psi}_{i,h}^{n-1} - \psi_{i,h}^{n-1}\|_{C_h^0}, \tag{47}$$

where q_0 is defined by (14) and c_9 is a constant independent of h. By virtue of (43) and (47), we obtain

$$\|\widetilde{\psi}_{i,h}^n - \psi_{i,h}^n\|_{C_h^0} \leq c_{10}h^4\big(1 + q_0 + q_0^2 + \cdots + q_0^{n-1}\big) \leq c_{11}h^4, \tag{48}$$

where c_{10} and c_{11} are constants independent of h. According to (37), it follows that

$$\|\psi_i^1\|_{C^0} \leq \bigg(1 - \frac{\xi}{b}\bigg)\big(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\big), \tag{49}$$

$$\left\|\psi_i^n-\psi_i^{n-1}\right\|_{C^0}\le|B_i||\alpha|\sum_{k=1}^{M}|\rho_k|\left\|\psi_i^{n-1}-\psi_i^{n-2}\right\|_{C^0},\quad i=1,2,\dots,M,\tag{50}$$

where φ is defined by (31). From (49) and (50), we have

$$\left\|\psi_i^n-\psi_i^{n-1}\right\|_{C^0}\le q_1^n\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right),\quad i=1,2,\dots,M,$$

where $q_1=1-\frac{\xi}{b}$. Moreover, for any $m=1,2,\dots,$ we obtain

$$\left\|\psi_i^{n+m}-\psi_i^n\right\|_{C^0}\le q_1^{n+1}\left(\frac{1-q_1^m}{1-q_1}\right)\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right),\quad i=1,2,\dots,M.\tag{51}$$

Since

$$\left\|\psi_i^n-\psi_i\right\|_{C^0}\le\left\|\psi_i^{n+m}-\psi_i^n\right\|_{C^0}+\left\|\psi_i^{n+m}-\psi_i\right\|_{C^0},\quad i=1,2,\dots,M,\tag{52}$$

by taking the limit as $m\to\infty$, from (51) and (52), it follows that

$$\left\|\psi_i^n-\psi_i\right\|_{C^0}\le\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right),\quad i=1,2,\dots,M.\tag{53}$$

From (48) and (53), we have

$$\left\|\widetilde{\psi}_{i,h}^n-\psi_{i,h}\right\|_{C_h^0}\le c_{11}h^4+\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right),\quad i=1,2,\dots,M.\tag{54}$$

Let $U_h(x,y)$ be the solution of the system of grid equations

$$U_h=B_hU_h\quad\text{on }R_h,\qquad U_h=\tau\quad\text{on }\gamma_h^2,\qquad U_h=0\quad\text{on }\gamma_h^1\cup\gamma_h^3,\tag{55}$$

$$U_h=f_h\quad\text{on }\gamma_h^4,\tag{56}$$

which approximates problem (35)–(36) when f_h is the trace of f on $[0,a]_h$. Since τ, μ, φ, and ψ_i, $i=1,2,\dots,M$, belong to $C^{4,\lambda}$, $0<\lambda<1$, on the interval $0\le x\le 1$, and the $(2m)$th order derivatives vanish at the endpoints for $m=0,1,2$ (see [20]), by [22], we have

$$\max_{(x,y)\in\overline{R}_h}|U_h-U|\le c_{12}h^4,\tag{57}$$

where U is the solution of problem (35)–(36) and c_{12} is a constant independent of h. In view of inequalities (39) and (54), we obtain

$$\left\|\widetilde{f}_h^n-f_h\right\|_{C_h^0}\le c_{13}h^4+q_0\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right),\tag{58}$$

where q_0 is defined by (14) and c_{13} is a constant independent of h. By the grid maximum principle and from (58), we have

$$\max_{(x,y)\in\overline{R}_h}\left|\widetilde{u}_h^n-U_h\right|\le c_{13}h^4+q_0\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0}+\|\mu\|_{C^0}\right),\tag{59}$$

where $\widetilde{u}_h^n$ is the solution of problem (24)–(25) and U_h is the solution of problem (55)–(56). According to estimates (57) and (59), the following inequality holds:

$$\max_{(x,y)\in\overline{R}_h} \left|\widetilde{u}_h^n - U\right| \le c_{14}h^4 + q_0\frac{q_1^{n+1}}{1-q_1}\left(\|\varphi\|_{C^0} + \|\mu\|_{C^0}\right), \tag{60}$$

where U is the solution of problem (35)–(36) and c_{14} is a constant independent of h.

Using the estimate (60) and by the maximum principle for the Laplace equation with the truncation error of Simpson's rule, which is order of $O(h^4)$, we obtain the final estimate

$$\begin{aligned}\max_{(x,y)\in\overline{R}_h} \left|\widetilde{u}_h^n - u\right| &\le \max_{(x,y)\in\overline{R}_h} \left|\widetilde{u}_h^n - U\right| + \max_{(x,y)\in\overline{R}_h} |U-u| \\ &\le c_1h^4 + q_0\frac{q_1^{n+1}}{1-q_1}c^*, \end{aligned}\tag{61}$$

where u is the solution of problem (1)–(2), c_1 is a constant independent of h, and $c^* = \|\varphi\|_{C^0} + \|\mu\|_{C^0}$. □

Remark 4 In (61), the right-hand side is of order $O(h^4)$, when

$$\frac{q_1^{n+1}}{1-q_1} \approx h^4. \tag{62}$$

From (62) it follows that

$$n = \max\left\{\left[\frac{\ln h^4(1-q_1)}{\ln q_1}\right], 1\right\},$$

where $[a]$ is the integer part of a.

5 Numerical experiments

Let

$$R = \{(x,y) : 0 < x < 1, 0 < y < 2\}.$$

Problem 1

$$\begin{aligned} &\triangle u = 0 \quad \text{on } R, \qquad u(0,y) = u(1,y) = 0, \quad 0 \le y \le 2, \\ &u(x,2) = 100e^{-\pi}\sin\pi x, \quad 0 \le x \le 1, \\ &u(x,0) = \frac{1}{400}\int_{\frac{1}{8}}^{2} u(x,y)\,dy, \quad 0 < x < 1. \end{aligned}$$

Problem 2

$$\begin{aligned} &\triangle u = 0 \quad \text{on } R, \qquad u(0,y) = u(1,y) = 0, \quad 0 \le y \le 2, \\ &u(x,2) = x^{\frac{121}{30}}\left(\tan^{-1}x - \frac{\pi}{4}\right), \quad 0 \le x \le 1, \\ &u(x,0) = \frac{1}{250}\int_{\frac{1}{4}}^{2} u(x,y)\,dy, \quad 0 < x < 1. \end{aligned}$$

Table 1 Solutions on the line $y = 0$ of Problem 1

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
1.06874E–003	1.06873E–003	1.06874E–003	1.06877E–003
2.09641E–003	2.09639E–003	2.09641E–003	2.09647E–003
3.04351E–003	3.04350E–003	3.04352E–003	3.04361E–003
3.87366E–003	3.87364E–003	3.87366E–003	3.87378E–003
4.55494E–003	4.55491E–003	4.55495E–003	4.55508E–003
5.06118E–003	5.06115E–003	5.06119E–003	5.06134E–003
5.37292E–003	5.37289E–003	5.37293E–003	5.37309E–003
5.47818E–003	5.47815E–003	5.47819E–003	5.47835E–003
5.37292E–003	5.37289E–003	5.37293E–003	5.37309E–003
5.06118E–003	5.06115E–003	5.06119E–003	5.06134E–003
4.55494E–003	4.55491E–003	4.55495E–003	4.55508E–003
3.87366E–003	3.87364E–003	3.87366E–003	3.87378E–003
3.04351E–003	3.04350E–003	3.04352E–003	3.04361E–003
2.09641E–003	2.09639E–003	2.09641E–003	2.09647E–003
1.06874E–003	1.06873E–003	1.06874E–003	1.06877E–003

Table 2 Solutions on the line $y = 0$ of Problem 2

$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
−2.69158E–006	−2.68953E–006	−2.68153E–006	−2.64961E–006
−5.51443E–006	−5.51067E–006	−5.49500E–006	−5.43245E–006
−8.61713E–006	−8.61191E–006	−8.58921E–006	−8.49847E–006
−1.21399E–005	−1.21335E–005	−1.21047E–005	−1.19893E–005
−1.61725E–005	−1.61651E–005	−1.61313E–005	−1.59957E–005
−2.07100E–005	−2.07019E–005	−2.06644E–005	−2.05138E–005
−2.56153E–005	−2.56069E–005	−2.55671E–005	−2.54074E–005
−3.05943E–005	−3.05858E–005	−3.05454E–005	−3.03827E–005
−3.51857E–005	−3.51775E–005	−3.51378E–005	−3.49784E–005
−3.87703E–005	−3.87626E–005	−3.87253E–005	−3.85752E–005
−4.06024E–005	−4.05955E–005	−4.05620E–005	−4.04271E–005
−3.98689E–005	−3.98629E–005	−3.98345E–005	−3.97198E–005
−3.57837E–005	−3.57787E–005	−3.57564E–005	−3.56664E–005
−2.77381E–005	−2.77337E–005	−2.77183E–005	−2.76563E–005
−1.55532E–005	−1.55474E–005	−1.55393E–005	−1.55078E–005

The exact solutions of Problems 1 and 2 are unknown. The approximate values of Problems 1 and 2 on the line $y = 0$ obtained by the proposed method are given in Tables 1 and 2, respectively. According to repeated digits, for the decreasing mesh steps $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$, it follows that the maximum error on this line decreases as $O(h^4)$. To obtain these results, 14 iterations are run for the construction of $\widetilde{f}_h^n$ with the successive error which is less than 10^{-16}.

Problem 3

$$\triangle u = 0 \quad \text{on } R, \qquad u(0,y) = u(1,y) = 0, \quad 0 \le y \le 2,$$
$$u(x,2) = e^{2\pi} \sin \pi x, \quad 0 \le x \le 1,$$
$$u(x,0) = \frac{1}{100} \int_{\frac{1}{16}}^{2} u(x,y)\,dy + \mu(x), \quad 0 < x < 1,$$

where $u = e^{\pi y} \sin \pi x$ is the exact solution, $\mu(x) = [1 + \frac{\alpha}{\pi}(1 - e^{2\pi})] \sin \pi x$.

Table 3 Maximum errors for the solution of Problem 3

h	Max error	Order of reduction
1/16	$1.40629393 \times 10^{-9}$	
1/32	$8.77042882 \times 10^{-11}$	16.03449
1/64	$5.47739631 \times 10^{-12}$	16.01203
1/128	$3.42279360 \times 10^{-13}$	16.00270

Table 4 CPU times (in seconds) for Problem 1

h	Discrete Fourier	Gauss–Seidel with reducing	Gauss–Seidel without reducing
1/16	0.10125	0.13325	0.65250
1/32	1.58375	2.27125	6.70625
1/64	19.87500	25.15375	81.11175
1/128	284.72625	467.22025	1325.14725

Table 5 CPU times (in seconds) for Problem 2

h	Discrete Fourier	Gauss–Seidel with reducing	Gauss–Seidel without reducing
1/16	0.19115	0.23565	0.71300
1/32	2.00135	3.97115	8.12375
1/64	26.6875	37.35625	90.72425
1/128	355.62775	580.22315	1798.54315

Table 6 CPU times (in seconds) for Problem 3

h	Discrete Fourier	Gauss–Seidel with reducing	Gauss–Seidel without reducing
1/16	0.11375	0.12125	0.62500
1/32	1.28437	2.18375	5.78125
1/64	17.96875	24.35625	79.23375
1/128	278.82815	443.0125	1243.84875

In Table 3 for Problem 3, the maximum error for each step $h = \frac{1}{2^k}$, $k = 4, 5, 6, 7$ and the reduction orders are given. From the third column it follows that the convergence order is $O(h^4)$.

In Tables 4, 5, and 6, the results of the CPU times (in seconds), when solving Problems 1, 2, and 3, respectively, are given. In columns 2 and 3, the CPU times for the realization of the proposed approaches by the discrete Fourier method and by the Gauss–Seidel method are given. For the construction of the local function $\widetilde{f}_h^n$ for Problems 1 and 2, just 14 iterations are used. Problem 3 needs 11 iterations. In column 4, the Gauss–Seidel method is used to solve the given problems without reducing to the Dirichlet problem. From these results it follows that the discrete Fourier method, which cannot be used on the problem without reducing to the Dirichlet problem, is faster than others. The third and fourth columns show that for the method which is applicable for both approaches (as Gauss–Seidel), the CPU times with reducing are less than the CPU times without reducing to the Dirichlet problem.

As it follows from Tables 4–6, the CPU times for Problems 1 and 3 in Tables 4 and 6 are less than those for Problem 2 in Table 5. This takes place because of low smoothness of the boundary function in Problem 2.

6 Conclusion

A new constructive method for the approximate solution of the nonlocal boundary value for Laplace's equation with integral boundary condition is given. In the proposed method, the system of finite-difference equations is defined as the 9-point solution of the Dirichlet problem by constructing the function on the side of the rectangle where the nonlocal boundary condition was given. This function is defined by using the nth term of the convergent simplest fixed point iteration (18) for the solution of the nonlinear system of (21). A uniform estimate for the error of the approximate solution of the nonlocal problem by using the nth term for $n = \max\{[(\ln h^4(1-q_1))/\ln q_1], 1\}$ is of order $O(h^4)$, where h is the step size.

The proposed method gives an opportunity to solve nonlocal problems by using different fast algorithms constructed for the local Dirichlet problem by many authors (see [6] and the references therein).

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