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# Almost sure exponential stabilization of neural networks by aperiodically intermittent control based on delay observations

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## Abstract

This paper is concerned with almost sure exponential stabilization of neural networks by intermittent control based on delay observations. By the stochastic comparison principle and Itô's formula, a sufficient criterion is derived, under which unstable neural networks can be stabilized by stochastic intermittent control based on delay observations. The range of intermittent rate is given, and the upper bound of time delay can be solved from a transcend equation. Finally, two examples are provided to demonstrate the feasibility and validity of our proposed methods.

**Keywords:** Exponential stabilization; Intermittent control; Itô's formula; Delay observations

## 1 Introduction

In the past decade, the Hopfield neural networks have been thoroughly investigated. Nowadays they are widely applied in several areas, such as image processing, communication engineering, and optimization. These applications mainly depend on the asymptotic behavior of the neural networks [1–5], especially their stability, and therefore more and more authors study the stability of neural networks. The readers can refer to [6–10] and their references.

Actually, many practical problems are related to the unstable neural networks, which cannot be directly applied in engineering unless they are stabilized in advance. Hence, various control strategies have been proposed in order to stabilize the unstable neural networks, including intermittent control [11–15], pinning control [16], impulsive control [17], finite-time control [18], and adaptive control [19]. Meanwhile, noise disturbance is ubiquitous in the real world. As a result, an increasing number of authors have revealed the positive impact of the white noise on the systems in recent decades [20–24]. For example, Mao showed that noise can suppress an explosive solution for population systems in [25]. Also, some scholars have paid attention to the stochastic stabilization for neural networks. Shen and Wang have utilized the white noise to stabilize the unstable networks in [26]. The readers can refer to [27–29] for more details.

In the past decades, more and more scholars have realized that there is time delay  $\tau$  between the observation time of the state and the arrival time of the feedback control. Thus, time-delay feedback control has attracted more and more researchers' attention

[30–34]. It was Guo and Mao who first integrated the delay feedback control strategy with the stochastic stabilization theory. Guo and Mao showed that the differential system is stabilized by delay feedback control provided the time delay is no more than an upper bound in [22]. Nevertheless, there are distinct characteristics in neural networks. It is significant to investigate the stochastic delay stabilization for neural networks based on the network characteristics. To the best of our knowledge, the stochastic delay stabilization for neural networks has scarcely been investigated yet. Accordingly, tackling this issue constitutes the first motivation of this paper.

Moreover, the intermittent control strategy has attracted some scholars [11–13, 35–40]. The networks are controlled by white noise during working time, and the white noise is removed from the networks during rest time. The corresponding controlled system can be regarded as a switching of a closed-loop subsystem during working time and an open-loop subsystem during rest time. Intermittent control has its advantages over classic continuous control. We cut costs by reducing the excessive wear of the controller due to long time work. Presently there are several results applying intermittent control to networks [11, 12, 37]. Then a question arises naturally: Can we integrate the intermittent control strategy with stochastic delay stabilization strategy? To date, there are few results available on this topic since the simultaneous presence of the delay feedback control and intermittent noise complicates the problem. As a result, the proposed approaches in existing literature cannot be directly adopted. Thus, overcoming the difficulties stemming from delay feedback control and intermittent noise is the second motivation.

Summarizing the above statements, this paper focuses on stochastic intermittent stabilization based on delay feedback control for neural networks. Sufficient conditions for almost sure exponential stabilization are obtained provided that the time delay is bounded by  $\tau_0$  and intermittent rate  $\phi$  satisfies  $2\lambda_+(-D + |\bar{A}|K) < \sigma^2(1 - \phi)$  ( $D, \bar{A}, K$ , and  $\sigma$  will be defined in Sect. 2, see [41] for more details). The main contribution of this paper lies in three aspects as follows. (1) The stochastic delay stabilization for neural networks has been investigated based on neural networks characteristic. (2) By the stochastic comparison principle and Itô’s formula, the stochastic intermittent stabilization based on delay feedback control can be obtained. (3) We succeed in overcoming the difficulties mainly arising from the simultaneous presence of the intermittent noise and delay feedback control.

## 2 Preliminary

Throughout this paper, unless otherwise specified, let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions. Let  $\tau > 0$ , and denote by  $C = C([-\tau, 0]; R^n)$  the family of continuous functions  $\xi$  from  $[-\tau, 0]$  to  $R^n$  with the norm  $\|\xi\| = \sup_{-\tau \leq \theta \leq 0} |\xi(\theta)| < \infty$ . Denote by  $L^2_{\mathcal{F}_0}([-\tau, 0]; R^n)$  the family of all  $\mathcal{F}_0$ -measurable  $C([-\tau, 0]; R^n)$  valued random variables  $\zeta = \{\zeta(\theta) : -\tau \leq \theta \leq 0\}$  such that  $\sup_{-\tau \leq \theta \leq 0} E|\zeta(\theta)|^2 < \infty$ , where  $E|\cdot|$  stands for the mathematical expectation operator with respect to the given probability measure  $\mathcal{P}$ . Let  $G = (g_{ij})_{n \times n}$ . Denote  $\bar{G} = (\bar{g}_{ij})_{n \times n}$  with  $\bar{g}_{ii} = \max\{g_{ii}, 0\}$ ,  $\bar{g}_{ij} = g_{ij}$ , and  $|G| = (|g_{ij}|)_{n \times n}$ .

Consider the unstable neural networks as follows:

$$\dot{x}(t) = -Dx(t) + Af(x(t)), \tag{1}$$

where  $D = \text{diag}(d_1, d_2, \dots, d_n)$ ,  $A = (a_{ij})_{n \times n}$ ,  $f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))^T$ ,  $f_i(\cdot) : R \rightarrow R$  is an activation function.  $x(t) \in R^n$ , the variable  $x_i(t)$  represents the voltage on the input of the  $i$ th neuron. Consider the following neural networks by aperiodically intermittent control based on delay state observations:

$$\begin{cases} dx(t) = (-Dx(t) + Af(x)) dt + \Sigma h(t)x(t - \tau) dB(t), \\ h(t) = \begin{cases} 1, & t \in [t_i, s_i], \\ 0, & t \in (s_i, t_{i+1}), i = 0, 1, 2, \dots \end{cases} \end{cases} \tag{2}$$

where  $\Sigma = \text{diag}(\sigma, \dots, \sigma)$ ,  $B(t)$  is a scalar Brown motion,  $\Sigma h(t)x(t - \tau) dB(t)$  is an aperiodically intermittent controller.  $t_k - t_{k-1} > 0$  is the  $k$ th time interval length. The feedback control is imposed on the networks in the time interval  $[t_k, s_k)$ , while the control is removed in the rest of the interval  $[s_k, t_{k+1})$ . Set  $\phi_k = (t_{k+1} - t_k)^{-1}(t_{k+1} - s_k)$ , and  $\phi = \limsup_{k \rightarrow +\infty} \phi_k$  is the intermittence rate.

In order to analyze the asymptotic behavior for networks (2), we define the auxiliary networks without delay observations:

$$dy(t) = (-Dy(t) + Af(y)) dt + \Sigma h(t)y(t) dB(t). \tag{3}$$

The key technique in this paper is the comparison principle. The  $p$ th moment difference between networks (2) and (3) is estimated. The whole frame is based on two basic assumptions.

**Assumption 1** For each  $i = 1, 2, \dots, n$ , there exists  $\kappa_i$  such that

$$0 < \frac{f_i(u) - f_i(v)}{u - v} \leq \kappa_i, \quad u, v \in R.$$

Denote  $K = \text{diag}(\kappa_1, \dots, \kappa_n)$ .

**Assumption 2**  $\lambda_+(-D + |\bar{A}|K) = \sup_{|x|=1, x \in R_+^n} x^T(-D + |\bar{A}|K)x > 0$ .

*Remark 1* If  $\lambda_+(-D + |\bar{A}|K) < 0$ , calculating the derivative of  $|x(t)|^2$  along networks (1) gives

$$\begin{aligned} D^+ |x(t)|^2 &= 2x^T(t)(-D + Af(x(t)))x(t) \\ &= -2x^T(t)Dx(t) + 2 \sum_{i=1}^n a_{ii}^+ \kappa_i |x_i^2| + 2 \sum_{i=1}^n \sum_{j \neq i, j=1}^n |a_{ij}| \kappa_j |x_i| |x_j| \\ &= 2([x]^+)^T |(-D + \bar{A}|K)[x]^+ = \lambda_+(-D + |\bar{A}|K) |x(t)|^2 < 0, \end{aligned}$$

where  $[x]^+ = (|x_1|, |x_2|, \dots, |x_n|)^T$ , and  $\bar{a}_{ii} = \max\{a_{ii}, 0\}$ ,  $\bar{a}_{ij} = |a_{ij}|$ . This implies networks (1) are stable.

### 3 Main results

The main results will be presented in this section.

**Theorem 3.1** *Let Assumption 1 and  $2\lambda_+(-D + |\bar{A}|K) < \sigma^2(1 - \phi)$  hold. System (2) can be stabilized by stochastic intermittent control  $\sigma h(t)X(t - \tau) dB(t)$  provided  $\tau < \tau_0$ , where  $\tau_0$  is the solution of (23).*

*Proof* We divide the proof into three steps for convenience. The main aim of step 1 is to show the stability of system (3). The moment estimation on  $E|x(t) - y(t)|^p$  has been performed in step 2. We will show the stabilization by stochastic intermittent control with time delay in step 3.

*Step 1.* We will show that system (3) is  $p$ th moment exponentially stable if  $p \in (0, 1 - 2\lambda_+(-D + |\bar{A}|K)\sigma^{-2}(1 - \phi)^{-1})$  is sufficiently small. Applying Itô's formula to  $V = |y(t)|^p$  yields

$$dV = \left\{ -p|y|^{p-2}y^T D y + p|y|^{p-2}y^T A f(y) + \frac{1}{2}\sigma^2 h^2(t)p(p-1)|y|^p \right\} dt + p\sigma h(t)|y|^p dB(t). \tag{4}$$

By similar computations in Remark 1, we have

$$LV \leq p|y|^{p-2}|y|^T(-D + |\bar{A}|K)|y| + \frac{1}{2}\sigma^2 h^2(t)p(p-1)|y|^p \leq p\left(\lambda_+(-D + |\bar{A}|K) + \frac{1}{2}\sigma^2 h(t)(p-1)\right)|y|^p.$$

For convenience, denote  $\lambda_1 = \lambda_+(-D + |\bar{A}|K)$ , then (4) can be written as

$$d|y(t)|^p \leq \left(\lambda_1 p + \frac{1}{2}\sigma^2 h(t)p(p-1)\right)|y(t)|^p dt + p\sigma h(t)|y(t)|^p dB(t).$$

It follows from the stochastic comparison principle that

$$|y(t)|^p \leq |y(t_0)|^p \exp\left\{\left(\lambda_1 p - \frac{1}{2}p\sigma^2\right)(t - t_0) + p\sigma(B(t) - B(t_0))\right\}, \quad t_0 \leq t \leq s_0. \tag{5}$$

Note that  $t = s_0$ ,

$$|y(s_0)|^p \leq |y(t_0)|^p \exp\left\{\left(\lambda_1 p - \frac{1}{2}p\sigma^2\right)(s_0 - t_0) + p\sigma(B(s_0) - B(t_0))\right\}.$$

It is obvious that  $h(t) = 0, s_0 < t \leq t_1$ , then we obtain

$$|y(t)|^p \leq |y(s_0)|^p \exp\left\{\left(\lambda_1 p - \frac{1}{2}p\sigma^2\right)(t - s_0)\right\} \leq |y(t_0)|^p \exp\left\{\left(\lambda_1 p - \frac{1}{2}p\sigma^2\right)(s_0 - t_0) + \lambda_1 p(t - s_0) + p\sigma(B(s_0) - B(t_0))\right\}, \quad s_0 < t \leq t_1. \tag{6}$$

Applying Itô's formula to  $V_1(t) = e^{p\sigma B(t)}$  yields

$$dV_1(t) = \frac{1}{2}p^2\sigma^2 V_1 dt + p\sigma V_1(t) dB(t).$$

Simple computations show that  $EV_1 = e^{\frac{1}{2}p^2\sigma^2t}$ . Taking expectation on both sides of (5) yields

$$E|y(t)|^p \leq |y(t_0)|^p \exp\left\{\left(\lambda_1 p + \frac{1}{2}\sigma^2 p(p-1)\right)(t-t_0)\right\}, \quad t_0 \leq t \leq s_0.$$

In the same way, using (6) yields

$$E|y(t)|^p \leq |y(t_0)|^p \exp\left\{\left(\lambda_1 p + \frac{1}{2}\sigma^2 p(p-1)\right)(s_0-t_0) + \lambda_1 p(t-s_0)\right\}, \quad s_0 < t \leq t_1.$$

Denote  $\alpha_1 = \lambda_1 p + \frac{1}{2}\sigma^2 p(p-1)$ ,  $\alpha_2 = \lambda_1 p$ . For  $t_i \leq t < s_i$ , we can readily verify that

$$\begin{aligned} E|y(t)|^p &\leq |y(t_0)|^p \exp\left\{\alpha_1 \sum_{k=0}^{i-1} (s_k - t_k) + \alpha_2 \sum_{k=0}^{i-1} (t_{k+1} - s_k) + \alpha_1(t-t_i)\right\} \\ &= |y(t_0)|^p \exp\left\{\alpha_1 \sum_{k=0}^{i-1} (1-\phi_k)(t_{k+1} - t_k) + \alpha_2 \sum_{k=0}^{i-1} \phi_k(t_{k+1} - t_k) + \alpha_1(t-t_i)\right\}. \end{aligned}$$

It follows from the definition of  $\phi$  that, for any  $\varepsilon > 0$ , there exists a positive integer  $N > 0$  such that  $\phi_k < \phi + \varepsilon$  for any  $k > N$ . Consequently, for  $t_i \leq t < s_i$ , we have

$$\begin{aligned} E|y(t)|^p &\leq |y(t_0)|^p \exp\left\{C + \sum_{k=N+1}^{i-1} (\alpha_1(1-\phi-\varepsilon) + \alpha_2(\phi+\varepsilon))(t_{k+1} - t_k) + \alpha_1(t-t_i)\right\}, \end{aligned} \tag{7}$$

where  $C$  is a constant. Similarly, for  $s_i \leq t < t_{i+1}$ , we get

$$\begin{aligned} E|y(t)|^p &\leq |y(t_0)|^p \exp\left\{\alpha_1 \sum_{k=0}^i (s_k - t_k) + \alpha_2 \sum_{k=0}^{i-1} (t_{k+1} - s_k) + \alpha_2(t-s_i)\right\} \\ &= |y(t_0)|^p \exp\left\{\alpha_1 \sum_{k=0}^i (1-\phi_k)(t_{k+1} - t_k) + \alpha_2 \sum_{k=0}^{i-1} \phi_k(t_{k+1} - t_k) + \alpha_2(t-s_i)\right\} \\ &= |y(t_0)|^p \exp\left\{C + \sum_{k=N+1}^{i-1} (\alpha_1(1-\phi-\varepsilon) + \alpha_2(\phi+\varepsilon))(t_{k+1} - t_k) \right. \\ &\quad \left. + \alpha_1(1-\phi_k)(t_{i+1} - t_i) + \alpha_2(t-s_i)\right\}. \end{aligned} \tag{8}$$

Combining (7) and (8), for every  $\varepsilon > 0$ ,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log E|y(t)|^p \leq -(\alpha_1(1-\phi-\varepsilon) + \alpha_2(\phi+\varepsilon)). \tag{9}$$

Letting  $\varepsilon \rightarrow 0$  yields

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log E|y(t)|^p \leq -\gamma = p\left(\lambda_1 - \frac{1}{2}\sigma^2(1-\phi)(1-p)\right) < 0.$$

Then we can claim that there exists a positive real number  $T_1 > 0$  such that, for any  $t - t_0 > T_1$ ,

$$E|y(t)|^p \leq E|y(t_0)|^p e^{-0.5\gamma(t-t_0)}, \quad t - t_0 > T_1. \tag{10}$$

*Step 2.* The main aim now is to estimate the  $p$ th moment for solution process  $x(t)$  and the difference process  $x(t) - y(t)$  between networks (2) and (3). Applying Itô's formula to  $|x(t)|^2$  yields

$$\begin{aligned} |x(t)|^2 &\leq |x(t_0)|^2 + \int_{t_0}^t (2x^T(s)(-D + |\bar{A}|K)x(s) + \sigma^2 h^2(t)x^T(s - \tau)x(s - \tau)) \, ds \\ &\quad + 2\sigma \int_{t_0}^t h(s)x^T(s)x(s - \tau) \, dB(s). \end{aligned}$$

Taking expectations on both sides, we have

$$E|x(t)|^2 \leq E|x(t_0)|^2 + 2\lambda_1 \int_{t_0}^t E|x(s)|^2 \, ds + \sigma^2 \int_{t_0}^t E|x(s - \tau)|^2 \, ds.$$

Taking supremum on  $[t_0 - \tau, t]$  gives

$$\begin{aligned} \sup_{t_0 - \tau \leq u \leq t} E|x(u)|^2 &\leq E|\zeta|^2 + \sup_{t_0 \leq u \leq t} E|x(t)|^2 \\ &\leq 2E|\zeta|^2 + 2\lambda_1 \int_{t_0}^t \left( \sup_{t_0 \leq u \leq s} E|x(u)|^2 \right) \, ds + \sigma^2 \int_{t_0}^t \left( \sup_{t_0 - \tau \leq u \leq s} E|x(u)|^2 \right) \, ds. \end{aligned}$$

Note that the right term of the above inequality is monotonically increasing for  $t \geq t_0$ , then we obtain

$$\sup_{t_0 - \tau \leq u \leq t} E|x(t)|^2 \leq 2E|\zeta|^2 + (2\lambda_1 + \sigma^2) \int_{t_0}^t \left( \sup_{t_0 - \tau \leq u \leq s} E|x(u)|^2 \right) \, ds.$$

The Gronwall inequality then gives

$$\sup_{t_0 - \tau \leq u \leq t} E|x(u)|^2 \leq 2e^{(2\lambda_1 + \sigma^2)(t-t_0)} E\|\zeta\|^2. \tag{11}$$

By the Burkholder–Davis–Gundy inequality and the Hölder inequality, we obtain

$$\begin{aligned} &E\left( \sup_{0 \leq u \leq \tau} |x(t + u) - x(t)|^2 \right) \\ &\leq 2E\left\{ \sup_{0 \leq u \leq \tau} \left| \int_t^{t+u} [-Dx(t) + Af(x(s))] \, ds \right|^2 \right\} + 2E\left\{ \sup_{0 \leq u \leq \tau} \left| \int_t^{t+u} \sigma x(s - \tau) \, dB(s) \right|^2 \right\} \\ &\leq 2\lambda_2^2 \tau \int_t^{t+\tau} E|x(s)|^2 \, ds + 8\sigma^2 \int_t^{t+\tau} E|x(s - \tau)|^2 \, ds. \end{aligned}$$

Using (11) yields

$$E\left( \sup_{0 \leq u \leq \tau} |x(t + u) - x(t)|^2 \right)$$

$$\leq [4\tau(\lambda_2^2\tau \exp\{(2\lambda_1 + \sigma^2)\tau\} + 4\sigma^2) \exp\{(2\lambda_1 + \sigma^2)(t - t_0)\}]E\|\zeta\|^2, \tag{12}$$

where  $\lambda_2 = \sup_{|x|=1} x^T(D^2 + K^T(x)A^TAK(x) - 0.5DAK(x) - 0.5K^T(x)A^TD)x$ , with  $K(x) = \text{diag}(f(x_1/x_1, f(x_2/x_2), \dots, f(x_n/x_n))$ . It follows from the Hölder inequality that

$$E\left(\sup_{0 \leq u \leq \tau} |x(t+u) - x(t)|^p\right) \leq F_1(\tau, p) \exp\{(p\lambda_1 + 0.5p\sigma^2)(t - t_0)\}E\|\zeta\|^p,$$

where  $F_1(\tau, p) = [4\tau(\lambda_2^2\tau \exp\{(2\lambda_1 + \sigma^2)\tau\} + 4\sigma^2)]^{\frac{p}{2}}$ . Now we estimate the expectation  $E(\sup_{t_0 \leq u \leq t} |x(u)|^2)$ . The element inequality gives

$$|x(t)|^2 \leq 3|x(t_0)|^2 + 3\lambda_2^2(t - t_0) \int_{t_0}^t |x(s)|^2 ds + 3\sigma^2 \left| \int_{t_0}^t |x(s - \tau)|^2 dB(s) \right|^2.$$

Taking supremum on  $[t_0, t]$  and expectations on both sides yields

$$E\left(\sup_{t_0 \leq u \leq t} |x(u)|^2\right) \leq 3|x(t_0)|^2 + 3\lambda_2^2(t - t_0) \int_{t_0}^t E|x(s)|^2 ds + 12\sigma^2 \int_{t_0}^t E|x(s - \tau)|^2 ds.$$

Together with (11), we have

$$E\left(\sup_{t_0 \leq u \leq t} |x(u)|^2\right) \leq \left(3 + \frac{6(\lambda_2^2(t - t_0) + 4\sigma^2)}{2\lambda_1 + \sigma^2} (\exp\{(2\lambda_1 + \sigma^2)(t - t_0)\} - 1)\right)E\|\zeta\|^2.$$

The Hölder inequality then gives

$$\begin{aligned} E\left(\sup_{t_0 \leq u \leq t} |x(u)|^p\right) &\leq \left(3 + \frac{6(\lambda_2^2(t - t_0) + 4\sigma^2)}{2\lambda_1 + \sigma^2} (\exp\{(2\lambda_1 + \sigma^2)(t - t_0)\} - 1)\right)^{\frac{p}{2}} E\|\zeta\|^p \\ &:= F_2(\tau, p, t - t_0)E|\zeta|^p. \end{aligned} \tag{13}$$

Next we estimate the  $p$ th moment difference between networks (2) and (3). By Itô's formula and the Hölder inequality, we get

$$\begin{aligned} &E|x(t) - y(t)|^2 \\ &= E \int_{t_0+\tau}^t \{2(x(s) - y(s))[-D(x(s) - y(s)) + A(f(x(s)) - f(y(s)))] \\ &\quad + \sigma^2 h^2(t)(x(s - \tau) - y(s))\} ds \\ &\leq 2(\lambda_1 + \sigma^2) \int_{t_0+\tau}^t E|x(s) - y(s)|^2 ds + 2\sigma^2 \int_{t_0+\tau}^t E|x(s - \tau) - x(s)|^2 ds. \end{aligned} \tag{14}$$

Instituting (12) to (14) gives

$$\begin{aligned} E|x(t) - y(t)|^2 &\leq 2(\lambda_1 + \sigma^2) \int_{t_0+\tau}^t E|x(s) - y(s)|^2 ds \\ &\quad + 2\sigma^2 F_1(\tau, 2)(2\lambda_1 + \sigma^2)^{-1} \exp\{(2\lambda_1 + \sigma^2)(t - \tau - t_0)\}. \end{aligned} \tag{15}$$

It follows from the Gronwall inequality that

$$E|x(t) - y(t)|^2 \leq (2\lambda_1 + \sigma^2)^{-1} F_1(\tau, 2) \exp\{(2\lambda_1 + \sigma^2)(t - \tau - t_0)\} \times [2\sigma^2 + 4(\lambda_1 + \sigma^2)(\exp\{\sigma^2(t - \tau - t_0)\} - 1)] E\|\zeta\|^2. \tag{16}$$

It follows from the Hölder inequality that

$$E|x(t) - y(t)|^p \leq F_3(\tau, p, t - t_0) E\|\zeta\|^p,$$

where

$$F_3(\tau, p, t - t_0) = ((2\lambda_1 + \sigma^2)^{-1} F_1(\tau, 2) \exp\{(2\lambda_1 + \sigma^2)(t - \tau - t_0)\} \times [2\sigma^2 + 4(\lambda_1 + \sigma^2)(\exp\{\sigma^2(t - \tau - t_0)\} - 1)])^{\frac{p}{2}}. \tag{17}$$

*Step 3.* Let  $x(t) = x(t, t_0, \zeta)$ ,  $y(t_0 + \tau + T) = y(t_0 + \tau + T, t_0 + \tau, x(t_0 + \tau))$  for simplicity. Taking  $T = \max\{T_1, \frac{2}{\gamma} \log(\frac{2^{2.5p}}{\epsilon})\}$  with  $\epsilon \in (0, 1)$ , assertion (10) gives that

$$E|y(t_0 + \tau + T)|^p \leq E|x(t_0 + \tau)|^p e^{\frac{1}{2}p(\alpha - \frac{1}{2}\sigma^2(1-\phi))T} \leq e^{-\frac{1}{2}\gamma T} (2e^{(2\alpha + \sigma^2)\tau})^{\frac{p}{2}} E\|\zeta\|^p. \tag{18}$$

The elementary inequality  $(x + y)^p \leq 2^p(x^p + y^p)$  for  $x, y \geq 0$  yields

$$E|x(t_0 + \tau + T)|^p \leq 2^p E|y(t_0 + \tau + T)|^p + 2^p E|x(t_0 + \tau + T) - y(t_0 + \tau + T)|^p.$$

It follows from (17) and (18) that

$$E|x(t_0 + \tau + T)|^p \leq 2^p (e^{-\frac{1}{2}\gamma T} (2e^{(2\alpha + \sigma^2)\tau})^{\frac{p}{2}} + F_3(\tau, p, T + \tau)) E\|\zeta\|^p. \tag{19}$$

Using the elementary inequality and (19),

$$\begin{aligned} E|x(t_0 + 2\tau + T)|^p &\leq 2^p E|x(t_0 + \tau + T)|^p \\ &\quad + 2^p E\left(\sup_{0 \leq \mu \leq \tau} |x(t_0 + \tau + T + \mu) - x(t_0 + \tau + T)|\right) \\ &\leq 2^p E|x(t_0 + \tau + T)|^p + 2^p F_1(\tau, p, T + \tau) E\|\zeta\|^p. \end{aligned} \tag{20}$$

Using (19) and (20), we have

$$E|x(t_0 + 2\tau + T)|^p \leq (\epsilon e^{(\alpha + \frac{1}{2}\sigma^2)p\tau} + 2^p F_1(\tau, p, T + \tau) + 4^p F_3(\tau, p, T + \tau)) E\|\zeta\|^p \tag{21}$$

$$:= G(\tau, \epsilon, p, T) E\|\zeta\|^p. \tag{22}$$

Note that, for given  $\epsilon \in (0, 1)$ ,  $G(\tau, \epsilon, p, T)$  is a monotonously increasing function,  $G(0, \epsilon, p, T) = \epsilon < 1$ , and  $G(0, \epsilon, p, T) \rightarrow \infty$  as  $\tau \rightarrow \infty$ . Now we claim that there exists a unique  $\tau_0$  to the following equation:

$$\epsilon e^{(\alpha + \frac{1}{2}\sigma^2)p\tau} + 2^p F_1(\tau, p, T + \tau) + 4^p F_3(\tau, p, T + \tau) = 1. \tag{23}$$

Note the left item of (23), it is monotonically increasing when we think of it as a function with the independent variable  $\tau$ , and it is equal to  $\epsilon$  if  $\tau = 0$ . As a result, equation (23) has a unique solution if  $\tau_0 > 0$ . Determine  $\tau \in (0, \tau_0)$ , and  $\zeta \in \mathcal{L}^2_{\mathcal{F}_{\tau_0}}(\Omega, C([- \tau, 0]; R^n))$  is an arbitrary initial value. From  $\tau < \tau_0$  we can verify that

$$\epsilon e^{(\alpha + \frac{1}{2}\sigma^2)p\tau} + 2^p F_1(\tau, p, T + \tau) + 4^p F_3(\tau, p, T + \tau) < 1$$

and therefore we can find a suitable constant  $c > 0$  such that

$$\epsilon e^{(\alpha + \frac{1}{2}\sigma^2)p\tau} + 2^p F_1(\tau, p, T + \tau) + 4^p F_3(\tau, p, T + \tau) = e^{-c(T+2\tau)}.$$

We obtain from (21) that

$$E|x(t_0 + 2\tau + T)|^p \leq e^{-c(T+2\tau)} E\|\zeta\|^p. \tag{24}$$

Next we discuss the solution  $x(t)$  for  $t \geq t_0 + 2\tau + T$ . We know that there is a unique solution to networks (2) for  $t > t_0 - \tau$ . In other words, we can regard  $x(t_0 + 2\tau + T)$  as the initial value of  $x(t)$  at  $t = t_0 + 2\tau + T$ . By (24) we have

$$E|x(t_0 + 2(2\tau + T))|^p \leq e^{-c(T+2\tau)} E|x(t_0 + 2\tau + T)|^p.$$

Analyzing (24) and the equation above, we get

$$E|x(t_0 + 2(2\tau + T))|^p \leq e^{-2c(T+2\tau)} E\|\zeta\|^p.$$

After repeated iteration we have

$$E|x(t_0 + c(2\tau + T))|^p \leq e^{-nc(T+2\tau)} E\|\zeta\|^p, \quad n = 1, 2, \dots \tag{25}$$

Moreover, we can verify that it is established if  $n = 0$ . Using (13) and (25), we have

$$E\left(\sup_{t_0+n\mathcal{E} \leq t \leq t_0+(n+1)\mathcal{E}} |x(t)|^p\right) \leq F_2 E|x(t_0 + n\mathcal{E})|^p \leq F_2 e^{-nc\mathcal{E}} E\|\zeta\|^p, \quad n = 0, 1, \dots, \tag{26}$$

where  $\mathcal{E} = T + 2\tau$  and  $F_2$  is defined by (13). Using Markov’s inequality and (26),

$$\begin{aligned} P\left(\sup_{t_0+n\mathcal{E} \leq t \leq t_0+(n+1)\mathcal{E}} |x(t)|^p \geq e^{-\frac{1}{2}nc\mathcal{E}}\right) &\leq e^{-\frac{1}{2}nc\mathcal{E}} E\left(\sup_{t_0+n\mathcal{E} \leq t \leq t_0+(n+1)\mathcal{E}} |x(t)|^p\right) \\ &\leq F_2 e^{-\frac{1}{2}nc\mathcal{E}} E\|\zeta\|^p, \quad n = 0, 1, \dots \end{aligned}$$

Using the Borel–Cantelli lemma, we can verify that, for almost every  $\omega$ , there exists an integer  $N_0 = N_0(\omega)$  such that

$$\sup_{t_0+n\mathcal{E} \leq t \leq t_0+(n+1)\mathcal{E}} |x(t)|^p < e^{-\frac{1}{2}nc\mathcal{E}}, \quad N \geq N_0.$$

That is,

$$\sup_{t_0+n\mathcal{E} \leq t \leq t_0+(n+1)\mathcal{E}} \frac{\log |x(t)|}{t} < \sup_{t_0+n\mathcal{E} \leq t \leq t_0+(n+1)\mathcal{E}} \frac{\log |x(t)|}{n\mathcal{E}} < -\frac{c}{2p}.$$

For almost every  $\omega$ , we get

$$\limsup_{t \rightarrow +\infty} \frac{\log |x(t)|}{t} < -\frac{c}{2p} < 0$$

as desired. □

*Remark 2* Shen and Wang studied the stabilization of recurrent neural networks by continuous noise (see [26]). Compared to the existing results, we show that neural networks can be stabilized by intermittent noise with time delay.

Step 1 of Theorem 3.1 implies a sufficient condition on stabilization by stochastic intermittent control without delay observations as follows.

**Corollary 3.2** *Let Assumption 1 and  $2\lambda_+(-D + |\bar{A}|K) < \sigma^2(1 - \phi)$  hold. Networks (2) can be stabilized by stochastic intermittent control  $h(t)\Sigma x(t) dB(t)$ .*

*Remark 3* Guo and Mao have discussed almost sure stabilization of delay differential systems by delay feedback control in [22]. Nevertheless, there is distinct characteristics in delay neural networks. We can make full use of the network characteristics. Thus, it is desirable to derive the stabilization condition. In this study the neural networks are stabilized by aperiodically intermittent noise based on delay observations. Comparing with the results in [22], we further integrate the intermittent control strategy.

When  $\psi = 0$ , the white noise is continuous and Theorem 3.1 implies a criterion on stabilization by delay feedback control.

**Corollary 3.3** *Let Assumption 1 and  $2\lambda_+(-D + |\bar{A}|K) < \sigma^2$  hold. Networks (2) can be stabilized by delay feedback control  $\Sigma x(t - \tau) dB(t)$  provided  $\tau > \tau_0$ , where  $\tau_0$  is the solution to (23).*

### 4 Numerical example

A numerical example is presented in this section. We verify that the theorem above is available.

*Example 1* Consider two-neural networks:

$$dx(t) = (-Dx(t) + Af(x)) dt, \tag{27}$$

where  $x(t) = (x_1(t), x_2(t))^T$ ,  $f(x) = \tanh x$ , and the other parameters in networks (27) are selected as follows:

$$D = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad A = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.6 \end{pmatrix}.$$

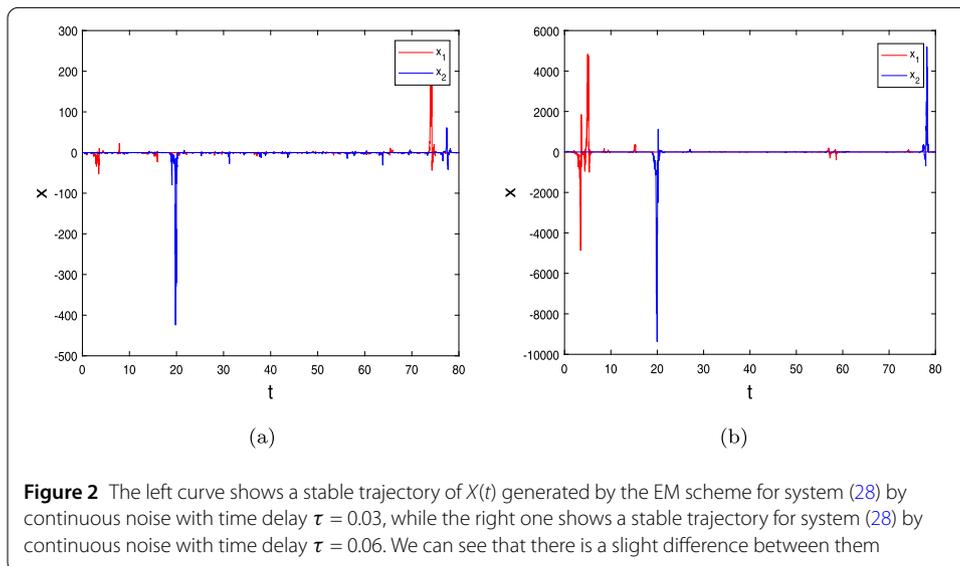
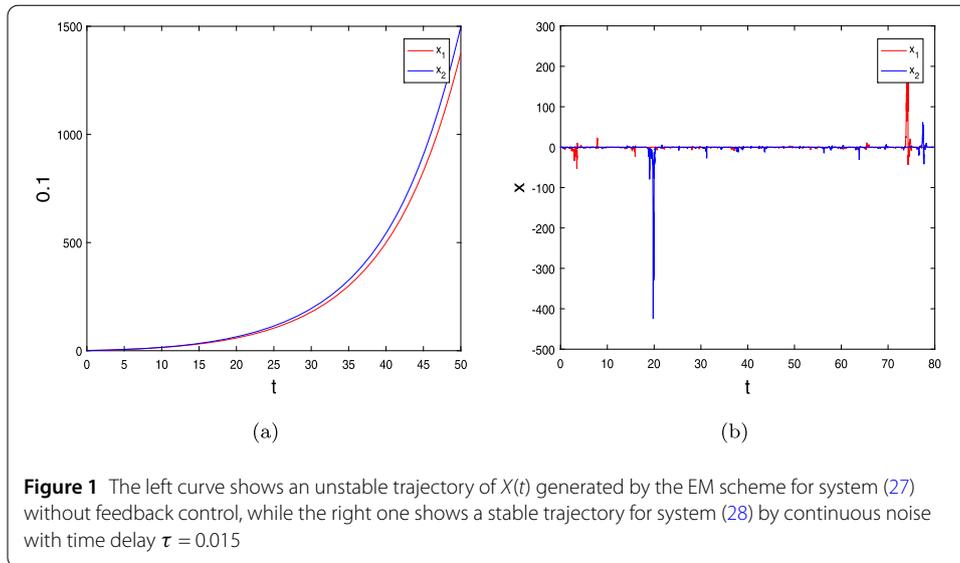
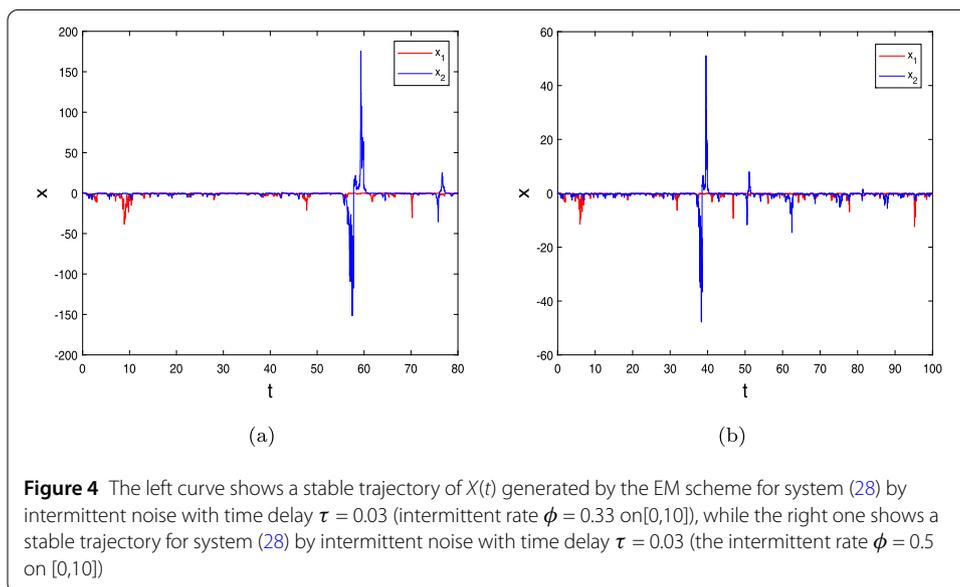
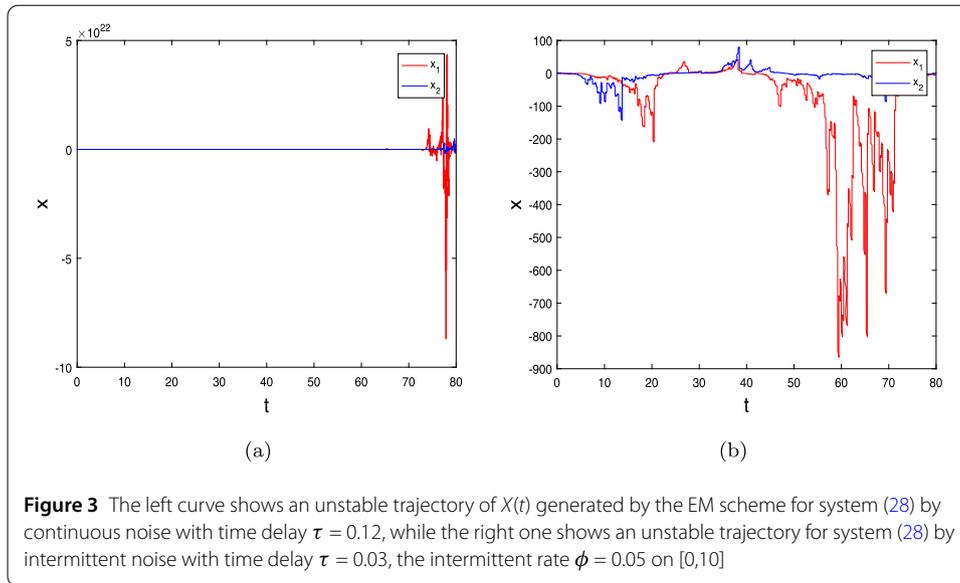


Figure 1(a) shows that networks (27) are unstable. The controller  $\Sigma h(t)x(t - \tau) dB(t)$  is designed. That is,

$$\begin{aligned}
 dx(t) &= (-Dx(t) + Af(x)) dt + \Sigma h(t)x(t - \tau) dB(t), \\
 h(t) &= \begin{cases} 1, & t \in [t_i, s_i], \\ 0, & t \in (s_i, t_{i+1}], i = 0, 1, 2, \dots, \end{cases} \tag{28}
 \end{aligned}$$

where  $\tau = 0.015, \Sigma = 0.2I$ .

We take aperiodic controlled intervals  $[0, 0.1] \cup [2.0, 2.1] \cup [4.0, 4.1] \cup [6.0, 6.1] \cup [8.0, 8.1] \cup \dots$ , or  $[0, 0.33] \cup [1.00, 1.34] \cup [2.00, 2.33] \cup [3.00, 3.33] \cup [4.00, 4.33] \cup [5.00, 5.34] \cup [6.00, 6.33] \cup [7.00, 7.33] \cup [8.0, 8.33] \cup [9.00, 9.34] \cup \dots$ , or  $[0, 0.9] \cup [2.0, 3.1] \cup [4.0, 5.1] \cup [6.0, 6.9] \cup [8.0, 9.0] \cup \dots$ , the intermittent rates are 0.05, 0.33, 0.5 respectively.



We draw four figures by Matlab. The networks are stabilized by continuous white noise with delay observations (the time delay is 0.015 (Fig. 1b), 0.03 (Fig. 2a), 0.06 (Fig. 2b), respectively), while they are not stabilized if time delay is 0.12 (Fig. 3a). We see easily that the bigger time delay is better owing to less observation interval; however, the networks cannot be stabilized by continuous noise if the time delay is big enough. We fix time delay  $\tau = 0.03$  and switch the control strategy from continuous noise to intermittent noise, then we change the intermittent rate  $\phi$  ( $\phi = 0.05$  (Fig. 3b), 0.33 (Fig. 4a), 0.5 (Fig. 4b), respectively). The networks are stabilized when  $\phi = 0.33$  or  $\phi = 0.5$ , while they are unstable when  $\phi = 0.05$ . That is, the networks are unstable if the intermittent rate is small enough. The networks work best when  $\phi = 0.5$ .

The numerical example shows that the proposed methods are practical and efficient. We observe the states less frequently and cut the costs by reducing the controlled time com-

pared with the algorithm proposed by Mao (see [24]). The neural networks are stabilized by aperiodic intermittent control with delay observations.

## 5 Conclusions

In this study, we have investigated the exponential stabilization for neural networks by aperiodically intermittent control based on delay observations. First of all, by using the stochastic comparison principle, Itô's formula, and the sequence analysis technique, we show that the unstable neural networks can be stabilized by aperiodically intermittent noise. Secondly, in terms of the characteristic of neural networks, we show that the networks are exponentially stabilized based on delay observations. Finally, a numerical example is provided to illustrate the superiority and effectiveness of the proposed approaches.

### Acknowledgements

The authors would like to show their great thanks to Professor Jifeng Chu for his great and consistent support, without him this work would not be possible.

### Funding

The research work is supported by Fundamental Research Funds for the Central Universities (Grant No. 2018B19914); the National Science Foundation of China (Grant Nos. 61773152); the Chinese Postdoctoral Science Foundation (Grant No. 2016M601698, 2017T100318); the Jiangsu Province Postdoctoral Science Foundation (Grant No. 1701078B); and the project funded by the Qing Lan Project of Jiangsu Province, China.

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

The authors have made the same contribution. All authors read and approved the final manuscript.

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Received: 7 November 2018 Accepted: 25 July 2019 Published online: 22 August 2019

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