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Iterative positive solutions to a coupled fractional differential system with the multistrip and multipoint mixed boundary conditions

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Abstract

Using the monotone iterative technique, we investigate the existence of iterative positive solutions to a coupled system of fractional differential equations supplemented with multistrip and multipoint mixed boundary conditions. It is worth mentioning that the nonlinear terms of the system depend on the lower fractional-order derivatives of the unknown functions and the boundary conditions involve the combination of the multistrip fractional integral and the multipoint value of the unknown functions in [0, 1].

MSC: 26A33; 34B15; 34B18

Keywords: Fractional differential system; Multistrip and multipoint mixed boundary conditions; Green's function; Monotone iterative technique

1 Introduction

Fractional differential equations have attracted more and more scholars' attention since they are more widely used and realistic than integer-order differential equations. In the past few years the fractional boundary value problems are found to be popular in the research community because of their numerous applications in many disciplinary areas, such as optics, thermal, mechanics, control theory, nuclear physics, economics, signal and image processing, medicine, and so on [1–4]. To meet the practical application needs, many different theoretical approaches have been taken to study the existence, uniqueness, and multiplicity of solutions to fractional-order boundary value problems, for instance, the method of upper and lower solutions [5–9], the fixed point theory [10–13], the monotone iterative technique [14–19], the coincidence degree theory [20–22], etc. In comparison, the monotone iterative technique has more advantages, such as it not only proves the existence of positive solutions but also can obtain approximate solutions that can meet different accuracy requirements.

Meanwhile, recently, coupled fractional differential systems have also aroused great interest and developed rapidly. Many researchers established the existence of solutions by the class methods [22–25]. However, to the best of our knowledge, only few papers applied the monotone iterative technique to discuss the boundary value problem of coupled fractional differential systems [26–28]. To get more extensive results, different from the existing literature, we consider a generalized model that includes the nonlinear terms of the system depending on the lower fractional-order derivatives of the unknown functions and the boundary conditions involving a combination of the multistrip fractional integral and linear multipoint values of the unknown functions in $[0,1]$.

Based on these considerations, we investigate the existence of iterative positive solutions to the following fractional differential systems:

$$\begin{cases} D_t^{\alpha_1}u(t)+f_1(t,u(t),v(t),D_t^{\gamma_1}u(t),D_t^{\gamma_2}v(t))=0, & t\in(0,1),\\ D_t^{\alpha_2}v(t)+f_2(t,u(t),v(t),D_t^{\gamma_1}u(t),D_t^{\gamma_2}v(t))=0, & t\in(0,1),\end{cases} \tag{1.1}$$

with the coupled fractional-order integral and discrete mixed boundary conditions

$$\begin{cases} u(0)=u'(0)=0, & u(1)=\sum_{i=1}^m \lambda_{1i}I_t^{\beta_1}v(\xi_i)+\sum_{j=1}^n b_jv(\eta_j),\\ v(0)=v'(0)=0, & v(1)=\sum_{i=1}^m \lambda_{2i}I_t^{\beta_2}u(\xi_i)+\sum_{j=1}^n b_ju(\eta_j),\end{cases} \tag{1.2}$$

where $2<\alpha_k\leq 3$, $0<\gamma_k<\alpha_k-2$, $1<\beta_k\leq 2$ for $k=1,2$; $\lambda_{1i}>0$, $\lambda_{2i}>0$, $0<\xi_i<1$ for $i=1,2,\ldots,m$, $b_j\geq 0$, $0<\eta_j<1$ for $j=1,2,\ldots,n$, and $D_t^{\alpha_k}$ and $D_t^{\gamma_k}$ are the standard Riemann–Liouville fractional derivatives of orders α_k and γ_k for $k=1,2$.

The coupled multistrip and multipoint mixed boundary conditions in (1.2) represent the value of unknown function $u(t)$ at the right end point $t=1$, which is equal to the sum of the values of the Riemann–Liouville fractional integral of the unknown function $v(t)$ on the subinterval $[0,\xi_i]$ $(i=1,2,\ldots,m)$ and the linear combination of discrete values of the unknown function $v(t)$ at η_j $(j=1,2,\ldots,n)$.

To apply the monotone iterative technique, we construct a concrete form of initial iterative function vector, which is a fractional power function vector satisfying the multiconstraints from the cone, the monotonicity of the complete continuous operator T, and the monotonicity of the lower fractional-order derivatives of T. The initial function vector of concise form could make the iteration process concise and effective.

2 Preliminaries

In this section, we present here the definitions, some lemmas from the theory of fractional calculus, and some auxiliary results for the proof of our main results.

Definition 2.1 ([1]) The Riemann–Liouville fractional integral of order $\alpha>0$ of a function $y:(0,\infty)\to\mathbb{R}$ is given by

$$\left(I_t^\alpha y\right)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds,\quad t>0,$$

provided that the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the Euler gamma function defined as $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}\,dt$ for $\alpha>0$.

Definition 2.2 ([1]) The Riemann–Liouville fractional derivative of order $\alpha\geq 0$ for a function $y:(0,\infty)\to\mathbb{R}$ is given by

$$\left(D_t^\alpha y\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t y(s)(t-s)^{n-\alpha-1}\,ds,\quad t>0,$$

where $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined on $[0,\infty)$. The notation $[\alpha]$ stands for the largest integer not greater than α. We also denote the Riemann–Liouville fractional derivative of y by $D_t^\alpha y(t)$. If $\alpha = m \in \mathbb{N}$, then $D_t^m y(t) = y^{(m)}(t)$ for $t > 0$, and if $\alpha = 0$, then $D_t^0 y(t) = y(t)$ for $t > 0$.

Lemma 2.1 *Let $\alpha > 0$ and $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, that is, n is the smallest integer greater than or equal to α. Then the solutions of the fractional differential equation $D_t^\alpha u(t) = 0$, $0 < t < 1$, are*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad 0 < t < 1,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants.

Lemma 2.2 *Let $\alpha > 0$, let n be the smallest integer greater than or equal to α ($n-1 < \alpha \le n$), and let $y \in L^1(0,1)$. The solutions of the fractional equation $D_t^\alpha u(t) + y(t) = 0$, $0 < t < 1$, are*

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_1 t^{\alpha-1} + \cdots + c_n t^{\alpha-n}, \quad 0 < t < 1,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants.

Remark 2.1 The following properties are useful for our discussion:

(i) As a basic example, we quote for $\alpha > -1$,

$$D_t^\gamma t^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma+1)} t^{\alpha-\gamma};$$

(ii) $D_t^\gamma I_t^\gamma u(t) = u(t)$ for $u \in L^1(0,1)$, $\gamma > 0$;

(iii) Assume that $u \in L^1(0,1)$ with $\gamma > 0$, then

$$I_t^\gamma \left(D_t^\gamma u(t) \right) = u(t) + m_1 t^{\gamma-1} + m_2 t^{\gamma-2} + \cdots + m_n t^{\gamma-n}$$

for some $m_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to γ.

For convenience, we denote

$$\begin{cases} l_1 = \sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(\beta_2)} \int_0^{\xi_i} (\xi_i - s)^{\beta_2-1} s^{\alpha_1-1}\, ds + \sum_{j=1}^n b_j \eta_j^{\alpha_1-1}, \\ l_2 = \sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i - s)^{\beta_1-1} s^{\alpha_2-1}\, ds + \sum_{j=1}^n b_j \eta_j^{\alpha_2-1}. \end{cases} \tag{2.1}$$

In the forthcoming analysis, we always need the following assumptions:

(F_1) $2 < \alpha_k < 3$, $1 < \beta_k \le 2$, and $0 < \gamma_k < \alpha_k - 2$ for $k = 1, 2$;

(F_2) $0 < \eta_j, \xi_i < 1$, $b_j \ge 0$, and $\lambda_{1i}, \lambda_{2i} > 0$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$;

(F_3) $1 - l_1 l_2 > 0$, where l_1, l_2 are defined by (2.1);

(F_4) $f_i : [0,1] \times [0,+\infty)^4 \to [0,+\infty)$, $i = 1, 2$, are continuous functions.

Subject to BVP (1.1) and (1.2), we consider the corresponding linear boundary value problem as follows and establish expressions of the corresponding Green's functions.

Lemma 2.3 *Assume that* (F_1)–(F_3) *hold. For* $h_1, h_2 \in L^1(0,1)$, *the fractional differential system*

$$\begin{cases} D_t^{\alpha_1} u(t) + h_1(t) = 0, & t \in (0,1), \\ D_t^{\alpha_2} v(t) + h_2(t) = 0, & t \in (0,1), \end{cases} \tag{2.2}$$

with boundary conditions (1.2) *has the integral representation*

$$\begin{cases} u(t) = \int_0^1 K_1(t,s)h_1(s)\,ds + \int_0^1 H_1(t,s)h_2(s)\,ds, \\ v(t) = \int_0^1 K_2(t,s)h_2(s)\,ds + \int_0^1 H_2(t,s)h_1(s)\,ds, \end{cases} \tag{2.3}$$

where

$$\begin{aligned} K_1(t,s) &= g_1(t,s) + \frac{l_2 t^{\alpha_1-1}}{1-l_1l_2}\left[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(\beta_2)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_2-1}g_1(\tau,s)d\tau + \sum_{j=1}^n b_j g_1(\eta_j,s)\right], \\ H_1(t,s) &= \frac{t^{\alpha_1-1}}{1-l_1l_2}\left[\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(\beta_1)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_1-1}g_2(\tau,s)\,d\tau + \sum_{j=1}^n b_j g_2(\eta_j,s)\right], \end{aligned} \tag{2.4}$$

$$\begin{aligned} K_2(t,s) &= g_2(t,s) + \frac{l_1 t^{\alpha_2-1}}{1-l_1l_2}\left[\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(\beta_1)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_1-1}g_2(\tau,s)d\tau + \sum_{j=1}^n b_j g_2(\eta_j,s)\right], \\ H_2(t,s) &= \frac{t^{\alpha_2-1}}{1-l_1l_2}\left[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(\beta_2)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_2-1}g_1(\tau,s)\,d\tau + \sum_{j=1}^n b_j g_1(\eta_j,s)\right], \end{aligned} \tag{2.5}$$

and for $k = 1,2$,

$$g_k(t,s) = \frac{1}{\Gamma(\alpha_k)}\begin{cases} t^{\alpha_k-1}(1-s)^{\alpha_k-1} - (t-s)^{\alpha_k-1}, & 0 \le s \le t \le 1, \\ t^{\alpha_k-1}(1-s)^{\alpha_k-1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.6}$$

Proof From Lemma 2.2 we can reduce (2.2) and (1.2) to the following equivalent integral equations:

$$\begin{cases} u(t) = -\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}h_1(s)\,ds + c_{11}t^{\alpha_1-1} + c_{12}t^{\alpha_1-2} + c_{13}t^{\alpha_1-3}, \\ v(t) = -\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}h_2(s)\,ds + c_{21}t^{\alpha_2-1} + c_{22}t^{\alpha_2-2} + c_{23}t^{\alpha_2-3}, \end{cases} \tag{2.7}$$

where c_{12}, c_{13}, c_{22}, c_{23} are constants.

From $u(0) = u'(0) = v(0) = v'(0) = 0$ we have $c_{12} = c_{13} = c_{22} = c_{23} = 0$. Further, we use the right-hand side boundary conditions of (1.2) to reduce (2.7) to

$$\begin{cases} u(t) = t^{\alpha_1-1}[\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(\beta_1)}\int_0^{\xi_i}(\xi_i-s)^{\beta_1-1}v(s)\,ds + \sum_{j=1}^n b_j v(\eta_j)] + \int_0^1 g_1(t,s)h_1(s)\,ds, \\ v(t) = t^{\alpha_2-1}[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(\beta_2)}\int_0^{\xi_i}(\xi_i-s)^{\beta_2-1}u(s)\,ds + \sum_{j=1}^n b_j u(\eta_j)] + \int_0^1 g_2(t,s)h_2(s)\,ds. \end{cases} \tag{2.8}$$

Then we can get

$$\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(\beta_1)}\int_0^{\xi_i}(\xi_i-s)^{\beta_1-1}v(s)\,ds + \sum_{j=1}^n b_j v(\eta_j)$$

$$
\begin{aligned}
&= \left[\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(\beta_2)} \int_0^{\xi_i} (\xi_i - s)^{\beta_2-1} u(s)\,ds + \sum_{j=1}^{n} b_j u(\eta_j)\right] \\
&\quad \times \left[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i - s)^{\beta_1-1} s^{\alpha_2-1}\,ds + \sum_{j=1}^{n} b_j \eta_j^{\alpha_2-1}\right] \\
&\quad + \sum_{j=1}^{n} b_j \int_0^1 g_2(\eta_j, s) h_2(s)\,ds \\
&\quad + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i - \tau)^{\beta_1-1} \int_0^1 g_2(\tau, s) h_2(s)\,ds\,d\tau;
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
&\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(\beta_2)} \int_0^{\xi_i} (\xi_i - s)^{\beta_2-1} u(s)\,ds + \sum_{j=1}^{n} b_j u(\eta_j) \\
&\quad = \left[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i - s)^{\beta_1-1} v(s)\,ds + \sum_{j=1}^{n} b_j v(\eta_j)\right] \\
&\quad\quad \times \left[\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(\beta_2)} \int_0^{\xi_i} (\xi_i - s)^{\beta_2-1} s^{\alpha_1-1}\,ds + \sum_{j=1}^{n} b_j \eta_j^{\alpha_1-1}\right] \\
&\quad\quad + \sum_{j=1}^{n} b_j \int_0^1 g_1(\eta_j, s) h_1(s)\,ds \\
&\quad\quad + \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(\beta_2)} \int_0^{\xi_i} (\xi_i - \tau)^{\beta_2-1} \int_0^1 g_1(\tau, s) h_1(s)\,ds\,d\tau.
\end{aligned} \tag{2.10}
$$

Combining (2.1), (2.9), and (2.10), we can see that

$$
\begin{aligned}
&\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i - s)^{\beta_1-1} v(s)\,ds + \sum_{j=1}^{n} b_j v(\eta_j) \\
&\quad = \frac{1}{1 - l_1 l_2} \left[\left(\sum_{j=1}^{n} b_j \int_0^1 g_2(\eta_j, s) h_2(s)\,ds\right.\right. \\
&\quad\quad \left. + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i - \tau)^{\beta_1-1} \int_0^1 g_2(\tau, s) h_2(s)\,ds\,d\tau\right) \\
&\quad\quad + l_2 \left(\sum_{j=1}^{n} b_j \int_0^1 g_1(\eta_j, s) h_1(s)\,ds\right. \\
&\quad\quad \left.\left. + \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(\beta_2)} \int_0^{\xi_i} (\xi_i - \tau)^{\beta_2-1} \int_0^1 g_1(\tau, s) h_1(s)\,ds\,d\tau\right)\right],
\end{aligned} \tag{2.11}
$$

where l_k ($k = 1, 2$) are defined by (2.1). From (2.9) and (2.11) we have

$$
\begin{aligned}
u(t) &= \frac{t^{\alpha_1-1}}{1 - l_1 l_2} \left(\sum_{j=1}^{n} b_j \int_0^1 g_2(\eta_j, s) h_2(s)\,ds\right. \\
&\quad \left. + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i - \tau)^{\beta_1-1} \int_0^1 g_2(\tau, s) h_2(s)\,ds\,d\tau\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{l_2 t^{\alpha_1-1}}{1-l_1 l_2}\left(\sum_{j=1}^{n} b_j \int_0^1 g_1(\eta_j,s)h_1(s)\,ds\right. \\
&\left.+ \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(\beta_2)} \int_0^{\xi_i} (\xi_i-\tau)^{\beta_2-1} \int_0^1 g_1(\tau,s)h_1(s)\,ds\,d\tau\right) \\
&+ \int_0^1 g_1(t,s)h_1(s)\,ds \\
={}& \int_0^1 \left[g_1(t,s) + \frac{l_2 t^{\alpha_1-1}}{1-l_1 l_2}\left(\sum_{j=1}^{n} b_j g_1(\eta_j,s) + \sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(\beta_2)} \int_0^{\xi_i} (\xi_i-\tau)^{\beta_2-1} g_1(\tau,s)\,d\tau\right)\right] \\
&\times h_1(s)\,ds \\
&+ \int_0^1 \frac{t^{\alpha_1-1}}{1-l_1 l_2}\left(\sum_{j=1}^{n} b_j g_2(\eta_j,s) + \sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i-\tau)^{\beta_1-1} g_2(\tau,s)\,d\tau\right) h_2(s)\,ds \\
={}& \int_0^1 K_1(t,s)h_1(s)\,ds + \int_0^1 H_1(t,s)h_2(s)\,ds.
\end{aligned}
$$

Similarly, we obtain

$$
v(t) = \int_0^1 K_2(t,s)h_2(s)\,ds + \int_0^1 H_2(t,s)h_1(s)\,ds,
$$

where $K_2(t,s)$ and $H_2(t,s)$ are given by (2.5).

This completes the proof of the lemma. □

Moreover, according to (2.3) and Remark 2.1, the fractional-order derivative of the solution (2.3) can be expressed as

$$
\begin{cases}
D_t^{\gamma_1} u(t) = \int_0^1 K_3(t,s)h_1(s)\,ds + \int_0^1 H_3(t,s)h_2(s)\,ds, \\
D_t^{\gamma_2} v(t) = \int_0^1 K_4(t,s)h_2(s)\,ds + \int_0^1 H_4(t,s)h_1(s)\,ds,
\end{cases}
\tag{2.12}
$$

where

$$
\begin{aligned}
K_3(t,s) ={}& D_t^{\gamma_1} g_1(t,s) + \frac{l_2 t^{\alpha_1-\gamma_1-1}\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)(1-l_1 l_2)}\left[\sum_{i=1}^{m} \frac{\lambda_{2i}}{\Gamma(\beta_2)} \int_0^{\xi_i} (\xi_i-\tau)^{\beta_2-1} g_1(\tau,s)\,d\tau\right. \\
&\left.+ \sum_{j=1}^{n} b_j g_1(\eta_j,s)\right], \\
H_3(t,s) ={}& \frac{\Gamma(\alpha_1) t^{\alpha_1-\gamma_1-1}}{\Gamma(\alpha_1-\gamma_1)(1-l_1 l_2)}\left[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i-\tau)^{\beta_1-1} g_2(\tau,s)\,d\tau\right. \\
&\left.+ \sum_{j=1}^{n} b_j g_2(\eta_j,s)\right], \qquad (2.13) \\
K_4(t,s) ={}& D_t^{\gamma_2} g_2(t,s) + \frac{l_1 t^{\alpha_2-\gamma_2-1}\Gamma(\alpha_2)}{\Gamma(\alpha_2-\gamma_2)(1-l_1 l_2)}\left[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i-\tau)^{\beta_1-1} g_2(\tau,s)\,d\tau\right.
\end{aligned}
$$

$$+\sum_{j=1}^{n} b_j g_2(\eta_j, s)\Bigg],$$

$$H_4(t,s) = \frac{\Gamma(\alpha_2)t^{\alpha_2-\gamma_2-1}}{\Gamma(\alpha_2-\gamma_2)(1-l_1l_2)}\Bigg[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(\beta_2)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_2-1}g_1(\tau,s)\,d\tau + \sum_{j=1}^{n} b_j g_1(\eta_j,s)\Bigg],$$

and for $k,j = 1,2$,

$$D_t^{\gamma_j} g_k(t,s) = \frac{1}{\Gamma(\alpha_k-\gamma_j)}\begin{cases} t^{\alpha_k-\gamma_j-1}(1-s)^{\alpha_k-1} - (t-s)^{\alpha_k-\gamma_j-1}, & 0 \le s \le t \le 1, \\ t^{\alpha_k-\gamma_j-1}(1-s)^{\alpha_k-1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.14}$$

Lemma 2.4 *Assume that* (F_1) *holds. Then the functions* $g_k(t,s)$ *and* $D_t^{r_j} g_k(t,s)$, $k,j = 1,2$, *defined by* (2.6) *and* (2.14) *have the following properties*:

(1) $0 \le g_k(t,s) \le \frac{1}{\Gamma(\alpha_k)} t^{\alpha_k-1}[(1-s)^{\alpha_k-1}+1]$ *for* $t,s \in [0,1]$;

(2) $|D_t^{r_j} g_k(t,s)| \le \frac{1}{\Gamma(\alpha_k-\gamma_j)} t^{\alpha_k-\gamma_j-1}[(1-s)^{\alpha_k-1}+1]$ *for* $t,s \in [0,1]$.

Proof (1) For $0 \le s \le t \le 1$, we have

$$\begin{aligned} g_k(t,s) &= \frac{1}{\Gamma(\alpha_k)}\big(t^{\alpha_k-1}(1-s)^{\alpha_k-1} - (t-s)^{\alpha_k-1}\big) \\ &= \frac{1}{\Gamma(\alpha_k)}\big((t-ts)^{\alpha_k-1} - (t-s)^{\alpha_k-1}\big) \\ &\ge 0, \\ g_k(t,s) &= \frac{1}{\Gamma(\alpha_k)}\big(t^{\alpha_k-1}(1-s)^{\alpha_k-1} - (t-s)^{\alpha_k-1}\big) \\ &\le \frac{1}{\Gamma(\alpha_k)} t^{\alpha_k-1}(1-s)^{\alpha_k-1} \\ &\le \frac{1}{\Gamma(\alpha_k)} t^{\alpha_k-1}\big[(1-s)^{\alpha_k-1}+1\big]. \end{aligned}$$

For $0 \le t \le s \le 1$, we have

$$0 \le g_k(t,s) = \frac{1}{\Gamma(\alpha_k)} t^{\alpha_k-1}(1-s)^{\alpha_k-1} \le \frac{1}{\Gamma(\alpha_k)} t^{\alpha_k-1}\big[(1-s)^{\alpha_k-1}+1\big].$$

(2) For $0 \le s \le t \le 1$, we get

$$\begin{aligned} \big|D_t^{r_j} g_k(t,s)\big| &= \left|\frac{1}{\Gamma(\alpha_k-\gamma_j)}\big[t^{\alpha_k-\gamma_j-1}(1-s)^{\alpha_k-1} - (t-s)^{\alpha_k-\gamma_j-1}\big]\right| \\ &\le \frac{1}{\Gamma(\alpha_k-\gamma_j)}\big[t^{\alpha_k-\gamma_j-1}(1-s)^{\alpha_k-1} + (t-s)^{\alpha_k-\gamma_j-1}\big] \\ &\le \frac{1}{\Gamma(\alpha_k-\gamma_j)}\big[t^{\alpha_k-\gamma_j-1}(1-s)^{\alpha_k-1} + t^{\alpha_k-\gamma_j-1}\big] \\ &= \frac{1}{\Gamma(\alpha_k-\gamma_j)} t^{\alpha_k-\gamma_j-1}\big[(1-s)^{\alpha_k-1}+1\big]. \end{aligned}$$

For $0 \le t \le s \le 1$, we get

$$\begin{aligned}\left|D_t^{\gamma_j} g_k(t,s)\right| &= \frac{1}{\Gamma(\alpha_k-\gamma_j)} t^{\alpha_k-\gamma_j-1}(1-s)^{\alpha_k-1} \\ &\le \frac{1}{\Gamma(\alpha_k-\gamma_j)} t^{\alpha_k-\gamma_j-1}\bigl[(1-s)^{\alpha_k-1}+1\bigr].\end{aligned}$$

This completes the proof of the lemma. □

For convenience, we denote

$$\varrho_1 = \frac{1}{\Gamma(\alpha_1)}\left[1+\frac{l_2}{1-l_1l_2}\left(\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(\beta_2)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_2-1}\tau^{\alpha_1-1}\,d\tau + \sum_{j=1}^n b_j\eta_j^{\alpha_1-1}\right)\right], \tag{2.15}$$

$$\varrho_2 = \frac{1}{\Gamma(\alpha_2)}\left[1+\frac{l_1}{1-l_1l_2}\left(\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(\beta_1)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_1-1}\tau^{\alpha_2-1}\,d\tau + \sum_{j=1}^n b_j\eta_j^{\alpha_2-1}\right)\right], \tag{2.16}$$

$$\rho_1 = \frac{1}{\Gamma(\alpha_2)(1-l_1l_2)}\left[\sum_{i=1}^m \frac{\lambda_{1i}}{\Gamma(\beta_1)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_1-1}\tau^{\alpha_2-1}\,d\tau + \sum_{j=1}^n b_j\eta_j^{\alpha_2-1}\right], \tag{2.17}$$

$$\rho_2 = \frac{1}{\Gamma(\alpha_1)(1-l_1l_2)}\left[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(\beta_2)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_2-1}\tau^{\alpha_1-1}\,d\tau + \sum_{j=1}^n b_j\eta_j^{\alpha_1-1}\right]. \tag{2.18}$$

Lemma 2.5 *Assume that* (F_1)–(F_3) *hold. Then for* $(t,s) \in [0,1]\times[0,1]$, *the functions* $K_i(t,s)$ *and* $H_i(t,s)$, $i=1,2,3,4$, *defined by* (2.4), (2.5), *and* (2.13) *satisfy the following inequalities*:

(1) $$\begin{aligned} &0 \le K_1(t,s) \le t^{\alpha_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\varrho_1, \\ &0 \le K_2(t,s) \le t^{\alpha_2-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\varrho_2, \\ &\left|K_3(t,s)\right| \le \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)} t^{\alpha_1-\gamma_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\varrho_1, \\ &\left|K_4(t,s)\right| \le \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-\gamma_2)} t^{\alpha_2-\gamma_2-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\varrho_2;\end{aligned}$$

(2) $$\begin{aligned} &0 \le H_1(t,s) \le t^{\alpha_1-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\rho_1, \\ &0 \le H_2(t,s) \le t^{\alpha_2-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\rho_2, \\ &0 \le H_3(t,s) \le \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)} t^{\alpha_1-\gamma_1-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\rho_1, \\ &0 \le H_4(t,s) \le \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-\gamma_2)} t^{\alpha_2-\gamma_2-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\rho_2.\end{aligned}$$

Proof (1) According to (F_3), Lemma 2.4, and the definition of $K_i(t,s)$, we obtain

$$\begin{aligned} 0 \le K_1(t,s) &= g_1(t,s) + \frac{l_2 t^{\alpha_1-1}}{1-l_1l_2}\left[\sum_{i=1}^m \frac{\lambda_{2i}}{\Gamma(\beta_2)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_2-1}g_1(\tau,s)\,d\tau + \sum_{j=1}^n b_j g_1(\eta_j,s)\right] \\ &\le \frac{1}{\Gamma(\alpha_1)} t^{\alpha_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\end{aligned}$$

$$\begin{aligned}
&+\frac{l_2 t^{\alpha_1-1}}{1-l_1l_2}\Biggl[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(\beta_2)\Gamma(\alpha_1)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_2-1}\tau^{\alpha_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\,d\tau\\
&+\frac{1}{\Gamma(\alpha_1)}\sum_{j=1}^{n}b_j\eta_j^{\alpha_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\Biggr]\\
={}&t^{\alpha_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\frac{1}{\Gamma(\alpha_1)}\Biggl[1+\frac{l_2}{1-l_1l_2}\Biggl(\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(\beta_2)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_2-1}\tau^{\alpha_1-1}\,d\tau\\
&+\sum_{j=1}^{n}b_j\eta_j^{\alpha_1-1}\Biggr)\Biggr]\\
={}&t^{\alpha_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\varrho_1
\end{aligned}$$

and

$$\begin{aligned}
\bigl|K_3(t,s)\bigr| ={}& \Biggl|D_t^{\gamma_1}g_1(t,s)+\frac{l_2t^{\alpha_1-\gamma_1-1}\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)(1-l_1l_2)}\Biggl[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(\beta_2)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_2-1}g_1(\tau,s)\,d\tau\\
&+\sum_{j=1}^{n}b_jg_1(\eta_j,s)\Biggr]\Biggr|\\
\le{}&\frac{1}{\Gamma(\alpha_1-\gamma_1)}t^{\alpha_1-\gamma_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\\
&+\frac{l_2t^{\alpha_1-\gamma_1-1}\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)(1-l_1l_2)}\Biggl[\sum_{i=1}^{m}\frac{\lambda_{2i}}{\Gamma(\beta_2)\Gamma(\alpha_1)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_2-1}\\
&\times\tau^{\alpha_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\,d\tau+\frac{1}{\Gamma(\alpha_1)}\sum_{j=1}^{n}b_j\eta_j^{\alpha_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\Biggr]\\
\le{}&\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)}t^{\alpha_1-\gamma_1-1}\bigl[(1-s)^{\alpha_1-1}+\bigr]\varrho_1,
\end{aligned}$$

where ϱ_1 is defined by (2.15).

Similarly, we get

$$\begin{aligned}
&0\le K_2(t,s)\le t^{\alpha_2-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\varrho_2,\\
&\bigl|K_4(t,s)\bigr|\le\frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-\gamma_2)}t^{\alpha_2-\gamma_2-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\varrho_2,
\end{aligned}$$

where ϱ_2 is defined by (2.16).

(2) According to (F_3), Lemma 2.4, and the definition of $H_k(t,s)$ $(k=1,2,3,4)$, we infer that

$$\begin{aligned}
0\le H_1(t,s)={}&\frac{t^{\alpha_1-1}}{1-l_1l_2}\Biggl[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(\beta_1)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_1-1}g_2(\tau,s)\,d\tau+\sum_{j=1}^{n}b_jg_2(\eta_j,s)\Biggr]\\
\le{}&\frac{t^{\alpha_1-1}}{1-l_1l_2}\Biggl[\sum_{i=1}^{m}\frac{\lambda_{1i}}{\Gamma(\beta_1)\Gamma(\alpha_2)}\int_0^{\xi_i}(\xi_i-\tau)^{\beta_1-1}\tau^{\alpha_2-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{1}{\Gamma(\alpha_2)} \sum_{j=1}^{n} b_j \eta_j^{\alpha_2-1} \bigl[(1-s)^{\alpha_2-1}+1\bigr] \Biggr] \\
&= t^{\alpha_1-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr] \frac{1}{\Gamma(\alpha_2)(1-l_1 l_2)} \Biggl[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i-\tau)^{\beta_1-1}\tau^{\alpha_2-1}\,d\tau \\
&\quad + \sum_{j=1}^{n} b_j \eta_j^{\alpha_2-1} \Biggr] \\
&= t^{\alpha_1-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\rho_1
\end{aligned}
$$

and

$$
\begin{aligned}
&0 \le H_3(t,s) \\
&= \frac{\Gamma(\alpha_1)t^{\alpha_1-\gamma_1-1}}{\Gamma(\alpha_1-\gamma_1)(1-l_1 l_2)} \Biggl[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)} \int_0^{\xi_i} (\xi_i-\tau)^{\beta_1-1} g_2(\tau,s)\,d\tau + \sum_{j=1}^{n} b_j g_2(\eta_j,s) \Biggr] \\
&\le \frac{\Gamma(\alpha_1)t^{\alpha_1-\gamma_1-1}}{\Gamma(\alpha_1-\gamma_1)(1-l_1 l_2)} \Biggl[\sum_{i=1}^{m} \frac{\lambda_{1i}}{\Gamma(\beta_1)\Gamma(\alpha_2)} \int_0^{\xi_i} (\xi_i-\tau)^{\beta_1-1}\tau^{\alpha_2-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr] d\tau \\
&\quad + \frac{1}{\Gamma(\alpha_2)} \sum_{j=1}^{n} b_j \eta_j^{\alpha_2-1} \bigl[(1-s)^{\alpha_2-1}+1\bigr] \Biggr] \\
&= \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)} t^{\alpha_1-\gamma_1-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\rho_1,
\end{aligned}
$$

where ρ_1 is defined by (2.17). Analogously, we get

$$
\begin{aligned}
&0 \le H_2(t,s) \le t^{\alpha_2-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\rho_2, \\
&0 \le H_4(t,s) \le \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-\gamma_2)} t^{\alpha_2-\gamma_2-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\rho_2,
\end{aligned}
$$

where ρ_2 is defined by (2.18).

This completes the proof of the lemma. □

Let $X = \{u | u \in C[0,1] \text{ and } D_t^{\gamma_1} u \in C[0,1]\}$ be endowed with the norm

$$
\|u\| = \max\bigl\{\|u\|_0, \bigl\|D_t^{\gamma_1} u\bigr\|_0\bigr\},
$$

where $\|u\|_0 = \max_{t\in[0,1]} |u(t)|$ and $\|D_t^{\gamma_1} u\|_0 = \max_{t\in[0,1]} |D_t^{\gamma_1} u(t)|$. Also let $Y = \{v | v \in C[0,1] \text{ and } D_t^{\gamma_2} v \in C[0,1]\}$ be endowed with the norm

$$
\|v\| = \max\bigl\{\|v\|_0, \bigl\|D_t^{\gamma_2} v\bigr\|_0\bigr\},
$$

where $\|v\|_0 = \max_{t\in[0,1]} |v(t)|$ and $\|D_t^{\gamma_1} v\|_0 = \max_{t\in[0,1]} |D_t^{\gamma_1} v(t)|$. We introduce the product space $(X \times Y, \|(u,v)\|)$ endowed with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$ and define a partial order over the product space:

$$
\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \ge \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}
$$

if $u_1(t) \ge u_2(t)$, $v_1(t) \ge v_2(t)$, $D_t^{\gamma_1} u_1(t) \ge D_t^{\gamma_1} u_2(t)$, and $D_t^{\gamma_2} v_1(t) \ge D_t^{\gamma_2} v_2(t)$, $t \in [0,1]$.

Lemma 2.6 $(X \times Y, \|(u,v)\|)$ *is a Banach space.*

Proof Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in the space $(X, \|\cdot\|)$. It is obvious that $\{u_n\}_{n=1}^{\infty}$ and $\{D_t^{\gamma_1} u_n\}_{n=1}^{\infty}$ are Cauchy sequences in the space $C[0,1]$. Let us assume that $\{u_n\}_{n=1}^{\infty}$ and $\{D_t^{\gamma_1} u_n\}_{n=1}^{\infty}$ uniformly converge on $[0,1]$ to $u \in C[0,1]$ and $w \in C[0,1]$. Now we should prove that $w = D_t^{\gamma_1} u$.

For $t \in [0,1]$, we have

$$\begin{aligned}\left|I_t^{\gamma_1} D_t^{\gamma_1} u_n(t) - I_t^{\gamma_1} w(t)\right| &\le \frac{1}{\Gamma(\gamma_1)} \int_0^t (t-s)^{\gamma_1 - 1} \left|D_t^{\gamma_1} u_n(s) - w(s)\right| ds \\ &\le \frac{1}{\Gamma(\gamma_1 + 1)} \max_{s \in [0,1]} \left|D_t^{\gamma_1} u_n(s) - w(s)\right|.\end{aligned}$$

In view of the convergence of $\{u_n\}_{n=1}^{\infty}$, we obtain that $\lim_{n\to\infty} I_t^{\gamma_1} D_t^{\gamma_1} u_n(t) = I_t^{\gamma_1} w(t)$ uniformly for $t \in [0,1]$. Otherwise, by Remark 2.1 we have $I_t^{\gamma_1} D_t^{\gamma_1} u_n(t) = u_n(t)$ for $t \in [0,1]$. These two facts imply that

$$\lim_{n\to\infty} I_t^{\gamma_1} D_t^{\gamma_1} u_n(t) = \lim_{n\to\infty} u_n(t) = I_t^{\gamma_1} w(t) \quad \text{for } t \in [0,1].$$

Combining this with $\lim_{n\to\infty} u_n(t) = u(t)$ for $t \in [0,1]$, we have

$$u(t) = I_t^{\gamma_1} w(t) \quad \text{for } t \in [0,1]. \tag{2.19}$$

Taking the γ_1th-order derivatives of both sides of equation (2.19), in consequence, we have

$$D_t^{\gamma_1} I_t^{\gamma_1} w(t) = D_t^{\gamma_1} u(t) \quad \text{for } t \in [0,1].$$

By Remark 2.1 this leads to

$$w(t) = D_t^{\gamma_1} u(t) \quad \text{for } t \in [0,1],$$

which proves that $(X, \|\cdot\|)$ is a Banach space.

In the same way, we can prove that $(Y, \|\cdot\|)$ is a Banach space. Moreover, the product space $(X \times Y, \|(u,v)\|)$ is also a Banach space with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$.

The proof of the lemma is completed. □

Further, we define the cone $P \subset X \times Y$ by $P = \{(u,v) \in X \times Y : u(t) \ge 0, v(t) \ge 0, D_t^{\gamma_1} u(t) \ge 0, D_t^{\gamma_2} v(t) \ge 0, t \in [0,1]\}$. For all $(u,v) \in P$, in view of Lemma 2.3 and (F_4), let $T: P \to P$ be the operator defined by

$$T(u,v)(t) = \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix},$$

where

$$T_1(u,v)(t) = \int_0^1 K_1(t,s) f_{1(u,v)}(s)\, ds + \int_0^1 H_1(t,s) f_{2(u,v)}(s)\, ds, \tag{2.20}$$

$$T_2(u,v)(t) = \int_0^1 K_2(t,s) f_{2(u,v)}(s)\,ds + \int_0^1 H_2(t,s) f_{1(u,v)}(s)\,ds, \tag{2.21}$$

and for convenience, we set

$$\begin{cases} f_{1(u,v)}(s) \triangleq f_1(s,u(s),v(s),D_t^{\gamma_1}u(s),D_t^{\gamma_2}v(s)), \\ f_{2(u,v)}(s) \triangleq f_2(s,u(s),v(s),D_t^{\gamma_1}u(s),D_t^{\gamma_2}v(s)). \end{cases}$$

Also, from (2.12) we have

$$D_t^{\gamma_1} T_1(u,v)(t) = \int_0^1 K_3(t,s) f_{1(u,v)}(s)\,ds + \int_0^1 H_3(t,s) f_{2(u,v)}(s)\,ds, \tag{2.22}$$

$$D_t^{\gamma_2} T_2(u,v)(t) = \int_0^1 K_4(t,s) f_{2(u,v)}(s)\,ds + \int_0^1 H_4(t,s) f_{1(u,v)}(s)\,ds. \tag{2.23}$$

Lemma 2.7 *The operator $T : P \to P$ is completely continuous.*

Proof By the continuity of the functions $K_1(t,s)$–$K_4(t,s)$, $H_1(t,s)$–$H_4(t,s)$, and f_1 and f_2 the operator T is continuous.

Then we show T is uniformly bounded. Let $\Omega \subset P$ be bounded. There exists a positive constant M satisfying the inequality

$$\max\bigl\{\bigl|f_{1(u,v)}(t)\bigr|, \bigl|f_{2(u,v)}(t)\bigr|\bigr\} \le M, \quad \forall (u,v) \in \Omega. \tag{2.24}$$

For any $(u,v) \in \Omega$, combining Lemma 2.5 with (2.20), (2.22), and (2.24), we get

$$\begin{aligned} \bigl|T_1(u,v)(t)\bigr| &\le \left| M \int_0^1 K_1(t,s)\,ds + M \int_0^1 H_1(t,s)\,ds \right| \\ &\le M \int_0^1 \bigl\{ t^{\alpha_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\varrho_1 + t^{\alpha_1-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\rho_1 \bigr\}\,ds \\ &\le M \int_0^1 \bigl\{ \bigl[(1-s)^{\alpha_1-1}+1\bigr]\varrho_1 + \bigl[(1-s)^{\alpha_2-1}+1\bigr]\rho_1 \bigr\}\,ds \end{aligned}$$

and

$$\begin{aligned} &\bigl|D_t^{\gamma_1} T_1(u,v)(t)\bigr| \\ &\quad\le \left| M \int_0^1 K_3(t,s)\,ds + M \int_0^1 H_3(t,s)\,ds \right| \\ &\quad\le \frac{M\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)} \int_0^1 \bigl\{ t^{\alpha_1-\gamma_1-1}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\varrho_1 + t^{\alpha_1-\gamma_1-1}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\rho_1 \bigr\}\,ds \\ &\quad\le \frac{M\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)} \int_0^1 \bigl\{ \bigl[(1-s)^{\alpha_1-1}+1\bigr]\varrho_1 + \bigl[(1-s)^{\alpha_2-1}+1\bigr]\rho_1 \bigr\}\,ds, \end{aligned}$$

which implies that $\|T_1(u,v)\|$ is uniformly bounded. Further, we get that $\|T_2(u,v)\|$ is also uniformly bounded. Thus it follows from the above inequalities that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. For any $(u,v) \in \Omega$ and $t_1, t_2 \in [0,1]$, in view of Lemma 2.5, (2.20), (2.22), and (2.24), we infer that

$$\begin{aligned}
&\bigl|T_1(u,v)(t_2) - T_1(u,v)(t_1)\bigr| \\
&\quad \leq \left|M\int_0^1 \bigl(K_1(t_2,s) - K_1(t_1,s)\bigr)\,ds\right| + \left|M\int_0^1 \bigl(H_1(t_2,s) - H_1(t_1,s)\bigr)\,ds\right| \\
&\quad \leq M\int_0^1 \bigl\{\bigl|\bigl(t_2^{\alpha_1-1} - t_1^{\alpha_1-1}\bigr)\bigl[(1-s)^{\alpha_1-1} + 1\bigr]\bigr|\varrho_1 \\
&\qquad + \bigl|\bigl(t_2^{\alpha_1-1} - t_1^{\alpha_1-1}\bigr)\bigl[(1-s)^{\alpha_2-1} + 1\bigr]\bigr|\rho_1\bigr\}\,ds.
\end{aligned}$$

Applying the mean value theorem, we have the inequality

$$t_2^{\alpha-1} - t_1^{\alpha-1} \leq (\alpha-1)|t_2 - t_1|.$$

This implies that

$$\begin{aligned}
&\bigl|T_1(u,v)(t_2) - T_1(u,v)(t_1)\bigr| \\
&\quad \leq M\int_0^1 \bigl\{(\alpha_1-1)\bigl[(1-s)^{\alpha_1-1} + 1\bigr]\varrho_1 + (\alpha_1-1)\bigl[(1-s)^{\alpha_2-1} + 1\bigr]\rho_1\bigr\}\,ds|t_2 - t_1| \\
&\quad \to 0 \quad \text{as } t_2 \to t_1.
\end{aligned}$$

Besides, we find

$$\begin{aligned}
&\bigl|D_t^{\gamma_1}T_1(u,v)(t_2) - D_t^{\gamma_1}T_1(u,v)(t_1)\bigr| \\
&\quad \leq \left|M\int_0^1 \bigl(K_3(t_2,s) - K_3(t_1,s)\bigr)\,ds\right| + \left|M\int_0^1 \bigl(H_3(t_2,s) - H_3(t_1,s)\bigr)\,ds\right| \\
&\quad \leq \frac{M\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)}\int_0^1 \bigl\{(\alpha_1-\gamma_1-1)\bigl[(1-s)^{\alpha_1-1} + 1\bigr]\varrho_1 \\
&\qquad + (\alpha_1-\gamma_1-1)\bigl[(1-s)^{\alpha_2-1} + 1\bigr]\rho_1\bigr\}\,ds|t_2 - t_1| \\
&\quad \to 0 \quad \text{as } t_2 \to t_1.
\end{aligned}$$

Therefore T_1 is equicontinuous for all $(u,v) \in \Omega$, and thus T_2 is equicontinuous for all $(u,v) \in \Omega$.

As a consequence, the operator $T(u,v)$ is equicontinuous for all $(u,v) \in \Omega$. By the Arzelà–Ascoli theorem the operator $T(u,v)$ is completely continuous. □

3 Existence of monotone iterative positive solutions

Now, based on Lemmas 2.5–2.7, we will show that there exist positive extremal solutions for BVP (1.1)–(1.2) by the monotone iterative method.

Theorem 3.1 *Assume that* (F_1)–(F_4) *hold. Let* A_1, A_2, *and* l *be three positive constants satisfying*

$$\begin{aligned} l = \max\biggl\{&A_1\int_0^1 \varrho_1\bigl[(1-s)^{\alpha_1-1}+1\bigr]\,ds + A_2\int_0^1\bigl[(1-s)^{\alpha_2-1}+1\bigr]\rho_1\,ds, \\ &A_1\int_0^1 \rho_2\bigl[(1-s)^{\alpha_1-1}+1\bigr]\,ds + A_2\int_0^1 \varrho_2\bigl[(1-s)^{\alpha_2-1}+1\bigr]\,ds\biggr\}, \end{aligned} \tag{3.1}$$

(S1) *For* $t\in[0,1]$, $f_j(t,x_1,x_2,x_3,x_4)$ *is increasing in* $x_i\in[0,l]$ $(i=1,2,3,4)$ *for* $j=1,2$;

(S2) $\max_{0\le t\le 1} f_i(t,l,l,l,l)\le A_i$, $f_i(t,0,0,0,0)\not\equiv 0$, $0\le t\le 1$, *for* $i=1,2$.

Then BVP (1.1)–(1.2) *has positive solutions* (u^*,v^*) *and* (w^*,z^*) *satisfying* $0\le\|(u^*,v^*)\|\le l$, *and* $\lim_{n\to\infty}(u_n,v_n)=(u^*,v^*)$,

$$(u_n,v_n)=T(u_{n-1},v_{n-1})=\begin{pmatrix} T_1(u_{n-1},v_{n-1})\\ T_2(u_{n-1},v_{n-1})\end{pmatrix},\quad n=1,2,\ldots,$$

and

$$\begin{pmatrix} u_0(t)\\ v_0(t)\end{pmatrix}=\begin{pmatrix} t^{\alpha_1-1}\int_0^1\{A_1\varrho_1[(1-s)^{\alpha_1-1}+1]+A_2\rho_1[(1-s)^{\alpha_2-1}+1]\}\,ds\\ t^{\alpha_2-1}\int_0^1\{A_1\rho_2[(1-s)^{\alpha_1-1}+1]+A_2\varrho_2[(1-s)^{\alpha_2-1}+1]\}\,ds\end{pmatrix}; \tag{3.2}$$

$0\le\|(w^*,z^*)\|\le l$, *and* $\lim_{n\to\infty}(w_n,z_n)=(w^*,z^*)$,

$$(w_n,z_n)=T(w_{n-1},z_{n-1})=\begin{pmatrix} T_1(w_{n-1},z_{n-1})\\ T_2(w_{n-1},z_{n-1})\end{pmatrix},\quad n=1,2,\ldots,$$

and

$$\begin{pmatrix} w_0(t)\\ z_0(t)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}. \tag{3.3}$$

Proof Denote $P_l=\{(u,v)\in P\,|\,\|(u,v)\|\le l\}$, where l is introduced by (3.1). In the following, we first prove that $T:P_l\to P_l$. Let $(u,v)\in P_l$. Then for $t\in[0,1]$, we have

$$\begin{aligned} &0\le u(t)\le\|u\|\le l,\qquad 0\le\bigl|D_t^{\gamma_1}u(t)\bigr|\le\|u\|\le l,\\ &0\le v(t)\le\|v\|\le l,\qquad 0\le\bigl|D_t^{\gamma_2}v(t)\bigr|\le\|v\|\le l. \end{aligned}$$

So, for $0\le t\le 1$, $i=1,2$, by (S1) and (S2) we get

$$0\le f_i\bigl(t,u(t),v(t),D_t^{\gamma_1}u(t),D_t^{\gamma_2}v(t)\bigr)\le\max_{0\le t\le 1}\bigl\{f_i(t,l,l,l,l)\bigr\}\le A_i. \tag{3.4}$$

Consequently, for $t\in[0,1]$, in view of Lemma 2.5 and (3.4), we have

$$\begin{aligned} T_1(u,v)(t)=&\int_0^1 K_1(t,s)f_1\bigl(s,u(s),v(s),D_t^{\gamma_1}u(s),D_t^{\gamma_2}v(s)\bigr)\,ds\\ &+\int_0^1 H_1(t,s)f_2\bigl(s,u(s),v(s),D_t^{\gamma_1}u(s),D_t^{\gamma_2}v(s)\bigr)\,ds \end{aligned}$$

$$
\begin{aligned}
&\leq A_1 \int_0^1 \left[(1-s)^{\alpha_1-1}+1\right]\varrho_1\,ds + A_2 \int_0^1 \left[(1-s)^{\alpha_2-1}+1\right]\rho_1\,ds \leq l,\\
\left|D_t^{\gamma_1} T_1(u,v)(t)\right| &= \left|\int_0^1 K_3(t,s) f_1\bigl(s,u(s),v(s),D_t^{\gamma_1}u(s),D_t^{\gamma_2}v(s)\bigr)\,ds\right.\\
&\quad \left.+ \int_0^1 H_3(t,s) f_2\bigl(s,u(s),v(s),D_t^{\gamma_1}u(s),D_t^{\gamma_2}v(s)\bigr)\,ds\right|\\
&\leq A_1 \int_0^1 \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)}\left[(1-s)^{\alpha_1-1}+1\right]\varrho_1\,ds\\
&\quad + A_2 \int_0^1 \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1-\gamma_1)}\left[(1-s)^{\alpha_2-1}+1\right]\rho_1\,ds\\
&\leq A_1 \int_0^1 \left[(1-s)^{\alpha_1-1}+1\right]\varrho_1\,ds + A_2 \int_0^1 \left[(1-s)^{\alpha_2-1}+1\right]\rho_1\,ds \leq l,
\end{aligned}
$$

and, further,

$$
\begin{aligned}
T_2(u,v)(t) &= \int_0^1 K_2(t,s) f_2\bigl(s,u(s),v(s),D_t^{\gamma_1}u(s),D_t^{\gamma_2}v(s)\bigr)\,ds\\
&\quad + \int_0^1 H_2(t,s) f_1\bigl(s,u(s),v(s),D_t^{\gamma_1}u(s),D_t^{\gamma_2}v(s)\bigr)\,ds\\
&\leq A_2 \int_0^1 \left[(1-s)^{\alpha_2-1}+1\right]\varrho_2\,ds + A_1 \int_0^1 \left[(1-s)^{\alpha_1-1}+1\right]\rho_2\,ds \leq l,\\
\left|D_t^{\gamma_2} T_2(u,v)(t)\right| &= \left|\int_0^1 K_4(t,s) f_2\bigl(s,u(s),v(s),D_t^{\gamma_1}u(s),D_t^{\gamma_2}v(s)\bigr)\,ds\right.\\
&\quad \left.+ \int_0^1 H_4(t,s) f_1\bigl(s,u(s),v(s),D_t^{\gamma_1}u(s),D_t^{\gamma_2}v(s)\bigr)\,ds\right|\\
&\leq A_2 \int_0^1 \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-\gamma_2)}\left[(1-s)^{\alpha_2-1}+1\right]\varrho_2\,ds\\
&\quad + A_1 \int_0^1 \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_2-\gamma_2)}\left[(1-s)^{\alpha_1-1}+1\right]\rho_2\,ds\\
&\leq A_2 \int_0^1 \left[(1-s)^{\alpha_2-1}+1\right]\varrho_2\,ds + A_1 \int_0^1 \left[(1-s)^{\alpha_1-1}+1\right]\rho_2\,ds \leq l.
\end{aligned}
$$

As a result, we obtain

$$
\left\|T(u,v)\right\| = \left\{\max_{0\leq t\leq 1}\left|T_1u(t)\right|, \max_{0\leq t\leq 1}\left|D_t^{\gamma_1}T_1u(t)\right|, \max_{0\leq t\leq 1}\left|T_2v(t)\right|, \max_{0\leq t\leq 1}\left|D_t^{\gamma_2}T_2v(t)\right|\right\} \leq l,
$$

and thus $T: P_l \to P_l$.

According to (3.2) and (3.3), it is obvious that $(u_0,v_0),(w_0,z_0) \in P_l$. Using the completely continuous operator T, we define the sequences $\{(u_n,v_n)\}$ and $\{(w_n,z_n)\}$ as $(u_n,v_n) = T(u_{n-1},v_{n-1})$, $(w_n,z_n) = T(w_{n-1},z_{n-1})$ for $n = 1,2,\ldots$. Since $T: P_l \to P_l$, we get that $(u_n,v_n),(w_n,z_n) \in P_l$ for $n = 1,2,\ldots$.

Hence we prove that there exist (u^*,v^*) and (w^*,z^*) satisfying $\lim_{n\to\infty}(u_n,v_n) = (u^*,v^*)$ and $\lim_{n\to\infty}(w_n,z_n) = (w^*,z^*)$, which are monotone positive solutions of problem (1.1)–(1.2).

For $t \in [0,1]$, by the definition of the iterative scheme we have

$$\begin{aligned} & T_1(u_0,v_0)(t) \\ &\quad = \int_0^1 K_1(t,s)f_{1(u_0,v_0)}(s)\,ds + \int_0^1 H_1(t,s)f_{2(u_0,v_0)}(s)\,ds \\ &\quad \leq t^{\alpha_1-1}\int_0^1 \bigl\{A_1\varrho_1\bigl[(1-s)^{\alpha_1-1}+1\bigr] + A_2\rho_1\bigl[(1-s)^{\alpha_2-1}+1\bigr]\bigr\}\,ds \\ &\quad = u_0(t) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & T_2(u_0,v_0)(t) \\ &\quad = \int_0^1 K_2(t,s)f_{2(u_0,v_0)}(s)\,ds + \int_0^1 H_2(t,s)f_{1(u_0,v_0)}(s)\,ds \\ &\quad \leq t^{\alpha_2-1}\int_0^1 \bigl\{A_1\rho_2\bigl[(1-s)^{\alpha_1-1}+1\bigr] + A_2\varrho_2\bigl[(1-s)^{\alpha_2-1}+1\bigr]\bigr\}\,ds \\ &\quad = v_0(t). \end{aligned} \tag{3.6}$$

From (3.5) and (3.6) we get

$$\begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix} = \begin{pmatrix} T_1(u_0,v_0)(t) \\ T_2(u_0,v_0)(t) \end{pmatrix} \leq \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix}. \tag{3.7}$$

Next, we discuss the monotonicity of the fractional derivative of (u,v). From (3.2) we obtain

$$\begin{aligned} D_t^{\gamma_1}u_0(t) &= \int_0^1 \frac{\Gamma(\alpha_1)t^{\alpha_1-\gamma_1-1}}{\Gamma(\alpha_1-\gamma_1)}\bigl\{A_1\varrho_1\bigl[(1-s)^{\alpha_1-1}+1\bigr] + A_2\rho_1\bigl[(1-s)^{\alpha_2-1}+1\bigr]\bigr\}\,ds \geq 0, \\ D_t^{\gamma_2}v_0(t) &= \int_0^1 \frac{\Gamma(\alpha_2)t^{\alpha_2-\gamma_2-1}}{\Gamma(\alpha_2-\gamma_2)}\bigl\{A_1\rho_2\bigl[(1-s)^{\alpha_1-1}+1\bigr] + A_2\varrho_2\bigl[(1-s)^{\alpha_2-1}+1\bigr]\bigr\}\,ds \geq 0. \end{aligned}$$

Hence we have

$$\begin{aligned} & \bigl|D_t^{\gamma_1}u_1(t)\bigr| = \bigl|D_t^{\gamma_1}T_1(u_0,v_0)(t)\bigr| \\ &\quad = \left|\int_0^1 K_3(t,s)f_{1(u_0,v_0)}(s)\,ds + \int_0^1 H_3(t,s)f_{2(u_0,v_0)}(s)\,ds\right| \\ &\quad \leq A_1\int_0^1 \frac{\Gamma(\alpha_1)t^{\alpha_1-\gamma_1-1}}{\Gamma(\alpha_1-\gamma_1)}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\varrho_1\,ds \\ &\qquad + A_2\int_0^1 \frac{\Gamma(\alpha_1)t^{\alpha_1-\gamma_1-1}}{\Gamma(\alpha_1-\gamma_1)}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\rho_1\,ds \\ &\quad = \bigl|D_t^{\gamma_1}u_0(t)\bigr| \end{aligned} \tag{3.8}$$

and

$$\bigl|D_t^{\gamma_2}v_1(t)\bigr| = \bigl|D_t^{\gamma_2}T_2(u_0,v_0)(t)\bigr|$$

$$\begin{aligned}
&= \left|\int_0^1 K_4(t,s)f_{2(u_0,v_0)}(s)\,ds + \int_0^1 H_4(t,s)f_{1(u_0,v_0)}(s)\,ds\right| \\
&\le A_2\int_0^1 \frac{\Gamma(\alpha_2)t^{\alpha_2-\gamma_2-1}}{\Gamma(\alpha_2-\gamma_2)}\bigl[(1-s)^{\alpha_2-1}+1\bigr]\varrho_2\,ds \\
&\quad + A_1\int_0^1 \frac{\Gamma(\alpha_2)t^{\alpha_2-\gamma_2-1}}{\Gamma(\alpha_2-\gamma_2)}\bigl[(1-s)^{\alpha_1-1}+1\bigr]\rho_2\,ds \\
&= \bigl|D_t^{\gamma_2}v_0(t)\bigr|. \qquad (3.9)
\end{aligned}$$

Combining (3.8) and (3.9), we easily to see that

$$\begin{pmatrix} |D_t^{\gamma_1}u_1(t)| \\ |D_t^{\gamma_2}v_1(t)| \end{pmatrix} = \begin{pmatrix} |D_t^{\gamma_1}T_1(u_0,v_0)(t)| \\ |D_t^{\gamma_2}T_2(u_0,v_0)(t)| \end{pmatrix} \le \begin{pmatrix} |D_t^{\gamma_1}u_0(t)| \\ |D_t^{\gamma_2}v_0(t)| \end{pmatrix}. \qquad (3.10)$$

Thus, for $0 \le t \le 1$, from (3.7), (3.10), and (S1) we do the second iteration

$$\begin{aligned}
&\begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} T_1(u_1,v_1)(t) \\ T_2(u_1,v_1)(t) \end{pmatrix} \le \begin{pmatrix} T_1(u_0,v_0)(t) \\ T_2(u_0,v_0)(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix}, \\
&\begin{pmatrix} |D_t^{\gamma_1}u_2(t)| \\ |D_t^{\gamma_2}v_2(t)| \end{pmatrix} = \begin{pmatrix} |D_t^{\gamma_1}T_1(u_1,v_1)(t)| \\ |D_t^{\gamma_2}T_2(u_1,v_1)(t)| \end{pmatrix} \le \begin{pmatrix} |D_t^{\gamma_1}T_1(u_0,v_0)(t)| \\ |D_t^{\gamma_2}T_2(u_0,v_0)(t)| \end{pmatrix} = \begin{pmatrix} |D_t^{\gamma_1}u_1(t)| \\ |D_t^{\gamma_2}v_1(t)| \end{pmatrix}.
\end{aligned}$$

By induction, for $n = 0, 1, 2, \dots$, we have

$$\begin{pmatrix} u_{n+1}(t) \\ v_{n+1}(t) \end{pmatrix} \le \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix}, \qquad \begin{pmatrix} |D_t^{\gamma_1}u_{n+1}(t)| \\ |D_t^{\gamma_2}v_{n+1}(t)| \end{pmatrix} \le \begin{pmatrix} |D_t^{\gamma_1}u_n(t)| \\ |D_t^{\gamma_2}v_n(t)| \end{pmatrix} \quad \text{for } 0 \le t \le 1.$$

So we assert that $(u_n, v_n) \to (u^*, v^*)$ and $T(u^*, v^*) = (u^*, v^*)$, since T is completely continuous and $(u_{n+1}, v_{n+1}) = T(u_n, v_n)$.

For the sequence $\{(w_n, z_n)\}_{n=1}^{\infty}$, we apply a similar argument. For $t \in [0, 1]$, we have

$$\begin{aligned}
\begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix} &= \begin{pmatrix} T_1(w_0,z_0)(t) \\ T_2(w_0,z_0)(t) \end{pmatrix} \\
&= \begin{pmatrix} \int_0^1 K_1(t,s)f_{1(w_0,z_0)}(s)\,ds + \int_0^1 H_1(t,s)f_{2(w_0,z_0)}(s)\,ds \\ \int_0^1 K_2(t,s)f_{2(w_0,z_0)}(s)\,ds + \int_0^1 H_2(t,s)f_{1(w_0,z_0)}(s)\,ds \end{pmatrix} \\
&\ge \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix}; \\
\begin{pmatrix} |D_t^{\gamma_1}w_1(t)| \\ |D_t^{\gamma_2}z_1(t)| \end{pmatrix} &= \begin{pmatrix} |D_t^{\gamma_1}T_1(w_0,z_0)(t)| \\ |D_t^{\gamma_2}T_2(w_0,z_0)(t)| \end{pmatrix} \\
&= \begin{pmatrix} |\int_0^1 K_3(t,s)f_{1(w_0,z_0)}(s)\,ds + \int_0^1 H_3(t,s)f_{2(w_0,z_0)}(s)\,ds| \\ |\int_0^1 K_4(t,s)f_{2(w_0,z_0)}(s)\,ds + \int_0^1 H_4(t,s)f_{1(w_0,z_0)}(s)\,ds| \end{pmatrix} \\
&\ge \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} |D_t^{\gamma_1}w_0(t)| \\ |D_t^{\gamma_2}z_0(t)| \end{pmatrix}.
\end{aligned}$$

To summarize, for $0 \le t \le 1$, we have

$$\begin{pmatrix} w_2(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} T_1(w_1, z_1)(t) \\ T_2(w_1, z_1)(t) \end{pmatrix} \ge \begin{pmatrix} T_1(w_0, z_0)(t) \\ T_2(w_0, z_0)(t) \end{pmatrix} = \begin{pmatrix} w_1(t) \\ z_1(t) \end{pmatrix},$$
$$\begin{pmatrix} |D_t^{\gamma_1} w_2(t)| \\ |D_t^{\gamma_2} z_2(t)| \end{pmatrix} = \begin{pmatrix} |D_t^{\gamma_1} T_1(w_1, z_1)(t)| \\ |D_t^{\gamma_2} T_2(w_1, z_1)(t)| \end{pmatrix} \ge \begin{pmatrix} |D_t^{\gamma_1} T_1(w_0, z_0)(t)| \\ |D_t^{\gamma_2} T_2(w_0, z_0)(t)| \end{pmatrix} = \begin{pmatrix} |D_t^{\gamma_1} w_1(t)| \\ |D_t^{\gamma_2} z_1(t)| \end{pmatrix}.$$

Analogously, for $n = 1, 2, \ldots$, we have

$$\begin{pmatrix} w_{n+1}(t) \\ z_{n+1}(t) \end{pmatrix} \ge \begin{pmatrix} w_n(t) \\ z_n(t) \end{pmatrix}, \qquad \begin{pmatrix} |D_t^{\gamma_1} w_{n+1}(t)| \\ |D_t^{\gamma_2} z_{n+1}(t)| \end{pmatrix} \ge \begin{pmatrix} |D_t^{\gamma_1} w_n(t)| \\ |D_t^{\gamma_2} z_n(t)| \end{pmatrix} \quad \text{for } 0 \le t \le 1.$$

So we can assert that $(w_n, z_n) \to (w^*, z^*)$ and $T(w^*, z^*) = (w^*, z^*)$, since T is completely continuous and $(w_{n+1}, z_{n+1}) = T(w_n, z_n)$.

Consequently, there exist (u^*, v^*) and (w^*, z^*) in P_l that are nonnegative extremal solutions of BVP (1.1)–(1.2). From (S2) it is obvious that $(u^*, v^*)(t) > 0$ and $(w^*, z^*)(t) > 0$ for $t \in [0, 1]$, since zero is not a solution of problem (1.1). The proof is completed. □

Example 3.1 For $t \in [0, 1]$, consider the following fractional differential system:

$$\begin{cases} D_t^{2.5} u(t) + (\frac{1}{2} t^2 + \frac{1}{10} u(t) + \frac{1}{32} \sin(v(t)) + \frac{1}{50} D_t^{0.3} u(t) + \frac{1}{D_t^{0.2} v(t) - \frac{1}{2}} + 2) = 0, \\ D_t^{2.5} v(t) + (\frac{1}{3} t + \frac{1}{4\pi} \sin(2\pi u(t)) + \frac{1}{18} v(t) + \frac{1}{5} (D_t^{0.3} u(t))^2 + D_t^{0.2} v(t) + \frac{10}{11}) = 0, \end{cases} \tag{3.11}$$

with the coupled integral and discrete mixed boundary conditions

$$\begin{cases} u(0) = u'(0) = 0, & u(1) = \sum_{i=1}^{2} \lambda_{1i} I_t^{1.5} v(\xi_i) + \sum_{i=1}^{2} b_i v(\eta_i), \\ v(0) = v'(0) = 0, & v(1) = \sum_{i=1}^{2} \lambda_{2i} I_t^{1.2} u(\xi_i) + \sum_{i=1}^{2} b_i u(\eta_i). \end{cases} \tag{3.12}$$

In this model, we set

$$\lambda_{11} = \frac{1}{2}, \qquad \lambda_{21} = \frac{2}{7}, \qquad \xi_1 = \frac{1}{4}, \qquad b_1 = \frac{1}{3}, \qquad \eta_1 = \frac{1}{3},$$
$$\lambda_{12} = \frac{1}{4}, \qquad \lambda_{22} = \frac{3}{7}, \qquad \xi_2 = \frac{3}{4}, \qquad b_2 = \frac{2}{3}, \qquad \eta_2 = \frac{2}{3}.$$

It is obvious that (F_1) and (F_2) hold. By calculation we get

$$l_1 = 0.4920, \qquad l_2 = 0.4521,$$
$$\varrho_1 = 0.9675, \qquad \varrho_2 = 0.9675,$$
$$\rho_1 = 0.4374, \qquad \rho_2 = 0.4760.$$

Setting $A_1 = A_2 = 5$, we get $l = \max\{A_1 \int_0^1 \varrho_1 [(1-s)^{\alpha_1 - 1} + 1]\, ds + A_2 \int_0^1 [(1-s)^{\alpha_2 - 1} + 1] \rho_1\, ds, A_1 \int_0^1 \rho_2 [(1-s)^{\alpha_1 - 1} + 1]\, ds + A_2 \int_0^1 \varrho_2 [(1-s)^{\alpha_2 - 1} + 1]\, ds\}$. Then all the hypotheses of Theorem 3.1 are satisfied with $l = 11$. Hence, BVP (3.11)–(3.12) has monotone positive solutions (u^*,v^*) and (w^*,z^*), which satisfy, for $t\in[0,1]$,

$$\begin{pmatrix} u_0(t)\\ v_0(t)\end{pmatrix} = \begin{pmatrix} t^{\alpha_1-1}\{A_1\int_0^1 \varrho_1[(1-s)^{\alpha_1-1}+1]\,ds + A_2\int_0^1[(1-s)^{\alpha_2-1}+1]\rho_1\,ds\}\\ t^{\alpha_2-1}\{A_1\int_0^1 \rho_2[(1-s)^{\alpha_1-1}+1]\,ds + A_2\int_0^1[(1-s)^{\alpha_2-1}+1]\varrho_2\,ds\}\end{pmatrix} = \begin{pmatrix} 9.8343t^{1.5}\\ 10.1045t^{1.5}\end{pmatrix}$$

and

$$\begin{pmatrix} w_0(t)\\ z_0(t)\end{pmatrix} = \begin{pmatrix} 0\\ 0\end{pmatrix}.$$

For $n=1,2,\ldots$, the two iterative schemes are

$$\begin{aligned} &\begin{pmatrix} u_0(t)\\ v_0(t)\end{pmatrix} = \begin{pmatrix} 9.8343t^{1.5}\\ 10.1045t^{1.5}\end{pmatrix}, \qquad \ldots,\\ &\begin{pmatrix} u_{n+1}(t)\\ v_{n+1}(t)\end{pmatrix} = \begin{pmatrix} T_1(u_n,v_n)(t)\\ T_2(u_n,v_n)(t)\end{pmatrix} = \begin{pmatrix} \int_0^1 K_1(t,s)f_{1(u_n,v_n)}(s)\,ds + \int_0^1 H_1(t,s)f_{2(u_n,v_n)}(s)\,ds\\ \int_0^1 K_2(t,s)f_{2(u_n,v_n)}(s)\,ds + \int_0^1 H_2(t,s)f_{1(u_n,v_n)}(s)\,ds\end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} &\int_0^1 K_1(t,s)f_{1(u_n,v_n)}(s)\,ds + \int_0^1 H_1(t,s)f_{2(u_n,v_n)}(s)\,ds\\ &\quad= \frac{1}{\Gamma(2.5)}\int_0^1\biggl\{t^{1.5}(1-s)^{1.5} + \frac{l_2t^{1.5}}{1-l_1l_2}\biggl[\frac{2}{7\Gamma(1.2)}\int_0^{\frac14}\biggl(\frac14-\tau\biggr)^{0.2}\tau^{1.5}(1-s)^{1.5}d\tau\\ &\qquad-\frac{2}{7\Gamma(1.2)}\int_s^{\frac14}\biggl(\frac14-\tau\biggr)^{0.2}(\tau-s)^{1.5}\,d\tau + \frac{3}{7\Gamma(1.2)}\int_0^{\frac34}\biggl(\frac34-\tau\biggr)^{0.2}\tau^{1.5}(1-s)^{1.5}d\tau\\ &\qquad-\frac{3}{7\Gamma(1.2)}\int_s^{\frac34}\biggl(\frac34-\tau\biggr)^{0.2}(\tau-s)^{1.5}\,d\tau + \frac13\cdot\frac13^{1.5}(1-s)^{1.5} + \frac23\cdot\frac23^{1.5}(1-s)^{1.5}\biggr]\biggr\}\\ &\qquad\times\biggl(\frac12s^2 + \frac{1}{10}u_n(s) + \frac{1}{32}\sin\bigl(v_n(s)\bigr) + \frac{1}{50}D_t^{0.3}u_n(s) + \frac{1}{D_t^{0.2}v_n(s)-\frac12} + 2\biggr)\,ds\\ &\qquad-\frac{1}{\Gamma(2.5)}\int_0^t\biggl\{(t-s)^{1.5} + \frac{l_2t^{1.5}}{1-l_1l_2}\biggl[\frac13\biggl(\frac13-s\biggr)^{1.5} + \frac23\biggl(\frac23-s\biggr)^{1.5}\biggr]\biggr\}\biggl(\frac12s^2 + \frac{1}{10}u_n(s)\\ &\qquad+\frac{1}{32}\sin\bigl(v_n(s)\bigr) + \frac{1}{50}D_t^{0.3}u_n(s) + \frac{1}{D_t^{0.2}v_n(s)-\frac12} + 2\biggr)\,ds\\ &\qquad+\frac{1}{\Gamma(2.5)}\int_0^1\frac{t^{1.5}}{1-l_1l_2}\biggl[\frac{1}{2\Gamma(1.5)}\int_0^{\frac14}\biggl(\frac14-\tau\biggr)^{0.5}\tau^{1.5}(1-s)^{1.5}\,d\tau\\ &\qquad-\frac{1}{2\Gamma(1.5)}\int_s^{\frac14}\biggl(\frac14-\tau\biggr)^{0.5}(\tau-s)^{1.5}\,d\tau + \frac{1}{4\Gamma(1.5)}\int_0^{\frac34}\biggl(\frac34-\tau\biggr)^{0.5}\tau^{1.5}(1-s)^{1.5}d\tau\\ &\qquad-\frac{1}{4\Gamma(1.5)}\int_s^{\frac34}\biggl(\frac34-\tau\biggr)^{0.5}(\tau-s)^{1.5}\,d\tau + \frac13\cdot\frac13^{1.5}(1-s)^{1.5} + \frac23\cdot\frac23^{1.5}(1-s)^{1.5}\biggr]\\ &\qquad\times\biggl(\frac13s + \frac{1}{4\pi}\sin\bigl(2\pi u_n(s)\bigr) + \frac{1}{18}v_n(t) + \frac15\bigl(D_t^{0.3}u_n(t)\bigr)^2 + D_t^{0.2}v_n(t) + \frac{10}{11}\biggr)\,ds \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(2.5)}\int_0^t \frac{t^{1.5}}{1-l_1l_2}\left[\frac{1}{3}\left(\frac{1}{3}-s\right)^{1.5}+\frac{2}{3}\left(\frac{2}{3}-s\right)^{1.5}\right] \\
&\times\left(\frac{1}{3}s+\frac{1}{4\pi}\sin\bigl(2\pi u_n(s)\bigr)+\frac{1}{18}v_n(t)\right. \\
&\left.+\frac{1}{5}\bigl(D_t^{0.3}u_n(t)\bigr)^2+D_t^{0.2}v_n(t)+\frac{10}{11}\right)ds,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 K_2(t,s)f_{2(u_n,v_n)}(s)\,ds+\int_0^1 H_2(t,s)f_{1(u_n,v_n)}(s)\,ds \\
&=\frac{1}{\Gamma(2.5)}\int_0^1\left\{t^{1.5}(1-s)^{1.5}+\frac{l_1t^{1.5}}{1-l_1l_2}\left[\frac{1}{2\Gamma(1.5)}\int_0^{\frac{1}{4}}\left(\frac{1}{4}-\tau\right)^{0.5}\tau^{1.5}(1-s)^{1.5}d\tau\right.\right. \\
&-\frac{1}{2\Gamma(1.5)}\int_s^{\frac{1}{4}}\left(\frac{1}{4}-\tau\right)^{0.5}(\tau-s)^{1.5}\,d\tau+\frac{1}{4\Gamma(1.5)}\int_0^{\frac{3}{4}}\left(\frac{3}{4}-\tau\right)^{0.5}\tau^{1.5}(1-s)^{1.5}d\tau \\
&\left.\left.-\frac{1}{4\Gamma(1.5)}\int_s^{\frac{3}{4}}\left(\frac{3}{4}-\tau\right)^{0.5}(\tau-s)^{1.5}\,d\tau+\frac{1}{3}\cdot\frac{1}{3}^{1.5}(1-s)^{1.5}+\frac{2}{3}\cdot\frac{2}{3}^{1.5}(1-s)^{1.5}\right]\right\} \\
&\times\left(\frac{1}{3}s+\frac{1}{4\pi}\sin\bigl(2\pi u_n(s)\bigr)+\frac{1}{18}v_n(t)+\frac{1}{5}\bigl(D_t^{0.3}u_n(t)\bigr)^2+D_t^{0.2}v_n(t)+\frac{10}{11}\right)ds \\
&-\frac{1}{\Gamma(2.5)}\int_0^t\left\{(t-s)^{1.5}+\frac{l_1t^{1.5}}{1-l_1l_2}\left[\frac{1}{3}\left(\frac{1}{3}-s\right)^{1.5}+\frac{2}{3}\left(\frac{2}{3}-s\right)^{1.5}\right]\right\} \\
&\times\left(\frac{1}{3}s+\frac{1}{4\pi}\sin\bigl(2\pi u_n(s)\bigr)\right. \\
&\left.+\frac{1}{18}v_n(t)+\frac{1}{5}\bigl(D_t^{0.3}u_n(t)\bigr)^2+D_t^{0.2}v_n(t)+\frac{10}{11}\right)ds \\
&+\frac{1}{\Gamma(2.5)}\int_0^1\frac{t^{1.5}}{1-l_1l_2}\left[\frac{2}{7\Gamma(1.2)}\int_0^{\frac{1}{4}}\left(\frac{1}{4}-\tau\right)^{0.2}\tau^{1.5}(1-s)^{1.5}\,d\tau\right. \\
&-\frac{2}{7\Gamma(1.2)}\int_s^{\frac{1}{4}}\left(\frac{1}{4}-\tau\right)^{0.2}(\tau-s)^{1.5}\,d\tau+\frac{3}{7\Gamma(1.2)}\int_0^{\frac{3}{4}}\left(\frac{3}{4}-\tau\right)^{0.2}\tau^{1.5}(1-s)^{1.5}d\tau \\
&\left.-\frac{3}{7\Gamma(1.2)}\int_s^{\frac{3}{4}}\left(\frac{3}{4}-\tau\right)^{0.2}(\tau-s)^{1.5}\,d\tau+\frac{1}{3}\cdot\frac{1}{3}^{1.5}(1-s)^{1.5}+\frac{2}{3}\cdot\frac{2}{3}^{1.5}(1-s)^{1.5}\right] \\
&\times\left(\frac{1}{2}s^2+\frac{1}{10}u_n(s)+\frac{1}{32}\sin\bigl(v_n(s)\bigr)+\frac{1}{50}D_t^{0.3}u_n(s)+\frac{1}{D_t^{0.2}v_n(s)-\frac{1}{2}}+2\right)ds \\
&-\frac{1}{\Gamma(2.5)}\int_0^t\frac{t^{1.5}}{1-l_1l_2}\left[\frac{1}{3}\left(\frac{1}{3}-s\right)^{1.5}+\frac{2}{3}\left(\frac{2}{3}-s\right)^{1.5}\right] \\
&\times\left(\frac{1}{2}s^2+\frac{1}{10}u_n(s)+\frac{1}{32}\sin\bigl(v_n(s)\bigr)\right. \\
&\left.+\frac{1}{50}D_t^{0.3}u_n(s)+\frac{1}{D_t^{0.2}v_n(s)-\frac{1}{2}}+2\right)ds,
\end{aligned}
$$

and

$$
\begin{pmatrix} w_0(t) \\ z_0(t) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \ldots,
$$

$$\begin{pmatrix} w_{n+1}(t) \\ z_{n+1}(t) \end{pmatrix} = \begin{pmatrix} \int_0^1 K_1(t,s)f_{1(w_n,z_n)}(s)\,ds + \int_0^1 H_1(t,s)f_{2(w_n,z_n)}(s)\,ds \\ \int_0^1 K_2(t,s)f_{2(w_n,z_n)}(s)\,ds + \int_0^1 H_2(t,s)f_{1(w_n,z_n)}(s)\,ds \end{pmatrix},$$

where

$$\begin{aligned}
&\int_0^1 K_1(t,s)f_{1(w_n,z_n)}(s)\,ds + \int_0^1 H_1(t,s)f_{2(w_n,z_n)}(s)\,ds \\
&\quad= \frac{1}{\Gamma(2.5)}\int_0^1 \biggl\{ t^{1.5}(1-s)^{1.5} + \frac{l_2 t^{1.5}}{1-l_1 l_2}\biggl[\frac{2}{7\Gamma(1.2)}\int_0^{\frac{1}{4}} \biggl(\frac{1}{4}-\tau\biggr)^{0.2} \tau^{1.5}(1-s)^{1.5}d\tau \\
&\qquad - \frac{2}{7\Gamma(1.2)}\int_s^{\frac{1}{4}} \biggl(\frac{1}{4}-\tau\biggr)^{0.2} (\tau-s)^{1.5}\,d\tau + \frac{3}{7\Gamma(1.2)}\int_0^{\frac{3}{4}} \biggl(\frac{3}{4}-\tau\biggr)^{0.2} \tau^{1.5}(1-s)^{1.5}d\tau \\
&\qquad - \frac{3}{7\Gamma(1.2)}\int_s^{\frac{3}{4}} \biggl(\frac{3}{4}-\tau\biggr)^{0.2} (\tau-s)^{1.5}\,d\tau + \frac{1}{3}\cdot\frac{1}{3}^{1.5}(1-s)^{1.5} + \frac{2}{3}\cdot\frac{2}{3}^{1.5}(1-s)^{1.5} \biggr]\biggr\} \\
&\qquad \times \biggl(\frac{1}{2}s^2 + \frac{1}{10}w_n(s) + \frac{1}{32}\sin\bigl(z_n(s)\bigr) + \frac{1}{50}D_t^{0.3}w_n(s) + \frac{1}{D_t^{0.2}z_n(s)-\frac{1}{2}} + 2 \biggr)\,ds \\
&\qquad - \frac{1}{\Gamma(2.5)}\int_0^t \biggl\{ (t-s)^{1.5} + \frac{l_2 t^{1.5}}{1-l_1 l_2}\biggl[\frac{1}{3}\biggl(\frac{1}{3}-s\biggr)^{1.5} + \frac{2}{3}\biggl(\frac{2}{3}-s\biggr)^{1.5} \biggr]\biggr\} \biggl(\frac{1}{2}s^2 + \frac{1}{10}w_n(s) \\
&\qquad + \frac{1}{32}\sin\bigl(z_n(s)\bigr) + \frac{1}{50}D_t^{0.3}w_n(s) + \frac{1}{D_t^{0.2}z_n(s)-\frac{1}{2}} + 2 \biggr)\,ds \\
&\qquad + \frac{1}{\Gamma(2.5)}\int_0^1 \frac{t^{1.5}}{1-l_1 l_2}\biggl[\frac{1}{2\Gamma(1.5)}\int_0^{\frac{1}{4}} \biggl(\frac{1}{4}-\tau\biggr)^{0.5} \tau^{1.5}(1-s)^{1.5}\,d\tau \\
&\qquad - \frac{1}{2\Gamma(1.5)}\int_s^{\frac{1}{4}} \biggl(\frac{1}{4}-\tau\biggr)^{0.5} (\tau-s)^{1.5}\,d\tau + \frac{1}{4\Gamma(1.5)}\int_0^{\frac{3}{4}} \biggl(\frac{3}{4}-\tau\biggr)^{0.5} \tau^{1.5}(1-s)^{1.5}d\tau \\
&\qquad - \frac{1}{4\Gamma(1.5)}\int_s^{\frac{3}{4}} \biggl(\frac{3}{4}-\tau\biggr)^{0.5} (\tau-s)^{1.5}\,d\tau + \frac{1}{3}\cdot\frac{1}{3}^{1.5}(1-s)^{1.5} + \frac{2}{3}\cdot\frac{2}{3}^{1.5}(1-s)^{1.5} \biggr] \\
&\qquad \times \biggl(\frac{1}{3}s + \frac{1}{4\pi}\sin\bigl(2\pi w_n(s)\bigr) + \frac{1}{18}z_n(t) + \frac{1}{5}\bigl(D_t^{0.3}w_n(t)\bigr)^2 + D_t^{0.2}z_n(t) + \frac{10}{11} \biggr)\,ds \\
&\qquad - \frac{1}{\Gamma(2.5)}\int_0^t \frac{t^{1.5}}{1-l_1 l_2}\biggl[\frac{1}{3}\biggl(\frac{1}{3}-s\biggr)^{1.5} + \frac{2}{3}\biggl(\frac{2}{3}-s\biggr)^{1.5} \biggr] \\
&\qquad \times \biggl(\frac{1}{3}s + \frac{1}{4\pi}\sin\bigl(2\pi w_n(s)\bigr) + \frac{1}{18}z_n(t) \\
&\qquad + \frac{1}{5}\bigl(D_t^{0.3}w_n(t)\bigr)^2 + D_t^{0.2}z_n(t) + \frac{10}{11} \biggr)\,ds, \\
&\int_0^1 K_2(t,s)f_{2(w_n,z_n)}(s)\,ds + \int_0^1 H_2(t,s)f_{1(w_n,z_n)}(s)\,ds \\
&\quad= \frac{1}{\Gamma(2.5)}\int_0^1 \biggl\{ t^{1.5}(1-s)^{1.5} + \frac{l_1 t^{1.5}}{1-l_1 l_2}\biggl[\frac{1}{2\Gamma(1.5)}\int_0^{\frac{1}{4}} \biggl(\frac{1}{4}-\tau\biggr)^{0.5} \tau^{1.5}(1-s)^{1.5}d\tau \\
&\qquad - \frac{1}{2\Gamma(1.5)}\int_s^{\frac{1}{4}} \biggl(\frac{1}{4}-\tau\biggr)^{0.5} (\tau-s)^{1.5}\,d\tau + \frac{1}{4\Gamma(1.5)}\int_0^{\frac{3}{4}} \biggl(\frac{3}{4}-\tau\biggr)^{0.5} \tau^{1.5}(1-s)^{1.5}d\tau \\
&\qquad - \frac{1}{4\Gamma(1.5)}\int_s^{\frac{3}{4}} \biggl(\frac{3}{4}-\tau\biggr)^{0.5} (\tau-s)^{1.5}\,d\tau + \frac{1}{3}\cdot\frac{1}{3}^{1.5}(1-s)^{1.5} + \frac{2}{3}\cdot\frac{2}{3}^{1.5}(1-s)^{1.5} \biggr]\biggr\}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{3}s + \frac{1}{4\pi}\sin\bigl(2\pi w_n(s)\bigr) + \frac{1}{18}z_n(t) + \frac{1}{5}\bigl(D_t^{0.3}w_n(t)\bigr)^2 + D_t^{0.2}z_n(t) + \frac{10}{11}\right) ds \\
&- \frac{1}{\Gamma(2.5)}\int_0^t \left\{(t-s)^{1.5} + \frac{l_1 t^{1.5}}{1-l_1 l_2}\left[\frac{1}{3}\left(\frac{1}{3}-s\right)^{1.5} + \frac{2}{3}\left(\frac{2}{3}-s\right)^{1.5}\right]\right\} \\
&\times \left(\frac{1}{3}s + \frac{1}{4\pi}\sin\bigl(2\pi w_n(s)\bigr)\right. \\
&\left. + \frac{1}{18}z_n(t) + \frac{1}{5}\bigl(D_t^{0.3}w_n(t)\bigr)^2 + D_t^{0.2}z_n(t) + \frac{10}{11}\right) ds \\
&+ \frac{1}{\Gamma(2.5)}\int_0^1 \frac{t^{1.5}}{1-l_1 l_2}\left[\frac{2}{7\Gamma(1.2)}\int_0^{\frac{1}{4}}\left(\frac{1}{4}-\tau\right)^{0.2}\tau^{1.5}(1-s)^{1.5}\, d\tau\right. \\
&- \frac{2}{7\Gamma(1.2)}\int_s^{\frac{1}{4}}\left(\frac{1}{4}-\tau\right)^{0.2}(\tau-s)^{1.5}\, d\tau + \frac{3}{7\Gamma(1.2)}\int_0^{\frac{3}{4}}\left(\frac{3}{4}-\tau\right)^{0.2}\tau^{1.5}(1-s)^{1.5} d\tau \\
&\left. - \frac{3}{7\Gamma(1.2)}\int_s^{\frac{3}{4}}\left(\frac{3}{4}-\tau\right)^{0.2}(\tau-s)^{1.5}\, d\tau + \frac{1}{3}\cdot\frac{1}{3}^{1.5}(1-s)^{1.5} + \frac{2}{3}\cdot\frac{2}{3}^{1.5}(1-s)^{1.5}\right] \\
&\times \left(\frac{1}{2}s^2 + \frac{1}{10}w_n(s) + \frac{1}{32}\sin\bigl(z_n(s)\bigr) + \frac{1}{50}D_t^{0.3}w_n(s) + \frac{1}{D_t^{0.2}z_n(s)-\frac{1}{2}} + 2\right) ds \\
&- \frac{1}{\Gamma(2.5)}\int_0^t \frac{t^{1.5}}{1-l_1 l_2}\left[\frac{1}{3}\left(\frac{1}{3}-s\right)^{1.5} + \frac{2}{3}\left(\frac{2}{3}-s\right)^{1.5}\right] \\
&\times \left(\frac{1}{2}s^2 + \frac{1}{10}w_n(s) + \frac{1}{32}\sin\bigl(z_n(s)\bigr)\right. \\
&\left. + \frac{1}{50}D_t^{0.3}w_n(s) + \frac{1}{D_t^{0.2}z_n(s)-\frac{1}{2}} + 2\right) ds.
\end{aligned}
$$

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