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Existence of positive solutions of boundary value problems for high-order nonlinear conformable differential equations with p-Laplacian operator

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Abstract

In this paper, we study the existence of positive solutions for boundary value problems of high-order conformable differential equations involving the p-Laplacian operator. By applying properties of the Green's function and the Guo–Krasnosel'skii fixed point theorem, some sufficient conditions for the existence of at least one positive solution are established. In addition, we demonstrate the effectiveness of the main result by using an example.

Keywords: p-Laplacian operator; Boundary value problem; Conformable fractional derivative; Fixed point theorems

1 Introduction

In this paper, we study the high-order conformable differential equations with p-Laplacian operator as follows:

$$\begin{cases} T_\alpha^{0+}(\varphi_p(T_\alpha^{0+}u(t))) = f(t,u(t),T_\alpha^{0+}u(t)), \quad 0 \le t \le 1; \\ u^{(i)}(0) = 0, \qquad [\varphi_p(T_\alpha^{0+}u)]^{(i)}(0) = 0, \quad i = 0,1,2,\ldots,n-2; \\ [T_\beta^{0+}u(t)]_{t=1} = 0, \quad m-1 < \beta \le m; \\ [T_\beta^{0+}(\varphi_p(T_\alpha^{0+}u(t)))]_{t=1} = 0, \quad 1 \le m \le n-1; \end{cases} \tag{1.1}$$

where $n-1 < \alpha \le n$, φ_p is the p-Laplacian operator, $p > 1$, and $\varphi_p(s) = |s|^{p-2}s$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in C([0,1] \times [0,+\infty) \times (-\infty,0],[0,+\infty))$, T_α is a new fractional derivative called "the conformable fractional derivative", it was defined by Khalil in 2014 (see [1]). Namely, for a function $f : (0,\infty) \to R$, the conformable fraction derivative of order $0 < \alpha < 1$ of f at $t > 0$ was defined by $T_\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$, and the fraction derivative at 0 is defined as $(Tf)(0) = \lim_{\varepsilon\to 0^+}(T_\alpha f)(t)$. The new definition satisfies the major properties of the usual derivative. By means of the properties of the Green's function and fixed point theorems on cone, we establish conditions that ensure the existence of positive solutions for boundary value problems (1.1).

In general, two types of fractional derivatives, namely Riemann–Liouville and Caputo, are famous. Mathematicians prefer the Riemann–Liouville fractional derivative because it is amenable to many mathematical manipulations, while physicists and engineers prefer the Caputo fractional derivative. In addition, there are also two discrete definitions called Grunwald–Letnikov fractional derivative and Riesz fractional derivative which are used in numerical mathematics. We know that all fractional derivative definitions satisfy the property that the fractional derivative is linear, this is the only property inherited from the first derivative by all of the fractional derivative definitions. However, the Riemann–Liouville derivative and the Caputo derivative do not obey the Leibniz rule and chain rule, which sometimes prevents us from applying these derivatives to the ordinary physical system with standard Newton derivative. The following are some of the setbacks of the Riemann –Liouville and Caputo fractional derivative definitions:

(i) The Riemann–Liouville derivative does not satisfy ${}^{R}_{a}D^{\alpha}_{t}(1) = 0$ (${}^{C}_{a}D^{\alpha}_{t}(1) = 0$ for the Caputo derivative) if α is not a natural number;

(ii) The Riemann–Liouville and Caputo derivatives do not satisfy the known formula of the derivative of the product of two functions: ${}^{R(C)}_{a}D^{\alpha}_{t}(fg) = f{}^{R(C)}_{a}D^{\alpha}_{t}(g) + g{}^{R(C)}_{a}D^{\alpha}_{t}(f)$;

(iii) The Riemann–Liouville and Caputo derivatives do not satisfy the known formula of the derivative of the quotient of two functions: ${}^{R(C)}_{a}D^{\alpha}_{t}(\frac{f}{g}) = \frac{g{}^{R(C)}_{a}D^{\alpha}_{t}(f)-f{}^{R(C)}_{a}D^{\alpha}_{t}(g)}{g^{2}}$;

(iv) The Riemann–Liouville and Caputo derivatives do not satisfy the chain rule: ${}^{R(C)}_{a}D^{\alpha}_{t}(f \circ g) = f^{(\alpha)}(g(t))g^{(\alpha)}(t)$;

(v) The Riemann–Liouville and Caputo derivatives do not satisfy: ${}^{R(C)}_{a}D^{\alpha}_{t}{}^{R(C)}_{a}D^{\beta}_{t} = {}^{R(C)}_{a}D^{\alpha+\beta}_{t}$ in general;

(vi) The Caputo derivative definition assumes that the function is differentiable.

On the contrary, the conformable fractional derivative is a well-behaved simple fractional derivative definition depending just on the basic limit definition of the derivative. The new definition seems to be a natural extension of the usual derivative, and it satisfies the first four properties mentioned above and the mean value theorem. Recently, in [2], the author pointed out a major flaw of "the conformable fractional derivative" defined in [3] and uncovered the real source of this conformability, that is, the conformable α-derivative is not a fractional derivative, the term "conformable" is supposedly attributed to the properties this proposed definition provides.

Fractional calculus has been applied to various areas of engineering, physics, chemistry, and biology. There are a large number of literature works and monographs that deal with all sorts of problems in fractional calculus (see [4–12]). On the other hand, for studying the turbulent flow in a porous medium, Leibenson [13] introduced the differential equation models with p-Laplacian operator. So, differential equations with p-Laplacian operator have since been applied in many fields of physics and natural phenomena, see [14–20] and the references therein.

Lu *et al.* [16] studied the following Riemann–Liouville fractional differential equations boundary problems with p-Laplacian operator:

$$\begin{cases} D^{\beta}_{0+}(\varphi_{p}(D^{\alpha}_{0+}u(t))) = f(t,u(t)), \quad 0 \le t \le 1; \\ u(0) = u'(0) = u'(1) = 0; \\ D^{\alpha}_{0+}u(0) = D^{\alpha}_{0+}u(1) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\varphi_p^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, D_{0+}^{α}, D_{0+}^{β} are standard Riemann–Liouville fractional derivatives. By the properties of Green's function, the Guo–Krasnosel'skii fixed-point theorem on cone, and the upper and lower solutions method, some new results on the existence of positive solutions were obtained.

Chen *et al.* [17] investigated the Caputo fractional differential equation boundary value problems with p-Laplacian operator at resonance:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}x(t))) = f(t, x(t), D_{0+}^{\alpha}x(t)), \quad t \in [0,1]; \\ D_{0+}^{\alpha}x(0) = D_{0+}^{\alpha}x(1) = 0, \end{cases}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, D_{0+}^{α} and D_{0+}^{β} are standard Caputo fractional derivatives. By using the coincidence degree theory, the existence of solutions for the boundary value problem was obtained.

In [20], Liu and Jia studied the following integral boundary value problems of fractional p-Laplacian equation with mixed fractional derivatives:

$$\begin{cases} D_{0+}^{\alpha}(\varphi_p({}^{C}D_{0+}^{\beta}u(t))) + f(t, u(t), {}^{C}D_{0+}^{\beta}u(t)) = 0, \quad 0 \leq t \leq 1; \\ {}^{C}D_{0+}^{\beta}(0) = u'(0) = 0, \qquad u(0) = \int_0^1 g_0(s)u(s)\,ds; \\ D_{0+}^{\alpha-1}(\varphi_p({}^{C}D_{0+}^{\beta}u(1))) = \int_0^1 g_1(s)u(s)\varphi_p({}^{C}D_{0+}^{\beta}u(s))\,ds, \end{cases}$$

where φ_p is the p-Laplacian operator, $1 < \alpha, \beta \leq 2$, ${}^{C}D^{\beta}$ is the standard Caputo fractional derivative operator and D^{α} is the standard Riemann–Liouville fractional derivative operator. By using the generalization of Leggett–Williams fixed point theorem, some new results on the existence of multiple positive solutions to the boundary value problems were obtained.

To the best of our knowledge, there are few studies that consider the existence of positive solutions on high-order fractional differential equations with p-Laplacian operator, especially for conformable differential equations. In this paper, we investigate the existence of positive solutions for boundary value problem of conformable differential p-Laplacian equation systems on $n-1 < \alpha \leq n$ by using the Guo–Krasnosel'skii fixed point theorem. Our work established novel results which contribute to the existing literature and knowledge by improving on existing equations. At the end of this paper, we demonstrate the effectiveness of the main results by one example.

2 Preliminaries

Definition 2.1 ([1]) The conformable fractional derivative starting from a of a function $f : [a, \infty) \to R$ of order $0 < \alpha < 1$ is defined by

$$\left(T_{\alpha}^{a}f\right)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \epsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon},$$

when $a = 0$, we write T_{α}. If $(T_{\alpha}f)(t)$ exists on $[a, b]$, then $(T_{\alpha}^{a}f)(a) = \lim_{t \to a^{+}}(T_{\alpha}^{a}f)(t)$.

The conformable fractional integral starting from a of a function $f : [a, \infty) \to R$ is defined by

$$\left(I_{\alpha}^{a}f\right)(t) = I_{1}^{a}\left((t-a)^{\alpha-1}f(t)\right) = \int_{a}^{t} \frac{f(x)}{(x-a)^{1-\alpha}}\,dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$.

Definition 2.2 ([3]) Let $\alpha \in (n, n+1)$, the conformable fractional derivative starting from a of a function $f : [a, +\infty) \to R$ of order α, where $f^{(n)}(t)$ exists, is defined by

$$\big(T_\alpha^a f\big)(t) = \big(T_{\alpha-n}^a f^{(n)}(t)\big).$$

Let $\alpha \in (n, n+1)$, the conformable fractional integral of order α starting at a is defined by

$$\begin{aligned}\big(I_\alpha^a f\big)(t) &= I_{n+1}^a\big((t-a)^{\alpha-n-1} f(t)\big)\\ &= \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\alpha-n-1} f(x)\,dx.\end{aligned}$$

Definition 2.3 ([14]) Let $p > 1$, the p-Laplacian operator is given by

$$\varphi_p(x) = |x|^{p-2}x.$$

Obviously, φ_p is continuous, increasing, invertible and its inverse operator is φ_q, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.1 ([1]) *Let $\alpha \in (n, n+1]$ and f be a continuous function defined in $[a, +\infty)$, one has $T_\alpha I_\alpha f(t) = f(t)$ for $t > a$.*

Lemma 2.2 ([3]) *Let $\alpha \in (n, n+1]$ and $f : [a, +\infty)$ be $(n+1)$ times differentiable for $t > a$, we have*

$$I_\alpha^a T_\alpha^a f(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}.$$

Lemma 2.3 *Let $g \in C[0,1]$ be given, then the conformable fractional boundary value problem*

$$\begin{cases} T_\alpha^{0+} u(t) + g(t) = 0, & n-1 < \alpha \le n;\\ u^{(i)}(0) = 0, & i = 0, 1, 2, \ldots, n-2;\\ [T_\beta^{0+} u(t)]_{t=1} = 0, & m-1 < \beta \le m, 1 \le m \le n-1; \end{cases} \tag{2.1}$$

has a unique positive solution

$$u(t) = \int_0^1 G(t,s) g(s)\,ds, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\Gamma(n)} \begin{cases} s^{\alpha-n}[(1-s)^{n-m-1} t^{n-1} - (t-s)^{n-1}], & 0 \le s \le t \le 1;\\ (1-s)^{n-m-1} s^{\alpha-n} t^{n-1}, & 0 \le t \le s \le 1; \end{cases} \tag{2.3}$$

is the Green's function for this problem.

Proof In view of Lemma 2.2, Definition 2.2, and the boundary values, we have

$$\begin{aligned} I_{\alpha}^{0+}T_{\alpha}^{0+}u(t) &= u(t) - u(0) - u'(0)t - \frac{u''(0)}{2!}t^2 - \cdots \\ &\quad - \frac{u^{(n-2)}(0)}{(n-2)!}t^{n-2} - \frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1} \\ &= u(t) - \frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1}, \\ I_{\alpha}^{0+}g(t) &= \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}s^{\alpha-n}g(s)\,ds, \end{aligned}$$

so we have

$$u(t) = \frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1} - \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}s^{\alpha-n}g(s)\,ds.$$

Assume $1 < \beta \leq 2$, from the above equation, we have

$$\begin{aligned} T_{\beta}^{0+}u(t) &= T_{\beta-1}^{0+}\left[\frac{d}{dt}\frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1} - \frac{d}{dt}\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}s^{\alpha-n}g(s)\,ds\right] \\ &= \frac{u^{(n-1)}(0)}{(n-3)!}t^{n-\beta-1} - \frac{t^{2-\beta}}{(n-3)!}\int_0^t (t-s)^{n-3}s^{\alpha-n}g(s)\,ds. \end{aligned}$$

Let $t = 1$ in the above equation, by the condition $[T_{\beta}^{0+}u(t)]_{t=1} = 0$, we can get

$$\begin{aligned} &\frac{1}{(n-3)!}u^{(n-1)}(0) - \frac{1}{(n-3)!}\int_0^1 (1-s)^{n-3}s^{\alpha-n}g(s)\,ds = 0, \\ &u^{(n-1)}(0) = \int_0^1 (1-s)^{n-3}s^{\alpha-n}g(s)\,ds. \end{aligned}$$

So, by implication,

$$u(t) = \frac{1}{(n-1)!}\int_0^1 (1-s)^{n-3}s^{\alpha-n}t^{n-1}g(s)\,ds - \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}s^{\alpha-n}g(s)\,ds.$$

Through the same logical deduction, when $2 < \beta \leq 3$, we have

$$u(t) = \frac{1}{(n-1)!}\int_0^1 (1-s)^{n-4}s^{\alpha-n}t^{n-1}g(s)\,ds - \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}s^{\alpha-n}g(s)\,ds.$$

If $m - 1 < \beta \leq m$, it is easy to know that

$$\begin{aligned} u(t) &= \frac{1}{(n-1)!}\int_0^1 (1-s)^{n-m-1}s^{\alpha-n}t^{n-1}g(s)\,ds - \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}s^{\alpha-n}g(s)\,ds \\ &= \int_0^1 G(t,s)g(s)\,ds, \end{aligned}$$

where

$$G(t,s)=\frac{1}{\Gamma(n)}\begin{cases}s^{\alpha-n}[(1-s)^{n-m-1}t^{n-1}-(t-s)^{n-1}], & 0\le s\le t\le 1;\\ (1-s)^{n-m-1}s^{\alpha-n}t^{n-1}, & 0\le t\le s\le 1;\end{cases}$$

is the Green's function for this problem. □

Lemma 2.4 *If $g\in C[0,1]$ is given, then the conformable fractional boundary value problem*

$$\begin{cases}T_\alpha^{0+}(\varphi_p(T_\alpha^{0+}u(t)))=g(t), \quad n-1<\alpha\le n;\\ u^{(i)}(0)=0, \qquad [\varphi_p(T_\alpha^{0+}u)]^{(i)}(0)=0, \quad i=0,1,2,\dots,n-2;\\ [T_\beta^{0+}u(t)]_{t=1}=0, \quad m-1<\beta\le m;\\ [T_\beta^{0+}(\varphi_p(T_\alpha^{0+}u(t)))]_{t=1}=0, \quad 1\le m\le n-1;\end{cases}\tag{2.4}$$

has a unique positive solution

$$u(t)=\int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)g(\tau)\,d\tau\right)ds,\tag{2.5}$$

where $G(t,s)$ is given in (2.3).

Proof Applying the operator I_α^{0+} on both sides of (2.4), we have

$$\begin{aligned}&I_\alpha^{0+}T_\alpha^{0+}\bigl(\varphi_p\bigl(T_\alpha^{0+}u(t)\bigr)\bigr)\\ &\quad=\varphi_p\bigl(T_\alpha^{0+}u(t)\bigr)-\bigl(\varphi_p\bigl(T_\alpha^{0+}u\bigr)\bigr)(0)-\bigl(\varphi_p\bigl(T_\alpha^{0+}u\bigr)\bigr)'(0)t-\frac{(\varphi_p(T_\alpha^{0+}u))''(0)}{2!}t^2\\ &\qquad-\cdots-\frac{(\varphi_p(T_\alpha^{0+}u))^{(n-2)}(0)}{(n-2)!}t^{n-2}-\frac{(\varphi_p(T_\alpha^{0+}u))^{(n-1)}(0)}{(n-1)!}t^{n-1}\\ &\quad=\varphi_p\bigl(T_\alpha^{0+}u(t)\bigr)-\frac{(\varphi_p(T_\alpha^{0+}u))^{(n-1)}(0)}{(n-1)!}t^{n-1}\\ &\quad=\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}s^{\alpha-n}g(s)\,ds,\end{aligned}$$

so we have

$$\varphi_p\bigl(T_\alpha^{0+}u(t)\bigr)=\frac{(\varphi_p(T_\alpha^{0+}u))^{(n-1)}(0)}{(n-1)!}t^{n-1}+\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}s^{\alpha-n}g(s)\,ds.$$

Assume $1<\beta\le 2$, applying the operator T_β^{0+} on both sides of the equation above, we have

$$\begin{aligned}&T_\beta^{0+}\bigl(\varphi_p\bigl(T_\alpha^{0+}u(t)\bigr)\bigr)\\ &\quad=T_{\beta-1}^{0+}\left[\frac{d}{dt}\frac{(\varphi_p(T_\alpha^{0+}u))^{(n-1)}(0)}{(n-1)!}t^{n-1}+\frac{d}{dt}\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}s^{\alpha-n}g(s)\,ds\right]\\ &\quad=\frac{(\varphi_p(T_\alpha^{0+}u))^{(n-1)}(0)}{(n-3)!}t^{n-\beta-1}+\frac{t^{2-\beta}}{(n-3)!}\int_0^t(t-s)^{n-3}s^{\alpha-n}g(s)\,ds.\end{aligned}$$

Letting $t = 1$ in the above equation, by the condition $[T_\beta^{0+}(\varphi_p(T_\alpha^{0+}u(t)))]_{t=1} = 0$, we can get

$$\frac{1}{(n-3)!}\big(\varphi_p\big(T_\alpha^{0+}u\big)\big)^{(n-1)}(0) + \frac{1}{(n-3)!}\int_0^1 (1-s)^{n-3}s^{\alpha-n}g(s)\,ds = 0,$$
$$\big(\varphi_p\big(T_\alpha^{0+}u\big)\big)^{(n-1)}(0) = -\int_0^1 (1-s)^{n-3}s^{\alpha-n}g(s)\,ds.$$

By implication, we know

$$\begin{aligned}\varphi_p\big(T_\alpha^{0+}u(t)\big) &= -\frac{1}{(n-1)!}\int_0^1 (1-s)^{n-3}s^{\alpha-n}t^{n-1}g(s)\,ds \\ &\quad + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}s^{\alpha-n}g(s)\,ds.\end{aligned}$$

Through the same logical deduction, if $2 < \beta \le 3$, we have

$$\begin{aligned}\varphi_p\big(T_\alpha^{0+}u(t)\big) &= -\frac{1}{(n-1)!}\int_0^1 (1-s)^{n-4}s^{\alpha-n}t^{n-1}g(s)\,ds \\ &\quad + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}s^{\alpha-n}g(s)\,ds.\end{aligned}$$

If $m-1 < \beta \le m$, it is easy to know that

$$\begin{aligned}\varphi_p\big(T_\alpha^{0+}u(t)\big) &= -\frac{1}{(n-1)!}\int_0^1 (1-s)^{n-m-1}s^{\alpha-n}t^{n-1}g(s)\,ds \\ &\quad + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}s^{\alpha-n}g(s)\,ds \\ &= -\int_0^1 G(t,s)g(s)\,ds.\end{aligned}$$

Applying the operator φ_q on both sides of the equation above, we get

$$T_\alpha^{0+}u(t) + \varphi_q\int_0^1 G(t,s)g(s)\,ds = 0.$$

Letting $\widetilde{g}(t) = \varphi_q\int_0^1 G(t,s)g(s)\,ds$, thus, the conformable fractional differential equation boundary value problem (2.4) is equivalent to the problem

$$\begin{cases} T_\alpha^{0+}u(t) + \widetilde{g}(t) = 0, & n-1 < \alpha \le n; \\ u^{(i)}(0) = 0, & 0 \le i \le n-2; \\ [T_\beta^{0+}u(t)]_{t=1} = 0, & m-1 < \beta \le m, 1 \le m \le n-1. \end{cases} \tag{2.6}$$

By using Lemma 2.3, we know that the conformable fractional differential equation boundary value problem (2.6) has a unique solution

$$u(t) = \int_0^1 G(t,s)\widetilde{g}(s)\,ds = \int_0^1 G(t,s)\varphi_q\bigg(\int_0^1 G(s,\tau)g(\tau)\,d\tau\bigg)\,ds. \tag{2.7}$$

This constitutes the complete proof.

By Lemma 2.4, we can easily know that

$$T_{\alpha}^{0+}u(t) = -\varphi_q\left(\int_0^1 G(t,s)g(s)\,ds\right). \tag{2.8}$$

□

Lemma 2.5 *The Green's function* (2.3) *has the following properties*:

$$(1-s)^{n-m-1}\big[1-(1-s)^m\big]s^{\alpha-n}t^{n-1} \le \Gamma(n)G(t,s) \le (1-s)^{n-m-1}s^{\alpha-n}t^{n-1}.$$

Proof Evidently, the right inequality holds. So, we only need to prove the left inequality. For convenience, let us put

$$G_1(t,s) = \frac{1}{\Gamma(n)}\big[s^{\alpha-n}(1-s)^{n-m-1}t^{n-1} - s^{\alpha-n}(t-s)^{n-1}\big], \quad 0 \le s \le t \le 1;$$

and

$$G_2(t,s) = \frac{1}{\Gamma(n)}s^{\alpha-n}(1-s)^{n-m-1}t^{n-1}, \quad 0 \le t \le s \le 1.$$

If $0 \le s \le t \le 1$, then we have $0 \le t-s \le t-ts = (1-s)t$, and thus $(t-s)^{n-1} \le (1-s)^{n-1}t^{n-1}$. Hence, if $0 \le s \le t \le 1$,

$$\begin{aligned}
\Gamma(n)G_1(t,s) &= s^{\alpha-n}\big[(1-s)^{n-m-1}t^{n-1} - (t-s)^{n-1}\big] \\
&\ge s^{\alpha-n}\big[(1-s)^{n-m-1}t^{n-1} - (1-s)^{n-1}t^{n-1}\big] \\
&= (1-s)^{n-m-1}\big[1-(1-s)^m\big]s^{\alpha-n}t^{n-1}.
\end{aligned}$$

If $0 \le t \le s \le 1$, we have

$$\begin{aligned}
\Gamma(n)G_2(t,s) &= s^{\alpha-n}(1-s)^{n-m-1}t^{n-1} \\
&\ge s^{\alpha-n}\big[(1-s)^{n-m-1} - (1-s)^{n-1}\big]t^{n-1} \\
&= (1-s)^{n-m-1}\big[1-(1-s)^m\big]s^{\alpha-n}t^{n-1}.
\end{aligned}$$

Therefore, the proof is done. □

Lemma 2.6 *Let $G(t,s)$ be as given in the statement of Lemma* 2.2. *Then we find that*:

(i) *$G(t,s)$ is a continuous function on the unit square* $[0,1] \times [0,1]$;

(ii) *$G(t,s) \ge 0$ for each* $(t,s) \in [0,1] \times [0,1]$.

Proof That property (i) holds is trivial, it is clear that $G_1(t,s)$ and $G_2(t,s)$ are continuous on their domains and that $G_1(s,s) = G_2(s,s)$, whence (i) follows.

By Lemma 2.3, we can know that

$$G(t,s) \ge \frac{1}{\Gamma(n)}(1-s)^{n-m-1}\big[1-(1-s)^m\big]s^{\alpha-n}t^{n-1} \ge 0 \quad \forall s,t \in [0,1].$$

Thus, (ii) holds, and the proof is completed. □

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, that is, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. θ denotes the zero element of E.

A nonempty closed convex set $P \subset E$ is a cone if it satisfies:

(I_1) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$;

(I_2) $x \in P, -x \in P \Rightarrow x = \theta$.

Putting $\dot{P} := \{x \in P \mid x \text{ is an interior point of } P\}$, a cone P is said to be solid if its interior $\dot{P}$ is nonempty. Moreover, P is called normal if there exists a constant M such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq M\|y\|$; in this case M is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 < x < x_2\}$ is called the order interval between x_1 and x_2.

Lemma 2.7 ([21]) *Let P be a normal cone in a real Banach space E, $\langle v_0, u_0\rangle \in E$, and $T : \langle v_0, u_0\rangle \to \langle v_0, u_0\rangle$ be an increasing operator. If T is completely continuous, then T has a fixed point $u^* \in \langle v_0, u_0\rangle$.*

Lemma 2.8 ([22]) *Let E be an ordered Banach space, $P \subset E$ is a cone, and suppose that Ω_1, Ω_2 are bounded open sunsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $\Phi : P \to P$ be a completely continuous operator such that either*

(i) *$\|\Phi u\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|\Phi u\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$ or*

(ii) *$\|\Phi u\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|\Phi u\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.*

Then Φ has a fixed point in $P \cap \overline{\Omega}_2 \setminus \Omega_1$.

3 Main results

We denote that $E = C^\alpha[0,1] := \{u \mid u \in C[0,1], T_\alpha^{0^+} u \in C[0,1]\}$ and endowed with the norm $\|u\|_\alpha = \max\{\|u\|_\infty, \|T_\alpha^{0^+} u\|_\infty\}$, where $\|u\|_\infty = \max_{0\leq t\leq 1}|u(t)|$ and $\|T_\alpha^{0^+} u\|_\infty = \max_{0\leq t\leq 1}|T_\alpha^{0^+} u(t)|$. Then $(E, \|\cdot\|_\alpha)$ is a Banach space. Let $P = \{u \in E \mid u(t) \geq 0, T_\alpha^{0^+} u(t) \leq 0\}$, then P is a cone on the space E.

Define the operator $\Phi : P \to E$ by

$$(\Phi u)(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\big(\tau, u(\tau), T_\alpha^{0^+} u(\tau)\big)\,d\tau\right)ds. \tag{3.1}$$

For any $u \in E$, it is easy to show that $\Phi u \in E$ and

$$T_\alpha^{0+}(\Phi u)(t) = -\varphi_q\left(\int_0^1 G(t,s)f\big(s, u(s), T_\alpha^{0^+} u(s)\big)\,ds\right). \tag{3.2}$$

Obviously, the function u is a positive solution of the boundary value problem (1.1) if and only if u is a fixed point of the operator Φ in P.

Lemma 3.1 *Assume that $f \in C([0,1] \times [0,+\infty) \times [-\infty,0), [0,+\infty))$, then the operator $\Phi : P \to P$ is completely continuous.*

Proof For given $u \in P$, by (3.1), (3.2), and Lemma 2.6, we can easily obtain $(\Phi u)(t) \in P$, which implies that $\Phi : P \to P$. Let $\{u_j\} \subset P$ and $\lim_{j\to\infty} u_j = u \in P$. Then there exists a constant $\gamma_0 > 0$ such that $\|u_j\|_\alpha \leq \gamma_0$ and $\|u\|_\alpha \leq \gamma_0$ for $j = 1, 2, \ldots$.

Since $f \in C([0,1] \times [0,+\infty) \times [-\infty,0),[0,+\infty))$, we can show that, for $(t,u,v) \in [0,1] \times [-\gamma_0,\gamma_0] \times [-\gamma_0,\gamma_0]$,

$$0 \le f(t,u,v) \le M_0,$$

where $M_0 = \max_{(t,u,v)\in[0,1]\times[-\gamma_0,\gamma_0]\times[-\gamma_0,\gamma_0]} f(t,u,v)$ and

$$\lim_{j\to\infty} f\bigl(t,u_j,T_\alpha^{0^+}u_j(t)\bigr) = f\bigl(t,u,T_\alpha^{0^+}u(t)\bigr) \quad \text{for } t \in [0,1].$$

From Lemma 2.5 and Lemma 2.6, we can get that, for $(t,s) \in [0,1] \times [0,1]$,

$$0 \le \frac{1}{\Gamma(n)}(1-s)^{n-m-1}\bigl[1-(1-s)^m\bigr]s^{\alpha-n}t^{n-1} \le G(t,s) \le \frac{1}{\Gamma(n)}(1-s)^{n-m-1}s^{\alpha-n}t^{n-1}.$$

It follows from the Lebesgue dominated convergence theorem, and we have

$$\begin{aligned}
\lim_{j\to\infty}(\Phi u_j)(t) &= \lim_{j\to\infty}\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)f\bigl(\tau,u_j(\tau),T_\alpha^{0^+}u_j(\tau)\bigr)\,d\tau\biggr)\,ds \\
&= \int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 \lim_{j\to\infty} G(s,\tau)f\bigl(\tau,u_j(\tau),T_\alpha^{0^+}u_j(\tau)\bigr)\,d\tau\biggr)\,ds \\
&= \int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)f\bigl(\tau,u(\tau),T_\alpha^{0^+}u(\tau)\bigr)\,d\tau\biggr)\,ds \\
&= (\Phi u)(t)
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\lim_{j\to\infty} T_\alpha^{0^+}(\Phi u_j)(t) &= -\varphi_q\biggl(\lim_{j\to\infty}\int_0^1 G(t,s)f\bigl(\tau,u_j(s),T_\alpha^{0^+}u_j(s)\bigr)\,ds\biggr) \\
&= -\varphi_q\biggl(\int_0^1 G(t,s)f\bigl(\tau,u(s),T_\alpha^{0^+}u(s)\bigr)\,ds\biggr) \\
&= T_\alpha^{0^+}(\Phi u)(t).
\end{aligned} \tag{3.4}$$

Equations (3.3) and (3.4) imply that $\lim_{j\to\infty}(\Phi u_j)(t) = (\Phi u)(t)$ is uniform on $[0,1]$. Hence, Φ is continuous.

Let $A \subset P$ be any bounded set, then there exists a constant $\gamma_1 > 0$ such that $\|u\|_\alpha \le \gamma_1$ for each $u \in A$, which implies that $|u(t)| \le \gamma_1$ and $|T_\alpha^{0^+}u(t)| \le \gamma_1$ for $t \in [0,1]$. Because f is continuous, there exists $M_1 > 0$ such that $0 \le f(t,u(t),T_\alpha^{0^+}u(t)) \le M_1$ for $t \in [0,1]$.

Let $L = \frac{M_1}{\Gamma(n)}\int_0^1 (1-s)^{n-m-1}s^{\alpha-n}\,ds$, then

$$\begin{aligned}
0 &\le \bigl|(\Phi u)(t)\bigr| = \biggl|\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 G(s,\tau)f\bigl(\tau,u(\tau),T_\alpha^{0^+}u(\tau)\bigr)\,d\tau\biggr)\,ds\biggr| \\
&\le \varphi_q\biggl(\frac{M_1}{\Gamma(n)}\int_0^1 (1-s)^{n-m-1}s^{\alpha-n}\,ds\biggr)\int_0^1 G(t,s)\,ds \\
&\le \frac{\int_0^1 (1-s)^{n-m-1}s^{\alpha-n}\,ds}{\Gamma(n)}\varphi_q(L)
\end{aligned}$$

and

$$\begin{aligned}0 \le \left|T_\alpha^{0^+}(\Phi u)(t)\right| &= \left|-\varphi_q\left(\int_0^1 G(t,s)f\left(s,u(s),T_\alpha^{0^+}u(s)\right)ds\right)\right| \\ &\le \left|-\varphi_q\left(\frac{M_1}{\Gamma(n)}\int_0^1 (1-s)^{n-m-1}s^{\alpha-n}\,ds\right)\right| \\ &= \varphi_q(L), \end{aligned} \tag{3.5}$$

which implies that $\Phi(A)$ is uniformly bounded in P.

Because $G(t,s)$ is continuous on $[0,1]\times[0,1]$, then $G(t,s)$ is uniformly continuous. Hence, for any $\varepsilon > 0$, there exists $\delta_1 > 0$, whenever $t_1, t_2 \in [0,1]$ and $|t_2 - t_1| < \delta_1$,

$$\left|G(t_2,s) - G(t_1,s)\right| < \frac{\varepsilon}{\varphi_q(L)+1}.$$

For any $u \in P$, we have

$$\begin{aligned}\left|(\Phi u)(t_2) - (\Phi u)(t_1)\right| &= \left|\int_0^1 G(t_2,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,u(\tau),T_\alpha^{0^+}u(\tau)\right)d\tau\right)ds\right. \\ &\quad \left. - \int_0^1 G(t_1,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,u(\tau),T_\alpha^{0^+}u(\tau)\right)d\tau\right)ds\right| \\ &\le \int_0^1 \left|G(t_2,s) - G(t_1,s)\right|\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,u(\tau),T_\alpha^{0^+}u(\tau)\right)d\tau\right)ds \\ &\le \varphi_q(L)\int_0^1 \left|G(t_2,s) - G(t_1,s)\right|ds \\ &< \varepsilon.\end{aligned}$$

Let $F: P \to P$, by $(Fu)(t) = \int_0^1 G(t,s)f(s,u(s),T_\alpha^{0^+}u(s))\,ds$, we have

$$0 \le (Fu)(t) = \int_0^1 G(t,s)f\left(s,u(s),T_\alpha^{0^+}u(s)\right)ds \le L.$$

In view of the fact that $\varphi_q(x)$ is continuous on $[0,L]$, we can get $\varphi_q(x)$ is uniformly continuous on $[0,L]$, and for $\varepsilon > 0$ above, there exists $\eta > 0$,

$$\left|\varphi_q(x_2) - \varphi_q(x_1)\right| < \varepsilon, \quad \text{whenever } x_1, x_2 \in [0,L] \text{ and } |x_2 - x_1| < \eta. \tag{3.6}$$

Because $G(t,s)$ is uniformly continuous, so for $\eta > 0$, there exists $\delta_2 > 0$, whenever $t_1, t_2 \in [0,1]$, $s \in [0,1]$, and $|t_2 - t_1| < \delta_2$, we have $|G(t_2,s) - G(t_1,s)| < \frac{\eta}{M+1}$. Hence,

$$\begin{aligned}\left|(Fu)(t_2) - (Fu)(t_1)\right| &= \left|\int_0^1 G(t_2,s)f\left(s,u(s),T_\alpha^{0^+}u(s)\right)ds\right. \\ &\quad \left. - \int_0^1 G(t_1,s)f\left(s,u(s),T_\alpha^{0^+}u(s)\right)ds\right| \\ &\le \int_0^1 \left|G(t_2,s) - G(t_1,s)\right|f\left(s,u(s),T_\alpha^{0^+}u(s)\right)ds \\ &\le M_1\frac{\eta}{M_1+1} < \eta. \end{aligned} \tag{3.7}$$

By (3.6) and (3.7), it is easy to see that

$$\begin{aligned}\left|\left(T_\alpha^{0+}\Phi u\right)(t_2)-\left(T_\alpha^{0+}\Phi u\right)(t_1)\right| &= \left|\varphi_q\left(\int_0^1 G(t_2,s)f\left(s,u(s),T_\alpha^{0^+}u(s)\right)ds\right)\right.\\ &\quad\left.-\varphi_q\left(\int_0^1 G(t_1,s)f\left(s,u(s),T_\alpha^{0^+}u(s)\right)ds\right)\right|\\ &= \left|\varphi_q\left(Fu(t_2)\right)-\varphi_q\left(Fu(t_1)\right)\right|\\ &< \varepsilon.\end{aligned}$$

Thus, $\Phi(A)$ is equicontinuous. By Arzela–Ascoli theorem, we can show that Φ is relatively compact. Therefore, Φ is completely continuous. □

For the convenience, we introduce the following notations:

$$\begin{aligned}A^{-1} &= \max_{0\le t\le 1}\frac{1}{\Gamma(n)}\int_0^1(1-s)^{n-m-1}s^{\alpha-n}t^{n-1}\varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right)ds,\\ B^{-1} &= \max_{0\le t\le 1}\frac{1}{\Gamma(n)}\int_0^1(1-s)^{n-m-1}\left[1-(1-s)^m\right]s^{\alpha-n}t^{n-1}\varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right)ds.\end{aligned}$$

Theorem 3.1 *Assume that the following assumptions hold*:

(H_1) $f\in C([0,1]\times[0,+\infty)\times[-\infty,0),[0,+\infty))$;

(H_2) *There exist two positive constants* $a>b$ *such that* $\phi(a)\le\varphi_p(aA)$, $\psi(b)\ge\varphi_p(bB)$ *for any* $t\in[0,1]$, *where*

$$\begin{aligned}\phi(l) &= \max\left\{f(t,u,v),(t,u,v)\in[0,1]\times[0,l]\times[-l,0]\right\},\\ \psi(l) &= \min\left\{f(t,u,v),(t,u,v)\in[0,1]\times[0,l]\times[-l,0]\right\};\end{aligned}$$

(H_3) $B^{-1}\le\max_{0\le t\le 1}|\varphi_q(\int_0^1 G(t,s)\,ds)|\le A^{-1}$ *for* $\forall t\in[0,1]$.

Then problem (1.1) *has at least one positive solution* $u\in P$ *such that* $b\le\|u\|\le a$.

Proof By Lemma 3.1, we know $\Phi:P\to P$ is completely continuous, and we only need to consider the existence of a fixed point of the operator Φ in P. Now, we separate the proof into the following two steps.

Step 1. Let $\Omega_a:=\{u\in P\mid \|u\|_\alpha<a\}$. For any $u\in\partial\Omega_a$, we have $\|u\|_\alpha=a$ and $f(t,u(t),T_\alpha^{0+}u(t))\le\phi(a)\le\varphi_p(aA)$ for $(t,u,v)\in[0,1]\times[0,a]\times[-a,0]$. Hence, we have

$$\begin{aligned}\|\Phi u\|_\infty &= \max_{0\le t\le 1}\left|(\Phi u)(t)\right|\le\max_{0\le t\le 1}\int_0^1\left|G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,u(\tau),T_\alpha^{0^+}u(\tau)\right)d\tau\right)\right|ds\\ &\le\max_{0\le t\le 1}\frac{1}{\Gamma(n)}\int_0^1(1-s)^{n-m-1}s^{\alpha-n}t^{n-1}\varphi_q\left(\int_0^1 G(s,\tau)\varphi_p(aA)\,d\tau\right)ds\\ &= aA\max_{0\le t\le 1}\frac{1}{\Gamma(n)}\int_0^1(1-s)^{n-m-1}s^{\alpha-n}t^{n-1}\varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right)ds\\ &= a\end{aligned}$$

and

$$\begin{aligned}\left\|T_\alpha^{0+}\Phi u\right\|_\infty &= \max_{0\le t\le 1}\left|\left(T_\alpha^{0+}\Phi u\right)(t)\right| = \max_{0\le t\le 1}\left|\varphi_q\left(\int_0^1 G(t,s)f\left(\tau,u(s),T_\alpha^{0^+}u(s)\right)ds\right)\right|\\ &\le \max_{0\le t\le 1}\left|\varphi_q\left(\int_0^1 G(t,s)\phi_p(aA)\,ds\right)\right|\\ &= aA\max_{0\le t\le 1}\left|\varphi_q\left(\int_0^1 G(t,s)\,ds\right)\right|\\ &\le a.\end{aligned}$$

So

$$\|\Phi u\|_\alpha \le \|u\|_\alpha,\quad \forall u\in\partial\Omega_a.$$

Step 2. Let $\Omega_b := \{u\in P \mid \|u\|_\alpha < b\}$. For any $u\in\partial\Omega_b$, we have $\|u\|_\alpha = b$ and $f(t,u(t), T_\alpha^{0+}u(t)) \ge \psi(b) \ge \varphi_p(bB)$ for $(t,u,v)\in[0,1]\times[0,b]\times[-b,0]$. Hence, we have

$$\begin{aligned}\|\Phi u\|_\infty &= \max_{0\le t\le 1}\left|(\Phi u)(t)\right| = \max_{0\le t\le 1}\left|\int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,u(\tau),T_\alpha^{0^+}u(\tau)\right)d\tau\right)ds\right|\\ &\ge \max_{0\le t\le 1}\frac{1}{\Gamma(n)}\int_0^1(1-s)^{n-m-1}\left[1-(1-s)^m\right]s^{\alpha-n}t^{n-1}\\ &\quad\times\varphi_q\left(\int_0^1 G(s,\tau)\varphi_p(bB)\,d\tau\right)ds\\ &= bB\max_{0\le t\le 1}\frac{1}{\Gamma(n)}\int_0^1(1-s)^{n-m-1}\left[1-(1-s)^m\right]s^{\alpha-n}t^{n-1}\varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right)ds\\ &= b\end{aligned}$$

and

$$\begin{aligned}\left\|T_\alpha^{0+}\Phi u\right\|_\infty &= \max_{0\le t\le 1}\left|\left(T_\alpha^{0+}\Phi u\right)(t)\right| = \max_{0\le t\le 1}\left|\varphi_q\left(\int_0^1 G(t,s)f\left(\tau,u(s),T_\alpha^{0^+}u(s)\right)ds\right)\right|\\ &\ge \max_{0\le t\le 1}\left|\varphi_q\left(\int_0^1 G(t,s)\varphi_p(bB)\,ds\right)\right|\\ &= bB\max_{0\le t\le 1}\left|\varphi_q\left(\int_0^1 G(t,s)\,ds\right)\right|\\ &\ge b.\end{aligned}$$

So

$$\|\Phi u\|_\alpha \ge \|u\|_\alpha,\quad \forall u\in\partial\Omega_b.$$

By Lemma 2.8, Φ has a fixed point in $P\cap\overline{\Omega}_a\setminus\Omega_b$, i.e., problem (1.1) has a positive solution u such that $b\le\|u\|\le a$. □

Theorem 3.2 *Let $f \in C([0,1] \times [0,+\infty) \times (-\infty,0])$, $f(t,x,y)$ is increasing in x and y. Let there exist ν_0, ω_0 satisfying $\Phi\nu_0 \geq \nu_0$, $\Phi\omega_0 \leq \omega_0$ for $0 \leq \nu_0 \leq \omega_0$, $0 \leq t \leq 1$. Then problem* (1.1) *has a positive solution u^* such that $\nu_0 \leq u^* \leq \omega_0$.*

Proof We only need to consider the fixed point of the operator Φ. Let ν, $\omega \in P$ be such that $\nu \leq \omega$ and $T_\alpha^{0^+}\nu \leq T_\alpha^{0^+}\omega$, then $f(t,\nu(t),T_\alpha^{0^+}\nu(t)) \leq f(t,\omega(t),T_\alpha^{0^+}\omega(t))$ for $t \in [0,1]$, we have

$$\begin{aligned}(\Phi\nu)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,\nu(\tau),T_\alpha^{0^+}\nu(\tau)\right)d\tau\right)ds \\ &\leq \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,\omega(\tau),T_\alpha^{0^+}\omega(\tau)\right)d\tau\right)ds \\ &= (\Phi\omega)(t).\end{aligned}$$

Hence Φ is an increasing operator. By the assumption $\Phi\nu_0 \geq \nu_0$, $\Phi\omega_0 \leq \omega_0$, we have $T : \langle\nu_0,\omega_0\rangle \to \langle\nu_0,\omega_0\rangle$. Since P is a normal cone and $\Phi : P \to P$ is completely continuous in view of Lemma 3.1, by Lemma 2.7, Φ has one fixed point $u^* \in \langle\nu_0,\omega_0\rangle$, which is the required positive solution. □

Theorem 3.3 *Let $f \in C([0,1] \times [0,+\infty) \times (-\infty,0])$, $f(t,x,y)$ is increasing in x and y for each $t \in [0,1]$. Further, if $0 < \lim_{\|u\|_\alpha\to\infty} f(t,u,T_\alpha^{0^+}u) < \infty$ for each $t \in [0,1]$, then problem* (1.1) *has a positive solution.*

Proof As $0 < \lim_{\|u\|_\alpha\to\infty} f(t,u,T_\alpha^{0^+}u) < \infty$, there exist positive constants H and R such that, for $\|u\|_\alpha \geq R$, $f(t,u,T_\alpha^{0^+}u) \leq H$ for $\forall t \in [0,1]$. Let $c = \max\{f(t,u,T_\alpha^{0^+}u) \mid 0 \leq \|u\|_\alpha \leq R, 0 \leq t \leq 1\}$, then we can know $f \leq c + H$.

Consider the following boundary value problem:

$$\begin{cases} T_\alpha^{0+}(\varphi_p(T_\alpha^{0+}u(t))) - (c+H) = 0, & 0 \leq t \leq 1 \\ u^{(i)}(0) = 0, \qquad [\varphi_p(T_\alpha^{0+}u)]^{(i)}(0) = 0, & i = 0,1,2,\ldots,n-2; \\ [T_\beta^{0+}u(t)]_{t=1} = 0, \qquad [T_\beta^{0+}(\varphi_p(T_\alpha^{0+}u(t)))]_{t=1} = 0. \end{cases} \tag{3.8}$$

By using Lemma 2.4, the solution of problem (3.8) is equivalent to the following integral equation:

$$\omega(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)(c+H)\,d\tau\right)ds.$$

Hence,

$$\omega(t) \geq \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,\omega(\tau),T_\alpha^{0^+}\omega(\tau)\right)d\tau\right)ds = (\Phi\omega)(t),$$

we have $\omega \geq \Phi\omega$.

On the other hand, for $\nu = 0$,

$$(\Phi\nu)(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 G(s,\tau)f\left(\tau,\nu(\tau),T_\alpha^{0^+}\nu(\tau)\right)d\tau\right)ds \geq 0.$$

Hence, we have $\Phi v \geq v$, and as a consequence of Theorem 3.2, problem (1.1) has a positive solution. □

4 Applications

To testify our results established in the previous section, we provide an adequate problem.

Example 4.1

$$\begin{cases} T_{1.5}^{0+}(\varphi_2(T_{1.5}^{0+}u(t))) = 2 + \sqrt{u} + \frac{t^2}{2}, \quad 0 \leq t \leq 1, \\ u(0) = [\varphi_2(T_{1.5}^{0+}u)](0) = 0, \\ [T_{1.5}^{0+}u](1) = (\varphi_2(T_{1.5}^{0+}u))'(1) = 0. \end{cases} \tag{4.1}$$

In system (4.1), we see that $\alpha = 1.5$, $\beta = 1$, $p = 2$, $q = 2$, $m = 1$, $n = 2$. In addition, we have

$$\begin{aligned} A^{-1} &= \max_{0\leq t\leq 1} \frac{1}{\Gamma(n)} \int_0^1 (1-s)^{n-m-1} s^{\alpha-n} t^{n-1} \varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right) ds \\ &= \max_{0\leq t\leq 1} \frac{1}{\Gamma(2)} \int_0^1 s^{-\frac{1}{2}} t \varphi_2\left(\int_0^1 G(s,\tau)\,d\tau\right) ds \\ &= \max_{0\leq t\leq 1} \frac{1}{\Gamma(2)} \int_0^1 s^{-\frac{1}{2}} t \varphi_2\left(\int_0^s \tau^{\frac{1}{2}}\,d\tau + s\int_s^1 \tau^{-\frac{1}{2}}\,d\tau\right) ds \\ &= \max_{0\leq t\leq 1} t \int_0^1 \left(2s^{\frac{1}{2}} - \frac{4}{3}s\right) ds \\ &= 0.67, \\ B^{-1} &= \max_{0\leq t\leq 1} \frac{1}{\Gamma(n)} \int_0^1 (1-s)^{n-m-1}\left[1-(1-s)^m\right] s^{\alpha-n} t^{n-1} \varphi_q\left(\int_0^1 G(s,\tau)\,d\tau\right) ds \\ &= \max_{0\leq t\leq 1} \frac{1}{\Gamma(2)} \int_0^1 \left[1-(1-s)\right] s^{-0.5} t \varphi_2\left(\int_0^1 G(s,\tau)\,d\tau\right) ds \\ &= \max_{0\leq t\leq 1} \int_0^1 s^{\frac{1}{2}} t \left(2s - \frac{4}{3}s^{\frac{3}{2}}\right) ds \\ &= 0.356, \end{aligned}$$

and

$$\begin{aligned} \varphi_q\left(\int_0^1 G(t,s)\,ds\right) &= \varphi_2\left(\int_0^t s^{\frac{1}{2}}\,ds + t\int_t^1 s^{-\frac{1}{2}}\,ds\right) \\ &= 2t - \frac{4}{3}t^{\frac{3}{2}}. \end{aligned}$$

From $\frac{d}{dt}\varphi_q(\int_0^1 G(t,s)\,ds) = 2 - 2\sqrt{t} \geq 0$, $\forall t \in [0,1]$, we know that $\varphi_q(\int_0^1 G(t,s)\,ds)$ is increasing on t, then we get $\max_{0\leq t\leq 1}\varphi_q(\int_0^1 G(t,s)\,ds) = 0.67$. So, we get $B^{-1} \leq \max_{0\leq t\leq 1}\varphi_q(\int_0^1 G(t,s)\,ds) \leq A^{-1}$ and $A = 1.5$, $B = 2.8$.

Besides, let $f(t,u) = 2 + \sqrt{u} + \frac{t^2}{2}$ and choose $a = 3$, $b = \frac{1}{2}$, we get

$$f(t,u) = 2 + \sqrt{u} + \frac{t^2}{2} \leq 3.5 \leq \varphi_2(aA) = 4.5, \quad \forall (t,u) \in [0,1]\times[0,3];$$

$$f(t,u) = 2 + \sqrt{u} + \frac{t^2}{2} \geq 2 \geq \varphi_2(bB) = 1.4, \quad \forall (t,u) \in [0,1] \times \left[0, \frac{1}{2}\right].$$

From the definitions of ϕ and ψ, we get $\phi(a) \leq \varphi_2(aA)$ and $\psi(b) \geq \varphi_2(bB)$. So, all the conditions of Theorem 3.1 are satisfied, then system (4.1) has at least one positive solution u such that $\frac{1}{2} \leq \|u\|_\alpha \leq 3$.

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