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Continuum-wise expansive homoclinic classes for robust dynamical systems

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Abstract

In the study, we consider continuum-wise expansiveness for the homoclinic class of a kind of C^1-robustly expansive dynamical system. First, we show that if the homoclinic class $H(p,f)$, which contains a hyperbolic periodic point p, is R-robustly continuum-wise expansive, then it is hyperbolic. For a vector field, if the homoclinic class $H(\gamma,X)$ does not include singularities and is R-robustly continuum-wise expansive, then it is hyperbolic.

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1 Introduction

1.1 Continuum-wise expansiveness for diffeomorphisms

Let M be a closed connected smooth Riemannian manifold. A point $x \in M$ is called a *periodic point* if there is $\pi(x) > 0$ such that $f^{\pi(x)}(x) = x$, where $\pi(x)$ is the period of x. A periodic point p with period $\pi(p) > 0$ is considered *hyperbolic* if the derivative $D_pf^{\pi(p)}$ has no eigenvalues with norm one. Let $\mathrm{Per}(f) = \{x \in M : x \text{ is a periodic point of } f\}$, and let $p \in \mathrm{Per}(f)$ be hyperbolic. Subsequently, there are C^r ($r \geq 1$) sets $W^s(p)$ and $W^u(p)$, which are called the *stable manifold* of p and the *unstable manifold* of p, respectively, such that $f^{i\pi(p)}(x) \to p$ (as $i \to \infty$) for $x \in W^s(p)$ and $f^{-i\pi(p)}(x) \to p$ (as $i \to \infty$) for $x \in W^u(p)$.

Let $p, q \in \mathrm{Per}(f)$ be hyperbolic. We say that p and q are *homoclinically related* if $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^u(p) \pitchfork W^s(q) \neq \emptyset$, and in such a case, we write $p \sim q$. Let us denote $H(p,f) = \overline{\{q \in \mathrm{Per}(f) : p \sim q\}}$. It is known that $H(p,f)$ is a closed, f-invariant, and transitive set. Here a closed f-invariant set Λ is *transitive* if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega limit set of x.

According to the result of Samle [27], if a diffeomorphism f satisfies Axiom A, that is, the nonwandering set $\Omega(f) = \overline{\mathrm{Per}(f)}$ is hyperbolic, then this set can be written as the finite disjoint union of closed f-invariant sets that are homoclinic classes of a periodic point inside them. An interesting problem is the hyperbolicity of homoclinic classes under various C^1-perturbations of expansiveness (see [13, 22, 23, 25, 26, 29]).

Let d be the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. A closed f-invariant set Λ ($\subset M$) is *expansive* for f if there is $e > 0$ such that, for any distinct points $x, y \in \Lambda$, there is $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) \geq e$.

Let $p \in \mathrm{Per}(f)$ be hyperbolic. Then there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of p such that, for any $g \in \mathcal{U}(f)$, $p_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is a unique hyperbolic periodic point of g, where p_g is said to be the *continuation* of p.

We say that the homoclinic class $H(p,f)$ is C^1*-robustly expansive* if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that, for any $g \in \mathcal{U}(f)$, $H(p_g,g)$ is expansive, where p_g is the continuation of p. Note that, in the definition, the expansive constant depends on $g \in \mathcal{U}(f)$.

A closed f-invariant set $\Lambda \subset M$ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_xf^n|_{E^s_x}\| \le C\lambda^n \quad \text{and} \quad \|D_xf^{-n}|_{E^u_x}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$.

Sambarino and Vieitez [25] proved that if the homoclinic class $H(p,f)$ is C^1-robustly expansive and germ expansive, then it is hyperbolic. Here $H(p,f)$ is *germ expansive* for f indicating that if there is $e > 0$ such that, for any $x \in H(p,f)$, $y \in M$ if $d(f^i(x),f^i(y)) < e$ for all $i \in \mathbb{Z}$, then $x = y$. We say that the homoclinic class $H(p,f)$ is C^1*-stably expansive* if there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of $H(p,f)$ such that, for any $g \in \mathcal{U}(f)$, $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is expansive, where Λ_g is the continuation of Λ. Lee and Lee [13] proved that if the homoclinic class $H(p,f)$ is C^1-stably expansive, then it is hyperbolic.

For obtaining the results, we use a general notion of expansiveness (continuum-wise expansive) and consider the hyperbolicity of the homoclinic class. Continuum-wise expansiveness is a general notion of expansiveness (see [11, Example 3.5]). A set A is *nondegenerate* if it is not reduced to a point. We say that $A \subset M$ is a *nontrivial continuum* if it is a compact connected nondegenerate subset of M.

Definition 1.1 Let $f : M \to M$ be a diffeomorphism. A closed f-invariant set Λ ($\subset M$) is said to be a *continuum-wise expansive* subset of f if there is a constant $e > 0$ such that, for any nondegenerate subcontinuum $A \subset \Lambda$, there is $n \in \mathbb{Z}$ such that

$$\operatorname{diam} f^n(A) \ge e,$$

where $\operatorname{diam} A = \sup\{d(x,y) : x, y \in A\}$ for any subset $A \subset \Lambda$.

Thus the constant e is called a *continuum-wise expansive constant* for f. In the definition a diffeomorphism f is *continuum-wise expansive* if $\Lambda = M$.

Das, Lee, and Lee [6] proved that if the homoclinic class $H(p,f)$ is C^1-robustly continuum-wise expansive and satisfies the chain condition, then $H(p,f)$ is hyperbolic. However, it is still an open question if the chain condition is omitted. Subsequently, we consider that the homoclinic class $H(p,f)$ is a type of C^1-robustly continuum-wise expansiveness. Let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1 topology. We call a subset $\mathcal{G} \subset \mathrm{Diff}(M)$ a *residual* subset if it contains a countable intersection of open and dense subsets of $\mathrm{Diff}(M)$. A dynamic property is called a C^1*-generic* property if it holds in a residual subset of $\mathrm{Diff}(M)$. Sambarino and Vieitez [26] proved that if the homoclinic class $H(p,f)$ is generically C^1-robustly expansive, then it is hyperbolic. Lee [17] proved that if a locally maximal homoclinic class $H(p,f)$ is homogeneous, then it is hyperbolic. Lee [16] proved that if a homoclinic class $H(p,f)$ is continuum-wise expansive, then it is hyperbolic. Using the C^1-generic condition, we define a type of C^1-robust expansiveness, which was introduced by Li [19].

Definition 1.2 Let p be a hyperbolic periodic point of f. We say that the homoclinic class $H(p,f)$ is *R-robustly* $\mathfrak{P}$ if there exist a C^1-neighborhood $\mathcal{U}(f)$ of f and a residual set $\mathcal{G} \subset \mathcal{U}(f)$ such that, for any $g \in \mathcal{G}$, $H(p_g,g)$ is $\mathfrak{P}$, where p_g is the continuation of p.

In the definition, $\mathfrak{P}$ is replaced by various types of expansiveness. Accordingly, we introduce a general type of expansiveness proposed by Morales and Sirvent [20]. For a Borel probability measure μ on M, we consider that f is μ-expansive if there is $e > 0$ such that $\mu(\Gamma_e(x)) = 0$ for all $x \in M$, where $\Gamma_e(x) = \{y \in M : d(f^i(x),f^i(y)) \leq e \text{ for all } i \in \mathbb{Z}\}$. We say that f is *measure expansive* if it is μ-expansive for every nonatomic Borel probability measure μ on M. According to Artigue and Carrasco [2], we know the following:

expansive $\Rightarrow$ measure expansive $\Rightarrow$ continuum-wise expansive.

Lee [17] proved that if the homoclinic class $H(p,f)$ is R-robustly measure expansive, then it is hyperbolic. We can obtain the results for the R-robustly expansive homoclinic classes. According to these results, the following is a general result of [17].

Theorem A *Let p be a hyperbolic periodic point of f. If the homoclinic class $H(p,f)$ is R-robustly continuum-wise expansive, then $H(p,f)$ is hyperbolic.*

1.2 Continuum-wise expansiveness for vector fields

Let M be defined as before, and let $\mathfrak{X}(M)$ denote the set of C^1-vector fields on M endowed with the C^1-topology. Thus every $X \in \mathfrak{X}(M)$ generates a C^1-flow $X_t : M \times \mathbb{R} \to M$, that is, a C^1-map such that $X_t : M \to M$ is a diffeomorphism satisfying (i) $X_0(x) = x$, (ii) $X_{t+s}(x) = X_t(X_s(x))$ for all $t, s \in \mathbb{R}$ and $x \in M$,, and (iii) it is generated by the vector field X if

$$\frac{d}{dt}X_t(x)\bigg|_{t=t_0} = X\big(X_{t_0}(x)\big)$$

for all $x \in M$ and $t \in \mathbb{R}$. A point $\sigma \in M$ is *singular* if $X_t(\sigma) = \sigma$ for all $t \in \mathbb{R}$. We denote by $\mathrm{Sing}(X)$ the set of all singular points of X. For any $x \in M$, if x is not a singular point, then it is a *regular point* of X. Let R_X be the set of all regular points of X. A *periodic orbit* of X is an orbit $\gamma = \mathrm{Orb}(p)$ such that $X_T(p) = p$ for some minimal $T > 0$. We denote by $\mathrm{Per}(X)$ the set of all periodic orbits of X. A point $x \in M$ is a *critical element* if it is either a singular point or a periodic point of X. Let $\mathrm{Crit}(X) = \mathrm{Sing}(X) \cup \mathrm{Per}(X)$ be the set of all critical elements of X. Let X_t be the flow of $X \in \mathfrak{X}(M)$. A closed X_t-invariant set Λ is considered *hyperbolic* for X_t if there are constants $C > 0$ and $\lambda > 0$ and a splitting $T_xM = E^s_x \oplus \langle X(x)\rangle \oplus E^u_x$ such that the tangent flow $DX_t : TM \to TM$ leaves the invariant continuous splitting and

$$\|DX_t|_{E^s_x}\| \leq Ce^{-\lambda t} \quad \text{and} \quad \|DX_{-t}|_{E^u_x}\| \leq Ce^{-\lambda t}$$

for $t > 0$ and $x \in \Lambda$, where $\langle X(x)\rangle$ is the subspace generated by $X(x)$.

An increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ is called a *reparameterization*. Let $\mathrm{Hom}(\mathbb{R})$ denote the set of all homeomorphisms of $\mathbb{R}$. Let $\mathrm{Rep}(\mathbb{R}) = \{h \in \mathrm{Hom}(\mathbb{R}) : h$ is a reparameterization$\}$. Bowen and Walters [4] introduced and studied expansiveness for vector fields. They showed that if a vector field X is expansive, then every singular point is isolated.

A closed invariant set $\Lambda \subset M$ is *expansive* of $X \in \mathfrak{X}(M)$ if, for every $\epsilon > 0$, there exist $\delta > 0$ and $h \in \mathrm{Hom}(\mathbb{R})$ such that, for any $x, y \in \Lambda$, if $d(X_t(x), X_{h(t)}(y)) \leq \delta$ for all $t \in \mathbb{R}$, then $y \in X_{(-\epsilon,\epsilon)}(x)$. If $\Lambda = M$, then X is called *expansive.*

Regarding the notion of expansiveness, Arbieto, Codeiro, and Pacifico [1] introduced and studied a general notion of expansiveness for vector fields. They proved that if a vector field X is continuum-wise expansive, then every singular point is isolated. Here we explain continuum-wise expansiveness for vector fields in further detail. For a subset A of M, $C^0(A, \mathbb{R})$ denotes the set of real continuous maps defined on A. We define

$$\mathcal{H}(A) = \big\{h : A \to \mathrm{Rep}(\mathbb{R}) : \\ \text{there is } x_h \in A \text{ with } h(x_h) = id, \text{ and } h(\cdot)(t) \in C^0(A, \mathbb{R}) \text{ for all } t \in \mathbb{R}\big\},$$

and if $t \in \mathbb{R}$ and $h \in \mathcal{H}(A)$, then

$$\mathcal{X}_h^t(A) = \big\{X_{h(x)(t)}(x) : x \in A\big\}.$$

For convenience, we set $h(x)(t) = h_x(t)$ for all $x \in A$ and $t \in \mathbb{R}$. Let Λ be a closed set of M. A set A is called *nondegenerate* if it is not reduced to a point. We say that $A \subset M$ is a *continuum* if it is a compact connected nondegenerate subset A of M.

Definition 1.3 Let $X \in \mathfrak{X}(M)$. We say that X is *continuum-wise expansive* if, for any $\epsilon > 0$, there is $\delta > 0$ such that if $A \subset M$ is a continuum and $h \in \mathcal{H}(A)$ satisfies

$$\mathrm{diam}\big(\mathcal{X}_h^t(A)\big) < \delta \quad \text{for all } t \in \mathbb{R},$$

then $A \subset X_{(-\epsilon,\epsilon)}(x)$ for some $x \in A$.

Let $\gamma \in \mathrm{Per}(X)$ be hyperbolic. We consider that the dimension of the stable manifold $W^s(\gamma)$ of γ is the *index* of γ, denoted by $\mathrm{index}(\gamma)$. The *homoclinic class* of X associated with a hyperbolic closed orbit γ, denoted by $H(\gamma, X)$, is defined as the closure of the transverse intersection of the stable and unstable manifolds of γ, that is,

$$H(\gamma, X) = \overline{W^s(\gamma) \pitchfork W^u(\gamma)},$$

where $W^s(\gamma)$ is the stable manifold of γ, and $W^u(\gamma)$ is the unstable manifold of γ. It is evident that it is closed, X_t-invariant, and transitive. Here, a closed invariant set Λ is *transitive* if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$.

For two hyperbolic closed orbits γ and η of X, we say that γ and η are *homoclinically related*, denoted by $\gamma \sim \eta$, if

$$W^s(\gamma) \pitchfork W^u(\eta) \neq \emptyset \quad \text{and} \quad W^s(\eta) \pitchfork W^u(\gamma) \neq \emptyset.$$

If γ and η are homoclinically related, then $\operatorname{index}(\eta) = \operatorname{index}(\gamma)$. Let $\gamma \in \operatorname{Per}(X)$ be hyperbolic. Thus there exist a C^1-neighborhood $\mathcal{U}(X)$ of X and a neighborhood U of γ such that, for any $Y \in \mathcal{U}(X)$, there is a unique hyperbolic periodic orbit $\gamma_Y = \bigcap_{t\in\mathbb{R}} Y_t(U)$. The hyperbolic periodic orbit γ_Y is called the *continuation* of γ with respect to Y.

We say that the homoclinic class $H(\gamma, X)$ is C^1*-robustly expansive* if there is a C^1-neighborhood $\mathcal{U}(X)$ of X such that, for any $Y \in \mathcal{U}(X)$, $H(\gamma_Y, Y)$ is expansive, where γ_Y is the continuation of γ.

A subset $\mathcal{G} \subset \mathfrak{X}^1(M)$ is called a *residual* subset if it contains a countable intersection of the open and dense subsets of $\mathfrak{X}^1(M)$. A dynamic property is called a C^1*-generic* property if it holds in a residual subset of $\mathfrak{X}(M)$.

Lee and Park [18] proved that, for a C^1-generic X, if an isolated homoclinic class $H(\gamma, X)$ is expansive, then it is hyperbolic. Here, a closed X_t-invariant set Λ is *isolated* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{t\in\mathbb{R}} X_t(U)$. We consider that a closed invariant set Λ is *germ expansive* if, for any $\epsilon > 0$, there is $\delta > 0$ such that, for any $x \in \Lambda$ and $y \in M$, there is $h \in \operatorname{Hom}(\mathbb{R})$ such that if $d(X_t(x), X_{h(t)}(y)) < \delta$ for all $t \in \mathbb{R}$, then $y \in X_{(-\epsilon,\epsilon)}(x)$. It is evident that, if Λ is expansive, then it is germ expansive. However, the converse is not true. Note that if Λ is isolated germ expansive, then Λ is expansive.

Gang [10] proved that if the homoclinic class $H(\gamma, X)$ is C^1-robustly expansive and $H(\gamma, X)$-germ expansive, then it is hyperbolic.

A vector field X has the *shadowing property* on Λ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any $(\delta, 1)$-pseudo orbit $\xi = \{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\} \subset \Lambda$, there exist $y \in M$ and $h \in \operatorname{Hom}(\mathbb{R})$ satisfying

$$d\bigl(X_{h(t)}(y), X_{t-s_i}(x_i)\bigr) < \epsilon$$

for any $s_i \leq t < s_{i+1}$, where s_i are defined as $s_0 = 0$, $s_n = \sum_{i=0}^{n-1} t_i$, and $s_{-n} = \sum_{i=-n}^{-1} t_i$, $n = 1, 2, \ldots$.

Lee, Lee, and Lee [14] proved that if the homoclinic class $H(\gamma, X)$ is C^1-robustly expansive and shadowable, then it is hyperbolic. According to the results, we consider the hyperbolicity of the homoclinic class $H(\gamma, X)$ under a type of C^1-robustly continuum-wise expansiveness.

Definition 1.4 Let $X \in \mathfrak{X}(M)$. We say that the homoclinic class $H(\gamma, X)$ is R-robustly continuum-wise expansive if there exist a C^1-neighborhood $\mathcal{U}(X)$ of X and a residual set $\mathcal{G} \subset \mathcal{U}(X)$ such that, for any $Y \in \mathcal{G}$, $H(\gamma_Y, Y)$ is continuum-wise expansive, where γ_Y is the continuation of γ.

Using this definition, we have the following theorem.

Theorem B *Let $X \in \mathfrak{X}(M)$ and $H_X(\gamma) \cap \operatorname{Sing}(X) = \emptyset$. If the homoclinic class $H(\gamma, X)$ is R-robustly continuum-wise expansive, then it is hyperbolic for X.*

2 Proof of Theorem A

Let M be defined as before, and let $f : M \to M$ be a diffeomorphism. For any $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called a δ*-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. For a given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a finite δ-pseudo-orbit $\{x_i\}_{i=0}^{n}$ ($n \geq 1$)

of f such that $x_0 = x$ and $x_n = y$. We write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set of points $\{x \in M : x \leftrightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{CR}(f)$. The chain recurrence class of f is the set of equivalent classes $\leftrightsquigarrow$ on $\mathcal{CR}(f)$. Let p be a hyperbolic periodic point of f. Denote $C(p,f) = \{x \in M : x \rightsquigarrow p \text{ and } p \rightsquigarrow x\}$, which is a closed invariant set.

It is known that $C(p,f)$ is a closed f-invariant set. Moreover, $H(p,f) \subset C(p,f)$. A closed small arc $\mathcal{I}$ of f is called a *simply periodic curve* if, for any $\epsilon > 0$,

(a) there is $k > 0$ such that $f^k(\mathcal{I}) = \mathcal{I}$,
(b) $0 < l(f^i(\mathcal{I})) < \epsilon$ for all $0 \le i < k$,
(c) the endpoints of $\mathcal{I}$ are hyperbolic, and
(d) $\mathcal{I}$ is normally hyperbolic,

where $l(A)$ denotes the length of A (see [29]). It is evident that $\mathcal{I}$ is not a point set.

Lemma 2.1 *There is a residual set $\mathcal{G}_1 \subset \mathrm{Diff}(M)$ such that, for any $f \in \mathcal{G}_1$, we have the following*:

(a) *f is Kupka–Smale, that is, every periodic point of f is hyperbolic, and the stable and unstable manifolds are transversal intersections* (*see* [24]).
(b) $H(p,f) = C(p,f)$ (*see* [3]).
(c) *if, for any C^1-neighborhood $\mathcal{U}(f)$ of f, there is $g \in \mathcal{U}(f)$ such that g has a simply periodic curve $\mathcal{I}$, then f has a simply periodic curve $\mathcal{J}$* (*see* [29]).

The following lemma is important for a C^1 perturbation property, which is called Franks' lemma.

Lemma 2.2 ([8]) *Let $\mathcal{U}(f)$ be a C^1-neighborhood of f. Then there exist $\epsilon > 0$ and a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that, for any $g \in \mathcal{U}_0(f)$, a set $\{x_1, x_2, \dots, x_N\}$, a neighborhood U of $\{x_1, x_2, \dots, x_N\}$, and a linear map $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le N$, there is $\widehat{g} \in \mathcal{U}(f)$ such that $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \dots, x_N\} \cup (M \setminus U)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \le i \le N$.*

For any hyperbolic $p \in \mathrm{Per}(f)$, we say that p is *weakly hyperbolic* if, for any $\eta > 0$, there is an eigenvalue μ of $D_pf^{\pi(p)}$ such that

$$(1-\eta)^{\pi(p)} < |\mu| < (1+\eta)^{\pi(p)}.$$

It is evident that if p is a weakly hyperbolic periodic point of f, then there is g C^1-close to f such that p_g is not hyperbolic for g.

Lemma 2.3 *Let $p \in \mathrm{Per}(f)$ be hyperbolic. If $q \in H(p,f) \cap \mathrm{Per}(f)$ with $q \sim p$ is weakly hyperbolic, then there is g C^1-close to f such that g has a simply periodic curve $\mathcal{L} \subset C(p_g, g)$.*

Proof Suppose that $q \in H(p,f) \cap \mathrm{Per}(f)$ with $q \sim p$ is weakly hyperbolic. According to Lemma 2.2, there is g C^1-close to f such that p_g is not hyperbolic. Thus $D_{p_g}g^{\pi(p_g)}$ has an eigenvalue μ such that $|\mu| = 1$. For simplicity, we may assume that p_g is a fixed point of g. Let E_{p_g} be the vector space associated with the eigenvalue μ. For the proof, we consider the case of $\mu \in \mathbb{R}$. Consider a nonzero vector v associated with μ. According to Lemma 2.2, there is g_1 C^1-close to g such that

(i) $g_1(p_g) = g(p_g) = p_g$, and

(ii) $g_1(\exp_{p_g}(v)) = \exp_{p_g} \circ D_{p_g} g \circ \exp_{p_g}^{-1}(\exp_p(v)) = \exp_{p_g}(v)$.

For any small $\beta > 0$, we set $E_{p_{g_1}}(\beta) = \{t \cdot v : -\beta/2 \leq t \leq \beta/2\}$. Thus we have a closed small curve $\mathcal{J}$ such that

(i) $\mathcal{J} = \exp_{p_{g_1}}(E_{p_{g_1}}(\beta))$ with $\operatorname{diam} \mathcal{J} = \beta$,

(ii) $g_1^{\pi(p_{g_1})}(\mathcal{J}) = \mathcal{J}$ is the identity map, and

(iii) $\mathcal{J}$ is normally hyperbolic.

It is evident that the identity map is contained in $C(p_{g_1}, g_1)$. As $g_1^{\pi(p_{g_1})}(\mathcal{J}) = \mathcal{J}$ is the identity map, by Lemma 2.2 again, there is h C^1-close to g such that h has a closed small curve $\mathcal{L} \subset C(p_h, h)$. Thus the curve $\mathcal{L}$ is such that $h^{\pi(p_h)}(\mathcal{L}) = \mathcal{L}$ is the identity map, $\operatorname{diam} \mathcal{L} = \beta$, $\mathcal{L}$ is normally hyperbolic, and the endpoints of $\mathcal{L}$ are hyperbolic. The closed small curve $\mathcal{L}$ is a simply periodic curve of h, which is contained in $C(p_h, h)$. □

Note that, by Lemma 2.3, there is g C^1-close to f such that g has a simply periodic curve $\mathcal{L} \subset C(p_g, g)$. However, the simply periodic curve $\mathcal{L}$ is not contained in $H(p_g, g)$ (see [25]). Let $\mathcal{WH}$ denote the set of all weakly hyperbolic periodic points of f.

Lemma 2.4 *If the homoclinic class $H(p,f)$ is R-robustly continuum-wise expansive, then $H(p,f) \cap \mathcal{WH} = \emptyset$.*

Proof Suppose that $H(p,f) \cap \mathcal{WH} \neq \emptyset$. Thus there is $q \in H(p,f) \cap \operatorname{Per}(f)$ with $q \sim p$ such that q is weakly hyperbolic. As $H(p,f)$ is R-robustly continuum-wise expansive and $q \in H(p,f) \cap \operatorname{Per}(f)$ with $q \sim p$ such that q is weakly hyperbolic, there is $g \in \mathcal{G}_1 \cap \mathcal{U}(f)$ such that $H(p_g, g) = C(p_g, g)$, and according to Lemma 2.3, there is $\beta > 0$ such that g has a simply periodic curve $\mathcal{J} \subset C(p_g, g)$ with $\operatorname{diam} \mathcal{J} = \beta/4$. As $C(p_g, g)$ is continuum-wise expansive, $\mathcal{J}$ is continuum-wise expansive. According to [12, Proposition 2.6], g is continuum-wise expansive if and only if g^n is continuum-wise expansive for any $n \in \mathbb{Z} \setminus \{0\}$. Consider $e = \beta$. By the definition of a simply periodic curve there is $k > 0$ such that

$$\operatorname{diam} g^{ki}(\mathcal{J}) = \operatorname{diam} \mathcal{J} < e$$

for all $i \in \mathbb{Z}$. By the definition of continuum-wise expansivity, $\mathcal{J}$ should be a point. As $\mathcal{J}$ is a simply periodic curve, this is a contradiction. □

The following was proven by Wang [28]. He considered the Lyapunov exponents of the periodic point in the homoclinic class $H(p,f)$.

Lemma 2.5 *There is a residual set $\mathcal{G}_2 \subset \operatorname{Diff}(M)$ such that, for any $f \in \mathcal{G}_2$, if $H(p,f)$ is not hyperbolic, then there is $q \in H(p,f) \cap \operatorname{Per}(f)$ with $q \sim p$ such that q is a weakly hyperbolic periodic point.*

Proof of Theorem A Let $\mathcal{U}(f)$ be a C^1-neighborhood of f, and let $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2$. As $H(p,f)$ is R-robustly continuum-wise expansive, $H(p_g, g)$ is continuum-wise expansive for any $g \in \mathcal{G} \cap \mathcal{U}(f)$. Assume that there is $g \in \mathcal{G} \cap \mathcal{U}(f)$ such that $H(p_g, g)$ is not hyperbolic. As $g \in \mathcal{G} \cap \mathcal{U}(f)$, there is $q \in H(p_g, g) \cap \operatorname{Per}(g) = C(p_g, g) \cap \operatorname{Per}(g)$ with $q \sim p_g$ such that q is a weakly hyperbolic point. According to Lemma 2.4, this is a contradiction. Thus, if $H(p,f)$ is R-robustly continuum-wise expansive, then, for any $g \in \mathcal{G} \cap \mathcal{U}(f)$, $H(p_g, g)$ is hyperbolic, and hence $H(p,f)$ is hyperbolic. □

3 Proof of Theorem B

Let M be defined as before, and let $X \in \mathfrak{X}(M)$. We denote by $T_pM(\delta)$ the ball $\{v \in T_pM : \|v\| \le \delta\}$. For every $x \in R_X$, let $N_x = \langle X(x)\rangle^{\perp} \subset T_xM$, and let $N_x(\delta)$ be the δ ball in N_x. We set $N_{x,r} = N_x \cap T_xM(r)$ $(r > 0)$ and $\mathcal{N}_{x,r_0} = \exp(N_x(r_0))$ for $x \in M$.

Let $\mathrm{Sing}(X) = \emptyset$, and let $N = \bigcup_{x\in R_X} N_x$. We define the linear Poincaré flow

$$P_t^X := \pi_x \circ D_xX_t,$$

where $\pi_x : T_xM \to N_x$ $(\subset N)$ is the natural projection along the direction of $X(x)$, and D_xX_t is the derivative map of X_t. The following is an important result to prove hyperbolicity.

Remark 3.1 ([7]) Let $\Lambda \subset M$ be a compact invariant set of X_t. Then Λ is a hyperbolic set of X_t if and only if the linear Poincaré flow restriction on Λ has a hyperbolic splitting $N_\Lambda = N^s \oplus N^u$.

Let $X \in \mathfrak{X}(M)$, and suppose $p \in \gamma \in \mathrm{Per}(X)$ $(X_T(p) = p)$, where $T > 0$ is the prime period. If $f : \mathcal{N}_{p,r_0} \to \mathcal{N}_p$ is the Poincaré map $(r_0 > 0)$, then $f(p) = p$. Accordingly, γ is hyperbolic if and only if p is a hyperbolic fixed point of f. The following is a vector field version of Franks' lemma.

Lemma 3.2 ([21]) *Let $X \in \mathfrak{X}(M)$, $p \in \gamma \in \mathrm{Per}(X)$ $(X_T(p) = p$, $T > 0)$, and let $f : \mathcal{N}_{p,r_0} \to \mathcal{N}_p$ be the Poincaré map for some $r_0 > 0$. Let $\mathcal{U}(X) \subset \mathfrak{X}(M)$ be a C^1-neighborhood of X, and let $0 < r \le r_0$ be given. Then there exist $\delta_0 > 0$ and $0 < \epsilon_0 < r/2$ such that, for an isomorphism $L : N_p \to N_p$ with $\|L - D_pf\| < \delta_0$, there is $Y \in \mathcal{U}(X)$ having the following properties*:

(a) $Y(x) = X(x)$ *if* $x \notin F_p(X_t, r, T/2)$,

(b) $p \in \gamma \in \mathrm{Per}(Y)$,

(c)

$$g(x) = \begin{cases} \exp_p \circ L \circ \exp_p^{-1}(x) & \textit{if } x \in B_{\epsilon_0/4}(p) \cap \mathcal{N}_{p,r}, \\ f(x) & \textit{if } x \notin B_{\epsilon_0}(p) \cap \mathcal{N}_{p,r}, \end{cases}$$

where $B_\epsilon(x)$ is a closed ball in M center at $x \in M$ with radius $\epsilon > 0$, $F_p(X_t, r, T/2) = \{X_t(y) : y \in \mathcal{N}_{x,r}$ and $0 \le t \le T\}$, and $g : \mathcal{N}_{p,r} \to \mathcal{N}_p$ is the Poincaré map defined by Y_t.

Remark 3.3 Let $\Lambda \subset M$ be a closed X_t-invariant set, and let Λ be continuum-wise expansive for X. If $\Lambda \cap \mathrm{Sing}(X) \ne \emptyset$, then $\Lambda \cap \mathrm{Sing}(X)$ is totally disconnected.

Proof Suppose that $\Lambda \cap \mathrm{Sing}(X)$ is not totally disconnected. Thus there is a set $\mathcal{C} \subset \Lambda \cap \mathrm{Sing}(X)$ such that $\mathcal{C}$ is closed and connected, that is, a nontrivial continuum. Let $\epsilon > 0$ be given. We assume that $\mathrm{diam}(\mathcal{C}) < \epsilon$. As $\mathcal{C} \subset \Lambda \cap \mathrm{Sing}(X)$, $X_t(\mathcal{C}) = \mathcal{C}$ for all $t \in \mathbb{R}$. Thus we know that

$$\mathrm{diam}\big(X_t(\mathcal{C})\big) = \mathrm{diam}(\mathcal{C}) < \epsilon$$

for all $t \in \mathbb{R}$. Thus $\mathcal{C}$ should be an orbit. This is a contradiction as $\mathcal{C}$ is a nontrivial continuum. $\square$

For any $x, y \in M$, we write $x \rightharpoonup y$ if, for any $\delta > 0$, there is a δ-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1\}_{i=1}^n \subset M$ such that $x_0 = x$ and $d(X_{t_{n-1}}(x_{n-1}), y) < \delta$. Similarly, $y \rightharpoonup x$. We can observe that x, y satisfy both conditions, and thus $x \rightleftharpoons y$. Thus we have an equivalence relation on the set $\mathcal{R}(X)$. Every equivalence class of $\rightleftharpoons$ is called a *recurrence class* of X. Let γ be a hyperbolic periodic point of X. For some $p \in \gamma$, let $C(\gamma, X) = \{x \in M : x \rightleftharpoons p$ denote the chain recurrence class of $X\}$. According to the definition, we can observe that $C(\gamma, X)$ is closed and X_t-invariant and that $H(\gamma, X) \subset C(\gamma, X)$. Bonatti and Crovisier [3] showed that, for a C^1-vector field X, the chain recurrence class $C(\gamma, X)$ is the homoclinic class $H(\gamma, X)$, which is a version of the vector field of diffeomorphisms. Note that if a vector field X does not contain singularities, then the C^1-generic results of diffeomorphisms can be used for C^1 generic vector fields (see [5, 9]).

Lemma 3.4 *There is a residual set $\mathcal{R}_1 \subset \mathfrak{X}(M)$ such that every $X \in \mathcal{R}_1$ satisfies the following conditions*:

(a) *X is Kupka–Smale, that is, every critical point is hyperbolic and its invariant manifolds intersect transversally* (*see* [12]).

(b) *the chain recurrence class $C(\gamma, X) = H(\gamma, X)$ for any $\gamma \in \mathrm{Per}(X)$* (*see* [3]).

We say that a vector field X is a *local star on $H(\gamma, X)$* if there is a C^1-neighborhood $\mathcal{U}(X)$ of X such that, for any $Y \in \mathcal{U}(X)$, every $\eta \in H(\gamma_Y, Y) \cap \mathrm{Crit}(Y)$ is hyperbolic, where γ_Y is the continuation of Y. Let $\mathcal{G}^*(H(\gamma, Y))$ denote the set of all vector fields satisfying the local star on $H(\gamma, X)$.

Proposition 3.5 *Let $H_X(\gamma) \cap \mathrm{Sing}(X) = \emptyset$, and let $\gamma \in \mathrm{Per}(X)$ be hyperbolic. If the homoclinic class $H(\gamma, X)$ is R-robustly continuum-wise expansive, then $X \in \mathcal{G}^*(H(\gamma, X))$.*

Proof Since $H_X(\gamma) \cap \mathrm{Sing}(X)$, we prove that if $H(\gamma, X)$ is R-robustly continuum-wise expansive, then every $\eta \in H_X(\gamma) \cap \mathrm{Per}(X)$ is hyperbolic. Suppose by contradiction that there exist $Y \in \mathcal{U}(X)$ and $\gamma \in H(\gamma_Y, Y) \cap \mathrm{Per}(Y)$ such that γ is not hyperbolic. Consider $p \in \gamma$ such that $Y_T(p) = p(T > 0)$, and let $f : \mathcal{N}_{p,r} \to \mathcal{N}_p$ (for some $r > 0$) be the Poincaré map associated with Y. As γ is not hyperbolic, p is not hyperbolic. Thus we assume that there is an eigenvalue λ of D_pf such that $|\lambda| = 1$. Let $\delta_0 > 0$ and $0 < \epsilon_0 < r/4$ be given by Lemma 3.2, and let $L : N_p \to N_p$ be a linear isomorphism with $\|L - D_pf\| < \delta_0$ such that $L = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$ with respect to some splitting $N_p = G_p \oplus H_p (= E_p^s \oplus E_p^u)$, where $A : G_p \to G_p$ has an eigenvalue λ such that $\dim G_p = 1$ if $\lambda \in \mathbb{R}$ or $\dim G_p = 2$ if $\lambda \in \mathbb{C}$ and $B : H_p \to H_p$ is hyperbolic. According to Lemmas 3.2 and 3.4, there exists $Z \in \mathcal{R}_1$ C^1-close to Y ($Z \in \mathcal{U}(X)$) such that

(a) $Z(x) = Y(x)$ if $x \notin F_p(Y, r_0, T)$,

(b) $p \in \gamma \in \mathrm{Per}(Z)$, and

(c)

$$g(x) = \begin{cases} \exp_p \circ L \circ \exp_p^{-1}(x) & \text{if } x \in B_{\epsilon_0/4}(p) \cap \mathcal{N}_{p,r_0}, \\ f(x) & \text{if } x \notin B_{\epsilon_0}(p) \cap \mathcal{N}_{p,r_0}. \end{cases}$$

Here $g : \mathcal{N}_{p,r_0} \to \mathcal{N}_p$ is the Poincaré map associated with Z. Consider a nonzero vector $u \in G_p$ such that $\|u\| \leq \epsilon_0/8$. Then we have

$$g(\exp_p(u)) = \exp_p \circ L \circ \exp_p^{-1}(\exp_p(u)) = \exp_p(u).$$

Case 1. $\dim G_p = 1$. We may assume that $\lambda = 1$ for simplicity (the other case is similar). We set an arc $\mathcal{I}_u = \{su : 0 \le s \le 1\}$ and $\exp_p(\mathcal{I}_u) = \mathcal{J}_p$. Then we know that

(a) $\mathcal{J}_p \subset B_{\epsilon_0}(p) \cap \mathcal{N}_{p,r_0}$, and

(b) $g|_{\mathcal{J}_p} : \mathcal{J}_p \to \mathcal{J}_p$ is the identity map.

Let $\operatorname{diam}(\mathcal{J}_p) = \epsilon_0/2$. As $g|_{\mathcal{J}_p} : \mathcal{J}_p \to \mathcal{J}_p$ is the identity map, according to Lemma 3.4, $\mathcal{J}_p \subset C(\gamma_Z, Z)$, and hence $g|_{\mathcal{J}_p} : \mathcal{J}_p \to \mathcal{J}_p$ is continuum-wise expansive. However, it is evident that the identity map $g|_{\mathcal{J}_p}$ is not continuum-wise expansive, a contradiction.

Case 2. $\dim G_p = 2$. According to Lemma 3.2, we can find $Z \in \mathcal{R}_1 \cap \mathcal{U}(X)$ such that $D_p g$ is a rational rotation. Thus there is $l \ne 0$ such that $D_p g^l$ has an eigenvalue of 1. As in the proof of case 1, we can derive a contradiction. □

We say that $p \in \gamma \in \operatorname{Per}(X)$ is a *weakly hyperbolic periodic point* if, for any $\delta > 0$, there is an eigenvalue λ of $D_p f$ such that

$$(1 - \delta) \le \lambda \le (1 + \delta),$$

where $f : \mathcal{N}_{p,r} \to \mathcal{N}_p$ is the Poincaré map associated with X. We introduce the concept of a vector field version of diffeomorphisms (see [29]). Let $\operatorname{Sing}(X) = \emptyset$. For any $\eta > 0$, we consider that a C^1-curve $\mathcal{J}$ is η-simply periodic for X if

(a) $\mathcal{J}$ is periodic with period T,

(b) the length of $X_t(\mathcal{J})$ is less than η for any $0 \le t \le T$, and

(c) $\mathcal{J}$ is normally hyperbolic.

Lemma 3.6 *For any $X \in \mathcal{R}_1$, if $p \in \eta \in H(\gamma, X) \cap \operatorname{Per}(X)$ with $\eta \sim \gamma$ is a weakly hyperbolic periodic point, then, for any C^1-neighborhood $\mathcal{U}(X)$ of X, there is $Y \in \mathcal{R}_1 \cap \mathcal{U}(X)$ such that f has an ϵ-simply periodic curve $\mathcal{J} \subset H(\gamma_Y, Y)$ for some $\epsilon > 0$, where $f : \mathcal{N}_{p,r} \to \mathcal{N}_p$ is the Poincaré map defined by Y.*

Proof Let $X \in \mathcal{R}_1$, and let $\mathcal{U}(X)$ be a C^1-neighborhood of X. Suppose that $p \in \eta \in H(\gamma, X) \cap \operatorname{Per}(X)$ with $\eta \sim \gamma$ is a weakly hyperbolic periodic point. As $\eta \sim \gamma$, we consider two points $x \in W^s(\eta) \pitchfork W^u(\gamma)$ and $y \in W^u(\eta) \pitchfork W^s(\gamma)$. Consider $Y \in \mathcal{R}_1 \cap \mathcal{U}(X)$; thus, we have $H(\gamma_Y, Y) = C(\gamma_Y, Y)$. Thus, as in the proof of [15, Proposition 4.1], there exist $\epsilon > 0$ and the Poincaré map $g : \mathcal{N}_{p,r} \to \mathcal{N}_p$ associated with Y such that

(i) the map g is defined by Y,

(ii) g has a closed arc $\mathcal{I}$ or a disc $\mathcal{D}$ such that $g_{|\mathcal{I}} : \mathcal{I} \to \mathcal{I}$ is the identity map, or $g_{|\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is a rotation map,

(iii) $0 < \operatorname{diam} \mathcal{I} \le \epsilon$ and $0 < \operatorname{diam} \mathcal{D} \le \epsilon$,

(iv) $Y_{-t}(x) \to \gamma$ and $Y_t(y) \to \gamma$ as $t \to \infty$, and $g^n(x) \to \mathcal{J}$ (or $\mathcal{D}$) and $g^n(y) \to \mathcal{I}$ (or $\mathcal{D}$) as $n \to \infty$, and

(v) $\mathcal{I} \subset C(\gamma_Y, Y)$ and $\mathcal{D} \subset C(\gamma_Y, Y)$.

As $H(\gamma_Y, Y) = C(\gamma_Y, Y)$, we have $\mathcal{I} \subset H(\gamma_Y, Y)$ and $\mathcal{D} \subset H(\gamma_Y, Y)$, and they are ϵ-simply periodic curves. □

Lemma 3.7 *If the homoclinic class $H(\gamma, X)$ is continuum-wise expansive, then there is no η-simply periodic curve $\mathcal{J} \subset H(\gamma, X)$.*

Proof Assume that there is an η-simply periodic curve $\mathcal{J} \subset H(\gamma, X)$. Thus there is $T > 0$ such that $X_T(\mathcal{J}) = \mathcal{J}$ and $\mathrm{diam}(X_t(\mathcal{J})) \leq \eta$ for any $0 \leq t \leq T$. It is evident that the curve $\mathcal{J}$ is a nontrivial continuum. As $X_T(\mathcal{J}) = \mathcal{J}$, $X_T(x) = x$ for all $x \in \mathcal{J}$. We define $h : \mathcal{J} \to \mathrm{Rep}(\mathbb{R})$ such that $h_x(t) = t$ for all $x \in \mathcal{J}$ and $t \in \mathbb{R}$. Thus, for all $t \in \mathbb{R}$, we have

$$\begin{aligned}\mathrm{diam}\big(\mathcal{X}_h^t(\mathcal{J})\big) &= \max\big\{d\big(X_{h_x(t)}(x), X_{h_y(t)}(y)\big) : x, y \in \mathcal{J}\big\}\\ &= \max\big\{d\big(X_t(x), X_t(y)\big) : x, y \in \mathcal{J}\big\} < \eta.\end{aligned}$$

If η is a continuum-wise expansive constant, then it is a contradiction as $\mathcal{J}$ contains no any single orbit of $x \in \mathcal{J}$. □

Lemma 3.8 *Let $\gamma \in \mathrm{Per}(X)$ be hyperbolic. If the homoclinic class $H(\gamma, X)$ is R-robustly continuum-wise expansive, then, for any $\eta \in H(\gamma, X) \cap \mathrm{Per}(X)$ with $\eta \sim \gamma$, $p \in \eta$ is not a weakly hyperbolic periodic point.*

Proof Suppose by contradiction that there is a hyperbolic $\eta \in H(\gamma, X) \cap \mathrm{Per}(X)$ with $\eta \sim \gamma$ such that $p \in \eta$ is a weakly hyperbolic periodic point. According to Lemma 3.6, there is $Y \in \mathcal{R}_1 \cap \mathcal{U}(X)$ such that f has an ϵ-simply periodic curve $\mathcal{J} \subset H(\gamma_Y, Y)$ for some $\epsilon > 0$, where $f : \mathcal{N}_{p,r} \to \mathcal{N}_p$ is the Poincaré map defined by Y. As $H(\gamma, X)$ is R-robustly continuum-wise expansive, according to Lemma 3.7, this is a contradiction. □

Let $p \in \gamma$ be a hyperbolic periodic point of X with period $\pi(p)$, and let $f : \mathcal{N}_{p,r} \to \mathcal{N}_p$ be the Poincaré map with respect to X. Subsequently, if $\mu_1, \mu_2, \ldots, \mu_d$ are the eigenvalues of $D_p f$, then

$$\lambda_i = \frac{1}{\pi(p)} \log|\mu_i|$$

for $i = 1, 2, \ldots, d$ are called the *Lyapunov exponents* of p. Wang [28] proved that, for a C^1-generic nonsingular vector field $X \in \mathfrak{X}(M)$, if a homoclinic class $H(\gamma, X)$ is not hyperbolic, then there is a periodic orbit $\mathrm{Orb}(q)$ of f that is homoclinically related to $\mathrm{Orb}(p)$ and has a Lyapunov exponent arbitrarily close to 0, which is a vector field version of the result of Wang [28]. Note that if a hyperbolic periodic orbit γ has a Lyapunov exponent arbitrarily close to 0, then there is a point $p \in \gamma$ such that p is a weakly hyperbolic periodic point of X. Thus, we can rewrite the result of Wang [28] as follows.

Lemma 3.9 *There is a residual set $\mathcal{R}_2 \subset \mathfrak{X}(M)$ such that, for any $X \in \mathcal{R}_2$, if $H(\gamma, X) \cap \mathrm{Sing}(X) = \emptyset$ and $H(\gamma, X)$ is not hyperbolic, then there is $\eta \in H(\gamma, X) \cap \mathrm{Per}(X)$ with $\eta \sim \gamma$ such that $p \in \eta$ is a weakly hyperbolic periodic point of X.*

Proof of Theorem B As $H(\gamma, X)$ is continuum-wise expansive, $H(\gamma, X) \cap \mathrm{Sing}(X) = \emptyset$. To derive a contradiction, we assume that $H(\gamma, X)$ is not hyperbolic. Consider $Y \cap \mathcal{U}(X) \cap \mathcal{R}_1 \cap \mathcal{R}_2$. Thus, according to Lemma 3.9, there is $\eta \in H(\gamma_Y, Y) \cap \mathrm{Per}(X)$ with $\eta \sim \gamma_Y$ such that $p \in \eta$ is a weakly hyperbolic periodic point. As $H(\gamma, X)$ is R-robustly measure expansive, according to Lemma 3.8, Y has no weakly hyperbolic periodic points, a contradiction. □

Remark 3.10 Let $\varphi \equiv X_1 : M \to M$ be a diffeomorphism, and let $p \in \gamma \in \mathrm{Per}(X)$ with $X_{\pi(p)}(p) = p$. We set $X_1(p) = p_1$. Then we define the homoclinic class $H_\varphi(p_1)$ that contains p_1. By assumption $H_X(\gamma) \cap \mathrm{Sing}(X) = \emptyset$. According to [1, Theorem 3.2], a vector field X is continuum-wise expansive if and only if a suspension map φ of X is continuum-wise expansive. Thus as in the proof of Theorem A, we have that the homoclinic class $H_\varphi(p_1)$ is hyperbolic if $H_\varphi(p_1)$ is R-robustly continuum-wise expansive.

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