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Solution of fractional differential equations in quasi-b-metric and b-metric-like spaces

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Abstract

In this article, using by α-admissible and α_{qs^p}-admissible mappings, solutions of some fractional differential equations are investigated in quasi-b-metric and b-metric-like spaces.

Keywords: Fractional differential equation; α_{qs^p}-admissible mappings; Quasi-b-metric and b-metric-like spaces

1 Introduction and preliminaries

Throughout this paper we denote the set of continuous functions, b-metric space, b-metric-like space, and quasi-b-metric space by $X = C(J)$, b-MS, b-MLS, and b-QMS, respectively, where $J = [0,1]$.

In [24], the authors presented a new class of α_{qs^p}-admissible mappings and proved some consequences in b-MLS. In 2016, Nawab Hussain et al. [10] stated some conclusions in ordered b-QMS.

The existence of a solution for problem

$$D^\kappa w(\eta) = h\bigl(\eta, w(\eta)\bigr) \quad \bigl(\eta \in [0,1], 1 < \kappa \leq 2\bigr) \tag{1}$$

has been studied widely by many authors.

In [6], Baleanu, Rezapour and Mohammadi studied Eq. (1) by α-ψ-contractions. Similar ideas have also been considered by some authors; see, for example, [2, 3, 8, 9, 14–16, 18–20], and the references therein.

In [1], the authors obtained some conclusions for α-ψ-Geraghty type mappings in b-MS. Recently in [4], Afshari, Kalantari and Baleanu obtained solutions of equation (1) by α-ψ-Geraghty type mappings in b-MS. In this paper, using α- and α_{qs^p}-admissible mappings, we find solutions for some fractional differential equations in b-MLS and b-QMS.

Definition 1.1 ([12, 17]) The Riemann–Liouville derivative for a continuous function h is defined by

$$D^\kappa h(\eta) = \frac{1}{\Gamma(m-\kappa)} \left(\frac{d}{d\eta}\right)^m \int_0^\eta \frac{h(\zeta)}{(\eta-\zeta)^{\kappa-m+1}}\, d\zeta \quad \bigl(m = [\kappa] + 1\bigr),$$

where the right-hand side is defined on $(0,\infty)$.

Definition 1.2 ([21]) Let $g : X \to X$, where X is nonempty, and $\alpha : X \times X \to [0,\infty)$ be given, then g is α-admissible if for $s,t \in X$, $\alpha(s,t) \geq 1$ implies $\alpha(gs,gt) \geq 1$.

Definition 1.3 ([5]) Let X be a nonempty set. The map $b_l : X \times X \to \mathbb{R}^+$ is said to be metric-like on X if for any $w,y,z \in X$, the following hold:
(i) $b_l(w,y) = 0$ implies $w = y$;
(ii) $b_l(w,y) = b_l(y,w)$;
(iii) $b_l(w,y) \leq s(b_l(w,z) + b_l(z,y))$.
The pair (X,b_l) called a *b-MLS*.

Let $\alpha : X \times X \to [0,\infty)$ and $p,q \geq 1$ be arbitrary constants, then $g : X \to X$ is α_{qs^p}-admissible if $\alpha(w,y) \geq qs^p$ implies $\alpha(gw,gy) \geq qs^p$ for all $w,y \in X$. We further consider the following properties:
(H_{s^p}) If $\{w_n\} \subseteq X$ with $w_n \to w \in X$ and $\alpha(w_n,w_{n+1}) \geq s^p$, then there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\alpha(w_{n_k},w) \geq s^p$ for all $k \in N$.
Let Θ be the set of all mappings $\gamma : [0,\infty) \to [0,1)$ such that $\gamma(t_n) \to 1$ implies that $t_n \to 0$.

Proposition 1.4 ([24]) *Let (X,b_l) be a complete b-MLS with parameter $s \geq 1$, let $g : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Suppose*
(i) *g is α_{s^p}-admissible*;
(ii) *There exists $\gamma \in \Theta$ such that*

$$\alpha(w,y)b_l(gw,gy) \leq \gamma\big(b_l(w,y)\big)b_l(w,y); \tag{2}$$

(iii) *There exists $w_0 \in X$ with $\alpha(w_0,gw_0) \geq s^p$*;
(iv) *Either g is continuous or property (H_{s^p}) is satisfied.*
Then g has a fixed point.

2 Main result

We endow X with

$$b_l(w,y) = \max_{t\in J}\big(\big|w(t)\big| + \big|y(t)\big|\big)^p, \tag{3}$$

for $w,y \in X$, where $p > 1$. Then (X,b_l) is a complete *b-MLS* with $s = 2^{p-1}$. Now we study the problem

$$-D^\kappa w(\eta) = f\big(\eta, w(\eta)\big), \quad \eta \in (0,1), \tag{4}$$

with the boundary condition (BC)

$$w(0) = w'(0) = w'(1) = 0, \quad 2 < \kappa < 3, \tag{5}$$

where $f \in C(J \times [0,+\infty),\mathbb{R})$ and D^κ is the Riemann–Liouville derivative.

Lemma 2.1 ([23]) *Given $f \in C(J \times X, \mathbb{R})$ and $2 < \kappa < 3$, the unique solution of* (4) *with* (BC) (5) *is given by* $w(\eta) = \int_0^1 G(\eta,\zeta) f(\zeta, w(\zeta))\,d\zeta$, *where*

$$G(\eta,\zeta) = \begin{cases} \frac{\eta^{\kappa-1}(1-\zeta)^{\kappa-2}-(\eta-\zeta)^{\kappa-1}}{\Gamma(\kappa)}, & 0 \le \zeta \le \eta \le 1, \\ \frac{\eta^{\kappa-1}(1-\zeta)^{\kappa-2}}{\Gamma(\kappa)}, & 0 \le \eta \le \zeta \le 1. \end{cases} \tag{6}$$

Lemma 2.2 ([23]) *The function $G(\eta,\zeta)$ defined by* (6) *satisfies the following condition*:

$$\frac{\eta^{\kappa-1}\zeta(1-\zeta)^{\kappa-2}}{\Gamma(\kappa)} \le G(\eta,\zeta) \le \frac{\zeta(1-\zeta)^{\kappa-2}}{\Gamma(\kappa)}, \quad 0 \le \eta, \zeta \le 1.$$

Theorem 2.3 *Suppose there exists $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that*

(i) *There exists $p > 1$ such that*

$$\begin{aligned} &\bigl|f\bigl(\eta, w(\eta)\bigr)\bigr| + \bigl|f\bigl(\eta, y(\eta)\bigr)\bigr| \\ &\quad \le \frac{1}{2^{p-1}} \Gamma(\kappa+1)(\kappa-1)\bigl(\gamma\bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr)^p\bigr)^{\frac{1}{p}} \bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr), \end{aligned}$$

for $w \in C(J)$, $\eta \in J$;

(ii) *Inequality $\varphi(w(\eta), y(\eta)) \ge 0$ implies*

$$\varphi\biggl(\int_0^1 G(\eta,\zeta) f\bigl(\zeta, w(\zeta)\bigr)\,d\zeta, \int_0^1 G(\eta,\zeta) f\bigl(\zeta, y(\zeta)\bigr)\,d\zeta\biggr) \ge 0;$$

(iii) *If $\{w_n\} \subseteq C(J)$, $w_n \to w$ in $C(J)$ and $\varphi(w_n, w_{n+1}) \ge 0$, then there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\varphi(w_{n_k}, w) \ge 0$ for all $k \in N$;*

(iv) *There exists $w_0 \in C(J)$ with $\varphi(w_0(\eta), \int_0^1 G(\eta,\zeta) f(\zeta, w_0(\zeta))\,d\zeta) \ge 0$.*

Then problem (4) *has at least one solution in (X, b_l).*

Proof By Lemma 2.1, $w \in C(J)$ is a solution of (4) if and only if it is a solution of $w(\eta) = \int_0^1 G(\eta,\zeta) f(\zeta, w(\zeta))\,d\zeta$. Define $T : C(J) \to C(J)$ by $Tw(\eta) = \int_0^1 G(\eta,\zeta) f(\zeta, w(\zeta))\,d\zeta$, for all $\eta \in J$. We find a fixed point of T. Observe that

$$\begin{aligned} &\bigl(\bigl|Tw(\eta)\bigr| + \bigl|Ty(\eta)\bigr|\bigr)^p \\ &\quad = \biggl(\biggl|\int_0^1 G(\eta,\zeta) f\bigl(\zeta, w(\zeta)\bigr)\,d\zeta\biggr| + \biggl|\int_0^1 G(\eta,\zeta) f\bigl(\zeta, y(\zeta)\bigr)\,d\zeta\biggr|\biggr)^p \\ &\quad \le \biggl[\int_0^1 G(\eta,\zeta)\bigl|f\bigl(\zeta, w(\zeta)\bigr)\bigr| + \int_0^1 G(\eta,\zeta)\bigl|f\bigl(\zeta, y(\zeta)\bigr)\bigr|\,d\zeta\biggr]^p \\ &\quad = \biggl[\int_0^1 G(\eta,\zeta)\bigl(\bigl|f\bigl(\zeta, w(\zeta)\bigr)\bigr| + \bigl|f\bigl(\zeta, y(\zeta)\bigr)\bigr|\bigr)\,d\zeta\biggr]^p \\ &\quad \le \biggl[\int_0^1 G(\eta,\zeta) \frac{1}{2^{p-1}} \Gamma(\kappa+1)(\kappa-1)\bigl(\gamma\bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr)^p\bigr)^{\frac{1}{p}} \bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr)\,d\eta\biggr]^p \\ &\quad \le \frac{1}{2^{p(p-1)}} \gamma\bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr)^p \bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr)^p, \end{aligned}$$

with $\varphi(w(\eta), y(\eta)) \geq 0$. Define $\alpha : C(J) \times C(J) \to [0, \infty)$ by

$$\alpha(w, y) = \begin{cases} 2^{p(p-1)}, & \varphi(w(\eta), y(\eta)) \geq 0, \eta \in J, \\ 0, & \text{else.} \end{cases}$$

So

$$\alpha(w, y) b_l(Tw, Ty) \leq \gamma\big(b_l(w, y)\big) b_l(w, y), \quad \gamma \in S.$$

Considering (ii), $\alpha(w, y) \geq 2^{p(p-1)} = s^p$ implies $\varphi(w(\eta), y(\eta)) \geq 0$ and $\varphi(T(w), T(y)) \geq 0$ implies $\alpha(T(w), T(y)) \geq 2^{p(p-1)} = s^p$, $w \in C(J)$. Thus, T is α-admissible. From (iv), there exists $w_0 \in C(J)$ with $\alpha(w_0, Tw_0) \geq 1$. By (iii) and Proposition 1.4, we notice that $w^* \in C(J)$ with $w^* = Tw^*$. □

Corollary 2.4 *Suppose that for $\eta \in J$ and $w, y \in C(J)$ there exists $p > 1$ such that*

$$\big|f\big(\eta, w(\eta)\big)\big| + \big|f\big(\eta, y(\eta)\big)\big| \leq \frac{45\sqrt{\pi}}{2^{p+3}} \big(\gamma\big(\big|w(\eta)\big| + \big|y(\eta)\big|\big)^p\big)^{\frac{1}{p}} \big(\big|w(\eta)\big| + \big|y(\eta)\big|\big),$$

also conditions (ii)–(v) *from Theorem 2.3 hold for f, where $G(\eta, \zeta)$ is given in (6). Then the problem*

$$-\frac{D^{\frac{5}{2}}}{D\eta} w(\eta) = f\big(\eta, w(\eta)\big), \quad \eta \in J, \tag{7}$$

where

$$w(0) = w'(0) = w'(1) = 0,$$

has at least one solution in (X, b_l).

Lemma 2.5 ([13]) *If $f \in C(J \times [0, \infty), \mathbb{R})$, then the problem*

$$\begin{aligned} &D^{\kappa}_{0+} z(\eta) + f\big(\eta, z(\eta)\big) = 0 \quad (0 < \eta < 1, 1 < \kappa < 2), \\ &z(0) = z(1) = 0. \end{aligned} \tag{8}$$

has a unique positive solution

$$z(\eta) = \int_0^1 G(\eta, \zeta) f\big(\zeta, z(\zeta)\big)\, d\zeta,$$

where $G(\eta, \zeta)$ is as follows:

$$G(\eta, \zeta) = \frac{1}{\Gamma(\kappa)} \begin{cases} (\eta(1-\zeta))^{\kappa-1} - (\eta - \zeta)^{\kappa-1}, & \zeta \leq \eta, \\ (\eta(1-\zeta))^{\kappa-1}, & \eta \leq \zeta. \end{cases} \tag{9}$$

Lemma 2.6 ([22]) *Function $G(\eta,\zeta)$ in Lemma 2.5 has the following feature*:

$$\frac{\kappa-1}{\Gamma(\kappa)}\eta^{\kappa-1}(1-\eta)(1-\zeta)^{\kappa-1}\zeta \le G(\eta,\zeta) \le \frac{1}{\Gamma(\kappa)}\eta^{\kappa-1}(1-\eta)^{\kappa-1}(1-\zeta)^{\kappa-2},$$

where $\eta,\zeta \in J$, $1<\kappa<2$.

From Theorem 2.11, we get the following result.

Corollary 2.7 *Suppose for $\eta \in J$ and $w,y \in C(J)$ there exists $p>1$ such that*

$$\bigl|f\bigl(\eta,w(\eta)\bigr)\bigr| + \bigl|f\bigl(\eta,y(\eta)\bigr)\bigr| \le \frac{1}{M2^{p-1}}\gamma\bigl(\bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr)^p\bigr)^{\frac{1}{p}}\bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr),$$

where $M = \sup_{\eta\in J}\int_0^1 G(\eta,\zeta)\,d\zeta$, also conditions (ii)–(iv) *from Theorem 2.3 are satisfied, where $G(\eta,\zeta)$ is given in (9). Then problem (8) has at least one solution.*

Example 2.8 Endow $X = C(J)$ with

$$b_l(w,y) = \max_{\eta\in J}\bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr)^2, \tag{10}$$

then (X,d) is a complete *b-MLS* with $s=2$.

Let $\varphi(w,y) = wy$ and $w_n(\eta) = \frac{\eta n^2}{n^2+1}$. We consider $f: J\times X \to \mathcal{R}^+$ and the following periodic boundary value problem for $w,y \in X$:

$$-D^{\frac{5}{2}}w(\eta) = f\bigl(\eta,w(\eta)\bigr), \quad \eta \in (0,1), \tag{11}$$

with the boundary condition (BC)

$$w(0) = w'(0) = w'(1) = 0,$$

where f satisfies in the following condition:

$$\bigl|f\bigl(\eta,w(\eta)\bigr)\bigr| + \bigl|f\bigl(\eta,y(\eta)\bigr)\bigr| \le \frac{45\sqrt{\pi}}{64}\bigl(\gamma\bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr)^2\bigr)^{\frac{1}{2}}\bigl(\bigl|w(\eta)\bigr| + \bigl|y(\eta)\bigr|\bigr).$$

If $w_0(\eta) = \eta$ then

$$\varphi\left(w_0(\eta), \int_0^1 G(\eta,\zeta)h\bigl(\zeta,w_0(\zeta)\bigr)\,d\zeta\right) \ge 0,$$

for all $\eta \in J$, also $\varphi(w(\eta),y(\eta)) = w(\eta)y(\eta) \ge 0$ implies that

$$\varphi\left(\int_0^1 G(\eta,\zeta)f\bigl(\zeta,w(\zeta)\bigr)\,d\zeta, \int_0^1 G(\eta,\zeta)f\bigl(\zeta,y(\zeta)\bigr)\,d\zeta\right) \ge 0.$$

It is obvious that condition (iii) in Theorem 2.4 holds. Hence, from Theorem 2.4 problem (7) has at least one solution.

Definition 2.9 ([11]) Let X be a nonempty set, $s \geq 1$, and suppose $q_b : X \times X \to [0,\infty)$, for all $w, y \in X$, satisfies the following:

(q_{b_1}) $q_b(w,y) = 0$ if and only if $w = y$;

(q_{b_2}) $q_b(w,y) \leq s(q_b(w,z) + q_b(z,y))$ for all $w, y, z \in X$.

The pair (X, q_b) is called a *b-QMS*.

Theorem 2.10 ([10]) *Let (X, q_b) be a complete b-QMS, $g : X \to X$, and suppose there exists $\alpha : X \times X \to [0,\infty)$ with*

$$\alpha(w,y)q_b(gw,gy) \leq kq_b(w,y), \tag{12}$$

for all $w, y \in X$, $k \in [0, s^{-1})$. Also assume

(i) *g is α-admissible*;

(ii) *There exists $w_0 \in X$ such that $\alpha(w_0, gw_0) \geq 1$*;

(iii) *If $w_n \to w$, then $\limsup_{n\to\infty} q_b(w_n, y) \geq q_b(w,y)$, for all $y \in X$*;

(iv) *If $\{w_n\} \subseteq X$, $\alpha(w_n, w_{n+1}) \geq 1$, for all $n \in N$, and $w_n \to w \in X$, then there exists $\{w_{n(k)}\}$ of $\{w_n\}$ with $\alpha(w_{n(k)}, w) \geq 1$, for $k \in N$.*

Then there exists $w \in X$ with $g(w) = w$.

Let $q_b : X \times X \to [0,\infty)$ be given by

$$q_b(w,y) = \begin{cases} \|(w-y)^2\|_\infty + \|w\|_\infty, & w, y \in X, w \neq y, \\ 0 & \text{otherwise}, \end{cases} \tag{13}$$

where

$$\|w\|_\infty = \sup_{\eta \in J} \bigl|w(\eta)\bigr|.$$

Then (X, q_b) is a complete *b-QMS* with $s = 2$, but (X, q_b) is not *b-MS*.

Theorem 2.11 *Suppose*

(i) *There exists $k \in [0, \frac{1}{2})$ such that $|f(\eta, w(\eta))| \leq k\Gamma(\kappa+1)(\kappa-1)\|w\|_\infty$, and*

$$\bigl|f\bigl(\eta, w(\eta)\bigr) - f\bigl(\eta, y(\eta)\bigr)\bigr| \leq k\Gamma(\kappa+1)(1-\kappa)\bigl\|(w-y)^2\bigr\|_\infty$$

for $w, y \in C(J)$, $\eta \in J$.

(ii) *Inequality $\varphi(w(\eta), y(\eta)) \geq 0$ implies*

$$\varphi\biggl(\int_0^1 G(\eta,\zeta)f\bigl(\zeta, w(\zeta)\bigr)\,d\zeta, \int_0^1 G(\eta,\zeta)f\bigl(\zeta, y(\zeta)\bigr)\,d\zeta\biggr) \geq 0;$$

(iii) *If $w_n \to w$, $w_n, w \in C(J)$, then*

$$\limsup_{n\to\infty}\bigl(\bigl\|(w_n - y)^2\bigr\|_\infty + \|w_n\|_\infty\bigr) \geq \bigl\|(w-y)^2\bigr\|_\infty + \|w\|_\infty$$

(iv) *If $\{w_n\} \subseteq C(J)$, $w_n \to w$ in $C(J)$ and $\varphi(w_n, w_{n+1}) \geq 0$ then there exists $\{w_{n(i)}\}$ of $\{w_n\}$, with $\varphi(w_{n(i)}, w) \geq 0$ for $i \in N$.*

(v) *There exists* $w_0 \in C(J)$ *with* $\varphi(w_0(\eta), \int_0^1 G(\eta,\zeta)f(\zeta,w_0(\zeta))\,d\zeta) \geq 0$.
Then problem (4) *has at least one solution.*

Proof By Lemma 2.1, $w \in C(J)$ is a solution of (4) if and only if it is a solution of $w(\eta) = \int_0^1 G(\eta,\zeta)f(\zeta,w(\zeta))\,d\zeta$. We define $T : C(J) \to C(J)$ by $Tw(\eta) = \int_0^1 G(\eta,\zeta)f(\zeta,w(\zeta))\,d\zeta$ for all $\eta \in J$. For $w \in C(J)$ with $\varphi(w(\eta),y(\eta)) \geq 0$ and $\eta \in J$, using (i), we have

$$\begin{aligned}
&\bigl|Tw(\eta) - Ty(\eta)\bigr|^2 + \bigl|Tw(\eta)\bigr| \\
&\quad = \left|\int_0^1 G(\eta,\zeta)\bigl(f\bigl(\zeta,w(\zeta)\bigr) - f\bigl(\zeta,y(\zeta)\bigr)\bigr)\,d\zeta\right|^2 \\
&\qquad + \left|\int_0^1 G(\eta,\zeta)f\bigl(\zeta,w(\zeta)\bigr)\,d\zeta\right| \\
&\quad \leq \left(\int_0^1 G(\eta,\zeta)\bigl|f\bigl(\zeta,w(\zeta)\bigr) - f\bigl(\zeta,y(\zeta)\bigr)\bigr|\,d\zeta\right)^2 + \int_0^1 G(\eta,\zeta)\bigl|f\bigl(\zeta,w(\zeta)\bigr)\bigr|\,d\zeta \\
&\quad \leq \left(\int_0^1 G(\eta,\zeta)k\Gamma(\kappa+1)(1-\kappa)\bigl\|(w-y)^2\bigr\|_\infty\,d\zeta\right)^2 \\
&\qquad + \int_0^1 G(\eta,\zeta)k\Gamma(\kappa+1)(1-\kappa)\|w\|_\infty\,d\zeta \\
&\quad \leq k\bigl(\bigl(\bigl\|(w-y)^2\bigr\|_\infty\bigr)^2 + \|w\|_\infty\bigr) = kq_b(w,y).
\end{aligned}$$

For $w \in C(J)$, $\eta \in J$ with $\varphi(w(\eta),y(\eta)) \geq 0$, we have

$$\bigl\|(Tw-Ty)^2\bigr\|_\infty + \|Tw\|_\infty \leq kq_b(w,y).$$

Define $\alpha : C(J) \times C(J) \to [0,\infty)$ by

$$\alpha(w,y) = \begin{cases} 1, & \varphi(w(\eta),y(\eta)) \geq 0, \eta \in J, \\ 0, & \text{else}. \end{cases}$$

Then we have

$$\alpha(w,y)q_b a(Tw,Ty) \leq q_b a(Tw,Ty) \leq kq_b(w,y),$$

from (ii); $\alpha(w,y) \geq 1$ implies $\varphi(w(\eta),y(\eta)) \geq 0$, and $\varphi(T(w),T(y)) \geq 0$ implies $\alpha(T(w), T(y)) \geq 1$, $w \in C(J)$.

Thus, T is α-admissible. From (v), there exists $w_0 \in C(J)$ with $\alpha(w_0,Tw_0) \geq 1$. By (iii), (iv) and Theorem 2.10, we find that $w^* \in C(J)$ with $w^* = Tw^*$. □

Corollary 2.12 *Suppose for* $\eta \in J$ *and* $w \in C(J)$ *there exists* $k \in [0,\frac{1}{2})$, $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$\begin{aligned}
&\bigl|f\bigl(\eta,w(\eta)\bigr)\bigr| \leq k\frac{45\sqrt{\pi}}{16}\|w\|_\infty, \\
&\bigl|f\bigl(\eta,w(\eta)\bigr) - f\bigl(\eta,y(\eta)\bigr)\bigr| \leq k\frac{45\sqrt{\pi}}{16}\bigl\|(w-y)^2\bigr\|_\infty.
\end{aligned} \tag{14}$$

Also assume that conditions (ii)–(v) *from Theorem* 2.11 *hold for* f, *where* $G(\eta,\zeta)$ *is given in* (6). *Then the problem*

$$-\frac{D^{\frac{5}{2}}}{D\eta}w(\eta)=f\big(\eta,w(\eta)\big),\quad \eta\in J,\qquad w(0)=w'(0)=w'(1)=0,$$

has at least one solution.

Proof By using Lemma 2.2,

$$0\leq\int_0^1 G(\eta,\zeta)\,d\zeta\leq\frac{16}{45\sqrt{\pi}},\quad \eta\in J. \tag{15}$$

By employing (14), (15) and in accordance with 2.11, we obtain

$$\big\|(Tw-Ty)^2\big\|_\infty+\|Tw\|_\infty\leq k\Big(\big(\big\|(w-y)^2\big\|_\infty\big)^2+\|w\|_\infty\Big)=kq_b(w,y).$$

The rest of proof is similar to that of Theorem 2.11. □

Corollary 2.13 *Suppose for* $\eta\in J$ *and* $w,y\in C(J)$ *there exist* $k\in[0,\frac{1}{2})$ *such that*

$$\big|f\big(\eta,w(\eta)\big)-f\big(\eta,y(\eta)\big)\big|\leq\frac{k}{M}\big\|(w-y)^2\big\|_\infty,\qquad \big|f\big(\eta,w(\eta)\big)\big|\leq\frac{k}{M}\|w\|_\infty,$$

$M=\sup_{\eta\in J}\int_0^1 G(\eta,\zeta)\,d\zeta$, *also conditions* (ii)–(iv) *from Theorem* 2.11 *are satisfied, where* $G(\eta,\zeta)$ *is given in* (9). *Then problem* (8) *has at least one solution.*

Definition 2.14 ([12, 17]) For a continuous function $h:[0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order κ is defined by

$${}^cD^\kappa h(\eta)=\frac{1}{\Gamma(m-\kappa)}\int_0^\eta(\eta-\zeta)^{m-\kappa-1}h^{(m)}(\zeta)\,d\zeta,$$

where $m-1<\kappa<m$, $m=[\kappa]+1$, and $[\kappa]$ denotes the integer part of κ.

We consider

$${}^cD^\kappa w(\eta)+f\big(\eta,w(\eta)\big)=0,\quad 0<\eta<1, 2<\kappa<3, \tag{16}$$

with boundary conditions (BC)

$$w(0)=w''(0)=0,\qquad w(1)=\lambda\int_0^1 w(\zeta)\,d\zeta. \tag{17}$$

Lemma 2.15 ([7]) *Let* $2<\kappa<3$, $\lambda\neq 0$ *and* $f\in C([0,T]\times X,\mathbb{R})$ *be given. Then Eq.* (16) *with* (*BC*) (17) *has a unique solution given by*

$$w(\eta)=\int_0^1 G(\eta,\zeta)f\big(\zeta,w(\zeta)\big)\,d\zeta,$$

where

$$G(\eta,\zeta)=\begin{cases}\frac{2\eta(1-\zeta)^{\kappa-1}(\kappa-\lambda+\lambda\zeta)-(2-\lambda)\kappa(\eta-\zeta)^{\kappa-1}}{(2-\lambda)\Gamma(\kappa+1)}, & 0\le\zeta\le\eta\le 1,\\ \frac{2\eta(1-\zeta)^{\kappa-1}(\kappa-\lambda+\lambda\zeta)}{(2-\lambda)\Gamma(\kappa+1)}, & 0\le\eta\le\zeta\le 1.\end{cases} \tag{18}$$

From Lemma 2.15 and Theorem 2.11, we get the following conclusion.

Corollary 2.16 *Suppose for $\eta\in J$ and $w,y\in C(J)$ there exists $k\in[0,\frac{1}{2})$, such that*

$$\bigl|f\bigl(\eta,w(\eta)\bigr)\bigr|\le\frac{k(2-\lambda)\Gamma(\kappa)}{2}\|w\|_\infty,$$
$$\bigl|f\bigl(\eta,w(\eta)\bigr)-f\bigl(\eta,y(\eta)\bigr)\bigr|\le\frac{k(2-\lambda)\Gamma(\kappa)}{2}\bigl\|(w-y)^2\bigr\|_\infty,$$

where $0<\lambda<2$; also suppose that conditions (ii)–(iv) *from Theorem* 2.11 *are satisfied, where $G(\eta,\zeta)$ is given in* (18). *Then* (16) *with* (*BC*) (17) *has at least one solution.*

Let (X,q_b) be given in (13). For

$${}^cD^\kappa w(\eta)=f\bigl(\eta,w(\eta)\bigr)\quad(\eta\in J,1<\kappa\le 2), \tag{19}$$

with

$$w(0)=0,\qquad w(1)=\int_0^\xi w(\zeta)\,d\zeta\quad(0<\xi<1),$$

where $f:J\times X\to\mathbb{R}$ is continuous, we have the following result.

Theorem 2.17 *Assume*

(i) *There exists $k\in[0,\frac{1}{2})$ such that $|f(\eta,w(\eta))|\le\frac{k}{2}\frac{\Gamma(\kappa+1)}{5}\|w\|_\infty$, and*

$$\bigl|f\bigl(\eta,w(\eta)\bigr)-f\bigl(\eta,y(\eta)\bigr)\bigr|\le\sqrt{\frac{k}{2}\frac{\Gamma(\kappa+1)}{5}}\bigl\|(w-y)^2\bigr\|_\infty$$

for $w\in C(J)$, $\eta\in J$.

(ii) *Inequality $\varphi(w(\eta),y(\eta))\ge 0$ implies $\varphi(T(w(\eta)),T(y(\eta)))\ge 0$, where $T:C(J)\to C(J)$ is defined by*

$$\begin{aligned}Tw(\eta):={}&\frac{1}{\Gamma(\kappa)}\int_0^1(\eta-\zeta)^{\kappa-1}f\bigl(\zeta,w(\zeta)\bigr)\,d\zeta\\&-\frac{2\eta}{(2-\xi^2)\Gamma(\kappa)}\int_0^1(1-\zeta)^{\kappa-1}f\bigl(\zeta,w(\zeta)\bigr)\,d\zeta\\&+\frac{2\eta}{(2-\xi^2)\Gamma(\kappa)}\int_0^\xi\biggl(\int_0^\zeta(\zeta-n)^{\kappa-1}f\bigl(n,w(n)\bigr)\,dn\biggr)\,d\zeta\quad(\eta\in J);\end{aligned}$$

(iii) *If $w_n\to w$, $w_n,w\in C(J)$, then*

$$\limsup_{n\to\infty}\bigl(\bigl\|(w_n-y)^2\bigr\|_\infty+\|w_n\|_\infty\bigr)\ge\bigl\|(w-y)^2\bigr\|_\infty+\|w\|_\infty;$$

(iv) *If* $\{w_n\} \subseteq C(J)$, $w_n \to w$ *in* $C(J)$ *and* $\varphi(w_n, w_{n+1}) \geq 0$ *then there exists* $\{w_{n(i)}\}$ *of* $\{w_n\}$, *with* $\varphi(w_{n(i)}, w) \geq 0$ *for* $i \in N$;

(v) *There exists* $w_0 \in C(J)$ *with* $\varphi(w_0(\eta), T(w_0(\eta))) \geq 0$.

Then (19) *has at least one solution.*

Proof Function $w \in C(J)$ is a solution of (19) if and only if it is a solution of

$$\begin{aligned} w(\eta) &= \frac{1}{\Gamma(\kappa)} \int_0^1 (\eta - \zeta)^{\kappa-1} f\big(\zeta, w(\zeta)\big)\, d\zeta - \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^1 (1-\zeta)^{\kappa-1} f\big(\zeta, w(\zeta)\big)\, d\zeta \\ &\quad + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^\xi \left(\int_0^\zeta (\zeta - n)^{\kappa-1} f\big(n, w(n)\big)\, dn \right) d\zeta \quad (\eta \in J). \end{aligned}$$

Then (19) is replaceable to get $w^* \in C(J)$, with $Tw^* = w^*$. Let $w \in C(J)$ with $\varphi(w(\eta), y(\eta)) \geq 0$, $\eta \in J$. By (i), we have

$$\begin{aligned} &\big|Tw(\eta) - Ty(\eta)\big|^2 + \big|Tw(\eta)\big| \\ &\quad = \left| \frac{1}{\Gamma(\kappa)} \int_0^1 (\eta - \zeta)^{\kappa-1} f\big(\zeta, w(\zeta)\big)\, d\zeta \right. \\ &\qquad - \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^1 (1-\zeta)^{\kappa-1} f\big(\zeta, w(\zeta)\big)\, d\zeta \\ &\qquad + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^\xi \left(\int_0^\zeta (\zeta - n)^{\kappa-1} f\big(n, w(n)\big)\, dn \right) d\zeta \\ &\qquad - \frac{1}{\Gamma(\alpha)} \int_0^1 (\eta - \zeta)^{\kappa-1} f\big(\zeta, y(\zeta)\big)\, d\zeta \\ &\qquad + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^1 (1-\zeta)^{\kappa-1} f\big(\zeta, y(\zeta)\big)\, d\zeta \\ &\qquad \left. - \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^\xi \left(\int_0^\zeta (\zeta - n)^{\kappa-1} f\big(n, y(n)\big)\, dn \right) d\zeta \right|^2 \\ &\qquad + \left| \frac{1}{\Gamma(\kappa)} \int_0^1 (\eta - \zeta)^{\kappa-1} f\big(\zeta, w(\zeta)\big)\, d\zeta - \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^1 (1-\zeta)^{\kappa-1} f\big(\zeta, w(\zeta)\big)\, d\zeta \right. \\ &\qquad \left. + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^\xi \left(\int_0^\zeta (\zeta - n)^{\kappa-1} f\big(n, w(n)\big)\, dn \right) d\zeta \right| \\ &\quad \leq \left[\frac{1}{\Gamma(\kappa)} \int_0^1 |\eta - \zeta|^{\kappa-1} \big| f\big(\zeta, w(\zeta)\big) - f\big(\zeta, y(\zeta)\big)\big|\, d\zeta \right. \\ &\qquad + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^1 |1-\zeta|^{\kappa-1} \big| f\big(\zeta, w(\zeta)\big) - f\big(\zeta, y(\zeta)\big)\big|\, d\zeta \\ &\qquad \left. + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^\xi \left| \int_0^\zeta |\zeta - n|^{\kappa-1} \big| f\big(n, w(n)\big) - f\big(n, y(n)\big)\big|\, dn \right| d\zeta \right]^2 \\ &\qquad + \frac{1}{\Gamma(\kappa)} \int_0^1 \big|(\eta - \zeta)\big|^{\kappa-1} \big| f\big(\zeta, w(\zeta)\big)\big|\, d\zeta \\ &\qquad + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^1 \big|(1-\zeta)\big|^{\kappa-1} \big| f\big(\zeta, w(\zeta)\big)\big|\, d\zeta \\ &\qquad + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)} \int_0^\xi \left(\int_0^\zeta \big|(\zeta - n)\big|^{\kappa-1} \big| f\big(n, w(n)\big)\big|\, dn \right) d\zeta \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{\Gamma(\kappa+1)}{5}\right)^2 \frac{k}{2}\|w-y\|_\infty^2 \biggl[\sup\biggl(\int_0^1 |\eta-\zeta|^{\kappa-1}\,d\zeta + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)}\int_0^1 |1-\zeta|^{\kappa-1}\,d\zeta \\
&\quad + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)}\int_0^\xi \biggl(\int_0^\zeta |\zeta-n|^{\kappa-1}\,dn\biggr)\,d\zeta\biggr)\biggr]^2 \\
&\quad + \frac{\Gamma(\kappa+1)}{5}\frac{k}{2}\|w-y\|_\infty \biggl[\sup\biggl(\int_0^1 |\eta-\zeta|^{\kappa-1}\,d\zeta + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)}\int_0^1 |1-\zeta|^{\kappa-1}\,d\zeta \\
&\quad + \frac{2\eta}{(2-\xi^2)\Gamma(\kappa)}\int_0^\xi \biggl(\int_0^\zeta |\zeta-n|^{\kappa-1}\,dn\biggr)\,d\zeta\biggr)\biggr] \leq k\bigl(\|w-y\|_\infty^2 + \|w-y\|_\infty\bigr)
\end{aligned}$$

for each $w, y \in C(J)$ with $\varphi(w(\eta), y(\eta)) \geq 0$, $\eta \in J$, and

$$\bigl\|(Tw-Ty)^2\bigr\|_\infty + \|Tw\|_\infty \leq kq_b(w,y).$$

Suppose $\alpha : C(J) \times C(J) \to [0,\infty)$ is defined by

$$\alpha(w,y) = \begin{cases} 1, & \varphi(w(\eta), y(\eta)) \geq 0, \eta \in J, \\ 0, & \text{else}, \end{cases}$$

then

$$\alpha(w,y)q_b(Tw,Ty) \leq q_b(Tw,Ty) \leq kq_b(w,y),$$

for $w, y \in C(J)$. By Theorem 2.10, the result is obtained by the process of the proof of Theorem 2.11. □

Here, we find a positive solution for

$$\frac{{}^cD^\kappa}{D\eta} w(\eta) = f\bigl(\eta, w(\eta)\bigr), \quad 0 < \kappa \leq 1, \eta \in J, \tag{20}$$

where

$$w(0) + \int_0^1 w(\zeta)\,d\zeta = w(1).$$

We note that ${}^cD^\nu$ is the Caputo derivative of order ν. We consider the Banach space of continuous functions on J endowed with the sup norm. We have the following lemma.

Lemma 2.18 ([7]) *Let $0 < \kappa \leq 1$ and $h \in C([0,T] \times X, \mathbb{R})$ be given. Then the equation*

$${}^cD^\kappa w(\eta) = f\bigl(\eta, w(\eta)\bigr) \quad \bigl(\eta \in [0,T], T \geq 1\bigr),$$

with

$$w(0) + \int_0^T w(\zeta)\,d\zeta = w(T),$$

has a unique solution given by

$$w(\eta) = \int_0^T G(\eta,\zeta) f\bigl(\zeta, w(\zeta)\bigr)\, d\zeta,$$

where $G(\eta,\zeta)$ *is defined by*

$$G(\eta,\zeta) = \begin{cases} \frac{-(T-\zeta)^{\kappa}+\kappa T(\eta-\zeta)^{\kappa-1}}{T\Gamma(\kappa+1)} + \frac{(T-\zeta)^{\kappa-1}}{T\Gamma(\kappa)}, & 0 \leq \zeta < \eta, \\ \frac{-(T-\zeta)^{\kappa}}{T\Gamma(\kappa+1)} + \frac{(T-\zeta)^{\kappa-1}}{T\Gamma(\kappa)}, & \eta \leq \zeta < T. \end{cases} \tag{21}$$

From Lemma 2.18 and Theorem 2.11, we get the following conclusion.

Corollary 2.19 *Assume*

(i) *There exists* $k \in [0, \frac{1}{2})$ *such that* $|f(\eta, w(\eta))| \leq \frac{51k}{80}\|w\|_\infty$, *and*

$$\bigl|f\bigl(\eta, w(\eta)\bigr) - f\bigl(\eta, y(\eta)\bigr)\bigr| \leq \frac{51k}{80}\bigl\|(w-y)^2\bigr\|_\infty$$

for $w, y \in C(J)$, $\eta \in J$.

Suppose that conditions (ii)–(iv) *from Theorem* 2.11 *are met, where* $G(\eta,\zeta)$ *is given in* (21), *then the following problem has at least one solution*:

$$^{c}D^{\frac{1}{2}} w(\eta) = f\bigl(\eta, w(\eta)\bigr) \quad \bigl(\eta \in [0,1]\bigr), \qquad w(0) + \int_0^1 w(\zeta)\, d\zeta = w(1).$$

Example 2.20 Let $X = C(J)$ and $q_b : X \times X \to [0,\infty)$ be given by

$$q_b(w,y) = \begin{cases} \|(w-y)^2\|_\infty + \|w\|_\infty, & w, y \in X, w \neq y, \\ 0, & \text{otherwise}. \end{cases} \tag{22}$$

Then (X, d) is a complete b-QMS with $s = 2$, but is not a b-metric space.

Let $\theta(w,y) = w^3 y^3$, $w_n(\eta) = \frac{\eta}{n^2+1}$. We consider $f : J \times [0,5] \to [0,5]$ and the periodic boundary value problem

$$^{c}D^{\frac{1}{2}} w(\eta) = f\bigl(\eta, w(\eta)\bigr) \quad (\eta \in J), \tag{23}$$

with

$$w(0) = 0, \qquad w(1) = \int_0^{\xi} w(\zeta)\, d\zeta \quad (0 < \xi < 1),$$

and suppose there exists $k \in [0, \frac{1}{2})$ such that f satisfies in the following condition:

$$\bigl|f\bigl(\eta, w(\eta)\bigr)\bigr| \leq \frac{51k}{80}\|w\|_\infty, \qquad \bigl|f\bigl(\eta, w(\eta)\bigr) - f\bigl(\eta, y(\eta)\bigr)\bigr| \leq \frac{51k}{80}\bigl\|(w-y)^2\bigr\|_\infty$$

when $\eta \in J$ and $w(\eta), y(\eta) \in [0,5]$. If $w_0(\eta) = \eta$, then

$$\theta\biggl(w_0(\eta), \int_0^1 G(\eta,\zeta) f\bigl(\zeta, y_0(\zeta)\bigr)\, d\zeta\biggr) \geq 0,$$

for all $\eta \in J$, also $\theta(w(\eta), y(\eta)) = w(\eta)^3 y(\eta)^3 \geq 0$ implies that

$$\theta\left(\int_0^1 G(\eta,\zeta)f(\zeta,w_(\zeta))d\zeta, \int_0^1 G(\eta,\zeta)f\bigl(\zeta,y(\zeta)\bigr)\,d\zeta\right) \geq 0.$$

It is obvious that conditions (iii) and (iv) in Corollary 2.19 hold. Hence, from Corollary 2.19 problem (23) has at least one solution.

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