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Blow-up for a degenerate and singular parabolic equation with a nonlocal source

Nitithorn Sukwong[1], Panumart Sawangtong[1,3*], Sanoe Koonprasert[1] and Wannika Sawangtong[2,3]

*Correspondence:
panumart.s@sci.kmutnb.ac.th
[1]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand
[3]Centre of Excellence in Mathematics, PERDO, Commission on Higher Education, Ministry of Education, Bangkok, Thailand
Full list of author information is available at the end of the article

Abstract

This article studies the blow-up phenomenon for a degenerate and singular parabolic problem. Conditions for the local and global existence of solutions for the problem are given. In the case that blow-up occurs, the blow-up set for the problem is investigated. Finally, the asymptotic behaviour of the solution when time converges to the blow-up time is studied.

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1 Introduction

In this paper, we consider the blow-up phenomenon of the following degenerate and singular parabolic equation with a nonlocal source:

$$\left.\begin{array}{ll} u_t = (x^\beta (u^m)_x)_x + \int_0^a u^p(x,t)\,dx, & (x,t) \in (0,a) \times (0,\infty), \\ u(0,t) = u(a,t) = 0, & t > 0, \\ u(x,0) = g(x), & x \in [0,a], \end{array}\right\} \tag{1}$$

where $\beta \in [0,1)$ and $p > m > 1$ and g satisfies the following hypotheses:

(H1) $g \in C^{2+\alpha}(0,a) \cap C[0,a]$ with $0 < \alpha < 1$,

(H2) $g > 0$ on $(0,a)$, $g(0) = g(a) = 0$, and $g'(0) > 0$ and $g'(a) < 0$,

(H3) $(x^\beta (g^m)')' + \int_0^a g^p(x)\,dx > 0$ for $x \in (0,a)$,

(H4) $\lim_{x\to 0^+} (x^\beta (g^m)')' = -\int_0^a g^p(x)\,dx$ and $\lim_{x\to a^-} (x^\beta (g^m)')' = -\int_0^a g^p(x)\,dx$,

(H5) $(x^\beta (g^m)')' \le 0$ for $x \in (0,a)$.

We note that the idea for constructing the function g satisfying the assumptions (H1)–(H5) is in the appendix of [14]. Since $\beta \in [0,1)$, coefficients of terms u_x, u_{xx} may tend to 0 or ∞ as x converges to 0^+. We thus can regard (1) as degenerate and singular. Let us introduce the definition of blow-up in a finite time.

Definition 1.1 The solution u of (1) is said to show blow-up at the point x_b in a finite time T_b (> 0) if there exists a sequence $\{(x_n,t_n)\}$ in $(0,a) \times (0,\infty)$ such that $(x_n,t_n) \to (x_b,T_b)$ as $n \to \infty$ and $\lim_{n\to\infty} u(x_n,t_n) = \infty$. The point x_b and the time T_b are called a blow-up point and blow-up time, respectively. Furthermore, we call the set of all blow-up points to be the blow-up set, which is denoted by S. If $S = [0, a]$, we say that the solution u of (1) shows global blow-up.

The first paper concerning the blow-up problem for the reaction-diffusion equation was written by Fujita [9]. He studied the Cauchy problem: $u_t - \Delta u = u^{1+\alpha}$, $\alpha > 0$ and shown that if $0 < N\alpha < 2$ (N is the space dimension), then the initial value problem had no non-trivial global solutions while if $N\alpha > 2$, there were non-trivial global solutions. In this second case, it was essential that the initial values were sufficiently small. After the publication of Fujita's paper, the blow-up phenomenon for the reaction-diffusion equations has been the object of intensive research. Degenerate parabolic equations with/without nonlocal source have been studied by under various types of initial and boundary conditions since the early 1970s by many researchers ([1, 3, 5, 10–12] and [15]).

In 1997, Aderson and Deng [2] studied the following problem:

$$\left.\begin{array}{l} u_t = ((u^m)_x + \varepsilon u^n)_x + au\|u\|_q^{p-1}, \quad (x,t) \in (0,1) \times (0,\infty), \\ u(0,t) = u(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in [0,1]. \end{array}\right\} \tag{2}$$

They showed that the solution of (2) blows up in a finite time for a sufficiently large data u_0 if $p > \max\{1, \max\{m,n\}\}$. They, however, did not consider the blow-up profile of the blow-up solution.

In 2001, Deng et al. [6] considered the following problem:

$$\left.\begin{array}{l} u_t = (u^m)_{xx} + a\int_{-l}^{l} u^q\,dx, \quad (x,t) \in (-l,l) \times (0,\infty), \\ u(-l,t) = u(l,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in [-l,l], \end{array}\right\} \tag{3}$$

with $l > 0$, $a > 0$ and $q > m > 1$. They established that, under certain conditions, the solution of (3) either exists globally or blows up completely in a finite time. Moreover, they obtained

$$C_1(T_b - t)^{-1/(q-1)} \le \max_{x\in[-l,l]} u(x,t) \le C_2(T_b - t)^{-1/(q-1)}.$$

In 2003, Liu et al. [14] studied the following problem:

$$\left.\begin{array}{l} u_t = x^\alpha (u^m)_{xx} + \int_0^a u^p\,dx - ku^q, \quad (x,t) \in (0,a) \times (0,\infty), \\ u(0,t) = u(a,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in [0,a], \end{array}\right\} \tag{4}$$

and, under some assumptions, they proved the local existence and uniqueness of a classical solution of (4) and obtained some sufficient conditions for blow-up in a finite time of a solution of (4). Furthermore, they showed that the blow-up set of the solution is the whole domain.

In 2003, Li et al. [13] considered the following problem:

$$\left.\begin{array}{l} u_t = \Delta(u^m) + au^p\int_\Omega u^q\,dx, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \end{array}\right\} \tag{5}$$

where $\Omega \subset R^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. They showed that the solution of (5) either exists globally or blows up in a finite time. Moreover, if $p+q > m$, then they showed

$$C_1(T_b - t)^{-1/(p+q-1)} \le \max_{x\in[-l,l]} u(x,t) \le C_2(T_b - t)^{-1/(p+q-1)}.$$

This paper is organized as follows. In the next section, we establish local existence and uniqueness of the solution of (1). We give some criteria for the solution of (1) to exist globally or blow up in a finite time in Sect. 3. The blow-up set and blow-up profile of the solution are presented in Sect. 4.

2 Local existence

Since (1) is degenerate and singular, the standard theory of parabolic type cannot be applied directly to obtain the existence and uniqueness of its classical solution. To investigate the local existence of the solution of (1), we need some transformation. Let $v = u^m$, $t = \frac{\tau}{ma^{\beta-2}}$ and $x = a\xi$ in (1). Then (1) becomes

$$\left.\begin{aligned} &v_\tau = v^r[(\xi^\beta v_\xi)_\xi + a^{3-\beta}\textstyle\int_0^1 v^q(\xi,\tau)\,d\xi], \quad (\xi,\tau)\in(0,1)\times(0,\infty),\\ &v(0,\tau) = v(1,\tau) = 0, \quad \tau > 0,\\ &v(\xi,0) = k(\xi), \quad \xi\in[0,1], \end{aligned}\right\} \tag{6}$$

where $0 < r = \frac{m-1}{m} < 1$, $q = \frac{p}{m} > 1$, $k = g^m$ and k satisfies the following:

(H1′) $k \in C^{2+\alpha}(0,1)\cap C[0,1]$ with $0<\alpha<1$,
(H2′) $k > 0$ on $(0,1)$, $k(0) = k(1) = 0$, and $k'(0) \ge 0$ and $k'(1) \le 0$,
(H3′) $(\xi^\beta k')' + a^{3-\beta}\int_0^1 k^q(\xi)\,d\xi > 0$ for $\xi\in(0,1)$,
(H4′) $\lim_{\xi\to 0^+}(\xi^\beta k'(\xi))' = -a^{3-\beta}\int_0^1 k^q(\xi)\,d\xi$ and $\lim_{\xi\to 1^-}(\xi^\beta k'(\xi))' = -a^{3-\beta}\int_0^1 k^q(\xi)\,d\xi$,
(H5′) $(\xi^\beta k')' \le 0$ for $\xi\in(0,1)$.

In the part of showing the local existence of problem (6), we need the following lemma.

Lemma 2.1 *Let b_i is bounded and continuous and $b_i(\xi,\tau)\ge 0$ on $[0,1]\times[0,T]$ for $i = 1,2,3,4$ and $d(\xi,\tau)\ge 0$ on $[0,1]\times[0,T]$ with $0 < T \le \infty$. Suppose that $w\in C^{2,1}((0,1)\times(0,T))\cap C([0,1]\times[0,T])$ satisfies*

$$w_\tau - d(\xi,\tau)\big(\xi^\beta w_\xi\big)_\xi \ge b_1 w_\xi + b_2 w + b_3\int_0^1 b_4 w(\xi,\tau)\,d\xi, \quad (\xi,\tau)\in(0,1)\times(0,T],$$

$$w(0,\tau)\ge 0, \qquad w(1,\tau)\ge 0, \quad \tau\in(0,T],$$

$$w(\xi,0)\ge 0, \quad \xi\in[0,1].$$

Then $w \ge 0$ on $[0,1]\times[0,T]$.

Proof Suppose that there exists a point (ξ_0,τ_0) in $(0,1)\times(0,T]$ such that $w(\xi_0,\tau_0) < 0$. Let $B_i = \max_{(\xi,\tau)\in[0,1]\times[0,T]} b_i(\xi,\tau)$ for $i = 1,2,3,4$ and let $w(\xi,\tau) = e^{c\tau}v(\xi,\tau)$ for $(\xi,\tau)\in[0,1]\times[0,T]$ where c is a positive constant and $c > B_2 + B_3B_4$. Then we have, for any $(\xi,\tau)\in(0,1)\times(0,T]$,

$$v_\tau - d(\xi,\tau)\big(\xi^\beta v_\xi\big)_\xi - b_1 v_\xi + (c-b_2)v - b_3\int_0^1 b_4 v(\xi,\tau)\,d\xi \ge 0.$$

It follows from $w(\xi_0,\tau_0)<0$ and $w=e^{c\tau}v$ that $v(\xi_0,\tau_0)<0$. Since v is non-negative on the parabolic boundary and $v\in C^{2,1}((0,1)\times(0,T))\cap C([0,1]\times[0,T])$, there exists a point (ξ_1,τ_1) in $(0,1)\times(0,T]$ such that v attains its negative minimum at the point (ξ_1,τ_1). This yields $v(\xi_1,\tau_1)<0$, $v_\tau(\xi_1,\tau_1)=0$, $v_\xi(\xi_1,\tau_1)=0$ and $v_{\xi\xi}(\xi_1,\tau_1)\geq 0$. By the second mean value theorem for integrals, we find that there exists a $\xi_2\in(0,1)$ such that

$$\int_0^1 b_4(\xi,\tau)v(\xi,\tau)\,d\xi = v(\xi_2,\tau)\int_0^1 b_4(\xi,\tau)\,d\xi \quad \text{for any } \tau\in(0,T].$$

It is clear that $v(\xi_2,\tau_1)\geq v(\xi_1,\tau_1)$. Let us consider that

$$\begin{aligned}
&v_\tau(\xi_1,\tau_1)-d(\xi_1,\tau_1)\bigl[\xi_1^\beta v_{\xi\xi}(\xi_1,\tau_1)+\beta\xi_1^{\beta-1}v_\xi(\xi_1,\tau_1)\bigr]-b_1(\xi_1,\tau_1)v_\xi(\xi_1,\tau_1)\\
&\qquad+\bigl(c-b_2(\xi_1,\tau_1)\bigr)v(\xi_1,\tau_1)-b_3(\xi_1,\tau_1)v(\xi_2,\tau_1)\int_0^1 b_4(\xi,\tau_1)\,d\xi\\
&\quad\leq (c-B_2)v(\xi_1,\tau_1)-B_3B_4v(\xi_1,\tau_1)\\
&\quad=-v(\xi_1,\tau_1)\bigl[-(c-B_2)+B_3B_4\bigr]\\
&\quad<0.
\end{aligned}$$

This contradiction implies that $w(\xi,\tau)\geq 0$ for any $(\xi,\tau)\in[0,1]\times[0,T]$. Hence, the proof of Lemma 2.1 is completed. □

Since (6) is also degenerate, we will prove the local existence of the solution of (6) by considering the following problem:

$$\left.\begin{aligned}
&(v_1)_\tau=(v_1+\delta)^r\bigl[(\xi^\beta(v_1)_\xi)_\xi+a^{3-\beta}\textstyle\int_0^1(v_1)^q(\xi,\tau)\,d\xi\bigr],\quad (\xi,\tau)\in(0,1)\times(0,\infty),\\
&v_1(0,\tau;\delta)=v_1(1,\tau;\delta)=0,\quad \tau>0,\\
&v_1(\xi,0;\delta)=k(\xi),\quad \xi\in[0,1],
\end{aligned}\right\}\tag{7}$$

where δ is a positive constant and $\delta<1$. We note that the function $v_1=v_1(x,t;\delta)$ depends on x, t and δ. Let ε be a positive constant and $\varepsilon<\delta$. To show the existence of the classical solution of (7), we have to use the function given by Dunford and Schwartz [7]. There exists a non-decreasing and continuously differentiable function ρ such that $\rho=0$ if $\xi\leq 0$ and $\rho=1$ if $\xi\geq 1$. Let

$$\rho(\xi;\varepsilon)=\begin{cases}0, & \xi\leq\varepsilon,\\ \rho(\frac{\xi}{\varepsilon}-1), & \varepsilon<\xi<2\varepsilon,\\ 1, & \xi\geq 2\varepsilon,\end{cases}$$

and let $k(\xi;\varepsilon)=\rho(\xi;\varepsilon)k(\xi)$. We note that

$$\frac{\partial}{\partial\varepsilon}k(\xi;\varepsilon)=\begin{cases}0, & \xi\leq\varepsilon,\\ -\frac{\xi}{\varepsilon^2}\rho'(\frac{\xi}{\varepsilon}-1)k(\xi), & \varepsilon<\xi<2\varepsilon,\\ 0, & \xi\geq 2\varepsilon.\end{cases}$$

By the non-decreasing property of ρ, we have $\frac{\partial}{\partial\varepsilon}k(\xi;\varepsilon)\leq 0$. It follows from $0\leq\rho\leq 1$ that $k(\xi)\geq k(\xi;\varepsilon)$ and $\lim_{\varepsilon\to 0}k(\xi;\varepsilon)=k(\xi)$.

We see that (7) is degenerate and singular. By the regularization technique again, we consider the problem:

$$\left.\begin{aligned}&(\nu_2)_\tau = (\nu_2+\delta)^r[(\xi^\beta(\nu_2)_\xi)_\xi + a^{3-\beta}\textstyle\int_\varepsilon^1(\nu_2)^q\,d\xi], \quad (\xi,\tau)\in(\varepsilon,1)\times(0,\infty),\\&\nu_2(0,\tau;\delta,\varepsilon) = \nu_2(1,\tau;\delta,\varepsilon) = 0, \quad \tau>0,\\&\nu_2(\xi,0;\delta,\varepsilon) = k(\xi;\varepsilon), \quad \xi\in[\varepsilon,1].\end{aligned}\right\} \tag{8}$$

Now, the function $\nu_2 = \nu_2(\xi,\tau;\delta,\varepsilon)$ depends on ξ, τ, δ and ε. It is clear that, since the zero function is a lower solution of (8), that is, $\nu_2 \geq 0$. The next lemma show that the solution ν_2 of (8) is non-decreasing in τ.

Lemma 2.2 *Let ε and δ be any positive real number with $\varepsilon<\delta<1$. If $\frac{\partial}{\partial\xi}(\xi^\beta\frac{\partial}{\partial\xi}k(\xi;\varepsilon)) + a^{3-\beta}\int_\varepsilon^1 k^q(\xi;\varepsilon)\,d\xi > 0$ for any $\xi\in(\varepsilon,1)$, then $(\nu_2)_\tau \geq 0$ for any $(\xi,\tau)\in[\varepsilon,1]\times[0,\infty)$.*

Proof Let $z=(\nu_2)_\tau$. Then we have, for any $(\xi,\tau)\in(\varepsilon,1)\times(0,\infty)$,

$$\begin{aligned}z_\tau &= r(\nu_2+\delta)^{-1}\big((\nu_2)_\tau\big)^2 + (\nu_2+\delta)^r\big(\xi^\beta z_\xi\big)_\xi\\&\quad + q(\nu_2+\delta)^r a^{3-\beta}\int_\varepsilon^1 \nu_2^{q-1}(\xi,\tau;\delta,\varepsilon)z(\xi,\tau)\,d\xi.\end{aligned}$$

Thus, the function z satisfies the following:

$$\begin{aligned}&z_\tau - (\nu_2+\delta)^r\big(\xi^\beta z_\xi\big)_\xi \geq q(\nu_2+\delta)^r a^{3-\beta}\int_\varepsilon^1 \nu_2^{q-1}z(\xi,\tau)\,d\xi, \quad (\xi,\tau)\in(\varepsilon,1)\times(0,\infty),\\&z(\varepsilon,\tau) = (\nu_2)_\tau(\varepsilon,\tau;\delta,\varepsilon) = 0, \qquad z(1,\tau) = (\nu_2)_\tau(1,\tau;\delta,\varepsilon) = 0, \quad \tau>0,\\&z(\xi,0) = \big(k(\xi;\varepsilon)+\delta\big)^r\left[\frac{d}{d\xi}\left(\xi^\beta\frac{d}{d\xi}k(\xi;\varepsilon)\right) + a^{3-\beta}\int_\varepsilon^1 k^q(\xi;\varepsilon)\,d\xi\right] > 0, \quad \xi\in[\varepsilon,1].\end{aligned}$$

Lemma 2.1 implies that $(\nu_2)_\tau \geq 0$ for any $(\xi,\tau)\in[\varepsilon,1]\times[0,\infty)$. □

The boundedness and monotonicity properties of ν_2 are shown in Lemma 2.3 and Lemma 2.4, respectively.

Lemma 2.3 *There exist a time τ_1 and a function $f\in C^1[0,\tau_1]$ such that, for all $\varepsilon,\delta>0$ with $\varepsilon<\delta<1$, (8) has a unique classical solution ν_2 and $0\leq\nu_2(\xi,\tau;\delta,\varepsilon)\leq f(\tau)$ for any $(\xi,\tau)\in[\varepsilon,1]\times[0,\tau_1]$.*

Proof Let us consider the following ordinary differential equation:

$$\left.\begin{aligned}&f'(\tau) = a^{3-\beta}f^q(\tau)(f(\tau)+1)^r, \quad \tau>0,\\&f(0) = \max\nolimits_{\xi\in[0,1]} k(\xi).\end{aligned}\right\} \tag{9}$$

Then there exists a positive constant τ_1 such that (9) has a unique positive solution f on $[0,\tau_1]$. We next show that, for all $\varepsilon,\delta>0$ with $\varepsilon<\delta<1$, $f(\tau)\geq\nu_2(\xi,\tau;\delta,\varepsilon)$ for any $(\xi,\tau)\in[\varepsilon,1]\times[0,\tau_1]$. Let $z(\xi,\tau)=f(\tau)-\nu_2(\xi,\tau;\delta,\varepsilon)$ for $(\xi,\tau)\in[\varepsilon,1]\times[0,\tau_1]$. We then consider that, for any $(\xi,\tau) \in (\varepsilon,1) \times (0,\tau_1]$,

$$\begin{aligned} z_\tau &\geq \big(f(\tau)+\delta\big)^r a^{3-\beta} \int_\varepsilon^1 f^q(\tau)\,d\xi - (\nu_2+\delta)^r \biggl[\big(\xi^\beta (\nu_2)_\xi\big)_\xi + a^{3-\beta} \int_\varepsilon^1 \nu_2^q \,d\xi \biggr] \\ &= \big(f(\tau)+\delta\big)^r \big(\xi^\beta z_\xi\big)_\xi + r\eta_1^{r-1}(\nu_2+\delta)^{-r}(\nu_2)_\tau z \\ &\quad + qa^{3-\beta}\big(f(\tau)+\delta\big)^r \int_\varepsilon^1 \eta_2^{q-1} z(\xi,\tau)\,d\xi, \end{aligned}$$

where η_1 and η_2 are some intermediate values between h and ν_2. Thus, the function z satisfies $z_\tau - (f(\tau)+\delta)^r(\xi^\beta z_\xi)_\xi \geq \frac{r\eta_1^{r-1}(\nu_2)_\tau}{(\nu_2+\delta)^r} z + qa^{3-\beta}(f(\tau)+\delta)^r \int_\varepsilon^1 \eta_2^{q-1} z(\xi,\tau)\,d\xi$ for any $(\xi,\tau) \in (\varepsilon,1) \times (0,\tau_1]$ and on the parabolic boundary:

$$\begin{aligned} &z(\varepsilon,\tau) = f(\tau) > 0, \qquad z(1,\tau) = f(\tau) > 0, \quad \tau \in (0,\tau_1], \\ &z(\xi,0) = f(0) - \nu_2(\xi,0;\delta,\varepsilon) = \max_{s\in[0,1]} k(s) - k(\xi;\varepsilon) \geq 0, \quad \xi \in [\varepsilon,1]. \end{aligned}$$

Lemma 2.1 ensures that $z \geq 0$, that is, $\nu_2 \leq f$ for any $(\xi,\tau) \in [\varepsilon,1] \times [0,\tau_1]$. By modifying the proof of Theorem A.1. in [6], we see that there exists a unique classical positive solution ν_2 of (8) and $0 \leq \nu_2 \leq f$ for all ε and δ. The proof of this lemma is completed. □

Lemma 2.4 *Let* $0 < \varepsilon_1 < \varepsilon_2 < \delta < 1$ *and assume that* $\nu_2(\xi,\tau;\delta,\varepsilon_1)$ *and* $\nu_2(\xi,\tau;\delta,\varepsilon_2)$ *are solutions of* (8). *Then* $\nu_2(\xi,\tau;\delta,\varepsilon_1) > \nu_2(\xi,\tau;\delta,\varepsilon_2)$ *for any* $(\xi,\tau) \in [\varepsilon_2,1] \times [0,\tau_1]$.

Proof Let $z(\xi,\tau) = \nu_2(\xi,\tau;\delta,\varepsilon_1) - \nu_2(\xi,\tau;\delta,\varepsilon_2)$ on $[\varepsilon_2,1] \times [0,\tau_1]$. We have, for any $(\xi,\tau) \in (\varepsilon_2,1) \times (0,\tau_1]$,

$$\begin{aligned} z_\tau &\geq \big(\nu_2(\xi,\tau;\delta,\varepsilon_1)+\delta\big)^r \biggl[\Big(\xi^\beta \big(\nu_2(\xi,\tau;\delta,\varepsilon_1)\big)_\xi\Big)_\xi + a^{3-\beta} \int_{\varepsilon_2}^1 \nu_2^q(\xi,\tau;\delta,\varepsilon_1)\,d\xi \biggr] \\ &\quad - \big(\nu_2(\xi,\tau;\delta,\varepsilon_2)+\delta\big)^r \biggl[\Big(\xi^\beta \big(\nu_2(\xi,\tau;\delta,\varepsilon_2)\big)_\xi\Big)_\xi + a^{3-\beta} \int_{\varepsilon_2}^1 \nu_2^q(\xi,\tau;\delta,\varepsilon_2)\,d\xi \biggr] \\ &= \big(\nu_2(\xi,\tau;\delta,\varepsilon_1)+\delta\big)^r \big(\xi^\beta z_\xi\big)_\xi + r\eta_3^{r-1}\big(\nu_2(\xi,\tau;\delta,\varepsilon_2)+\delta\big)^{-r}\big(\nu_2(\xi,\tau;\delta,\varepsilon_2)\big)_\tau z \\ &\quad + qa^{3-\beta}\big(\nu_2(\xi,\tau;\delta,\varepsilon_1)+\delta\big)^r \int_{\varepsilon_2}^1 \eta_4^{q-1} z(\xi,\tau)\,d\xi, \end{aligned}$$

where η_3 and η_4 are some intermediate values between $\nu_2(\xi,\tau;\delta,\varepsilon_1)$ and $\nu_2(\xi,\tau;\delta,\varepsilon_2)$. Then it follows from $\frac{\partial}{\partial\varepsilon} k(\xi;\varepsilon) \leq 0$ that the function z satisfies

$$\begin{aligned} &z_\tau - \big(\nu_2(\xi,\tau;\delta,\varepsilon_1)+\delta\big)^r \big(\xi^\beta z_\xi\big)_\xi \\ &\quad \geq \frac{r\eta_3^{r-1}(\nu_2(\xi,\tau;\delta,\varepsilon_2))_\tau}{(\nu_2(\xi,\tau;\delta,\varepsilon_2)+\delta)^r} z + qa^{3-\beta}\big(\nu_2(\xi,\tau;\delta,\varepsilon_1)+\delta\big)^r \int_{\varepsilon_2}^1 \eta_4^{q-1} z(\xi,\tau)\,d\xi, \\ &\quad (\xi,\tau) \in (\varepsilon_2,1) \times (0,\tau_1], \\ &z(\varepsilon_2,\tau) = \nu_2(\varepsilon_2,\tau;\delta,\varepsilon_1) \geq 0, \qquad z(1,\tau) = 0, \quad \tau \in (0,\tau_1], \\ &z(\xi,0) = k(\xi;\varepsilon_1) - k(\xi;\varepsilon_2) \geq 0, \quad \xi \in [\varepsilon,1]. \end{aligned}$$

By Lemma 2.1, we can conclude that $v_2(\xi,\tau;\delta,\varepsilon_1) > v_2(\xi,\tau;\delta,\varepsilon_2)$ for any $(\xi,\tau) \in [\varepsilon_2,1] \times [0,\tau_1]$. The proof is completed. □

From Lemma 2.3 and Lemma 2.4, we can construct the function v_1 which is a good candidate for the solution for (7), by

$$v_1(\xi,\tau;\delta) = \begin{cases} \lim_{\varepsilon\to 0^+} v_2(\xi,\tau;\delta,\varepsilon), & (\xi,\tau) \in (\varepsilon,1] \times [0,\tau_1], \\ 0, & (\xi,\tau) \in \{0\} \times [0,\tau_1], \end{cases} \tag{10}$$

for all $\delta > 0$. By modifying the proofs of Theorem 2.3 in [8] and Lemma 10 and Theorem 12 in [4], we obtain the existence result.

Theorem 2.5 *Assume that* (H1′)–(H3′) *hold. The function* $v_1(\xi,\tau;\delta)$ *given by* (10) *is a unique classical solution of* (7) *for any* $(\xi,\tau) \in [0,1] \times [0,\tau_1]$ *and* $\delta > 0$.

In the next step, we show the existence of solutions of (6). By using the same technique as in Lemma 2.2 and Lemma 2.3, we can show that the solution v_1 of (7) satisfies $\frac{\partial}{\partial\tau} v_1(x,t;\delta) \geq 0$ for all δ and $k(\xi) \leq v_1(\xi,\tau;\delta) \leq f(\tau)$ for any $(\xi,\tau) \in [0,1] \times [0,\tau_1]$ where the function f is given in Lemma 2.3. We give an additional property of v_1 which is the monotonicity property with respect to δ.

Lemma 2.6 *Let* $0 < \delta_1 < \delta_2 < 1$ *and suppose that* $v_1(\xi,\tau;\delta_1)$ *and* $v_1(\xi,\tau;\delta_2)$ *are solutions of* (7). *Then* $v_1(\xi,\tau;\delta_2) > v_1(\xi,\tau;\delta_1)$ *for any* $(\xi,\tau) \in [0,1] \times [0,\tau_1]$.

Proof Let $z = v_1(\xi,\tau;\delta_2) - v_1(\xi,\tau;\delta_1)$ for any $(\xi,\tau) \in [0,1] \times [0,\tau_1]$. By (7), we obtain, for any $(\xi,\tau) \in (0,1) \times (0,\tau_1]$,

$$\begin{aligned} z_\tau \geq{}& \bigl(v_1(\xi,\tau;\delta_2)+\delta_2\bigr)^r \biggl[\bigl(\xi^\beta \bigl(v_1(\xi,\tau;\delta_2)\bigr)_\xi\bigr)_\xi + a^{3-\beta} \int_0^1 v_1^q(\xi,\tau;\delta_2)\,d\xi \biggr] \\ &- \bigl(v_1(\xi,\tau;\delta_1)+\delta_2\bigr)^r \biggl[\bigl(\xi^\beta \bigl(v_1(\xi,\tau;\delta_1)\bigr)_\xi\bigr)_\xi + a^{3-\beta} \int_0^1 v_1^q(\xi,\tau;\delta_1)\,d\xi \biggr] \\ ={}& \bigl(v_1(\xi,\tau;\delta_2)+\delta_2\bigr)^r \bigl(\xi^\beta z_\xi\bigr)_\xi + r\eta_5^{r-1}\bigl(v_1(\xi,\tau;\delta_1)+\delta_1\bigr)^{-r}\bigl(v_1(\xi,\tau;\delta_1)\bigr)_\tau z \\ &+ qa^{3-\beta}\bigl(v_1(\xi,\tau;\delta_2)+\delta_2\bigr)^r \int_0^1 \eta_6^{q-1} z(\xi,\tau)\,d\xi, \end{aligned}$$

where η_5 and η_6 are some intermediate values between $v_1(\xi,\tau;\delta_1)$ and $v_1(\xi,\tau;\delta_2)$. Then the function z satisfies $z_\tau - (v_1(\xi,\tau;\delta_2)+\delta_2)^r(\xi^\beta z_\xi)_\xi \geq \frac{r\eta_5^{r-1}(v_1(\xi,\tau;\delta_1))_\tau}{(v_1(\xi,\tau;\delta_1)+\delta_1)^r} z + qa^{3-\beta}(v_1(\xi,\tau;\delta_2)+\delta_2)^r \int_0^1 \eta_6^{q-1} z(\xi,\tau)\,d\xi$ for $(\xi,\tau) \in (0,1) \times (0,\tau_1]$ and on the parabolic boundary: $z(0,\tau) = 0$, $z(1,\tau) = 0$ for $\tau \in (0,\tau_1]$ and $z(\xi,0) = 0$ for $\xi \in [0,1]$. By Lemma 2.1, we have $v_1(\xi,\tau;\delta_2) > v_1(\xi,\tau;\delta_1)$ for any $(\xi,\tau) \in [0,1] \times [0,\tau_1]$. □

By Lemma 2.6 and $k(\xi) \leq v_1(\xi,\tau;\delta) \leq f(\tau)$ for any $(\xi,\tau) \in [0,1] \times [0,\tau_1]$ and for all δ, we define the function v by

$$v(\xi,\tau) = \lim_{\delta\to 0} v_1(\xi,\tau;\delta), \quad (\xi,\tau) \in (0,1) \times (0,\tau_1]. \tag{11}$$

Based on Lemma 2.7 in [6], and Lemma 10 and Theorem 12 in [4], we get the following theorem.

Theorem 2.7 *Assume that* (H1′)–(H3′) *hold. The function* v *given by* (11) *is a unique classical solution of* (6) *on* $[0,1] \times [0,\tau_1]$ *for some positive constant* τ_1.

Note that by the transformations $v = u^m$, $t = \frac{\tau}{ma^{\beta-2}}$ and $x = a\xi$ and Theorem 2.7, we find the following.

Corollary 2.8 *Assume that* (H1)–(H3) *hold. Then there exists a time* $\widetilde{\tau}_1 > 0$ *such that* (1) *admits a unique non-negative classical solution on* $[0,a] \times [0,\widetilde{\tau}_1]$.

3 Blow-up in a finite time

The sufficient condition for the occurrence of blow-up in a finite time of (1) is given in this section. Let us consider the following problem:

$$\left.\begin{aligned} &-(\xi^\beta \varphi'(\xi))' = \lambda \varphi(\xi), \quad \xi \in (0,1), \\ &\varphi(0) = \varphi(1) = 0. \end{aligned}\right\} \tag{12}$$

From [4], the eigenvalue problem (12) is solvable. Denote the first eigenvalue of (12) by $\lambda_1 > 0$ and the corresponding eigenfunction by φ_1, with the normalization $\varphi_1 > 0$ in $(0,1)$ and $\max_{\xi\in[0,1]} \varphi_1(\xi) = 1$. The next theorem deals with the condition that guarantee for the occurrence of blow-up in a finite time depending on the value of the constant a.

Theorem 3.1 *Suppose that the function* k *satisfies* (H1′)–(H3′). *If the constant* a *satisfies*

$$a > \max\left\{\left(\frac{\lambda_1}{\int_0^1 \varphi_1(\xi)\,d\xi}\right)^{\frac{q}{3-\beta}}, \left(\frac{1}{\int_0^1 k^q(\xi)\,d\xi}\right)^{\frac{1}{3-\beta}}\right\},$$

then the solution v *of* (6) *blows up in a finite time.*

Proof Let $H(\tau) = \int_0^1 v^{1-r}(\xi,\tau)\varphi_1(\xi)\,d\xi$. We then have

$$\begin{aligned} \frac{1}{1-r}H'(\tau) &= \int_0^1 \left(\xi^\beta v_\xi\right)_\xi \varphi_1(\xi)\,d\xi + a^{3-\beta}\int_0^1 v^q\,d\xi \int_0^1 \varphi_1(\xi)\,d\xi \\ &= -\lambda_1 \int_0^1 v(\xi,\tau)\varphi_1(\xi)\,d\xi + a^{3-\beta}\int_0^1 v^q\,d\xi \int_0^1 \varphi_1(\xi)\,d\xi. \end{aligned}$$

From

$$\begin{aligned} \int_0^1 v(\xi,\tau)\varphi_1(\xi)\,d\xi &\le \frac{1}{a^{(3-\beta)/q}}\left(\int_0^1 a^{3-\beta}v^q\,d\xi\right)^{\frac{1}{q}} \left(\int_0^1 \varphi_1^{\frac{q}{q-1}}(\xi)\,d\xi\right)^{1-\frac{1}{q}} \\ &\le \frac{1}{a^{(3-\beta)/q}}\left(\int_0^1 a^{3-\beta}v^q\,d\xi\right)^{\frac{1}{q}}, \end{aligned}$$

we obtain

$$\frac{1}{1-r}H'(\tau) \ge -\frac{\lambda_1}{a^{(3-\beta)/q}}\left(\int_0^1 a^{3-\beta}v^q\,dx\right)^{\frac{1}{q}} + a^{3-\beta}\int_0^1 v^q\,dx \int_0^1 \varphi_1(\xi)\,d\xi.$$

From $v_\tau \geq 0$ and $a^{3-\beta}\int_0^1 k^q(\xi)\,d\xi \geq 1$, we obtain $a^{3-\beta}\int_0^1 v^q(\xi,\tau)\,d\xi \geq 1$. It follows that $(\int_0^1 a^{3-\beta}v^q\,d\xi)^{\frac{1}{q}} \leq a^{3-\beta}\int_0^1 v^q(\xi,\tau)\,d\xi$ with $q > 1$. Then

$$\begin{aligned}\frac{1}{1-r}H'(\tau) &\geq -\frac{\lambda_1 a^{3-\beta}}{a^{(3-\beta)/q}}\int_0^1 v^q(\xi,\tau)\,d\xi + a^{3-\beta}\int_0^1 v^q(\xi,\tau)\,d\xi\int_0^1 \varphi_1(\xi)\,d\xi \\ &= a^{3-\beta}\int_0^1 v^q(\xi,\tau)\,d\xi\left[-\frac{\lambda_1}{a^{(3-\beta)/q}} + \int_0^1 \varphi_1(\xi)\,d\xi\right].\end{aligned}$$

By the assumption that $\lambda_1 < a^{(3-\beta)/q}\int_0^1 \varphi_1(\xi)\,d\xi$, we have $-\frac{\lambda_1}{a^{(3-\beta)/q}} + \int_0^1 \varphi_1(\xi)\,d\xi \geq \eta_9$ where η_9 is a positive constant. Thus,

$$\frac{1}{1-r}H'(\tau) \geq \eta_9 a^{3-\beta}\int_0^1 v^q(\xi,\tau)\,d\xi.$$

Since

$$\int_0^1 v^{1-r}(\xi,\tau)\varphi_1(\xi)\,d\xi \leq \left(\int_0^1 v^q(\xi,\tau)\,d\xi\right)^{\frac{1-r}{q}}\left(\int_0^1 \varphi_1^{\frac{q}{q+r-1}}(\xi)\,d\xi\right)^{\frac{q+r-1}{q}},$$

we get

$$\int_0^1 v^q(\xi,\tau)\,d\xi \geq \left(\int_0^1 v^{1-r}(\xi,\tau)\varphi_1(\xi)\,d\xi\right)^{\frac{q}{1-r}}\Big/\left(\int_0^1 \varphi_1^{\frac{q}{q+r-1}}(\xi)\,d\xi\right)^{\frac{q+r-1}{1-r}}.$$

We then obtain

$$\begin{aligned}\frac{1}{1-r}H'(\tau) &\geq \eta_9 a^{3-\beta}\left(\int_0^1 v^{1-r}(\xi,\tau)\varphi_1(\xi)\,d\xi\right)^{\frac{q}{1-r}}\Big/\left(\int_0^1 \varphi_1^{\frac{q}{q+r-1}}(\xi)\,d\xi\right)^{\frac{q+r-1}{1-r}} \\ &\geq \eta_9 a^{3-\beta}H^{\frac{q}{1-r}}(\tau),\end{aligned}$$

that is,

$$\left(H^{1-q/(1-r)}(\tau)\right)' \leq \eta_9 a^{3-\beta}(1-r-q). \tag{13}$$

Integrating (13) over $(0,\tau)$, we get

$$H^{1-q/(1-r)}(\tau) - H^{1-q/(1-r)}(0) \leq \eta_9 a^{3-\beta}(1-r-q)\tau$$

or

$$H^{\frac{q}{1-r}-1}(\tau) \geq \frac{H^{\frac{q}{1-r}-1}(0)}{1-\eta_9 a^{3-\beta}(q+r-1)H^{\frac{q}{1-r}-1}(0)\tau}.$$

We can see that $H^{\frac{q}{1-r}-1}(\tau)$ exists for $\tau \in [0,T_b)$ but $H^{\frac{q}{1-r}-1}(\tau)$ is unbounded as τ converges to T_b where

$$T_b = \frac{H^{1-\frac{q}{1-r}}(0)}{\eta_9 a^{3-\beta}(q+r-1)} = \frac{1}{\eta_9 a^{3-\beta}(q+r-1)}\left(\int_0^1 k^{1-r}(\xi)\varphi_1(\xi)\,d\xi\right)^{\frac{-(q+r-1)}{1-r}}.$$

Therefore, H blows up in a finite time. This implies that v blows up in a finite time. Then the proof of this theorem is completed. □

By the transformation technique and Theorem 3.1, we obtain the following.

Corollary 3.2 *Suppose that g satisfies* (H1)–(H3). *Then the solution u of* (1) *blows up in a finite time if the constant a is sufficiently large.*

In the following, we show that under some conditions, the solution v of (6) can exist globally. To obtain the desired results, we need the following comparison theorem.

Lemma 3.3 *Let v be the solution of* (6) *and suppose that a non-negative function $w \in C^{2,1}((0,1) \times (0,T)) \cap C([0,1] \times [0,T])$ satisfies*

$$
\begin{aligned}
& w_\tau \geq (\leq)\ w^r\left[\left(\xi^\beta w_\xi\right)_\xi + a^{3-\beta}\int_0^1 w^q(\xi,\tau)\,d\xi\right], \quad (\xi,\tau) \in (0,1) \times (0,T], \\
& w(0,\tau) \geq (=)\ 0, \qquad w(1,\tau) \geq (=)\ 0, \quad \tau \in (0,T], \\
& w(\xi,0) \geq (\leq)\ k(\xi), \quad \xi \in [0,1].
\end{aligned}
$$

Then $w \geq (\leq)\ v$ on $[0,1] \times [0,T]$.

Proof We first consider in the case "$\geq$". Let $z(\xi,\tau) = w(\xi,\tau) - v(\xi,\tau)$ on $[0,1] \times [0,T]$. It is clear that, from Lemma 2.1 and (H2′), $v > 0$ in $(0,1) \times (0,T]$. We then have, for any $(0,1) \times (0,T]$,

$$
z_\tau = w^r\left(\xi^\beta z_\xi\right)_\xi + r\eta_7^{r-1}v^{-r}v_\tau z + qa^{3-\beta}w^r\int_0^1 \eta_8^{q-1}z(\xi,\tau)\,d\xi,
$$

where η_7 and η_8 are some intermediate values between w and v. Then the function z satisfies

$$
\begin{aligned}
& z_\tau - w^r\left(\xi^\beta z_\xi\right)_\xi = r\eta_7^{r-1}v^{-r}v_\tau z + qa^{3-\beta}w^r\int_0^1 \eta_8^{q-1}z(\xi,\tau)\,d\xi, \quad (\xi,\tau) \in (0,1) \times (0,T], \\
& z(0,\tau) \geq 0, \qquad z(1,\tau) \geq 0, \quad \tau \in (0,T], \\
& z(\xi,0) \geq 0, \quad \xi \in [0,1].
\end{aligned}
$$

Lemma 2.1 implies that $w(\xi,\tau) \geq v(\xi,\tau)$ for any $(\xi,\tau) \in [0,1] \times [0,T]$. By using the above technique, we can get the result in the case "$\leq$". The proof of this lemma is completed. □

Let us consider the following boundary value problem:

$$
-\left(\xi^\beta\psi'(\xi)\right)' = 1, \quad \xi \in (0,1) \quad \text{and} \quad \psi(0) = \psi(1) = 0.
$$

The solution ψ is given by $\psi(\xi) = \frac{1}{2-\beta}\xi^{1-\beta}(1-\xi)$ for $\xi \in (0,1)$. By direct computation, we obtain $\int_0^1 \psi^q(\xi)\,d\xi = \frac{B(q(1-\beta)+1,q+1)}{(2-\beta)^q}$ where $B(l,m)$ is the Beta function which is defined by $B(l,m) = \int_0^1 \xi^{l-1}(1-\xi)^{m-1}\,d\xi$. The following theorem deals with the global existence result.

Theorem 3.4 *Suppose that k satisfies* (H1′)–(H3′). *Then the solution v of* (6) *exists globally if a is small enough.*

Proof Let $z(\xi,\tau) = M_1\psi(\xi)$ on $[0,1] \times [0,\infty)$ where M_1 is a positive constant and $M_1\psi(\xi) \geq k$. We choose $a \leq (\frac{(2-\beta)^q}{M^{q-1}B(q(1-\beta)+1,q+1)})^{\frac{1}{3-\beta}}$ and then the function z satisfies

$$\begin{aligned} &z_\tau - z^r\left[\left(\xi^\beta z_\xi\right)_\xi + a^{3-\beta}\int_0^1 z^q(\xi,\tau)\,d\xi\right] \\ &\quad = M_1^r\psi^r(\xi)\left[M_1 - a^{3-\beta}M_1^q\frac{B(q(1-\beta)+1,q+1)}{(2-\beta)^q}\right] \end{aligned}$$

for any $(\xi,\tau) \in (0,1) \times (0,\infty)$. Thus, $z_\tau - z^r[(\xi^\beta z_\xi)_\xi + a^{3-\beta}\int_0^1 z^q(\xi,\tau)\,d\xi] \geq 0$ for $(\xi,\tau) \in (0,1) \times (0,\infty)$. Furthermore, $z(0,\tau) = z(1,\tau) = 0$ for $\tau > 0$ and $z(\xi,0) = M_1\psi(\xi) \geq k(\xi)$ for $\xi \in [0,1]$. Lemma 3.3 implies that $z \geq v$ on $[0,1] \times [0,\infty)$. We can conclude that the solution v of (6) exists globally. □

It follows from the transformation technique and Theorem 3.4 that we have the following.

Corollary 3.5 *Suppose that g satisfies* (H1)–(H3). *Then the solution u of* (1) *exists globally if a is small enough.*

4 Blow-up set and uniform blow-up profile

In this section, we assume that the solution u of (1) blows up at the blow-up time T_b. Then we discuss the set of blow-up points and the blow-up profile for the solution u of (1). From the assumptions (H1)–(H5), we know that there are a sufficiently small positive constant ε_1 and a non-negative function $h(\xi;\varepsilon)$ for $0 < \varepsilon \leq \varepsilon_1$ such that:

- (H1*) $h(\xi;\varepsilon) \in C^{2+\alpha}(\varepsilon,1-\varepsilon) \cap C[\varepsilon,1-\varepsilon]$ with $\alpha \in (0,1)$,
- (H2*) $h(\varepsilon;\varepsilon) = 0$ and $h(1-\varepsilon;\varepsilon) = 0$,
- (H3*) $h(\xi;\varepsilon) < k(\xi)$ for $\xi \in (\varepsilon,2\varepsilon) \cup (1-2\varepsilon,1-\varepsilon)$ and $h(\xi;\varepsilon) = k(\xi)$ for $\xi \in (2\varepsilon,1-2\varepsilon)$,
- (H4*) $(\xi^\beta h_\xi(\xi;\varepsilon))_\xi \leq 0$ for $\xi \in (\varepsilon,1-\varepsilon)$,
- (H5*) $h(\xi;\varepsilon)$ is non-increasing with respect to $\varepsilon \in (0,\varepsilon_1]$, $\lim_{\xi\to\varepsilon}(\xi^\beta h_\xi(\xi;\varepsilon))_\xi = -a^{3-\beta} \times \int_\varepsilon^{1-\varepsilon} h^q(\xi;\varepsilon)\,d\xi$ and $\lim_{\xi\to 1-\varepsilon}(\xi^\beta h_\xi(\xi;\varepsilon))_\xi = -a^{3-\beta}\int_\varepsilon^{1-\varepsilon} h^q(\xi;\varepsilon)\,d\xi$,
- (H6*) $(\xi^\beta h_\xi(\xi;\varepsilon))_\xi + a^{3-\beta}\int_\varepsilon^{1-\varepsilon} h^q(\xi;\varepsilon)\,d\xi \geq 0$ for $\varepsilon \in (0,\varepsilon_1]$ and $\xi \in (\varepsilon,1-\varepsilon)$.

It is obvious that $\lim_{\varepsilon\to 0} h(\xi;\varepsilon) = k(\xi)$. We next consider the following regularized problem:

$$\left.\begin{aligned} &w_\tau = (w+\delta)^r[(\xi^\beta w_\xi)_\xi + a^{3-\beta}\textstyle\int_\varepsilon^{1-\varepsilon} w^q(\xi,\tau;\delta,\varepsilon)\,d\xi], \\ &\quad (\xi,\tau) \in (\varepsilon,1-\varepsilon) \times (0,\infty), \\ &w(\varepsilon,\tau;\delta,\varepsilon) = w(1-\varepsilon,\tau;\delta,\varepsilon) = 0, \quad \tau > 0, \\ &w(\xi,0;\delta,\varepsilon) = h(\xi;\varepsilon), \quad \xi \in [\varepsilon,1-\varepsilon]. \end{aligned}\right\} \tag{14}$$

In the same way as before, it is not difficult to show that the regularized problem (14) has a unique positive solution w and

$$\lim_{\delta\to 0,\varepsilon\to 0} w(\xi,\tau;\delta,\varepsilon) = v(\xi,\tau),$$

where v is the solution of (6). To find the blow-up set and blow-up profile of the blow-up solution u of (1), we need the following lemma.

Lemma 4.1 *Assume that k satisfies* (H1′)–(H5′). *Before blow-up occurs*, $(\xi^\beta v_\xi)_\xi \le 0$ *for* $(\xi,\tau) \in (0,1) \times [0,T_b)$.

Proof Let ε and δ be positive constants with $\varepsilon < \delta < 1$. From $(\xi^\beta h_\xi)_\xi + a^{3-\beta}\int_\varepsilon^{1-\varepsilon} h^q(\xi;\varepsilon)\,d\xi \ge 0$ for $\xi \in (\varepsilon, 1-\varepsilon)$, we have $w_\tau \ge 0$ for $(\xi,\tau) \in (\varepsilon,1-\varepsilon)\times[0,T_b)$. Let $z(\xi,\tau) = (\xi^\beta w_\xi)_\xi$ for $(\xi,\tau) \in (\varepsilon,1-\varepsilon)\times[0,T_b)$. We consider that, for $(\xi,\tau)\in(\varepsilon,1-\varepsilon)\times(0,T_b)$,

$$
\begin{aligned}
&z_\tau - (w+\delta)^r\big(\xi^\beta z_\xi\big)_\xi - 2r(w+\delta)^{r-1}\xi^\beta w_\xi z_\xi - r(w+\delta)^{-1}w_\tau z \\
&\quad = r(r-1)(w+\delta)^{-2}\xi^\beta w_\tau (w_\xi)^2.
\end{aligned}
$$

This means that $z_\tau - (w+\delta)^r(\xi^\beta z_\xi)_\xi - 2r(w+\delta)^{r-1}\xi^\beta w_\xi z_\xi - r(w+\delta)^{-1}w_\tau z \le 0$. for $(\xi,\tau) \in (\varepsilon,1-\varepsilon)\times(0,T_b)$. Furthermore, we have

$$
z(\varepsilon,\tau) = \big(\xi^\beta w_\xi\big)_\xi\big|_{\xi=\varepsilon} = -a^{3-\beta}\int_\varepsilon^{1-\varepsilon} w^q(\xi,\tau;\delta,\varepsilon)\,d\xi < 0
$$

and

$$
z(1-\varepsilon,\tau) = \big(\xi^\beta w_\xi\big)_\xi\big|_{\xi=1-\varepsilon} = -a^{3-\beta}\int_\varepsilon^{1-\varepsilon} w^q(\xi,\tau;\delta,\varepsilon)\,d\xi < 0.
$$

It follows from (H4*) that $z(\xi,0) \le 0$ for $\xi \in [\varepsilon,1-\varepsilon]$. By applying Lemma 2.1, we obtain $z \le 0$ on $[\varepsilon,1-\varepsilon]\times[0,T_b)$. Since ε and δ are arbitrary, we have $(\xi^\beta v_\xi)_\xi \le 0$ for $(\xi,\tau) \in (0,1)\times[0,T_b)$. Hence, the proof of this lemma is completed. □

From Lemma 4.1, we obtain the following corollary.

Corollary 4.2 *Assume that g satisfies* (H1)–(H5). *Before blow-up occurs*, $(x^\beta u_x^m)_x \le 0$ *for* $(x,t) \in (0,a)\times[0,T_b)$.

The next theorem states about the set of blow-up point of the solution u of (1). By modifying techniques in [16], we obtain this result.

Theorem 4.3 *Assume that the solution u of* (1) *blows up in a finite time T_b. Then $S = [0,a]$.*

Proof Let ε be any positive constant. We construct functions ϕ and Φ by $\phi(t) = \int_0^a u^p(x,t)\,dx$ and $\Phi(t) = \int_0^t \phi(s)\,ds$. We set $M_2 = \inf_{x\in(\varepsilon,a-\varepsilon)}\mu(x)$ where μ is the unique positive solution of the following problem

$$
\begin{aligned}
&-\frac{d}{dx}\left(x^\beta \frac{d}{dx}\mu^m(x)\right) = 1, \quad x \in (0,a), \\
&\mu(0) = \mu(a) = 0.
\end{aligned}
$$

Corollary 4.2 yields, for $t \in (0,T_b)$,

$$
\int_0^a u^m(x,t)\,dx = -\int_0^a u^m(x,t)\frac{d}{dx}\left(x^\beta\frac{d}{dx}\mu^m(x)\right)dx \ge -M_2^m\int_\varepsilon^{a-\varepsilon}\big(x^\beta u_x^m\big)_x\,dx.
$$

We then obtain

$$0 \leq \lim_{t\to T_b} \frac{-M_2^m \int_\varepsilon^{a-\varepsilon} (x^\beta u_x^m)_x \, dx}{\phi(t)} \leq \lim_{t\to T_b} \frac{\int_0^a u^m(x,t)\, dx}{\int_0^a u^p(x,t)\, dx} = 0$$

and this implies that $\lim_{t\to T_b} \frac{\int_\varepsilon^{a-\varepsilon} (x^\beta u_x^m)_x \, dx}{\phi(t)} = 0$. As $\varepsilon \to 0$, we obtain

$$\lim_{t\to T_b} \frac{(x^\beta u_x^m)_x}{\phi(t)} = 0 \quad \text{for } x \in (0,a). \tag{15}$$

Integrating the first equation in (1) with respect to t from 0 to t, we have

$$u(x,t) - g(x) = \int_0^t \big(x^\beta u_x^m(x,s)\big)_x \, ds + \Phi(t). \tag{16}$$

Since u blows up at the finite time T_b, $\lim_{t\to T_b} u(x_b,t) = \infty$ for some $x_b \in (0,a)$, and then we obtain

$$\lim_{t\to T_b} u(x_b,t) - \lim_{t\to T_b} g(x_b) = \lim_{t\to T_b} \int_0^t \big(x_b^\beta u_x^m(x_b,s)\big)_x \, ds + \lim_{t\to T_b} \Phi(t)$$

or

$$\lim_{t\to T_b} \Phi(t) = \infty. \tag{17}$$

It follows from (15) and (17) that

$$\lim_{t\to T_b} \frac{\int_0^t (x^\beta u_x^m(x,s))_x \, ds}{\Phi(t)} = 0 \quad \text{for } x \in (0,a). \tag{18}$$

Let $\widetilde{x}$ be a fixed point in $(0,a)$. We have, by (16),

$$\lim_{t\to T_b} \frac{u(\widetilde{x},t)}{\Phi(t)} = \lim_{t\to T_b} \frac{g(\widetilde{x})}{\Phi(t)} + \lim_{t\to T_b} \frac{\int_0^t (\widetilde{x}^\beta u_x^m(\widetilde{x},s))_x \, ds}{\Phi(t)} + 1.$$

Equations (17) and (18) imply

$$\lim_{t\to T_b} \frac{u(\widetilde{x},t)}{\Phi(t)} = 1, \tag{19}$$

which means that the solution u of (1) blows up at the point $\widetilde{x}$. Since $\widetilde{x}$ is arbitrary in $(0,a)$, we can conclude that the solution u of (1) blows up everywhere in $(0,a)$. For $\widetilde{x} \in \{0,a\}$, we can always find a sequence $\{(x_n,t_n)\}$ in $(0,a)\times(0,T_b)$ such that $(x_n,t_n) \to \{\widetilde{x}, T_b\}$ and $\lim_{n\to\infty} u(x_n,t_n) = \infty$. Hence, the blow-up set is $[0,a]$. The proof of Theorem 4.3 is completed. □

Finally, we consider the uniform blow-up profile of the solution u of (1).

Theorem 4.4 *Assume that g satisfies* (H1)–(H5).
Then $u(x,t) \sim [a(p-1)(T_b-t)]^{-\frac{1}{p-1}}$ *for any* $x \in (0,a)$ *as* $t \to T_b$.

Proof Equation (19) tells us that, for any $x \in (0,a)$,

$$u(x,t) \sim \Phi(t) \quad \text{as } t \to T_b. \tag{20}$$

Then we get

$$\Phi'(t) = \int_0^a u^p(x,t)\,dx \sim a\Phi^p(t) \quad \text{as } t \to T_b. \tag{21}$$

Integrating (21) over (t, T_b), we have, by (17),

$$\Phi(t) \sim \left[a(p-1)(T_b - t)\right]^{-\frac{1}{p-1}} \quad \text{as } t \to T_b. \tag{22}$$

It follows from (20) and (22) that, as t approaches the blow-up time T_b, $u(x,t) \sim [a(p-1)(T_b - t)]^{-\frac{1}{p-1}}$ for any $x \in (0,a)$. □

5 Conclusion

In this paper, we study a degenerate and singular parabolic problem with a nonlocal term. We show that such a problem has a local classical solution. Furthermore, the conditions that the solution exists globally or blows up in finite time are given. Finally, we demonstrate the uniform blow-up profile of the blow-up solution.

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NS and PS formulated the research problem and wrote the paper. NS,PS and WS participated in the derivation of the mathematical results. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand. [2]Department of Mathematics, Faculty of Science, Mahidol University, Bangkok, Thailand. [3]Centre of Excellence in Mathematics, PERDO, Commission on Higher Education, Ministry of Education, Bangkok, Thailand.

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