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A mass-conserved domain decomposition method for the unsaturated soil flow water problem

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Abstract

In this paper, an efficient mass conservative domain decomposition method is developed for solving the unsaturated soil water flow over non-overlapping multiple sub-domains. In the first step, the predicted interface fluxes are computed by the semi-implicit flux scheme. In the second step, the interior water content and fluxes in the interiors are computed by the coupled solution and flux implicit scheme. The interface fluxes are finally recomputed by the implicit scheme. The important features of our proposed scheme are mass conserved and unconditionally stable by defining global fluxes which contain the diffusivity and hydraulic conductivity. We prove that the scheme has the error estimate of $O(\Delta t + (\Delta z)^2 + \frac{(\Delta t)^2}{(\Delta z)^{\frac{3}{2}}})$ in L^2 -norm.

Keywords: Unsaturated soil water; Water content; Mass conservative; Domain decomposition; Error estimate

1 Introduction

The unsaturated soil water flow (see [1–7]) is an important form of flow in porous media and is widely used in atmospheric science, soil science, agricultural engineering, environment engineering and groundwater hydrology, etc. Because of the nonlinearity of water content equations and the complexity of physical parameters and boundary conditions, it is very difficult and impossible to obtain its analytical solution. Some numerical schemes [5, 8–10] are been developed for solving the unsaturated soil water flow problem. Reference [5] presented the general difference methods for one-dimensional unsaturated soil water flow problem. Reference [9] proposed the efficient reduced-order finite volume element formulation based on proper orthogonal decomposition method for solving two-dimensional unsaturated soil water flow problem. By the Crank–Nicolson extrapolation method, [10] presented the time second-order proper orthogonal decomposition method. Reference [11] considered conforming finite element discretizations based on a multiscale formulation along with recently developed, local postprocessing schemes. Reference [12] proposed the enriched Galerkin method, which augments piecewise constant functions to the classical continuous Galerkin finite element method. By employing a coarse partition of the fine grids and multiscale basis function for mapping the fine-scale information to the coarse-scale unknowns, [13] proposed a multiscale locally conservative Galerkin (MsLCG) method to accurately simulate multiphase flow in heterogeneous and

fractured porous media. Reference [14] proposed an element based post-processing technique through which local conservation can be established. Using the property of local conservation at steady state conditions to define a numerical flux at element boundaries, Reference [15] proposed a locally conservative Galerkin (LCG) finite element method for two-phase flow simulations in heterogeneous porous media.

Recently, non-overlapping domain decomposition methods have been studied to solve large scale and nonlinear partial differential equations, and allow the reduction of the sizes of problems by decomposing domains into smaller sub-domains on which the sub-problems can be solved by multiple computers in parallel. The explicit–implicit domain decomposition (EIDD) algorithms over non-overlapping sub-domains as shown for solving linear parabolic equations in [16–20], etc. Reference [16] proposed the mixed/hybrid schemes, where the interior solutions were solved by the implicit schemes in sub-domains while interface solutions were solved by the explicit schemes on interfaces. Reference [21] developed the EIDD methods by either a multi-step explicit scheme or a high-order explicit scheme on the interfaces which relaxed the stability conditions. Due to the stability requirements, [22–24] proposed stable explicit–implicit domain decomposition methods, in which the explicit predictors are first used to get the interface values, then the interior values of sub-domains are solved by implicit schemes, and finally the interface values are corrected by implicit schemes. By combining with the operator splitting technique, [25, 26] developed an efficient explicit–implicit splitting domain decomposition method (S-DDM) for solving parabolic equations and for solving compressible contamination fluid flows in porous media over multiple non-overlapping block-divided domain decompositions. However, these previous explicit–implicit domain decomposition methods do not satisfy the physical law of mass conservation over the whole domain.

The numerical schemes that preserve the mass of the model (see [27–30], etc.) are important and also required for parallel computations in long time simulations. Reference [27] presented an explicit–implicit conservative domain decomposition procedure for parabolic equations, where the fluxes at the sub-domains interfaces were calculated by an average operator from the solutions at the previous time level. Combining the operator splitting technique and the solution-flux coupled scheme on staggered meshes, [28] developed the mass-preserving S-DDM scheme for solving parabolic equations over multiple block-divided domain, where the interface fluxes were computed by the semi-implicit flux scheme, the solutions and fluxes in the interiors of sub-domains were computed by the splitting one-dimensional implicit scheme, and further the interface fluxes were corrected on interfaces. Further, by the conservative modified-upwind technique, [29] proposed a mass-preserving and modified-upwind S-DDM scheme over non-overlapping multi-block sub-domains for solving time-dependent convection-diffusion equations. By the local multi-point weighted average schemes, [31] analyzed the new and efficient mass-conserving S-DDM scheme for solving two-dimensional variable coefficient parabolic equations. By the time extrapolation and the Crank–Nicolson method, [32–34] proposed the time second-order conservative domain decomposition method and they are conditionally stable.

References [35–37] analyzed the explicit–implicit domain decomposition algorithms for parallel approximation of semilinear parabolic problems and gave the existence and prior estimates of numerical solutions by the fixed point technique. References [38, 39] proposed the stability domain decomposition methods for nonlinear parabolic systems. But

the domain decomposition method for solving the nonlinear parabolic equations are not conserved. Although [40] presented a conservative domain decomposition method for nonlinear diffusion equations, the stability and error estimates were not given.

To date, there is few research on mass conservative domain decomposition method for the unsaturated soil water flow. Since soil water content is an important climate factor, and its seasonal change has an important influence on weather and climate at mid-high latitudes. Thus, it is an important task to develop and analyze the efficient domain decomposition methods, which is mass conservative, for solving the unsaturated soil water flow.

In this paper, we propose the unconditionally stable conservative domain decomposition method for solving the unsaturated soil water flow over non-overlapping multiple sub-domains. Three steps method is used to compute the water content on each sub-domain at every time step. Firstly, the predicted interface fluxes are computed by the semi-implicit flux scheme on the interfaces of sub-domains. Secondly, the interior solutions and fluxes in the interiors are computed by the solution and flux coupled implicit scheme. Finally, the interface fluxes are recomputed by the implicit (or explicit) scheme. The important features of our proposed scheme are that the defined global fluxes which contain diffusivity and hydraulic conductivity ensure the scheme mass conservative and unconditionally stable. We prove theoretically that our scheme preserves mass conservation and is unconditionally stable in discrete L^2 -norm. We prove that the scheme has the error estimate of $O(\Delta t + (\Delta z)^2 + \frac{(\Delta t)^2}{(\Delta z)^2})$ in L^2 -norm. Numerical experiments test the theoretical analysis.

The rest of this paper is organized as follows. In Sect. 2, the unsaturated soil water flow problem are considered. In Sect. 3, we propose the mass conservative domain decomposition scheme and prove it to satisfy mass conservation. In Sect. 4, the stability of our scheme is analyzed. In Sect. 5, we analyze the convergence and prove the error estimate in the discrete L^2 -norm. Numerical experiments are presented in Sect. 6.

2 The unsaturated soil water flow problem

Soil water content is an important climate factor, and its seasonal change has an important influence on weather and climate at mid-high latitudes. Hydraulic processes at surface and subsurface, such as precipitation, evaporation, and evapotranspiration, seepage of surface water, and capillary elevation of deep-level water, absorption in root zone and liquid moisture flow of groundwater, all can be reduced to unsaturated flow problems. In fact, in all studies (see [1–5]) of the unsaturated zone, the fluid motion is assumed to obey the classical Richards equation. Based on horizontal resolution of a general circulation model, liquid moisture flow in soil along horizontal direction may be ignored. The one-dimensional unsaturated soil water flow equations with the absorption rate of root are considered as

$$\begin{cases} \frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z} (D(\theta) \frac{\partial \theta}{\partial z} - K(\theta)) = S_r, & (x, t) \in (0, L) \times (0, T), \\ K(\theta) - D(\theta) \frac{\partial \theta}{\partial z} = q(t), & \text{if } z = 0 \text{ or } z = L, t > 0, \\ \theta(z, 0) = \theta_0(z), & x \in (0, L), \end{cases} \quad (1)$$

where $\theta(z, t)$ is soil moisture, $D(\theta)$ is the soil water diffusivity, $K(\theta)$ is the unsaturated hydraulic conductivity, $-S_r$ is the absorption rate of root zone, and $q(t)$ is the infiltration or evaporation rate. Reference [4] gave the nonoscillatory solution and evade a non-

physics solution for the unsaturated flow problem by using the mass-lumped finite element method. References [5, 8–10] proposed the generalized difference, mixed finite element methods, and the classical finite volume element scheme with the proper orthogonal decomposition (POD) for solving the unsaturated soil water flow problem. Considering the nonlinearity of water content equations and the complexity of physical parameters and boundary conditions, it is an important work to develop the mass-conserved domain decomposition method for solving the unsaturated water flow problem.

Assume that the domain Ω will be divided into multiple non-overlapping sub-domains Ω_α , and each sub-domain will be further discretized. Let Γ_α be the interface point sets of all points on $\{z_{i_\alpha+\frac{1}{2}}\}$, where α is the interface index.

For simplicity, define a uniform conforming mesh on domain $\Omega = [0, L]$ with $\Delta z = \frac{L}{I}$ and introduce the staggered mesh points z_i and $z_{i+\frac{1}{2}}$ as

$$z_{i+\frac{1}{2}} = ih, \quad i = 0, 1, \dots, I, \quad z_i = \left(i - \frac{1}{2}\right)h, \quad i = 1, 2, \dots, I.$$

The time interval $(0, T]$ is discretized uniformly by $t^n = n\Delta t$, $n = 0, 1, \dots, M$, where $\Delta t = \frac{T}{M}$. Let $\phi_i^n = \phi(z_i, t^n)$, $\phi_{i+\frac{1}{2}}^n = \phi(z_{i+\frac{1}{2}}, t^n)$ denote the function ϕ at the mesh points (z_i, t^n) and $(z_{i+\frac{1}{2}}, t^n)$. Define the following difference operators:

$$\delta_z \phi_{i+\frac{1}{2}}^n = \frac{\phi_{i+1,j}^n - \phi_{i,j}^n}{\Delta z}, \quad \delta_z \phi_i^n = \frac{\phi_{i+\frac{1}{2},j}^n - \phi_{i-\frac{1}{2},j}^n}{\Delta z}, \quad \partial_t \phi_i^{n+1} = \frac{\phi_i^{n+1} - \phi_i^n}{\Delta t}.$$

3 Conservative domain decomposition method

Let $q(z, t)$ be the flux of problem (1) defined as $q = D(\theta) \frac{\partial \theta}{\partial z} - K(\theta)$ which ensure that our proposed scheme is mass conservative. We approximate both the solution and its flux on the staggered meshes of sub-domains. The numerical solutions $\{\Theta_i^n\}$ and numerical fluxes $\{Q_{i+\frac{1}{2}}^n\}$ to denote the numerical approximations to solution $\theta(z_i, t^n)$ and fluxes $q(z_{i+\frac{1}{2}}, t^n)$, respectively. Let $\Theta_{i+\frac{1}{2}} = \frac{\Theta_i + \Theta_{i+1}}{2}$ and $f = S_r$.

Our mass-conserving domain decomposition method is described as follows.

Step 1: The predicted interface fluxes $\{\tilde{Q}_{i_\alpha+\frac{1}{2}}^{n+1}\}$ over Γ_α are computed as

$$\begin{aligned} \frac{\tilde{Q}_{i_\alpha+\frac{1}{2}}^{n+1} - Q_{i_\alpha+\frac{1}{2}}^n}{\Delta t} &= D_{i_\alpha+\frac{1}{2}}^{n+1} \frac{Q_{i_\alpha+\frac{3}{2}}^n - 2\tilde{Q}_{i_\alpha+\frac{1}{2}}^{n+1} + Q_{i_\alpha-\frac{1}{2}}^n}{(\Delta z)^2} + \partial_t D_{i_\alpha+\frac{1}{2}}^{n+1} \delta_z \Theta_{i_\alpha+\frac{1}{2}}^n \\ &\quad - \partial_t K_{i_\alpha+\frac{1}{2}}^{n+1} + D_{i_\alpha+\frac{1}{2}}^{n+1} \delta_z f_{i_\alpha+\frac{1}{2}}^{n+1}, \end{aligned} \quad (2)$$

where i_α is the location index number of the interface and $D_{i_\alpha+\frac{1}{2}}^n = D(\Theta_{i_\alpha+\frac{1}{2}}^n)$.

Step 2: The intermediate solutions $\{\Theta_i^{n+1}\}$ are computed by the coupled implicit scheme,

$$\begin{cases} \frac{\Theta_i^{n+1} - \Theta_i^n}{\Delta t} = \delta_z Q_i^{n+1} + f_i^{n+1}, & x_i \in \Omega_\alpha, \\ Q_{i+\frac{1}{2}}^{n+1} = D_{i+\frac{1}{2}}^{n+1} \delta_z \Theta_{i+\frac{1}{2}}^{n+1} - K_{i+\frac{1}{2}}^{n+1}, & x_{i+\frac{1}{2}} \in \Omega_\alpha, \\ Q_{i_\alpha+\frac{1}{2}}^{n+1} = \tilde{Q}_{i_\alpha+\frac{1}{2}}^{n+1}, & x_{i_\alpha+\frac{1}{2}} \in \Gamma_\alpha, \end{cases} \quad (3)$$

where $D_{i+\frac{1}{2}}^{n+1} = D(\Theta_{i+\frac{1}{2}}^{n+1})$ and $K_{i+\frac{1}{2}}^{n+1} = K(\Theta_{i+\frac{1}{2}}^{n+1})$.

Step 3: The interface fluxes $\{\tilde{Q}_{i_{\alpha}+\frac{1}{2}}^{n+1}\}$ are recomputed by

$$Q_{i_{\alpha}+\frac{1}{2}}^{n+1} = D_{i+\frac{1}{2}}^{n+1} \delta_z \Theta_{i_{\alpha}+\frac{1}{2}}^{n+1} - K_{i_{\alpha}+\frac{1}{2}}^{n+1}. \quad (4)$$

The boundary conditions are approximated by

$$Q_{\frac{1}{2}}^{n+1} = q_{\frac{1}{2}}^{n+1}, \quad Q_{I+\frac{1}{2}}^{n+1} = q_{I+\frac{1}{2}}^{n+1}, \quad (5)$$

and the initial values are computed by

$$\Theta_i^0 = \theta_0(x_i). \quad (6)$$

Next, we will give the theoretical analysis when $Q_{\frac{1}{2}}^{n+1} = Q_{I+\frac{1}{2}}^{n+1} = 0$, and it easily extends to more general boundary problems.

Theorem 1 *The scheme (2)–(6) satisfies mass conservation over the global domain with $f(\Theta) = 0$, i.e.,*

$$\sum_{i=1}^I \Theta_i^n \Delta z = \sum_{i=1}^I \Theta_i^0 \Delta z, \quad n = 1, \dots, M. \quad (7)$$

Proof Multiplying the first equation of (3) with Δz and summing i from 1 to I , respectively, we have

$$\sum_{i=1}^I \frac{\Theta_i^{n+1} - \Theta_i^n}{\Delta t} \Delta z = \sum_{i=1}^I \delta_z Q_i^{n+1} \Delta z. \quad (8)$$

Applying the boundary condition $Q_{\frac{1}{2}}^{n+1} = Q_{I+\frac{1}{2}}^{n+1} = 0$, we have

$$\sum_{i=1}^I \delta_z Q_i^{n+1} = 0. \quad (9)$$

Substituting (9) into (8), we obtain

$$\sum_{i=1}^I \Theta_i^{n+1} \Delta z - \sum_{i=1}^I \Theta_i^n \Delta z = 0. \quad (10)$$

We thus complete the proof. \square

Remark 3.1 The mass-conserved domain decomposition method (2)–(6) is proposed for solving unsaturated soil water flow problems. The three steps are used to compute the solutions $\{\Theta_i^{n+1}\}$ at time interval. Firstly, the predicted interface fluxes are computed by the semi-implicit flux scheme on the interfaces of sub-domains. Secondly, the interior solutions and fluxes in the interiors are computed by the solution and flux coupled implicit scheme. Finally, the interface fluxes are recomputed by the explicit scheme.

Remark 3.2 The important features of our proposed scheme are that the defined global fluxes which contain diffusivity and hydraulic conductivity ensure the scheme is mass conservative.

4 Stability

4.1 Assumptions

- (I) The problem (1) has a unique smooth solution $\theta(x, t)$ and satisfies the regularity condition, i.e.,

$$\theta \in C^0([0, T]; C^5(\Omega)) \cap C^2([0, T]; C^1(\Omega)). \quad (11)$$

- (II) One has a positive constant D_0 , such that, for any ξ and θ ,

$$(\xi, D(x, t, \theta)\xi) \geq D_0|\xi|^2. \quad (12)$$

- (III) The coefficients $D(x, t, \theta)$, $K(x, t, \theta)$, and $f(x, t, \theta)\xi$ are continuous with respect to x , t , and continuously differentiable with respect to θ , i.e.,

$$\max\{|D_{\theta\theta}|, |K_{\theta\theta}|, |f_{\theta}|\} \leq G. \quad (13)$$

When the conditions (I)–(III) hold, we provide the analysis of the stability and convergence of the scheme (2)–(6) as follows.

Lemma 1 If $\Theta = \{\Theta_i\}$ and $Q = \{Q_{i+\frac{1}{2}}\}$ satisfy the condition $Q_{\frac{1}{2}} = Q_{I+\frac{1}{2}} = 0$, we have

$$(\delta_z Q, \Theta) = -\|Q\|_{\frac{1}{D}}^2 + \sum_{i=1}^{I-1} \frac{1}{D_{i+\frac{1}{2}}} Q_{i+\frac{1}{2}} K_{i+\frac{1}{2}} \Delta z. \quad (14)$$

Proof Applying the boundary conditions $Q_{\frac{1}{2}} = 0$, $Q_{I+\frac{1}{2}} = 0$, we have

$$\begin{aligned} (\delta_z Q, \Theta) &= -\sum_{i=1}^{I-1} \frac{1}{D_{i+\frac{1}{2}}} Q_{i+\frac{1}{2}} (D_{i+\frac{1}{2}} \delta_z \Theta_{i+\frac{1}{2}}) \Delta z \\ &= -\sum_{i=1}^{I-1} \frac{1}{D_{i+\frac{1}{2}}} Q_{i+\frac{1}{2}} (Q_{i+\frac{1}{2}} + K_{i+\frac{1}{2}}) \Delta z \\ &= -\|Q\|_{\frac{1}{D}}^2 - \sum_{i=1}^{I-1} \frac{1}{D_{i+\frac{1}{2}}} Q_{i+\frac{1}{2}} K_{i+\frac{1}{2}} \Delta z. \end{aligned} \quad (15)$$

□

Without loss of generality, we will prove our scheme stability and convergence on two sub-domains. Equations (3) are rewritten as

$$\begin{cases} \frac{\Theta_i^{n+1} - \Theta_i^n}{\Delta t} = \frac{Q_{i+\frac{1}{2}}^{n+1} - Q_{i-\frac{1}{2}}^{n+1}}{\Delta z} + f_i^{n+1} + \frac{\bar{Q}_{i_1+\frac{1}{2}}^{n+1} - Q_{i_1+\frac{1}{2}}^{n+1}}{\Delta z}, & i = i_1, \\ \frac{\Theta_i^{n+1} - \Theta_i^n}{\Delta t} = \frac{Q_{i+\frac{1}{2}}^{n+1} - Q_{i-\frac{1}{2}}^{n+1}}{\Delta z} + f_i^{n+1} + \frac{Q_{i_1+\frac{1}{2}}^{n+1} - \bar{Q}_{i_1+\frac{1}{2}}^{n+1}}{\Delta z}, & i = i_1 + 1, \\ \frac{\Theta_i^{n+1} - \Theta_i^n}{\Delta t} = \frac{Q_{i+\frac{1}{2}}^{n+1} - Q_{i-\frac{1}{2}}^{n+1}}{\Delta z} + f_i^{n+1}, & i \neq i_1, i_1 + 1, \end{cases} \quad (16)$$

where $D_{i+\frac{1}{2}}^{n+1} = D(\Theta_{i+\frac{1}{2}}^{n+1})$ and $K_{i+\frac{1}{2}}^{n+1} = K(\Theta_{i+\frac{1}{2}}^{n+1})$.

Lemma 2 Let $\Theta = \{\Theta_i\}$ and $Q = \{Q_{i+\frac{1}{2}}\}$ be the solution of Scheme (2)–(6). We have

$$\begin{aligned} & \frac{1}{\Delta t} (\|\Theta^{n+1}\|^2 - \|\Theta^n\|^2) + \sum_{i=1, i \neq i_1}^{I-1} \frac{1}{2D_{i+\frac{1}{2}}^{n+1}} |Q_{i+\frac{1}{2}}^{n+1}|^2 \Delta z + \frac{1}{2D_{i_1+\frac{1}{2}}^{n+1}} |\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2 \Delta z \\ & + \frac{\Delta t}{\Delta z} (|Q_{i_1-\frac{1}{2}}^{n+1}|^2 + |Q_{i_1+\frac{3}{2}}^{n+1}|^2 - (|Q_{i_1-\frac{1}{2}}^n|^2 + |Q_{i_1+\frac{3}{2}}^n|^2)) \\ & \leq \|\Theta^{n+1}\|^2 + \|K^{n+1}\|_{\frac{1}{D_1}}^2 + \|f^{n+1}\|^2 + M \frac{(\Delta t)^2}{\Delta z} (|f_{i_1}^{n+1}|^2 + |f_{i_1+1}^{n+1}|^2), \end{aligned} \quad (17)$$

where $M > 0$ is a positive constant.

Proof Multiplying both sides of Eqs. (3) by $\Theta_i^{n+1} \Delta x$, respectively, and summing them up for i from 1 to I , we obtain

$$(\partial_t \Theta^{n+1}, \Theta^{n+1}) = (\delta_z Q^{n+1}, \Theta^{n+1}) + (f^{n+1}, \Theta^{n+1}) + R, \quad (18)$$

where

$$R = (Q_{i_1+\frac{1}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1}) \delta_z \Theta_{i_1+\frac{1}{2}}^{n+1} \Delta z. \quad (19)$$

Applying the definition, we have

$$\begin{aligned} R &= (Q_{i_1+\frac{1}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1}) \frac{D_{i_1+\frac{1}{2}}^{n+1} \delta Q_{i_1+\frac{1}{2}}^{n+1}}{D_{i_1+\frac{1}{2}}^{n+1}} \Delta z \\ &= (Q_{i_1+\frac{1}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1}) \frac{D_{i_1+\frac{1}{2}}^{n+1} \delta \Theta_{i_1+\frac{1}{2}}^{n+1}}{D_{i_1+\frac{1}{2}}^{n+1}} \Delta z = (Q_{i_1+\frac{1}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1}) \frac{Q_{i_1+\frac{1}{2}}^{n+1} + K_{i_1+\frac{1}{2}}^{n+1}}{D_{i_1+\frac{1}{2}}^{n+1}} \Delta z \\ &= \frac{\Delta z}{D_{i_1+\frac{1}{2}}^{n+1}} (|Q_{i_1+\frac{1}{2}}^{n+1}|^2 - |\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2) - \frac{\Delta z}{D_{i_1+\frac{1}{2}}^{n+1}} (Q_{i_1+\frac{1}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1}) \tilde{Q}_{i_1+\frac{1}{2}}^{n+1} \\ &\quad + \frac{\Delta z}{D_{i_1+\frac{1}{2}}^{n+1}} (Q_{i_1+\frac{1}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1}) K_{i_1+\frac{1}{2}}^{n+1}. \end{aligned} \quad (20)$$

Subtracting the first equation from the second equation of Eqs. (16), it is easy to obtain

$$\frac{\delta_z \Theta_{i_1+\frac{1}{2}}^{n+1} - \delta_z \Theta_{i_1+\frac{1}{2}}^n}{\Delta t} = \frac{Q_{i_1+\frac{3}{2}}^{n+1} - 2\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} + Q_{i_1-\frac{1}{2}}^{n+1}}{(\Delta z)^2} + \delta_z f_{i_1+\frac{1}{2}}^{n+1}. \quad (21)$$

Multiplying both sides of (21) with $D_{i_1+\frac{1}{2}}^{n+1}$, we get

$$\begin{aligned} & \frac{D_{i_1+\frac{1}{2}}^{n+1} \delta_z \Theta_{i_1+\frac{1}{2}}^{n+1} - D_{i_1+\frac{1}{2}}^n \delta_z \Theta_{i_1+\frac{1}{2}}^n}{\Delta t} \\ &= \partial_t D_{i_1+\frac{1}{2}}^{n+1} \delta_z \Theta_{i_1+\frac{1}{2}}^n + D_{i_1+\frac{1}{2}}^{n+1} \frac{Q_{i_1+\frac{3}{2}}^{n+1} - 2\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} + Q_{i_1-\frac{1}{2}}^{n+1}}{(\Delta z)^2} + D_{i_1+\frac{1}{2}}^{n+1} \delta_z f_{i_1+\frac{1}{2}}^{n+1}. \end{aligned} \quad (22)$$

Adding both sides of (22) with $-\partial_t K_{i_1+\frac{1}{2}}^{n+1}$, we have

$$\begin{aligned} \frac{Q_{i_1+\frac{1}{2}}^{n+1} - Q_{i_1+\frac{1}{2}}^n}{\Delta t} &= D_{i_1+\frac{1}{2}}^{n+1} \frac{Q_{i_1+\frac{3}{2}}^{n+1} - 2\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} + Q_{i_1-\frac{1}{2}}^{n+1}}{(\Delta z)^2} + \partial_t D_{i_1+\frac{1}{2}}^{n+1} \delta_z \Theta_{i_1+\frac{1}{2}}^n \\ &\quad - \partial_t K_{i_1+\frac{1}{2}}^{n+1} + D_{i_1+\frac{1}{2}}^{n+1} \delta_z f_{i_1+\frac{1}{2}}^{n+1}. \end{aligned} \quad (23)$$

Subtracting (2) from (23), we have

$$Q_{i_1+\frac{1}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1} = D_{i_1+\frac{1}{2}}^{n+1} \frac{\Delta t}{(\Delta z)^2} (Q_{i_1-\frac{1}{2}}^{n+1} + Q_{i_1+\frac{3}{2}}^{n+1}) - D_{i_1+\frac{1}{2}}^{n+1} \frac{\Delta t}{(\Delta z)^2} (Q_{i_1-\frac{1}{2}}^n + Q_{i_1+\frac{3}{2}}^n) \quad (24)$$

and

$$\begin{aligned} &(Q_{i_1+\frac{1}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1}) \tilde{Q}_{i_1+\frac{1}{2}}^{n+1} \\ &= \left(D_{i_1+\frac{1}{2}}^{n+1} \frac{\Delta t}{(\Delta z)^2} (Q_{i_1-\frac{1}{2}}^{n+1} + Q_{i_1+\frac{3}{2}}^{n+1}) - D_{i_1+\frac{1}{2}}^{n+1} \frac{\Delta t}{(\Delta z)^2} (Q_{i_1-\frac{1}{2}}^n + Q_{i_1+\frac{3}{2}}^n) \right) \tilde{Q}_{i_1+\frac{1}{2}}^{n+1}. \end{aligned} \quad (25)$$

Applying the complete square formula, we get

$$\frac{\Delta t}{(\Delta z)^2} Q_{i_1-\frac{1}{2}}^{n+1} \tilde{Q}_{i_1+\frac{1}{2}}^{n+1} = \frac{\Delta t}{2(\Delta z)^2} (|Q_{i_1-\frac{1}{2}}^{n+1}|^2 + |\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2) - \frac{\Delta t}{2} \left(\frac{\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} - Q_{i_1-\frac{1}{2}}^{n+1}}{\Delta z} \right)^2. \quad (26)$$

By the first equation of Eqs. (16), we have

$$\begin{aligned} \left(\frac{\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} - Q_{i_1-\frac{1}{2}}^{n+1}}{\Delta z} \right)^2 &= \left(f_{i_1}^{n+1} - \frac{\Theta_{i_1}^{n+1} - \Theta_{i_1}^n}{\Delta t} \right)^2 \\ &= \left(|f_{i_1}^{n+1}|^2 - 2f_{i_1}^{n+1} \frac{\Theta_{i_1}^{n+1} - \Theta_{i_1}^n}{\Delta t} + \left| \frac{\Theta_{i_1}^{n+1} - \Theta_{i_1}^n}{\Delta t} \right|^2 \right) \\ &= \left(|f_{i_1}^{n+1}|^2 - 2f_{i_1}^{n+1} \left(f_{i_1}^{n+1} + \frac{\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} - Q_{i_1-\frac{1}{2}}^{n+1}}{\Delta z} \right) + \left| \frac{\Theta_{i_1}^{n+1} - \Theta_{i_1}^n}{\Delta t} \right|^2 \right) \\ &= \left| \frac{\Theta_{i_1}^{n+1} - \Theta_{i_1}^n}{\Delta t} \right|^2 - |f_{i_1}^{n+1}|^2 - 2 \frac{\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} - Q_{i_1-\frac{1}{2}}^{n+1}}{\Delta z} f_{i_1}^{n+1}. \end{aligned} \quad (27)$$

By (26) and (27), we have

$$\begin{aligned} \frac{\Delta t}{(\Delta z)^2} Q_{i_1-\frac{1}{2}}^{n+1} \tilde{Q}_{i_1+\frac{1}{2}}^{n+1} &= \frac{\Delta t}{2(\Delta z)^2} (|Q_{i_1-\frac{1}{2}}^{n+1}|^2 + |\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2) - \frac{\Delta t}{2} \left| \frac{\Theta_{i_1}^{n+1} - \Theta_{i_1}^n}{\Delta t} \right|^2 \\ &\quad + \frac{\Delta t}{2} |f_{i_1}^{n+1}|^2 + \frac{\Delta t}{\Delta z} (\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} - Q_{i_1-\frac{1}{2}}^{n+1}) f_{i_1}^{n+1}. \end{aligned} \quad (28)$$

Similarly, we have

$$\begin{aligned} \frac{\Delta t}{(\Delta z)^2} Q_{i_1+\frac{3}{2}}^{n+1} \tilde{Q}_{i_1+\frac{1}{2}}^{n+1} &= \frac{\Delta t}{2(\Delta z)^2} (|Q_{i_1+\frac{3}{2}}^{n+1}|^2 + |\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2) - \frac{\Delta t}{2} \left| \frac{\Theta_{i_1+1}^{n+1} - \Theta_{i_1+1}^n}{\Delta t} \right|^2 \\ &\quad + \frac{\Delta t}{2} |f_{i_1+1}^{n+1}|^2 + \frac{\Delta t}{\Delta z} (Q_{i_1+\frac{3}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1}) f_{i_1+1}^{n+1}. \end{aligned} \quad (29)$$

By (28)–(29), we can obtain

$$\begin{aligned}
 & \frac{-1}{D_{i_1+\frac{1}{2}}^{n+1}}(Q_{i_1+\frac{1}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1})\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} \\
 &= \frac{\Delta t}{2} \left(\left| \frac{\Theta_{i_1}^{n+1} - \Theta_{i_1}^n}{\Delta t} \right|^2 + \left| \frac{\Theta_{i_1+1}^{n+1} - \Theta_{i_1+1}^n}{\Delta t} \right|^2 - |f_{i_1}^{n+1}|^2 - |f_{i_1+1}^{n+1}|^2 \right) \\
 &\quad - \frac{\Delta t}{2(\Delta z)^2} (|Q_{i_1-\frac{1}{2}}^{n+1}|^2 + |Q_{i_1+\frac{3}{2}}^{n+1}|^2 + 2|\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2) \\
 &\quad - \frac{\Delta t}{\Delta z} (f_{i_1}^{n+1}(\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} - Q_{i_1-\frac{1}{2}}^{n+1}) + f_{i_1+1}^{n+1}(Q_{i_1+\frac{3}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1})) \\
 &\quad + \frac{\Delta t}{(\Delta z)^2} (Q_{i_1-\frac{1}{2}}^n + Q_{i_1+\frac{3}{2}}^n)\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}. \tag{30}
 \end{aligned}$$

Applying the ϵ -inequality for the third terms of Eq. (30), we have

$$\begin{aligned}
 & \frac{\Delta t}{\Delta z} (f_{i_1}^{n+1}(\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} - Q_{i_1-\frac{1}{2}}^{n+1}) + f_{i_1+1}^{n+1}(Q_{i_1+\frac{3}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1})) \\
 & \leq \epsilon (|Q_{i_1+\frac{3}{2}}^{n+1}|^2 + |Q_{i_1-\frac{1}{2}}^{n+1}|^2 + 2|\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2) + M \frac{(\Delta t)^2}{(\Delta z)^2} (|f_{i_1}^{n+1}|^2 + |f_{i_1+1}^{n+1}|^2). \tag{31}
 \end{aligned}$$

For the fourth terms of Eq. (30), by the Hölder inequality, we obtain

$$\frac{\Delta t}{(\Delta z)^2} (Q_{i_1-\frac{1}{2}}^n + Q_{i_1+\frac{3}{2}}^n)\tilde{Q}_{i_1+\frac{1}{2}}^{n+1} \leq \frac{\Delta t}{2(\Delta z)^2} (|Q_{i_1-\frac{1}{2}}^n|^2 + |Q_{i_1+\frac{3}{2}}^n|^2 + 2|\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2). \tag{32}$$

Substituting (30)–(32) into (20), we obtain

$$\begin{aligned}
 R & \leq \frac{\Delta z}{D_{i_1+\frac{1}{2}}^{n+1}} (|Q_{i_1+\frac{1}{2}}^{n+1}|^2 - |\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2) + \frac{\Delta z}{D_{i_1+\frac{1}{2}}^{n+1}} (Q_{i_1+\frac{1}{2}}^{n+1} - \tilde{Q}_{i_1+\frac{1}{2}}^{n+1})K_{i_1+\frac{1}{2}}^{n+1} \\
 & \quad + \frac{\Delta t \Delta z}{2} \left(\left| \frac{\Theta_{i_1}^{n+1} - \Theta_{i_1}^n}{\Delta t} \right|^2 + \left| \frac{\Theta_{i_1+1}^{n+1} - \Theta_{i_1+1}^n}{\Delta t} \right|^2 \right) \\
 & \quad - \frac{\Delta t}{2\Delta z} (|Q_{i_1-\frac{1}{2}}^{n+1}|^2 + |Q_{i_1+\frac{3}{2}}^{n+1}|^2 - (|Q_{i_1-\frac{1}{2}}^n|^2 + |Q_{i_1+\frac{3}{2}}^n|^2)) \\
 & \quad + \epsilon (|Q_{i_1+\frac{3}{2}}^{n+1}|^2 + |Q_{i_1-\frac{1}{2}}^{n+1}|^2 + 2|\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2)\Delta x + M \frac{(\Delta t)^2}{\Delta z} (|f_{i_1}^{n+1}|^2 + |f_{i_1+1}^{n+1}|^2). \tag{33}
 \end{aligned}$$

Substituting (33) into Eq. (18) and applying Lemma 1, we have

$$\begin{aligned}
 & \frac{1}{2\Delta t} (\|\Theta^{n+1}\|^2 - \|\Theta^n\|^2) + \|Q^{n+1}\|_{\frac{1}{D_1}}^2 \\
 & \quad + \frac{\Delta t}{2\Delta z} (|Q_{i_1-\frac{1}{2}}^{n+1}|^2 + |Q_{i_1+\frac{3}{2}}^{n+1}|^2 - (|Q_{i_1-\frac{1}{2}}^n|^2 + |Q_{i_1+\frac{3}{2}}^n|^2)) \\
 & \leq \frac{\Delta z}{D_{i_1+\frac{1}{2}}^{n+1}} (|Q_{i_1+\frac{1}{2}}^{n+1}|^2 - |\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2) - \sum_{i=1, i \neq i_1}^{I-1} \frac{\Delta z}{D_{i+\frac{1}{2}}} Q_{i+\frac{1}{2}}^{n+1} K_{i+\frac{1}{2}}^{n+1} - \frac{\Delta z}{D_{i+\frac{1}{2}}} \tilde{Q}_{i+\frac{1}{2}}^{n+1} K_{i+\frac{1}{2}}^{n+1} \\
 & \quad + \epsilon (|Q_{i_1+\frac{3}{2}}^{n+1}|^2 + |Q_{i_1-\frac{1}{2}}^{n+1}|^2 + 2|\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2)\Delta z \\
 & \quad + M \frac{(\Delta t)^2}{\Delta z} (|f_{i_1}^{n+1}|^2 + |f_{i_1+1}^{n+1}|^2) + (f^{n+1}, \Theta^{n+1}). \tag{34}
 \end{aligned}$$

Applying the Hölder inequality, we further obtain

$$\begin{aligned}
 & \frac{1}{2\Delta t} (\|\Theta^{n+1}\|^2 - \|\Theta^n\|^2) + \frac{1}{2} \|Q^{n+1}\|_{\frac{1}{D_1}}^2 - \frac{\Delta z}{2D_{i_1+\frac{1}{2}}^{n+1}} (|Q_{i_1+\frac{1}{2}}^{n+1}|^2 - |\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2) \\
 & + \frac{\Delta t}{2\Delta z} (|Q_{i_1-\frac{1}{2}}^{n+1}|^2 + |Q_{i_1+\frac{3}{2}}^{n+1}|^2 - (|Q_{i_1-\frac{1}{2}}^n|^2 + |Q_{i_1+\frac{3}{2}}^n|^2)) \\
 & \leq \frac{1}{2} \|\Theta^{n+1}\|^2 + \frac{1}{2} \|K^{n+1}\|_{\frac{1}{D_1}}^2 + \frac{1}{2} \|f^{n+1}\|^2 \\
 & + \epsilon (|Q_{i_1+\frac{3}{2}}^{n+1}|^2 + |Q_{i_1-\frac{1}{2}}^{n+1}|^2 + 2|\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2) \Delta x + M \frac{(\Delta t)^2}{\Delta z} (|f_{i_1}^{n+1}|^2 + |f_{i_1+1}^{n+1}|^2). \quad (35)
 \end{aligned}$$

Letting $\epsilon \leq \frac{1}{8}$, we have

$$\begin{aligned}
 & \frac{1}{2\Delta t} (\|\Theta^{n+1}\|^2 - \|\Theta^n\|^2) + \sum_{i=1, i \neq i_1}^{I-1} \frac{1}{4D_{i+\frac{1}{2}}^{n+1}} |Q_{i+\frac{1}{2}}^{n+1}|^2 \Delta z + \frac{1}{4D_{i_1+\frac{1}{2}}^{n+1}} |\tilde{Q}_{i_1+\frac{1}{2}}^{n+1}|^2 \Delta z \\
 & + \frac{\Delta t}{2\Delta z} (|Q_{i_1-\frac{1}{2}}^{n+1}|^2 + |Q_{i_1+\frac{3}{2}}^{n+1}|^2 - (|Q_{i_1-\frac{1}{2}}^n|^2 + |Q_{i_1+\frac{3}{2}}^n|^2)) \\
 & \leq \frac{1}{2} \|\Theta^{n+1}\|^2 + \frac{1}{2} \|K^{n+1}\|_{\frac{1}{D_1}}^2 + \frac{1}{2} \|f^{n+1}\|^2 + M \frac{(\Delta t)^2}{\Delta z} (|f_{i_1}^{n+1}|^2 + |f_{i_1+1}^{n+1}|^2). \quad (36)
 \end{aligned}$$

Thus (17) is proved. \square

Summing (18) up with respect to n and applying the boundary condition, then we will obtain the stability theorem

Theorem 2 (Stability) *The scheme (2)–(6), is unconditionally stable in the sense of discrete L^2 -norm, i.e.,*

$$\begin{aligned}
 \|\Theta^{n+1}\|^2 & \leq \|\Theta^0\|^2 + \Delta t \sum_{l=0}^n (\|\Theta^{l+1}\|^2 + \|K^{l+1}\|_{\frac{1}{D_1}}^2 + \|f^{l+1}\|^2) \\
 & + M \Delta t \sum_{l=0}^n \frac{(\Delta t)^2}{\Delta z} (|f_{i_1}^{l+1}|^2 + |f_{i_1+1}^{l+1}|^2), \quad (37)
 \end{aligned}$$

where $M > 0$ is a positive constant.

Remark 4.1 From Theorem 2, our scheme is proved to be unconditionally stable, and dependent of solution of the problem on the initial condition θ^0 , absorption term S_r and hydraulic conductivity $K(\theta)$.

5 Error estimation

In this section, we will analyze the convergence and prove error estimate of the mass-conserving domain decomposition scheme in discrete L^2 -norm. Let $\theta_i^n = \theta(z_i, t^n)$, $\bar{\theta}_{i+\frac{1}{2}}^n = \frac{\theta_i^n + \theta_{i+1}^n}{2}$ and the fluxes $q_{i+\frac{1}{2}}^n = D(\bar{\theta}_{i+\frac{1}{2}}^n) \delta_z \theta_{i+\frac{1}{2}}^n - K(\bar{\theta}_{i+\frac{1}{2}}^n)$.

5.1 Truncation errors

We first give the truncation error equations of the scheme (2)–(6). Assume that $\theta \in C^0([0, T]; C^4(\Omega)) \cap C^2([0, T]; C^0(\Omega))$. We have the following truncation error equations:

$$\frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} = \delta_z q_i^{n+1} + f_i^{n+1} + o_i^{n+1}, \quad (38)$$

where $o = O(\Delta t + (\Delta z)^2)$. Now, we give the simple proof.

Proof From (38), we have

$$o_i^{n+1} = \frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} - \frac{q_{i+\frac{1}{2}}^{n+1} - q_{i-\frac{1}{2}}^{n+1}}{\Delta z} - f_i^{n+1}. \quad (39)$$

Applying the Taylor formula for the first term of the right-hand side of (39), i.e.,

$$\frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} = (\theta_t)_i^{n+1} - \frac{\Delta t}{2} (\theta_{tt})_i^{n+1}. \quad (40)$$

For the second term of (39), we have

$$\begin{aligned} q_{i+\frac{1}{2}}^{n+1} &= D(\bar{\theta}_{i+\frac{1}{2}}^{n+1}) \frac{\theta_{i+1}^{n+1} - \theta_i^{n+1}}{\Delta z} - K(\bar{\theta}_{i+\frac{1}{2}}^{n+1}) \\ &= D(\bar{\theta}_{i+\frac{1}{2}}^{n+1}) \left((\theta_z)_{i+\frac{1}{2}}^{n+1} + \frac{(\Delta z)^2}{24} (\theta_{zzz})_{t_0}^{n+1} \right) - K(\bar{\theta}_{i+\frac{1}{2}}^{n+1}) \\ &= (D(\theta_{i+\frac{1}{2}}^{n+1}) + D_\theta(\bar{\theta}_{i+\frac{1}{2}}^{n+1} - \theta_{i+\frac{1}{2}}^{n+1})) \left((\theta_z)_{i+\frac{1}{2}}^{n+1} + \frac{(\Delta z)^2}{24} (\theta_{zzz})_{t_0}^{n+1} \right) \\ &\quad - (K(\theta_{i+\frac{1}{2}}^{n+1}) + K_\theta(\bar{\theta}_{i+\frac{1}{2}}^{n+1} - \theta_{i+\frac{1}{2}}^{n+1})) \\ &= \left(D(\theta_{i+\frac{1}{2}}^{n+1}) + \frac{(\Delta z)^2}{8} D_\theta(\theta_{zz})_{t_1}^{n+1} \right) \left((\theta_z)_{i+\frac{1}{2}}^{n+1} + \frac{(\Delta z)^2}{24} (\theta_{zzz})_{t_0}^{n+1} \right) \\ &\quad - \left(K(\theta_{i+\frac{1}{2}}^{n+1}) + \frac{(\Delta z)^2}{8} K_\theta(\theta_{zz})_{t_2}^{n+1} \right) \\ &= D(\theta_{i+\frac{1}{2}}^{n+1}) (\theta_z)_{i+\frac{1}{2}}^{n+1} - K(\theta_{i+\frac{1}{2}}^{n+1}) + O((\Delta z)^2 + (\Delta z)^4). \end{aligned} \quad (41)$$

Similarly, we have

$$q_{i-\frac{1}{2}}^{n+1} = D(\bar{\theta}_{i-\frac{1}{2}}^{n+1}) (\theta_z)_{i-\frac{1}{2}}^{n+1} - K(\theta_{i-\frac{1}{2}}^{n+1}) + O((\Delta z)^2 + (\Delta z)^4). \quad (42)$$

Applying the Taylor formula, we have

$$\begin{aligned} K(\theta_{i+\frac{1}{2}}^{n+1}) &= K(\theta_i^{n+1}) + K_\theta(\theta_{i+\frac{1}{2}}^{n+1} - \theta_i^{n+1}) + \frac{1}{2} K_{\theta\theta}(\theta_{i+\frac{1}{2}}^{n+1} - \theta_i^{n+1})^2 \\ &= K(\theta_i^{n+1}) + K_\theta \left(\frac{\Delta z}{2} (\theta_z)_i^{n+1} \right) + \frac{(\Delta z)^2}{8} (\theta_{zz})_{t_3}^{n+1} + \frac{1}{2} K_{\theta\theta} \left(\frac{\Delta z}{2} (\theta_z)_{t_4}^{n+1} \right)^2 \\ &= K(\theta_i^{n+1}) + \frac{\Delta z}{2} (K_\theta \theta_z)_i^{n+1} + O((\Delta z)^2) \end{aligned} \quad (43)$$

and

$$K(\theta_{i-\frac{1}{2}}^{n+1}) = K(\theta_i^{n+1}) - \frac{\Delta z}{2}(K_\theta \theta_z)_i^{n+1} + O((\Delta z)^2). \quad (44)$$

By (41)–(44), we have

$$\begin{aligned} \frac{q_{i+\frac{1}{2}}^{n+1} - q_{i-\frac{1}{2}}^{n+1}}{\Delta z} &= \left(\frac{\partial}{\partial z} \left(D(\theta) \frac{\partial \theta}{\partial z} \right) \right)_i^{n+1} - (K_\theta \theta_z)_i^{n+1} \\ &\quad + \frac{(\Delta z)^2}{24} \left(\frac{\partial^3}{\partial z^3} \left(D(\theta) \frac{\partial \theta}{\partial z} \right) \right)_{i_3}^{n+1} + O((\Delta z)^2). \end{aligned} \quad (45)$$

Substituting (40) and (45) into (39), it follows that

$$\begin{aligned} o_i^{n+1} &= (\theta_t)_i^{n+1} - \left(\frac{\partial}{\partial z} \left(D(\theta) \frac{\partial \theta}{\partial z} \right) \right)_i^{n+1} + (K_\theta \theta_z)_i^{n+1} - f_i^{n+1} + O(\Delta t + (\Delta z)^2) \\ &= \left(\theta_t - \frac{\partial}{\partial z} \left(D(\theta) \frac{\partial \theta}{\partial z} \right) + \frac{\partial K}{\partial z} - f \right)_i^{n+1} + O(\Delta t + (\Delta z)^2) \\ &= O(\Delta t + (\Delta z)^2). \end{aligned} \quad (46)$$

Similarly, if $\theta \in C^0([0, T]; C^5(\Omega)) \cap C^2([0, T]; C^1(\Omega))$, we obtain $\delta_z o_{i+\frac{1}{2}}^{n+1} = O(\Delta t + (\Delta z)^2)$. \square

5.2 Convergence

Let

$$\begin{aligned} \rho_i^n &= \theta_i^n - \Theta_i^n, & \rho_{i+\frac{1}{2}}^n &= \bar{\theta}_{i+\frac{1}{2}}^n - \Theta_{i+\frac{1}{2}}^n, \\ \zeta_{i+\frac{1}{2}}^n &= q_{i+\frac{1}{2}}^n - Q_{i+\frac{1}{2}}^n, & i \neq i_1, & \quad \tilde{\zeta}_{i_1+\frac{1}{2}}^n = q_{i_1+\frac{1}{2}}^n - \tilde{Q}_{i_1+\frac{1}{2}}^n, \end{aligned} \quad (47)$$

for $i \neq i_1$, i.e.,

$$\begin{aligned} \zeta_{i+\frac{1}{2}}^n &= D(\bar{\theta}_{i+\frac{1}{2}}^n) \delta_z \theta_{i+\frac{1}{2}}^n - K(\bar{\theta}_{i+\frac{1}{2}}^n) - (D(\Theta_{i+\frac{1}{2}}^n) \delta_z \Theta_{i+\frac{1}{2}}^n - K(\Theta_{i+\frac{1}{2}}^n)) \\ &= (D(\bar{\theta}_{i+\frac{1}{2}}^n) - D(\Theta_{i+\frac{1}{2}}^n)) \delta_z \theta_{i+\frac{1}{2}}^n + D(\Theta_{i+\frac{1}{2}}^n) \delta_z \rho_{i+\frac{1}{2}}^n - (K(\bar{\theta}_{i+\frac{1}{2}}^n) - K(\Theta_{i+\frac{1}{2}}^n)) \\ &= D(\Theta_{i+\frac{1}{2}}^n) \delta_z \rho_{i+\frac{1}{2}}^n + D_\theta \delta_z \theta_{i+\frac{1}{2}}^n \rho_{i+\frac{1}{2}}^n - K_\theta \rho_{i+\frac{1}{2}}^n \end{aligned} \quad (48)$$

and, when $i = i_1$, we have

$$\begin{aligned} \tilde{\zeta}_{i_1+\frac{1}{2}}^{n+1} &= \zeta_{i_1+\frac{1}{2}}^n + \Delta t D(\Theta_{i_1+\frac{1}{2}}^{n+1}) \frac{\zeta_{i_1-\frac{1}{2}}^n - 2\zeta_{i_1+\frac{1}{2}}^{n+1} + \zeta_{i_1+\frac{3}{2}}^n}{(\Delta z)^2} + \Delta t \partial_t D(\Theta_{i_1+\frac{1}{2}}^{n+1}) \delta_z \rho_{i_1+\frac{1}{2}}^n \\ &\quad - \Delta t \partial_t (K_\theta \rho_{i_1+\frac{1}{2}}^{n+1}) + O\left((\Delta t)^2 + \Delta t (\Delta z)^2 + \frac{(\Delta t)^2}{(\Delta z)^2}\right) \\ &\quad + \Delta t (D_\theta \delta_z^2 q_{i_1+\frac{1}{2}}^{n+1} \rho_{i_1+\frac{1}{2}}^{n+1} + D_\theta \delta_z f_{i_1+\frac{1}{2}}^{n+1} \rho_{i_1+\frac{1}{2}}^{n+1} + \partial_t (D_\theta \rho_{i_1+\frac{1}{2}}^{n+1}) \delta_z \theta_{i_1+\frac{1}{2}}^n). \end{aligned} \quad (49)$$

Next, we give the proof of (49).

Proof Taking the derivative of q with respect to the time t , we have

$$\begin{aligned}\frac{\partial q}{\partial t} &= D(\theta)\delta_z\left(\frac{\partial\theta}{\partial t}\right) + \frac{\partial(D(\theta))}{\partial t}\delta_z\theta - \frac{\partial(K(\theta))}{\partial t} \\ &= D(\theta)\delta_z\left(\frac{\partial q}{\partial z} + f\right) + \frac{\partial(D(\theta))}{\partial t}\delta_z\theta - \frac{\partial(K(\theta))}{\partial t}.\end{aligned}\quad (50)$$

By the Taylor formula, we have

$$\begin{aligned}q_{i_1+\frac{1}{2}}^{n+1} - q_{i_1+\frac{1}{2}}^n &= \Delta t D(\theta_{i_1+\frac{1}{2}}^{n+1})\delta_z^2 q_{i_1+\frac{1}{2}}^{n+1} + \Delta t D(\theta_{i_1+\frac{1}{2}}^{n+1})\delta_z f_{i_1+\frac{1}{2}}^{n+1} + \Delta t \partial_t D(\theta_{i_1+\frac{1}{2}}^{n+1})\delta_z \theta_{i_1+\frac{1}{2}}^n \\ &\quad - \Delta t \partial_t K(\theta_{i_1+\frac{1}{2}}^{n+1}) + O((\Delta t)^2 + \Delta t(\Delta z)^2) \\ &= \Delta t D(\theta_{i_1+\frac{1}{2}}^{n+1})\delta_z^2 q_{i_1+\frac{1}{2}}^{n+1} + \Delta t D(\theta_{i_1+\frac{1}{2}}^{n+1})\delta_z f_{i_1+\frac{1}{2}}^{n+1} + \Delta t \partial_t D(\theta_{i_1+\frac{1}{2}}^{n+1})\delta_z \theta_{i_1+\frac{1}{2}}^n \\ &\quad - \Delta t \partial_t K(\theta_{i_1+\frac{1}{2}}^{n+1}) + O((\Delta t)^2 + \Delta t(\Delta z)^2) \\ &\quad + \Delta t D_\theta \delta_z^2 q_{i_1+\frac{1}{2}}^{n+1} \rho_{i_1+\frac{1}{2}}^{n+1} + \Delta t D_\theta \delta_z f_{i_1+\frac{1}{2}}^{n+1} \rho_{i_1+\frac{1}{2}}^{n+1} + \Delta t \partial_t (D_\theta \rho_{i_1+\frac{1}{2}}^{n+1})\delta_z \theta_{i_1+\frac{1}{2}}^n \\ &= \Delta t D(\theta_{i_1+\frac{1}{2}}^{n+1}) \frac{q_{i_1-\frac{1}{2}}^n - 2q_{i_1+\frac{1}{2}}^{n+1} + q_{i_1+\frac{3}{2}}^n}{(\Delta z)^2} + \Delta t D(\theta_{i_1+\frac{1}{2}}^{n+1})\delta_z f_{i_1+\frac{1}{2}}^{n+1} + O\left(\frac{(\Delta t)^2}{(\Delta z)^2}\right) \\ &\quad + \Delta t \partial_t D(\theta_{i_1+\frac{1}{2}}^{n+1})\delta_z \theta_{i_1+\frac{1}{2}}^n - \Delta t \partial_t K(\theta_{i_1+\frac{1}{2}}^{n+1}) + O((\Delta t)^2 + \Delta t(\Delta z)^2) \\ &\quad + \Delta t D_\theta \delta_z^2 q_{i_1+\frac{1}{2}}^{n+1} \rho_{i_1+\frac{1}{2}}^{n+1} + \Delta t D_\theta \delta_z f_{i_1+\frac{1}{2}}^{n+1} \rho_{i_1+\frac{1}{2}}^{n+1} + \Delta t \partial_t (D_\theta \rho_{i_1+\frac{1}{2}}^{n+1})\delta_z \theta_{i_1+\frac{1}{2}}^n,\end{aligned}\quad (51)$$

where we assume that $\|\Theta\|_\infty \leq C$. Subtracting (2) from (51), we complete the proof. \square

Subtracting the first equations of Eqs. (16) from Eqs. (39), we have

$$\begin{cases} \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{\zeta_{i+\frac{1}{2}}^{n+1} - \zeta_{i-\frac{1}{2}}^{n+1}}{\Delta z} + o_i^{n+1} + \frac{\tilde{\zeta}_{i_1+\frac{1}{2}}^{n+1} - \zeta_{i_1+\frac{1}{2}}^{n+1}}{\Delta z}, & i = i_1, \\ \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{\zeta_{i+\frac{1}{2}}^{n+1} - \zeta_{i-\frac{1}{2}}^{n+1}}{\Delta z} + o_i^{n+1} + \frac{\zeta_{i_1+\frac{1}{2}}^{n+1} - \tilde{\zeta}_{i_1+\frac{1}{2}}^{n+1}}{\Delta z}, & i = i_1 + 1, \\ \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{\zeta_{i+\frac{1}{2}}^{n+1} - \zeta_{i-\frac{1}{2}}^{n+1}}{\Delta z} + o_i^{n+1}, & i \neq i_1, i_1 + 1. \end{cases}\quad (52)$$

By (48), (49) and (52), we have

$$\begin{aligned}\zeta_{i_1+\frac{1}{2}}^{n+1} - \tilde{\zeta}_{i_1+\frac{1}{2}}^{n+1} &= D(\theta_{i_1+\frac{1}{2}}^{n+1}) \frac{\Delta t}{(\Delta z)^2} (\zeta_{i_1-\frac{1}{2}}^{n+1} + \zeta_{i_1+\frac{3}{2}}^{n+1}) - D(\theta_{i_1+\frac{1}{2}}^{n+1}) \frac{\Delta t}{(\Delta z)^2} (\zeta_{i_1-\frac{1}{2}}^n + \zeta_{i_1+\frac{3}{2}}^n) \\ &\quad + \Delta t \delta_z o_{i_1+\frac{1}{2}}^{n+1} + O\left((\Delta t)^2 + \Delta t(\Delta z)^2 + \frac{(\Delta t)^2}{(\Delta z)^2}\right) \\ &\quad - (\Delta t)^2 \partial_t (D_\theta \rho_{i_1+\frac{1}{2}}^{n+1}) (\partial_t \delta_z \theta_{i_1+\frac{1}{2}}^{n+1}) \\ &\quad + \Delta t (D_\theta \delta_z^2 q_{i_1+\frac{1}{2}}^{n+1} + D_\theta \delta_z f_{i_1+\frac{1}{2}}^{n+1} + \partial_t (\delta_z \theta_{i_1+\frac{1}{2}}^{n+1}) D_\theta) \rho_{i_1+\frac{1}{2}}^{n+1}.\end{aligned}\quad (53)$$

Let $\rho = \{\rho_i\}$ and $\zeta = \{\zeta_{i+\frac{1}{2}}\}$ be the solution of Scheme (52). Similar to the proof of Lemma 2, we obtain the following convergence.

Theorem 3 Assume that $\theta \in C^0([0, T]; C^5(\Omega)) \cap C^2([0, T]; C^1(\Omega))$ and satisfy the conditions (I)–(III). We have

$$\|\theta^n - \Theta^n\| \leq M \left(\Delta t + (\Delta z)^2 + \frac{(\Delta t)^2}{(\Delta z)^{\frac{3}{2}}} \right), \quad (54)$$

for $n \geq 1$, where $M > 0$ is a constant.

Next, applying the inverse estimation, we give the estimation of $\|\Theta\|_\infty$ as

$$\begin{aligned} \|\Theta\|_\infty &\leq \|\theta\|_\infty + \|\theta - \Theta\|_\infty \leq \|\theta\|_\infty + (\Delta z)^{-\frac{1}{2}} \|\theta - \Theta\| \\ &\leq \theta_s + M \left(\frac{\Delta t}{(\Delta z)^{\frac{1}{2}}} + (\Delta z)^{\frac{3}{2}} + \frac{(\Delta t)^2}{(\Delta z)^2} \right). \end{aligned} \quad (55)$$

If $\Delta t = o(\Delta z)$, we have $\|\Theta\|_\infty \rightarrow \theta_s (\Delta z \rightarrow 0)$.

Remark 5.1 From Theorem 3, it is clear that our scheme is convergent of $O(\Delta t + (\Delta z)^2 + \frac{(\Delta t)^2}{(\Delta z)^{\frac{3}{2}}})$. It reaches the first-order error of $O(\Delta t)$ in time step under mesh ratios $\Delta z = (\Delta t)^{\frac{2}{3}}$ and first-order error of $O(\Delta z)^2$ in space step under mesh ratios $\Delta t = (\Delta z)^2$.

6 Numerical experiments

We present numerical experiments to test our scheme to meet the properties as conservation, stability and error estimation in the first experiment. In the second experiment, we will test our efficiency.

6.1 Experiment 1

The nonlinear diffusion equations are described as follows:

$$\begin{cases} \frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z} (D(\theta) \frac{\partial \theta}{\partial z}) = e^{-2D_0 t} \cos 2x, & (x, t) \in (0, \pi) \times (0, T), \\ D(\theta) \frac{\partial \theta}{\partial z} = 0, & \text{if } z = 0 \text{ or } z = \pi, t > 0, \\ \theta(z, 0) = \cos x, & x \in (0, \pi), \end{cases} \quad (56)$$

Let the diffusion coefficient $D(\theta) = D_0 + \theta$. We can solve the exact solution $u = e^{-D_0 t} \cos x$. Assume that the domain is divided into two sub-domains in the following tables.

The space orders of convergence and mass errors of the scheme are shown in Table 1 and we take $\Delta t = \frac{1}{10,000}$. Let mass error = $|\text{Mass}_N - \text{Mass}_0|$.

$$\text{Mass}_0 = \sum_{i=1}^N \Theta_i^0 h, \quad \text{Mass}_N = \sum_{i=1}^N \Theta_i^n h + \Delta t \sum_{l=0}^{n-1} f_i^{l+1} h. \quad (57)$$

From Table 1, we can find that the space-order convergence is of second order in L^2 -norm. The errors of mass reach the accuracy of 10^{-15} , i.e. the machine precision. We test the time orders of convergence by taking $\Delta t = \frac{10}{\pi^2} h^2$ in Table 2. It is clear that the scheme is of first-order convergence in time step and shows mass conservation.

In Table 3, we give the stability condition and mass error of the scheme, where $h = \pi/100$. From Table 3, it is clear that we can find when $\Delta t = \frac{1}{100}$, the stability condition $D_{\max} \frac{\Delta t}{h^2} = 30$ and our scheme is still stable. Almost, our scheme keeps stable with the time increasing.

Table 1 The space order of convergence and mass errors of the scheme

t	h	$\pi/10$	$\pi/20$	$\pi/30$	$\pi/40$
0.1	L^2 -norm-	2.7344E-3	6.8471E-4	3.0517E-4	1.7291E-4
	order	–	1.9977	1.9931	1.9747
	mass error	5.6209E-15	3.0334E-15	4.1574E-16	1.100E-15
0.2	L^2 -norm-	3.5663E-3	8.9895E-4	4.0760E-4	2.3666E-4
	order	–	1.9881	1.9507	1.8898
	mass error	4.4101E-15	6.2306E-15	3.7128E-16	2.9389E-15
0.5	L^2 -norm-	3.7909E-3	9.7838E-4	4.5906E-4	2.7762E-4
	order	–	1.9541	1.8663	1.7482
	mass error	4.3134E-15	4.8924E-15	7.4806E-16	6.0204E-15

Table 2 The time order of convergence and mass errors of the scheme

t	Δt	1/250	1/1000	1/2000	1/4000
0.1	L^2 -norm-	1.0671E-3	2.5095E-4	1.2075E-4	6.3302E-5
	order	–	1.0441	1.0554	0.9317
	mass error	1.2228E-15	2.6583E-16	1.3142E-16	1.8414E-15
0.2	L^2 -norm-	1.4099E-3	3.7218E-4	1.7956E-4	9.7289E-5
	order	–	0.9608	1.0515	0.8841
	mass error	5.4347E-16	1.3609E-15	1.2970E-15	2.1003E-16
0.5	L^2 -norm-	1.5520E-3	4.5392E-4	2.1815E-4	1.2226E-4
	order	–	0.8868	1.0571	0.8354
	mass error	1.4268E-15	3.9946E-15	2.2446E-16	1.0600E-15

Table 3 The stability and mass errors of the scheme

t	Δt	1/100	1/200	1/500	1/1000
0.1	L^2 -norm-	8.9592E-3	2.1893E-3	5.3842E-4	2.5095E-4
	mass error	4.0854E-15	1.7581E-15	2.1940E-15	2.6583E-16
0.2	L^2 -norm-	8.0124E-3	2.1446E-3	7.0027E-4	3.7218E-4
	mass error	6.2661E-15	2.3825E-15	1.6151E-15	1.3609E-15
0.5	L^2 -norm-	4.9804E-3	1.2775E-3	7.4550E-4	4.5391E-2
	mass error	3.0825E-15	4.4886E-15	6.4891E-17	3.9946E-15

Table 4 The space order of convergence and mass errors of the scheme for $u_0(x) = \cos 4x$

t	h	$\pi/10$	$\pi/20$	$\pi/30$	$\pi/40$
0.01	L^2 -norm-	5.6678E-2	1.3473E-2	5.9676E-3	3.3558E-3
	order	–	2.0726	2.0085	2.0010
	mass error	6.7035E-15	4.1767E-15	5.6799E-18	1.5305E-15
0.2	L^2 -norm-	6.8241E-2	1.5387E-2	6.7852E-3	3.8226E-3
	order	–	2.1489	2.0194	1.9946
	mass error	7.9878E-15	2.4906E-15	2.2245E-15	1.8313E-15
0.5	L^2 -norm-	5.4733E-2	1.3158E-2	5.8475E-3	3.3109E-3
	order	–	2.0565	2.0002	1.9772
	mass error	8.4530E-15	9.1308E-15	4.9728E-15	1.7777E-15

In general, the time order of convergence of the scheme is tested for the high frequency problem for the initial condition $u_0(x) = \cos 4x$, where $f = 16e^{-64t} \cos 8x$ and the time step $\Delta = \frac{1}{100,000}$. In Table 4, the numerical results show our scheme can work efficiently for the high frequency problem.

6.2 Experiment 2

In the following, we will simulate the model (1), where the absorption rate of root zone $S_r = 0$, the infiltration or evaporation rate $D(\theta) = -\frac{bK_s\phi_s}{\theta_s}(\frac{\theta}{\theta_s})^{b+2}$ and $K(\theta) = K_s(\frac{\theta}{\theta_s})^{2b+3}$, $\theta_s = 0.54$, $\phi_s = -200$ mm, $K_s = 3.2 \times 10^{-3}$ mm/s, $b = 7.6$, $L = 100$ cm. The initial conditions are

$$\theta(z, 0) = \begin{cases} 0.54 \times 0.419 + 0.54 \times (1 - 0.419) \times \frac{10-z}{z}, & z \in [0, 10], \\ 0.54 \times 0.419, & z \in [10, 100]. \end{cases} \quad (58)$$

The boundary conditions are

$$\begin{cases} \theta = 0.54, & z = 0, \\ \frac{\partial \theta}{\partial z} = 0, & z = L. \end{cases} \quad (59)$$

The space step size is $h = 2$ mm and the time step size is $\Delta t = 36$ s. The surface of the water content at $t = 10, 20, 30$ and 40 h are given as follows. From Fig. 1, when the water content crosses the interface of the domain, there is no numerical oscillation and our scheme can simulate the problem well.

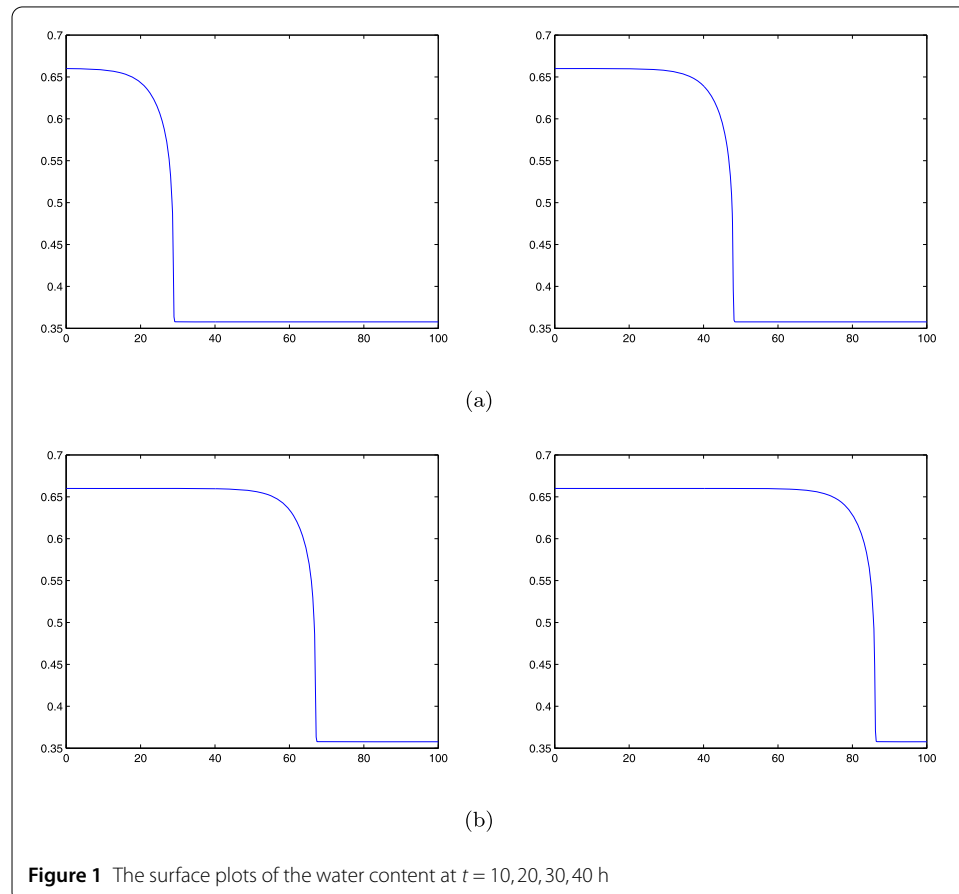


Table 5 The space order of convergence and mass errors of the scheme

t	h	$\pi/10$	$\pi/20$	$\pi/40$	$\pi/60$
$t = 0.01$	L^2 -norm-	7.2729E-4	1.8295E-4	4.5442E-5	1.9918E-5
	order	–	1.9911	2.0094	2.0342
	mass error	5.9008E-15	1.9856E-15	1.8239E-16	3.7283E-16
$t = 0.02$	L^2 -norm-	1.3843E-3	3.4802E-4	8.6450E-5	3.7907E-5
	order	–	1.9919	2.0092	2.0333
	mass error	1.5327E-14	3.6242E-15	1.4922E-15	4.8240E-16
$t = 0.05$	L^2 -norm-	3.0027E-3	7.5398E-4	1.8734E-4	8.2224E-5
	order	–	1.9937	2.0089	2.0309
	mass error	2.0386E-14	2.5691E-15	1.7195E-15	1.8908E-17

Table 6 The time order of convergence and mass errors of the scheme

t	Δt	1/250	1/1000	1/4000	1/16000
0.01	L^2 -norm-	6.0696E-3	7.0985E-5	1.7420E-5	4.3418E-6
	order	–	3.2090	1.0134	1.0022
	mass error	2.3822E-16	7.5664E-17	6.3878E-17	2.4554E-16
0.02	L^2 -norm-	5.2884E-4	1.3012E-4	3.2490E-5	8.1346E-6
	order	–	1.0115	1.0009	0.9989
	mass error	2.4307E-16	3.8470E-16	6.4907E-17	5.0431E-16
0.05	L^2 -norm-	5.4237E-3	2.7173E-4	6.8298E-5	1.7125E-5
	order	–	2.1595	0.9961	0.9979
	mass error	4.2901E-16	4.4524E-16	1.0077E-16	1.5782E-15

6.3 Experiment 3

In the subsection, two-dimensional nonlinear diffusion equations are considered as

$$\begin{cases}
 \frac{\partial \theta}{\partial t} - \nabla \cdot (D \nabla \theta) = e^{-4t} (\cos^2 y \cos 2x + \cos^2 x \cos 2y), \\
 (x, y, t) \in [0, \pi] \times [0, \pi] \times (0, T), \\
 D(\theta) \nabla \theta \cdot \vec{n} = 0, \quad x \in \partial \Omega, \\
 \theta(x, y, 0) = \cos x \cos y, \quad (x, y) \in \Omega,
 \end{cases} \quad (60)$$

where $D = 1 + \theta$. The operator splitting technique is used to solve the two-dimensional equations, (see [28, 29, 34]). In Tables 5 and 6, we take the uniform mesh partitions and the domain is divided into 2×2 sub-domains. The time step Δt is taken as $1/100,000$, and the space step h is taken as $\pi/10, \pi/20, \pi/40$ and $\pi/60$. We can find that our scheme is of second-order convergence in space. In Table 6, we take $\Delta t = \frac{10}{\pi^2} h^2$. By Table 6, we can find that the time-convergence order is first order when the time step Δt becomes small.

7 Conclusion

By computing the interface flux scheme by the semi-implicit scheme, an efficient mass conservative domain decomposition method is developed for solving the unsaturated soil water flow. By introducing the definition of the whole fluxes $q = D(\theta) \frac{\partial \theta}{\partial z} - K(\theta)$, and computing the interface fluxes explicitly on the interface, our scheme keeps mass conservation. We prove strictly our scheme meet mass conservation and unconditional stability. The scheme has the optional error estimate of $O(\Delta t + (\Delta z)^2)$ in L^2 -norm by the numerical experiment.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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