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Basic theory of differential equations with mixed perturbations of the second type on time scales

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Abstract

In this paper, we develop the theory of differential equations with mixed perturbations of the second type on time scales. We give an existence theorem for differential equations with mixed perturbations of the second type on time scales under Lipschitz condition. We also present some fundamental differential inequalities on time scales, which are utilized to investigate the existence of extremal solutions. We establish the comparison principle for differential equations with mixed perturbations of the second type on time scales. Our results in this paper extend and improve some well-known results.

Keywords: Mixed perturbations; Existence; Differential inequalities; Comparison principle; Time scales

1 Introduction

In this paper, we discuss the following differential equations with mixed perturbations of the second type on time scales (DETS):

$$\begin{cases} \left[\frac{u(t)-k(t,u(t))}{f(t,u(t))}\right]^{\Delta} = g(t,u(t)), & t \in J, \\ u(t_0) = u_0, \end{cases} \tag{1}$$

where $f \in C_{\mathrm{rd}}(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $k, g \in C_{\mathrm{rd}}(J \times \mathbb{R}, \mathbb{R})$.

Let $\mathbb{T}$ be a time scale, and let $J = [t_0, t_0 + a]_{\mathbb{T}} = [t_0, t_0 + a] \cap \mathbb{T}$ be a bounded interval in $\mathbb{T}$ for some $t_0 \in \mathbb{R}$ and $a > 0$. We denote by $C_{\mathrm{rd}}(J \times \mathbb{R}, \mathbb{R})$ the class of rd-continuous functions $g : J \times \mathbb{R} \to \mathbb{R}$. For basic definitions and useful lemmas from the theory of calculus on time scales, we refer to [1].

By a solution of the DETS (1) we mean a Δ-differentiable function u such that

(i) the function $t \mapsto \frac{u-k(t,u)}{f(t,u)}$ is Δ-differentiable for each $u \in \mathbb{R}$, and

(ii) u satisfies equations (1).

The theory of time scales has been drawn a lot of attention since 1988 (see [1–9]). In recent years the theory of nonlinear differential equations with perturbations has been a hot research topic; see [10–15]. Dhage [13] discussed the following first-order hybrid differential equation with mixed perturbations of the second type:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x(t)-k(t,x(t))}{f(t,x(t))}\right] = g(t,x(t)), & t \in [t_0, t_0+a], \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

where $[t_0, t_0 + a]$ is a bounded interval in $\mathbb{R}$ for some $t_0 \in \mathbb{R}$ and $a > 0$, $f \in C([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $k, g \in C([t_0, t_0 + a] \times \mathbb{R}, \mathbb{R})$. They developed the theory of hybrid differential equations with mixed perturbations of the second type and gave some original and interesting results.

As far as we know, there are no results for the DETS (1). From the works mentioned we consider the theory of DETS (1). We give an existence theorem for the DETS (1) under Lipschitz conditions. We also present some fundamental differential inequalities on time scales (DITS), which are utilized to investigate the existence of extremal solutions. We establish the comparison principle for the DETS (1). Our results in this paper extend and improve some well-known results.

The paper is organized as follows. In Sect. 2, we give an existence theorem for the DETS (1) under Lipschitz conditions by the fixed point theorem in Banach algebra due to Dhage. In Sect. 3, we establish some fundamental DITS to strict inequalities for the DETS (1). In Sect. 4, we present existence results of maximal and minimal solutions for HDTS. In Sect. 5, we prove the comparison principle for the DETS (1), which is followed by the conclusion in Sect. 6.

2 Existence result

In this section, we discuss the existence results for the DETS (1). We place the DETS (1) in the space $C_{\mathrm{rd}}(J, \mathbb{R})$ of rd-continuous functions defined on J with the supremum norm $\|\cdot\|$ defined as

$$\|u\| = \sup_{t\in J}\left|u(t)\right|$$

and the multiplication "$\cdot$" in $C_{\mathrm{rd}}(J, \mathbb{R})$ defined as

$$(u \cdot v)(t) = (uv)(t) = u(t)v(t)$$

for $u, v \in C_{\mathrm{rd}}(J, \mathbb{R})$. Clearly, $C_{\mathrm{rd}}(J, \mathbb{R})$ is a Banach algebra with respect to these norm and multiplication. By $L^1(J, \mathbb{R})$ we denote the space of Lebesgue Δ-integrable functions on J equipped with the norm $\|\cdot\|_{L^1}$ defined as

$$\|u\|_{L^1} = \int_{t_0}^{t_0+a}\left|u(s)\right|\Delta s.$$

The following fixed point theorem in a Banach algebra due to Dhage [16] is useful in the proofs of our main results.

Lemma 2.1 ([16]) *Let Q be a closed convex bounded subset of a Banach space P, and let $A, C : P \to P$ and $B : Q \to P$ be three operators such that*

(a) *A and C are Lipschitz with Lipschitz constants α and β, respectively,*
(b) *B is compact and continuous,*
(c) *$u = AuBv + Cu$ for all $v \in Q \Rightarrow u \in Q$, and*
(d) *$\alpha M + \beta < 1$, where $M = \|B(Q)\| = \sup\{\|B(u)\| : u \in Q\}$.*

Then the operator equation $AuBu + Cu = u$ has a solution in Q.

We present the following hypotheses.

(A_0) The function $u \mapsto \frac{u-k(t,u)}{f(t,u)}$ is increasing in $\mathbb{R}$ for all $t \in J$.

(A_1) There exist constants $L_1 > 0$ and $L_2 > 0$ such that

$$\bigl|f(t,u) - f(t,v)\bigr| \le L_1|u - v|$$

and

$$\bigl|k(t,u) - k(t,v)\bigr| \le L_2|u - v|$$

for all $t \in J$ and $u, v \in \mathbb{R}$. Moreover, $L \le M$.

(A_2) There exists a function $h \in L^1(J,\mathbb{R})$ such that

$$\bigl|g(t,u)\bigr| \le h(t), \quad t \in J,$$

for all $u \in \mathbb{R}$.

Lemma 2.2 *Suppose that (A_0) holds. Then for any $v \in L^1(J,\mathbb{R})$, the Δ-differentiable function u is a solution of the DETS*

$$\left[\frac{u(t) - k(t,u(t))}{f(t,u(t))}\right]^{\Delta} = v(t), \quad t \in J, \tag{2}$$

and

$$u(t_0) = u_0 \in \mathbb{R}, \tag{3}$$

if and only if u satisfies the integral equation

$$u(t) = k\bigl(t,u(t)\bigr) + f\bigl(t,u(t)\bigr)\left(\frac{u_0 - k(t_0,u_0)}{f(t_0,u_0)} + \int_{t_0}^{t} v(s)\Delta s\right), \quad t \in J. \tag{4}$$

Proof Let u be a solution of problem (2)–(3). Applying the Δ-integral to (2) from t_0 to t, we obtain

$$\left[\frac{u(t) - k(t,u(t))}{f(t,u(t))}\right] - \left[\frac{u_0 - k(t_0,u_0)}{f(t_0,u_0)}\right] = \int_{t_0}^{t} v(s)\Delta s,$$

that is,

$$u(t) = k\bigl(t,u(t)\bigr) + f\bigl(t,u(t)\bigr)\left(\frac{u_0 - k(t_0,u_0)}{f(t_0,u_0)} + \int_{t_0}^{t} v(s)\Delta s\right), \quad t \in J.$$

Thus (4) holds.

Conversely, suppose that u satisfies equation (4). By direct differentiation, applying the Δ-derivative to both sides of (4), we get that (2) is satisfied. Thus, substitute $t = t_0$ in (4) implies

$$\frac{u(t_0) - k(t_0, u(t_0))}{f(t_0, u(t_0))} = \frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}.$$

Since the map $u \mapsto \frac{u-k(t,u)}{f(t,u)}$ is increasing in $\mathbb{R}$ for $t \in J$, the map $u \mapsto \frac{u-k(t_0,u)}{f(t_0,u)}$ is injective in $\mathbb{R}$, and $u(t_0) = u_0$. Hence (3) also holds. □

Now we will give the following existence theorem for the DETS (1).

Theorem 2.1 *Suppose that* (A_0)*–*(A_2) *hold. If*

$$L_1\left(\left|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}\right| + \|h\|_{L^1}\right) + L_2 < 1, \tag{5}$$

then the DETS (1) *has a solution defined on* J.

Proof Set $U = C_{\mathrm{rd}}(J, \mathbb{R})$ and define the subset S of U by

$$S = \{u \in U \mid \|u\| \le N\},$$

where

$$N = \frac{F_0(|\frac{u_0-k(t_0,u_0)}{f(t_0,u_0)}| + \|h\|_{L^1}) + K_0}{1 - L_1(|\frac{u_0-k(t_0,u_0)}{f(t_0,u_0)}| + \|h\|_{L^1}) - L_2},$$

$F_0 = \sup_{t\in J} |f(t, 0)|$, and $K_0 = \sup_{t\in J} |k(t, 0)|$.

Clearly, S is a closed, convex, and bounded subset of the Banach space U. By Lemma 2.2 the DETS (1) is equivalent to the nonlinear integral equation

$$u(t) = k\bigl(t, u(t)\bigr) + f\bigl(t, u(t)\bigr)\left(\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} + \int_{t_0}^{t} g\bigl(s, u(s)\bigr)\Delta s\right), \quad t \in J. \tag{6}$$

Define three operators $A, C : U \to U$ and $B : S \to U$ by

$$Au(t) = f\bigl(t, u(t)\bigr), \quad t \in J, \tag{7}$$

$$Bu(t) = \frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} + \int_{t_0}^{t} g\bigl(s, u(s)\bigr)\Delta s, \quad t \in J, \tag{8}$$

and

$$Cu(t) = k\bigl(t, u(t)\bigr), \quad t \in J. \tag{9}$$

Then equation (6) is transformed into the operator equation as

$$Au(t)Bu(t) + Cu(t) = u(t), \quad t \in J.$$

Next, we prove that the operators A, B, and C satisfy all the conditions of Lemma 2.1.

First, we prove that A is a Lipschitz operator on U with Lipschitz constant L_1. Let $u, v \in U$. Then by (A_1)

$$\left|Au(t) - Av(t)\right| = \left|f\bigl(t, u(t)\bigr) - f\bigl(t, v(t)\bigr)\right| \le L_1\left|u(t) - v(t)\right| \le L_1\|u - v\|$$

for all $t \in J$. Taking the supremum over t, then we have

$$\|Au - Av\| \le L_1\|u - v\|$$

for all $u, v \in U$. This shows that A is a Lipschitz operator on U with Lipschitz constant L_1. Similarly, we can get that C is also a Lipschitz operator on U with Lipschitz constant L_2.

Next, we prove that B is a compact continuous operator from S into U. First, we prove that B is continuous on S. Let $\{u_n\}$ be a sequence in S converging to a point $u \in S$. Then by the Lebesgue dominated convergence theorem adapted to time scale we have

$$\begin{aligned}
\lim_{n\to\infty} Bu_n(t) &= \lim_{n\to\infty}\left(\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} + \int_{t_0}^{t} g\bigl(s, u_n(s)\bigr)\Delta s\right) \\
&= \frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} + \lim_{n\to\infty}\int_{t_0}^{t} g\bigl(s, u_n(s)\bigr)\Delta s \\
&= \frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} + \int_{t_0}^{t}\Bigl[\lim_{n\to\infty} g\bigl(s, u_n(s)\bigr)\Bigr]\Delta s \\
&= \frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} + \int_{t_0}^{t} g\bigl(s, u(s)\bigr)\Delta s \\
&= Bu(t)
\end{aligned}$$

for all $t \in J$. This shows that B is a continuous operator on S.

Next, we prove that B is a compact operator on S. It suffices to show that $B(S)$ is a uniformly bounded and equicontinuous set in U. Take arbitrary $u \in S$. Then by (A_2)

$$\begin{aligned}
\left|Bu(t)\right| &= \left|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} + \int_{t_0}^{t} g\bigl(s, u(s)\bigr)\Delta s\right| \\
&\le \left|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}\right| + \int_{t_0}^{t}\left|g\bigl(s, u(s)\bigr)\right|\Delta s \\
&\le \left|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}\right| + \int_{t_0}^{t} h(s)\Delta s \\
&\le \left|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}\right| + \|h\|_{L^1}
\end{aligned}$$

for all $t \in J$. Taking the supremum over t, we have

$$\|Bu\| \le \left|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}\right| + \|h\|_{L^1}$$

for all $u \in S$. This shows that B is uniformly bounded on S.

On the other hand, let $t_1, t_2 \in J$. Then for any $u \in S$, we get

$$\begin{aligned}
|Bu(t_1) - Bu(t_2)| &= \left| \int_{t_0}^{t_1} g(s, u(s))\Delta s - \int_{t_0}^{t_2} g(s, u(s))\Delta s \right| \\
&\leq \left| \int_{t_2}^{t_1} |g(s, u(s))| \Delta s \right| \\
&\leq \left| \int_{t_2}^{t_1} h(s)\Delta s \right| \\
&= |p(t_1) - p(t_2)|,
\end{aligned}$$

where $p(t) = \int_{t_0}^{t} h(s)\Delta s$. Since the function p is continuous on compact J, it is uniformly continuous. Hence, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|t_1 - t_2| < \delta \quad \Rightarrow \quad |Bu(t_1) - Bu(t_2)| < \varepsilon$$

for all $t_1, t_2 \in J$ and $u \in S$. This shows that $B(S)$ is an equicontinuous set in U. Now the set $B(S)$ is uniformly bounded and equicontinuous set in U, so it is compact by Arzelà–Ascoli theorem. Thus B is a compact operator on S.

Next, we show that (c) of Lemma 2.1 is satisfied. Let $u \in U$ and $v \in S$ be such that $u = AuBv + Cu$. Then by assumption (A_1) we have

$$\begin{aligned}
|u(t)| &\leq |Au(t)||Bv(t)| + |Cu(t)| \\
&= |f(t, u(t))| \left| \left(\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} + \int_{t_0}^{t} g(s, v(s))\Delta s \right) \right| + |k(t, u(t))| \\
&\leq \left[|f(t, u(t)) - f(t, 0)| + |f(t, 0)| \right] \\
&\quad \cdot \left(\left| \frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} \right| + \int_{t_0}^{t} |g(s, v(s))| \Delta s \right) \\
&\quad + |k(t, u(t)) - k(t, 0)| + |k(t, 0)| \\
&\leq \left[L_1 |u(t)| + F_0 \right] \left(\left| \frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} \right| + \int_{t_0}^{t} h(s)\Delta s \right) + L_2 |u(t)| + K_0 \\
&\leq \left[L_1 |u(t)| + F_0 \right] \left(\left| \frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} \right| + \|h\|_{L^1} \right) + L_2 |u(t)| + K_0.
\end{aligned}$$

Thus we get

$$|u(t)| \leq \frac{F_0(|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}| + \|h\|_{L^1}) + K_0}{1 - L_1(|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}| + \|h\|_{L^1}) - L_2}.$$

Taking the supremum over t, we have

$$\|u\| \leq \frac{F_0(|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}| + \|h\|_{L^1}) + K_0}{1 - L_1(|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}| + \|h\|_{L^1}) - L_2} = N.$$

This shows that (c) of Lemma 2.1 is satisfied.

Finally, we obtain

$$M = \|B(S)\| = \sup\{\|B(u)\| : u \in S\} \leq \left|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}\right| + \|h\|_{L^1},$$

and so

$$L_1 M + L_2 \leq L_1\left(\left|\frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)}\right| + \|h\|_{L^1}\right) + L_2 < 1.$$

Thus all the conditions of Lemma 2.1 are satisfied, and hence the operator equation $AuBu + Cu = u$ has a solution in S. Therefore the DETS (1) has a solution defined on J. □

3 Differential inequalities on time scales

In this section, we establish DITS for the DETS (1).

Theorem 3.1 *Suppose that* (A_0) *holds. Assume that there exist* Δ*-differentiable functions* v, w *such that*

$$\left[\frac{v(t) - k(t, v(t))}{f(t, v(t))}\right]^{\Delta} \leq g\bigl(t, v(t)\bigr), \quad t \in J, \tag{10}$$

and

$$\left[\frac{w(t) - k(t, w(t))}{f(t, w(t))}\right]^{\Delta} \geq g\bigl(t, w(t)\bigr), \quad t \in J, \tag{11}$$

one of the inequalities being strict. Then $v(t_0) < w(t_0)$ *implies*

$$v(t) < w(t) \tag{12}$$

for all $t \in J$.

Proof Assume that inequality (11) is strict. Suppose that the claim is false. Then there exists $t_1 \in J$, $t_1 > t_0$, such that $v(t_1) = w(t_1)$ and $v(t) < w(t)$ for $t_0 \leq t < t_1$.

Define

$$V(t) = \frac{v(t) - k(t, v(t))}{f(t, v(t))} \quad \text{and} \quad W(t) = \frac{w(t) - k(t, w(t))}{f(t, w(t))}$$

for all $t \in J$. Then we obtain $V(t_1) = W(t_1)$, and by (A_0) we have $V(t) < W(t)$ for all $t < t_1$.

Since $V(t_1) = W(t_1)$, we get

$$\frac{V(t_1 + h) - V(t_1)}{h} > \frac{W(t_1 + h) - W(t_1)}{h}$$

for sufficiently small $h < 0$. This inequality implies that

$$V^{\Delta}(t_1) \geq W^{\Delta}(t_1)$$

because of (A_0). Then we obtain

$$g\big(t_1,v(t_1)\big) \geq V^{\Delta}(t_1) \geq W^{\Delta}(t_1) > g\big(t_1,w(t_1)\big).$$

This is a contradiction with $v(t_1) = w(t_1)$. Hence inequality (12) is valid. □

The next result is concerned with nonstrict DITS, which needs a Lipshitz condition.

Theorem 3.2 *Suppose that the conditions of Theorem 3.1 hold with inequalities (10) and (11). Suppose that there exists a real number $K > 0$ such that*

$$g(t,u_1) - g(t,u_2) \leq K \sup_{t_0\leq s\leq t}\left(\frac{u_1(t)-k(t,u_1(t))}{f(t,u_1(t))} - \frac{u_2(t)-k(t,u_2(t))}{f(t,u_2(t))}\right), \quad t\in J \tag{13}$$

for all $u_1,u_2 \in \mathbb{R}$ with $u_1 \geq u_2$. Then $v(t_0) \leq w(t_0)$ implies $v(t) \leq w(t)$ for all $t \in J$.

Proof Let $\varepsilon > 0$ and $K > 0$ be given. Define

$$\frac{w_\varepsilon(t)-k(t,w_\varepsilon(t))}{f(t,w_\varepsilon(t))} = \frac{w(t)-k(t,w(t))}{f(t,w(t))} + \varepsilon e^{2L(t-t_0)},$$

so that we get

$$\frac{w_\varepsilon(t)-k(t,w_\varepsilon(t))}{f(t,w_\varepsilon(t))} > \frac{w(t)-k(t,w(t))}{f(t,w(t))} \quad\Rightarrow\quad w_\varepsilon(t) > w(t).$$

Let $W_\varepsilon(t) = \frac{w_\varepsilon(t)-k(t,w_\varepsilon(t))}{f(t,w_\varepsilon(t))}$, so that $W(t) = \frac{w(t)-k(t,w(t))}{f(t,w(t))}$ for $t\in J$. Then by (11) we obtain

$$W_\varepsilon^{\Delta}(t) = W^{\Delta}(t) + 2K\varepsilon e^{2L(t-t_0)} \geq g\big(t,w(t)\big) + 2L\varepsilon e^{2L(t-t_0)}.$$

Then from (13) we have

$$g\big(t,w_\varepsilon(t)\big) - g\big(t,w(t)\big) \leq K \sup_{t_0\leq s\leq t}\big(W_\varepsilon(s) - W(s)\big) = K\varepsilon e^{2L(t-t_0)}$$

for all $t\in J$, and thus

$$W_\varepsilon^{\Delta}(t) \geq g\big(t,w_\varepsilon(t)\big) - K\varepsilon e^{2L(t-t_0)} + 2K\varepsilon e^{2L(t-t_0)} > g\big(t,w_\varepsilon(t)\big),$$

that is,

$$\big[w_\varepsilon(t) - f\big(t,w_\varepsilon(t)\big)\big]^{\Delta} > g\big(t,w_\varepsilon(t)\big)$$

for all $t\in J$. Also, we get $w_\varepsilon(t_0) > w(t_0) > v(t_0)$. Hence Theorem 3.1 with $w = w_\varepsilon$ implies that $v(t) < w_\varepsilon(t)$ for all $t\in J$. By the arbitrariness of $\varepsilon > 0$, taking the limits as $\varepsilon \to 0$, we have $v(t) \leq w(t)$ for all $t \in J$. □

4 Existence of maximal and minimal solutions

In this section, we prove the existence of maximal and minimal solutions for the DETS (1) on $J = [t_0, t_0 + a]_{\mathbb{T}}$.

Definition 4.1 A solution r of the DETS (1) is said to be maximal if for any other solution u to the DETS (1), we have $u(t) \leq r(t)$ for all $t \in J$. Similarly, a solution ρ of the DETS (1) is said to be minimal if $\rho(t) \leq u(t)$ for all $t \in J$, where u is any solution of the DETS (1) on J.

We discuss the case of maximal solution only. Similarly, the case of minimal solution can be obtained with the same arguments with appropriate modifications. Given an arbitrary small number $\varepsilon > 0$, we discuss the following initial value problem of DETS:

$$\begin{cases} [\frac{u(t)-k(t,u(t))}{f(t,u(t))}]^{\Delta} = g(t,u(t)) + \varepsilon, \quad t \in J, \\ u(t_0) = u_0 + \varepsilon, \end{cases} \tag{14}$$

where $f \in C_{\mathrm{rd}}(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $k, g \in C_{\mathrm{rd}}(J \times \mathbb{R}, \mathbb{R})$.

An existence theorem for the DETS (14) can be stated as follows.

Theorem 4.1 *Suppose that* (A_0)–(A_2) *and inequality* (5) *hold. Then for every small number* $\varepsilon > 0$, *the DETS* (14) *has a solution defined on* J.

Proof By hypothesis, since

$$L_1 \left(\left| \frac{u_0 - k(t_0, u_0)}{f(t_0, u_0)} \right| + \|h\|_{L^1} \right) + L_2 < 1,$$

there exists $\varepsilon_0 > 0$ such that

$$L_1 \left(\left| \frac{u_0 + \varepsilon - k(t_0, u_0 + \varepsilon)}{f(t_0, u_0 + \varepsilon)} \right| + \|h\|_{L^1} + \varepsilon a \right) + L_2 < 1$$

for all $0 < \varepsilon \leq \varepsilon_0$. The rest of the proof is similar to that of Theorem 2.1, and we omit it. □

Our main existence theorem for maximal solution for the DETS (1) is the following:

Theorem 4.2 *Suppose that* (A_0)–(A_2) *and inequality* (5) *hold. Then the DETS* (1) *has a maximal solution defined on* J.

Proof Let $\{\varepsilon_n\}_0^{\infty}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, where ε_0 is a positive number satisfying the inequality

$$L_1 \left(\left| \frac{u_0 + \varepsilon_0 - k(t_0, u_0 + \varepsilon_0)}{f(t_0, u_0 + \varepsilon_0)} \right| + \|h\|_{L^1} + \varepsilon_0 a \right) + L_2 < 1.$$

Such a number ε_0 exists in view of inequality (5). Then for any solution x of the DETS (1), by Theorem 4.1 we get

$$x(t) < r(t, \varepsilon_n)$$

for all $t \in J$ and $n \in \mathbb{N} \cup \{0\}$, where $r(t, \varepsilon_n)$ defined on J is a solution of the DETS

$$\begin{cases} [\frac{u(t)-k(t,u(t))}{f(t,u(t))}]^{\Delta} = g(t,u(t)) + \varepsilon_n, \quad t \in J, \\ u(t_0) = u_0 + \varepsilon_n. \end{cases} \tag{15}$$

By Theorem 3.2, $\{r(t, \varepsilon_n)\}$ is a decreasing sequence of positive numbers, and thus the limit

$$r(t) = \lim_{n\to\infty} r(t, \varepsilon_n) \tag{16}$$

exists. We prove that the convergence in (16) is uniform on J. Next, we show that the sequence $\{r(t, \varepsilon_n)\}$ is equicontinuous in $C_{\mathrm{rd}}(J, \mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ be arbitrary. Then we have

$$\begin{aligned}
&\bigl|r(t_1, \varepsilon_n) - r(t_2, \varepsilon_n)\bigr| \\
&\quad = \biggl|\bigl[f\bigl(t_1, r(t_1, \varepsilon_n)\bigr)\bigr]\biggl(\frac{u_0 + \varepsilon_n - k(t_0, u_0 + \varepsilon_n)}{f(t_0, u_0 + \varepsilon_n)} + \int_{t_0}^{t_1} g\bigl(s, r(s, \varepsilon_n)\bigr)\Delta s \\
&\qquad + \int_{t_0}^{t_1} \varepsilon_n \Delta s\biggr) + k\bigl(t_1, r(t_1, \varepsilon_n)\bigr) - k\bigl(t_2, r(t_2, \varepsilon_n)\bigr) - \bigl[f\bigl(t_2, r(t_2, \varepsilon_n)\bigr)\bigr] \\
&\qquad \cdot \biggl(\frac{u_0 + \varepsilon_n - k(t_0, u_0 + \varepsilon_n)}{f(t_0, u_0 + \varepsilon_n)} + \int_{t_0}^{t_2} g\bigl(s, r(s, \varepsilon_n)\bigr)\Delta s + \int_{t_0}^{t_2} \varepsilon_n \Delta s\biggr)\biggr| \\
&\quad \le \bigl|k\bigl(t_1, r(t_1, \varepsilon_n)\bigr) - k\bigl(t_2, r(t_2, \varepsilon_n)\bigr)\bigr| + \biggl|\bigl[f\bigl(t_1, r(t_1, \varepsilon_n)\bigr)\bigr] \\
&\qquad \cdot \biggl(\frac{u_0 + \varepsilon_n - k(t_0, u_0 + \varepsilon_n)}{f(t_0, u_0 + \varepsilon_n)} + \int_{t_0}^{t_1} g\bigl(s, r(s, \varepsilon_n)\bigr)\Delta s + \int_{t_0}^{t_1} \varepsilon_n \Delta s\biggr) \\
&\qquad - \bigl[f\bigl(t_2, r(t_2, \varepsilon_n)\bigr)\bigr]\biggl(\frac{u_0 + \varepsilon_n - k(t_0, u_0 + \varepsilon_n)}{f(t_0, u_0 + \varepsilon_n)} + \int_{t_0}^{t_2} g\bigl(s, r(s, \varepsilon_n)\bigr)\Delta s \\
&\qquad + \int_{t_0}^{t_2} \varepsilon_n \Delta s\biggr)\biggr| \\
&\quad \le \bigl|k\bigl(t_1, r(t_1, \varepsilon_n)\bigr) - k\bigl(t_2, r(t_2, \varepsilon_n)\bigr)\bigr| + \biggl|\bigl[f\bigl(t_1, r(t_1, \varepsilon_n)\bigr)\bigr] \\
&\qquad \cdot \biggl(\frac{u_0 + \varepsilon_n - k(t_0, u_0 + \varepsilon_n)}{f(t_0, u_0 + \varepsilon_n)} + \int_{t_0}^{t_1} g\bigl(s, r(s, \varepsilon_n)\bigr)\Delta s + \int_{t_0}^{t_1} \varepsilon_n \Delta s\biggr) \\
&\qquad - \bigl[f\bigl(t_2, r(t_2, \varepsilon_n)\bigr)\bigr]\biggl(\frac{u_0 + \varepsilon_n - k(t_0, u_0 + \varepsilon_n)}{f(t_0, u_0 + \varepsilon_n)} + \int_{t_0}^{t_1} g\bigl(s, r(s, \varepsilon_n)\bigr)\Delta s \\
&\qquad + \int_{t_0}^{t_1} \varepsilon_n \Delta s\biggr)\biggr| + \biggl|\bigl[f\bigl(t_2, r(t_2, \varepsilon_n)\bigr)\bigr]\biggl(\frac{u_0 + \varepsilon_n - k(t_0, u_0 + \varepsilon_n)}{f(t_0, u_0 + \varepsilon_n)} \\
&\qquad + \int_{t_0}^{t_1} g\bigl(s, r(s, \varepsilon_n)\bigr)\Delta s + \int_{t_0}^{t_1} \varepsilon_n \Delta s\biggr) - \bigl[f\bigl(t_2, r(t_2, \varepsilon_n)\bigr)\bigr] \\
&\qquad \cdot \biggl(\frac{u_0 + \varepsilon_n - k(t_0, u_0 + \varepsilon_n)}{f(t_0, u_0 + \varepsilon_n)} + \int_{t_0}^{t_2} g\bigl(s, r(s, \varepsilon_n)\bigr)\Delta s + \int_{t_0}^{t_2} \varepsilon_n \Delta s\biggr)\biggr| \\
&\quad \le \bigl|k\bigl(t_1, r(t_1, \varepsilon_n)\bigr) - k\bigl(t_2, r(t_2, \varepsilon_n)\bigr)\bigr| + \bigl|f\bigl(t_1, r(t_1, \varepsilon_n)\bigr) - f\bigl(t_2, r(t_2, \varepsilon_n)\bigr)
\end{aligned}$$

$$\cdot \left|\left(\left|\frac{u_0+\varepsilon_n-k(t_0,u_0+\varepsilon_n)}{f(t_0,u_0+\varepsilon_n)}\right|+\|h\|_{L^1}+\varepsilon_n a\right)\right.$$
$$+F\left[\left|p(t_1)-p(t_2)\right|+|t_1-t_2|\varepsilon_n\right],$$

where $F=\sup_{(t,u)\in J\times[-N,N]}|f(t,u)|$ and $p(t)=\int_{t_0}^{t} h(s)\Delta s$.

Since f and k are continuous on the compact set $J\times[-N,N]$, they are uniformly continuous. Hence

$$\left|f\big(t_1,r(t_1,\varepsilon_n)\big)-f\big(t_2,r(t_2,\varepsilon_n)\big)\right|\to 0 \quad \text{as } t_1\to t_2$$

and

$$\left|k\big(t_1,r(t_1,\varepsilon_n)\big)-k\big(t_2,r(t_2,\varepsilon_n)\big)\right|\to 0 \quad \text{as } t_1\to t_2$$

uniformly for all $n\in\mathbb{N}$. Similarly, since the function p is continuous on the compact set J, it is uniformly continuous, and hence

$$\left|p(t_1)-p(t_2)\right|\to 0 \quad \text{as } t_1\to t_2.$$

Therefore we obtain

$$\left|r(t_1,\varepsilon_n)-r(t_2,\varepsilon_n)\right|\to 0 \quad \text{as } t_1\to t_2$$

uniformly for all $n\in\mathbb{N}$. Therefore

$$r(t,\varepsilon_n)\to r(t) \quad \text{as } n\to\infty$$

for all $t\in J$.

Next, we prove that the function $r(t)$ is a solution of the DETS (1) defined on J. Since $r(t,\varepsilon_n)$ is a solution of the DETS (15), we get

$$\begin{aligned} r(t,\varepsilon_n) &= \left[f\big(t,r(t,\varepsilon_n)\big)\right]\left(\frac{u_0+\varepsilon_n-k(t_0,u_0+\varepsilon_n)}{f(t_0,u_0+\varepsilon_n)}+\int_{t_0}^{t} g\big(s,r(s,\varepsilon_n)\big)\Delta s\right. \\ &\quad \left.+\int_{t_0}^{t}\varepsilon_n\Delta s\right)+k\big(t,r(t,\varepsilon_n)\big) \end{aligned} \tag{17}$$

for all $t\in J$. Taking the limit as $n\to\infty$ in (17) implies

$$r(t)=\left[f\big(t,r(t)\big)\right]\left(\frac{u_0-k(t_0,u_0)}{f(t_0,u_0)}+\int_{t_0}^{t} g\big(s,r(s)\big)\Delta s\right)+k\big(t,r(t)\big)$$

for all $t\in J$. Thus the function r is a solution of the DETS (1) on J. Finally, from inequality (15) it follows that $x(t)\le r(t)$ for all $t\in J$. Hence the DETS (1) has a maximal solution on J. □

5 Comparison theorems on time scales

The main problem of the DITS is to estimate a bound for the solution set for the DITS related to the DETS (1). In this section, we present the maximal and minimal solutions serving as bounds for the solutions of the related DITS to the DETS (1) on $J = [t_0, t_0 + a]_{\mathbb{T}}$.

Theorem 5.1 *Suppose that* (A_0)–(A_2) *and inequality* (5) *hold. Assume that there exists a* Δ*-differentiable function u such that*

$$\begin{cases} [\frac{x(t)-k(t,x(t))}{f(t,x(t))}]^{\Delta} \leq g(t,x(t)), \quad t \in J, \\ x(t_0) \leq u_0. \end{cases} \tag{18}$$

Then

$$x(t) \leq r(t) \tag{19}$$

for all $t \in J$, *where r is a maximal solution of the the DETS* (1) *on* J.

Proof Let $\varepsilon > 0$ be arbitrarily small. By Theorem 4.2, $r(t,\varepsilon)$ is a maximal solution of the DETS (14), the limit

$$r(t) = \lim_{\varepsilon \to 0} r(t,\varepsilon) \tag{20}$$

is uniform on J, and the function r is a maximal solution of the DETS (1) on J. Hence we have

$$\begin{cases} [\frac{r(t,\varepsilon)-k(t,r(t,\varepsilon))}{f(t,r(t,\varepsilon))}]^{\Delta} = g(t,r(t,\varepsilon)) + \varepsilon, \quad t \in J, \\ r(t_0,\varepsilon) = u_0 + \varepsilon. \end{cases}$$

By the above inequality it implies that

$$\begin{cases} [\frac{r(t,\varepsilon)-k(t,r(t,\varepsilon))}{f(t,r(t,\varepsilon))}]^{\Delta} > g(t,r(t,\varepsilon)), \quad t \in J, \\ r(t_0,\varepsilon) > u_0. \end{cases} \tag{21}$$

Now, applying Theorem 3.2 to inequalities (18) and (21), we conclude that $x(t) < r(t,\varepsilon)$ for all $t \in J$. Thus (20) implies that inequality (19) holds on J. □

Theorem 5.2 *Suppose that* (A_0)–(A_2) *and inequality* (5) *hold. Assume that there exists a* Δ*-differentiable function u such that*

$$\begin{cases} [\frac{y(t)-k(t,y(t))}{f(t,y(t))}]^{\Delta} \geq g(t,y(t)), \quad t \in J, \\ y(t_0) \geq u_0. \end{cases}$$

Then

$$\rho(t) \leq y(t)$$

for all $t \in J$, *where* ρ *is a minimal solution of the DETS* (1) *on* J.

Note that Theorem 5.1 is useful to prove the boundedness and uniqueness of the solutions for the DETS (1) on J. We have a following result.

Theorem 5.3 *Suppose that* (A_0)–(A_2) *and inequality* (5) *hold. Assume that there exists a function* $G : J \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$g(t,u_1) - g(t,u_2) \le G\left(t, \max_{s\in[t_0,t]}\left|\frac{u_1(s) - k(s,u_1(s))}{f(s,u_1(s))} - \frac{u_2(s) - k(s,u_2(s))}{f(s,u_2(s))}\right|\right), \quad t \in J,$$

for all $u_1, u_2 \in \mathbb{R}$ *with* $u_1 \ge u_2$. *If the identically zero function is the only solution of the differential equation*

$$m^{\Delta}(t) = G\big(t, m(t)\big), \quad t \in J, \qquad m(t_0) = 0, \tag{22}$$

then the DETS (1) *has a unique solution on* J.

Proof By Theorem 2.1 the DETS (1) has a solution defined on J. Suppose that there are two solutions x_1 and x_2 of the DETS (1) existing on J with $x_1 > x_2$. Define $m : J \to \mathbb{R}^+$ by

$$m(t) = \left|\frac{u_1(t) - k(t,u_1(t))}{f(t,u_1(t))} - \frac{u_2(t) - k(t,u_2(t))}{f(t,u_2(t))}\right|.$$

Since $(|x(t)|)^{\Delta} \le |x^{\Delta}(t)|$, we obtain that

$$\begin{aligned} m^{\Delta}(t) &\le \left|\left[\frac{u_1(t) - k(t,u_1(t))}{f(t,u_1(t))}\right]^{\Delta} - \left[\frac{u_2(t) - k(t,u_2(t))}{f(t,u_2(t))}\right]^{\Delta}\right| \\ &= \big|g(t,x_1) - g(t,x_2)\big| \\ &\le G\left(t, \left|\frac{u_1(t) - k(t,u_1(t))}{f(t,u_1(t))} - \frac{u_2(t) - k(t,u_2(t))}{f(t,u_2(t))}\right|\right) \\ &= G\big(t, m(t)\big) \end{aligned}$$

for $t \in J$ and $m(t_0) = 0$.

Now we apply Theorem 5.1 with $k(t,u) \equiv 0$ and $f(t,u) \equiv 1$ to get that $m(t) \le 0$ for all $t \in J$, where the identically zero function is the only solution of the DETS (22), which is a contradiction with $m(t) > 0$. This implies

$$\frac{u_1(t) - k(t,u_1(t))}{f(t,u_1(t))} = \frac{u_2(t) - k(t,u_2(t))}{f(t,u_2(t))}$$

for all $t \in J$. Then we have $x_1 = x_2$. □

Remark 5.1 When $k \equiv 0$, $f \equiv 1$, and $\mathbb{T} = \mathbb{R}$ in our results, we obtain the differential inequalities and other related results of Lakshmikantam and Leela [17] for the IVP of ordinary nonlinear differential equation

$$u'(t) = g\big(t, u(t)\big), \quad t \in [t_0, t_0 + a], \qquad u(t_0) = u_0.$$

Remark 5.2 The main results in this paper extend and improve some well-known results in [13] when $\mathbb{T} = \mathbb{R}$.

6 Conclusion

In this paper, we have developed the theory of the DETS (1). By the fixed point theorem in Banach algebra due to Dhage we have presented an existence theorem for the DETS (1) under Lipschitz conditions. We have also established some DITS for the DETS (1), which are used to investigate the existence of extremal solutions, and the comparison principle for the DETS (1). Our results in this paper extend and improve some well-known results.

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Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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