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Fractional Halanay inequality of order between one and two and application to neural network systems

Nasser-Eddine Tatar[1*]

*Correspondence:
tatarn@kfupm.edu.sa
[1]Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia

Abstract
We extend the (integer-order) Halanay inequality with distributed delay to the fractional-order case between one and two. The main feature is the passage from integer order to noninteger order between one and two. This case of order between one and two is more delicate than the case between zero and one because of several difficulties explained in this paper. These difficulties are encountered, in fact, in general differential equations. Here we show that solutions decay to zero as a power function in case the delay kernel satisfies a general (integral) condition. We provide a large class of admissible functions fulfilling this condition. The even more complicated nonlinear case is also addressed, and we obtain a local stability result of power type. Finally, we give an application to a problem arising in neural network theory and an explicit example.

Keywords: Hopfield neural network; Power-type stability; Caputo fractional derivative; fractional Halanay inequality

1 Introduction

The Halanay inequality is one of the most important inequalities used to prove the boundedness or stability of solutions of some functional differential equations. It contains a dissipative term, which tends to stabilize the system in an exponential manner, and a delayed term, which, on the contrary, usually has a destructive character. It is proved that when the dissipation coefficient dominates the discrete delayed term coefficient, then we get an exponential decay. Namely, we have the following [11]:

Lemma 1 *Assume that $w(t)$ is a nonnegative solution of*

$$w'(t) \le -Aw(t) + B \sup_{t-\tau \le s \le t} w(s), \quad \tau > 0, t \ge a.$$

If $0 < B < A$, then there exist $M > 0$ and $\alpha > 0$ such that

$$w(t) \le Me^{-\alpha(t-a)}, \quad t \ge a.$$

This inequality has been used in many engineering applications and extended to the variable delay and distributed delay cases [3, 13, 28, 32, 33, 38]:

$$w'(t) \leq -A(t)w(t) + B(t)\int_0^\infty k(s)w(t-s)\,ds, \quad t \geq 0.$$

It has been proved that solutions decay exponentially for kernels satisfying

$$\int_0^\infty e^{\beta s}k(s)\,ds < \infty$$

for some $\beta > 0$, provided that

$$B(t)\int_0^\infty k(s)\,ds \leq A(t) - b, \quad b > 0,\ t \geq 0.$$

Artificial neural networks (ANNs) are one of the many products of artificial intelligence. They have been applied successfully in many areas such as combinatorial optimization, cryptography, parallel computing, signal theory, image processing, biological, biomedical, medical (epidemiology), polymer composite, and geology [10, 12, 14–17, 20, 21, 27, 29, 36, 40]. In particular, in petroleum engineering, the characterization of a hydrocarbon reservoir depends on many static and dynamic parameters such as permeability, porosity, fluid saturation, and pressure in the reservoir. The lack of accuracy or the unavailability of certain parameters affect negatively the oil production performance. Unlike the existing conventional ways, ANNs have the ability of connecting input data to output without imposing a prior understanding of the fluid flow or the medium. They are also robust enough to deal with noisy, distorted, fuzzy, and even incomplete data [1, 4, 19, 31].

For material and processes that exhibit memory and hereditary effects, it has been shown that fractional derivatives describe better the phenomena [2, 5–8, 23].

Most of the existing results are concerned with the case of a fractional order between 0 and 1 and for the case of discrete delays only. Unfortunately, the arguments there do not work for the present case. For general fractional systems of order between zero and one, several stability results (including the Mittag–Leffler stability) have been obtained with explicit decay rates [7, 8, 13, 23–26, 35–37, 39, 43].

The stability for the linear system

$$D^\alpha x(t) = Ax(t), \quad t > t_0,$$

with $1 < \alpha < 2$, has been treated in [23, 42]. The stability in the cases of Riemann–Liouville and Caputo fractional derivatives has been established under the condition $|\arg(\mathrm{spec}(A))| > \alpha\pi/2$. In fact, the stability is of type $t^{-\alpha-1}$ in the case of Riemann–Liouville fractional derivative and of type $t^{-\alpha+1}$ in the case of Caputo fractional derivative.

For the equation

$$D^\alpha x(t) = Ax(t) + B(t)x(t), \quad t > t_0,$$

the zero solution is proved to be stable [41] if, in addition,

$$\int_{t_0}^\infty \|B(t)\|\,dt$$

is bounded, in case of both fractional derivatives. The stability is asymptotic if $\|B(t)\| = O(t-t_0)^\gamma$ or is bounded ($-1 < \gamma < 1-\alpha$). The authors in [36] assume that $\|B(t)\|$ is nondecreasing and $B(t) = O(t-t_0)^\theta$ ($\theta < -\alpha$).

The perturbed equation

$$D^\alpha x(t) = Ax(t) + f\bigl(x(t)\bigr), \quad t > t_0,$$

has been studied in [8, 24, 42], where asymptotic stability results are proved if

$$\lim_{\|x\|\to 0} \frac{\|f(x(t))\|}{\|x(t)\|} = 0, \quad t \geq t_0, \tag{1}$$

in addition to a condition on the spectrum of A.

We withdraw the attention of the reader to the work in [22], where the authors discussed a similar (control) problem and proved a "global" asymptotic stability result after noticing that the previous results were of "local" character because of condition (1).

The nonautonomous system

$$D^\alpha x(t) = Ax(t) + f\bigl(t, x(t)\bigr), \quad t > t_0,$$

has been the subject of investigation in [18, 30, 42]. Asymptotic stability results have been established under the following conditions: $f(t,x(t))$ is Lipschitz continuous, $\|f(t,x(t))\| \leq \gamma(t)\|x(t)\|$ with bounded $\int_{t_0}^\infty \gamma(t)\,dt$, and

$$\lim_{\|x\|\to 0} \frac{\|f(t,x(t))\|}{\|x(t)\|} = 0, \quad t \geq t_0.$$

Because of the size of the paper and our exclusive concern on the case $1 < \alpha < 2$, several references on the case $0 < \alpha < 1$ have not been reported here. We note here that the previously used arguments for the case $0 < \alpha < 1$ are not valid for $1 < \alpha < 2$. In particular, the use of the "one-sided" chain rule formula for fractional derivatives leads to uncontrollable terms and seems useless. We opted for the variation of parameters formula, but even in this framework, we faced considerable difficulties. The main difficulties were related to the sign of the involved Mittag–Leffler functions and also to the uniform boundedness of a convolution term. The formulas and properties found in the literature were not able to solve these difficulties. Then we have been forced to prove a new integral inequality, which may be useful in other contexts as well.

Our objective here is two-fold: we extend the distributed Halanay inequality from the integer-order case to the fractional-order case ($1 < \alpha < 2$) and from the linear case to the nonlinear case. We impose a general condition on the kernels and provide a class of admissible kernels, as an example, showing that this condition can be met. The decay we find is of power type. Once established, our results will be applied to a fractional neural network system of Hopfield type. Namely, we consider (discrete and distributed delayed) systems of the form

$$\begin{cases} D_C^\alpha x_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} g_j(x_j(t-\tau)) \\ \qquad\qquad + \sum_{j=1}^n d_{ij} \int_0^\infty k_j(s) h_j(x_j(t-s))\,ds + I_i, \quad t > 0, \\ x_i(t) = \chi_i(t), \quad t \leq 0, \end{cases}$$

for $i = 1, 2, \ldots, n$, $0 < \alpha < 1$, where

n	is the number of units in the network,
x_i	is the state of the ith neuron at time t,
$c_i > 0$	are the passive delay rates,
a_{ij}, b_{ij}, d_{ij}	are the connection weight matrices,
I_i	are external constant inputs,
f_j, g_j, h_j	are the signal transmission functions (activation functions),
k_j	is the delay feedback (delay kernel function),
$\tau > 0$	is the transmission delay, and
χ_i	is the prehistory of the ith state.

Our argument is flexible and may be applied to more general systems than this one. The next section contains some preliminaries. In Sect. 3, we extend the Halanay inequality to the order $1 < \alpha < 2$ and provide a large class of kernels for which our result applies. The nonlinear case is treated in Sect. 4. An application to a problem arising in neural network theory is given in Sect. 5 together with a numerical example.

2 Preliminaries

In this section, we give the definitions of the fractional integral and fractional derivative (of Riemann–Liouville and Caputo types) and the Mittag–Leffler functions.

Definition 2 The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad \alpha > 0,$$

for any measurable function f, provided that the right-hand side exists. Here $\Gamma(\alpha)$ is the usual gamma function.

Definition 3 The fractional derivative of order α in the sense of Caputo is defined by

$$D_C^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-\tau)^{n-\gamma-1} f^{(n)}(\tau)\, d\tau, \quad n = [\gamma] + 1, \gamma > 0,$$

whereas the fractional derivative of order α in the sense of Riemann–Liouville is defined by

$$D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\gamma-1} f(\tau)\, d\tau, \quad n = [\gamma] + 1, \gamma > 0,$$

provided that the integrals exist.

The one-parametric and two-parametric Mittag–Leffler functions $E_{\alpha}(z)$ and $E_{\alpha,\beta}(z)$ are defined by

$$E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \Re(\alpha) > 0,$$

and

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0,$$

respectively.

3 Fractional distributed Halanay inequality

Here we extend the standard (integer-order) Halanay inequality to the fractional case $1 < \alpha < 2$. We prove that the decay is of power type. Part of the difficulties encountered here is due to the fact that the properties of the Mittag–Leffler functions for $1 < \alpha < 2$ are different from those for $0 < \alpha < 1$, and therefore the methods used in the case $0 < \alpha < 1$ are not applicable anymore.

Theorem 4 *Let $u(t)$ be a nonnegative solution of*

$$\begin{cases} D_C^{\alpha} u(t) \le -au(t) + \int_0^t k(t-s)u(s)\,ds, & 1 < \alpha < 2,\ t > 0, \\ u(0) = u_0, \qquad u'(0) = u_1, \end{cases} \tag{2}$$

where $a > 0$, and k is a nonnegative summable function satisfying

$$t^{\alpha-1} \int_0^t (t-s)^{\alpha-1} \left| E_{\alpha,\alpha}\left(-a(t-s)^{\alpha}\right)\right| \left(\int_0^s k(s-\sigma)\sigma^{1-\alpha}\,d\sigma \right) ds < 1, \quad t > 0. \tag{3}$$

Then there exists a positive constant A such that

$$u(t) \le A/t^{\alpha-1}, \quad t > 0.$$

Proof We compare solutions of (2) to those of

$$\begin{cases} D_C^{\alpha} w(t) = -aw(t) + \int_0^t k(t-s)w(s)\,ds, & 1 < \alpha < 2,\ t > 0, \\ w(0) = w_0 = u_0, \qquad w'(0) = w_1 = u_1. \end{cases} \tag{4}$$

Applying the Laplace transform to (4), we obtain the variation-of-parameters formula (see [42] and [43])

$$\begin{aligned} w(t) &= E_{\alpha}\left(-at^{\alpha}\right)w_0 + tE_{\alpha,2}\left(-at^{\alpha}\right)w_1 \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-a(t-s)^{\alpha}\right) \left(\int_0^s k(s-\sigma)w(\sigma)\,d\sigma \right) ds, \quad t \ge 0. \end{aligned}$$

In view of the boundedness of $E_{\alpha,\beta}(-at^{\alpha})$, $0 < \alpha < 2$, $\beta > 0$, $a \ge 0$, $t \ge 0$ ([34, Thms. 1.4 and 1.6, pp. 33, 34]),

$$\left| E_{\alpha,\beta}\left(-at^{\alpha}\right)\right| \le M(\alpha,\beta)/at^{\alpha} \tag{5}$$

for some $M(\alpha,\beta) > 0$, and we may write

$$w(t) \le \left| E_{\alpha}\left(-at^{\alpha}\right)\right| w_0 + M_1(\alpha,a)t^{1-\alpha}|w_1|$$

$$+\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right|\left(\int_0^s k(s-\sigma)w(\sigma)\,d\sigma\right)ds,\quad t\geq 0,$$

or

$$\begin{aligned} &t^{\alpha-1}w(t) \\ &\quad\leq t^{\alpha-1}\left|E_\alpha\left(-at^\alpha\right)\right|w_0 + M_1(\alpha,a)|w_1| \\ &\qquad + t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right|\left(\int_0^s k(s-\sigma)w(\sigma)\,d\sigma\right)ds,\quad t\geq 0, \end{aligned} \tag{6}$$

where $M_1(\alpha,a) = M(\alpha,1)/a$ is coming from (5). Multiplying by $\sigma^{1-\alpha}\sigma^{\alpha-1}$ inside the inner integral in (6),

$$\begin{aligned} &t^{\alpha-1}w(t) \\ &\quad\leq t^{\alpha-1}\left|E_\alpha\left(-at^\alpha\right)\right|w_0 + M_1(\alpha,a)|w_1| \\ &\qquad + t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right|\left(\int_0^s k(s-\sigma)\sigma^{1-\alpha}\sigma^{\alpha-1}w(\sigma)\,d\sigma\right)ds,\quad t\geq 0 \end{aligned}$$

and taking the supremum, we find

$$\begin{aligned} &t^{\alpha-1}w(t) \\ &\quad\leq t^{\alpha-1}\left|E_\alpha\left(-at^\alpha\right)\right|w_0 + M_1(\alpha,a)|w_1| \\ &\qquad + t^{\alpha-1}\phi(t)\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right|\left(\int_0^s k(s-\sigma)\sigma^{1-\alpha}\,d\sigma\right)ds,\quad t\geq 0, \end{aligned} \tag{7}$$

where $\phi(t) := \sup_{0\leq\sigma\leq t}\sigma^{\alpha-1}w(\sigma)$. The expression $t^{\alpha-1}|E_\alpha(-at^\alpha)|$ is uniformly bounded (by $C_1 > 0$) nearby zero as $E_\alpha(-at^\alpha)$ is itself bounded, and it is also bounded far away from zero as $|E_\alpha(-at^\alpha)|$ is decaying as $t^{-\alpha}$ (see [34, 39]).

Assuming that

$$t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right|\left(\int_0^s k(s-\sigma)\sigma^{1-\alpha}\,d\sigma\right)ds \leq B < 1,$$

it follows from (7) that

$$t^{\alpha-1}w(t) \leq C_1 w_0 + M_1(\alpha,a)|w_1| + B\phi(t),\quad t > 0. \tag{8}$$

Then, taking supremum in (8), we find

$$(1-B)\phi(t) \leq C_1 w_0 + M_1(\alpha,a)|w_1|,\quad t > 0,$$

or

$$w(t) \leq \frac{C_1 w_0 + M_1(\alpha,a)|w_1|}{(1-B)t^{\alpha-1}},\quad t > 0.$$

This completes the proof. □

Lemma 5 *If $\nu \in C$ satisfies $\frac{\alpha\pi}{2} < |\arg(\nu)| \leq \pi$, then there exists a constant $A(\alpha,\nu) > 0$ (independent of t) such that*

$$\int_0^t \left|(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\nu(t-s)^\alpha\right)\right| ds < A(\alpha,\nu), \quad \forall t > 0.$$

Proof This lemma is proved in [9] when $0 < \alpha < 1$. The case $1 < \alpha < 2$ may be proved similarly. □

A class of admissible kernels. Condition (3) may be simplified considerably to

$$\int_0^t k(t-\sigma)\sigma^{1-\alpha}\, d\sigma \leq Ct^{1-\alpha}, \quad t \geq 0, \tag{9}$$

for some $C > 0$. To see this, we prove the following lemma.

Lemma 6 *For $1 < \alpha < 2$, we have*

$$t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right| s^{1-\alpha}\, ds \leq D, \quad a > 0, t \geq 0,$$

for some $D > 0$.

Proof Clearly,

$$\begin{aligned}
&t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right| s^{1-\alpha}\, ds \\
&\quad = t^{\alpha-1}\int_0^t (t-s)^{1-\alpha} s^{\alpha-1} |E_{\alpha,\alpha}(-as)^\alpha)|\, ds \\
&\quad = \int_0^1 (1-\xi)^{1-\alpha}\xi^{\alpha-1} t^{\alpha-1}\left|E_{\alpha,\alpha}\left(-at^\alpha\xi^\alpha\right)\right| t\, d\xi \\
&\quad = t^\alpha \int_0^1 (1-\xi)^{1-\alpha}\xi^{\alpha-1}\left|E_{\alpha,\alpha}\left(-at^\alpha\xi^\alpha\right)\right| d\xi, \quad t > 0,
\end{aligned}$$

where $\xi := s/t$ and $ds = t\, d\xi$. For $0 \leq \xi < 1/2$, we have

$$\begin{aligned}
&t^\alpha \int_0^{1/2} (1-\xi)^{1-\alpha}\xi^{\alpha-1}\left|E_{\alpha,\alpha}\left(-at^\alpha\xi^\alpha\right)\right| d\xi \\
&\quad \leq \max\left(1,2^{\alpha-1}\right)t^\alpha \int_0^{1/2} \xi^{\alpha-1}\left|E_{\alpha,\alpha}\left(-at^\alpha\xi^\alpha\right)\right| d\xi \\
&\quad \leq \max\left(1,2^{\alpha-1}\right)t \int_0^{1/2} (t\xi)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-at^\alpha\xi^\alpha\right)\right| d\xi,
\end{aligned}$$

and putting $\sigma := t\xi$ and $d\sigma := t\, d\xi$, we see that

$$\begin{aligned}
&t^\alpha \int_0^{1/2} (1-\xi)^{1-\alpha}\xi^{\alpha-1}\left|E_{\alpha,\alpha}\left(-at^\alpha\xi^\alpha\right)\right| d\xi \\
&\quad \leq \max\left(1,2^{\alpha-1}\right)t^\alpha \int_0^{1/2} \xi^{\alpha-1}\left|E_{\alpha,\alpha}\left(-at^\alpha\xi^\alpha\right)\right| d\xi
\end{aligned}$$

$$\leq \max\bigl(1, 2^{\alpha-1}\bigr) \int_0^{t/2} \sigma^{\alpha-1} \bigl|E_{\alpha,\alpha}\bigl(-a\sigma^\alpha\bigr)\bigr|\, d\sigma. \tag{10}$$

This last expression in (10) is bounded by Lemma 5.

For $1/2 \leq \xi < 1$, it is clear that

$$\begin{aligned} &t^\alpha \int_{1/2}^1 (1-\xi)^{1-\alpha} \xi^{\alpha-1} \bigl|E_{\alpha,\alpha}\bigl(-at^\alpha \xi^\alpha\bigr)\bigr|\, d\xi \\ &\quad \leq 2 \int_0^{1/2} (1-\xi)^{1-\alpha} t^\alpha \xi^\alpha \bigl|E_{\alpha,\alpha}\bigl(-at^\alpha \xi^\alpha\bigr)\bigr|\, d\xi, \end{aligned}$$

and, as the expression $t^\alpha \xi^\alpha |E_{\alpha,\alpha}(-at^\alpha \xi^\alpha)|$ is bounded by $M(\alpha,\alpha)/a$ (see (5)), we find

$$\begin{aligned} &t^\alpha \int_{1/2}^1 (1-\xi)^{1-\alpha} \xi^{\alpha-1} \bigl|E_{\alpha,\alpha}\bigl(-at^\alpha \xi^\alpha\bigr)\bigr|\, d\xi \\ &\quad \leq 2 \frac{M(\alpha,\alpha)}{a} \int_{1/2}^1 (1-\xi)^{1-\alpha}\, d\xi = \frac{2^{\alpha-1} M(\alpha,\alpha)}{(2-\alpha)a}. \end{aligned}$$

The lemma is proved. □

This lemma also gives us an idea about a class of kernels satisfying (9).

Example 7 Consider the class of nonnegative summable functions satisfying $0 \leq k(t) \leq C_2 t^{\alpha-1} |E_{\alpha,\alpha}(-bt^\alpha)|$ with C_2 and $b > 0$. This class encompasses, of course, the well-known class of kernels $k(t) = C_2 t^{\alpha-1} e^{-bt}$ used frequently in applications. By selecting appropriate constants C_2 and/or b we see that it satisfies all the requirements of the theorem.

4 Nonlinear case

Here we consider the nonlinear case. This case is not only important from the mathematical point of view, but it is also very useful in applications. In neural network theory, for instance, activation functions are usually assumed to be Lipschitz continuous, so that we can pass from the nonlinear case to the linear case and use the linear Halanay inequality. Therefore the present nonlinear case of Halanay inequality allows dealing with the non-Lipschitz case. The price to pay is that we obtain a local stability result.

The inequality of concern is

$$\begin{cases} D_C^\alpha u(t) \leq -au(t) + \int_0^t k(t-s) h(u(s))\, ds, & 1 < \alpha < 2, t > 0, \\ u(0) = u_0, \qquad u'(0) = u_1, \end{cases} \tag{11}$$

where h is a nonlinear function.

Theorem 8 *Assume that $u(t)$ is a solution of* (11), *$h(u) \leq u\tilde{h}(u)$ for some continuous nonnegative nondecreasing function $\tilde{h}(u)$, and $k(t)$ is a nonnegative summable function such that*

(i)

$$\int_0^t (t-s)^{\alpha-1} \bigl|E_{\alpha,\alpha}\bigl(-a(t-s)^\alpha\bigr)\bigr| \biggl(\int_0^s k(s-\sigma)\sigma^{1-\alpha}\, d\sigma\biggr)\, ds \leq B_1, \quad t > 0, \tag{12}$$

for some $B_1 > 0$ *and* $\varsigma > 0$ *such that* $B_1\tilde{h}(\varsigma) \le 1/2$, *and*

(ii)

$$\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right|\left(\int_0^s k(\sigma)\,d\sigma\right)ds \le B_2, \quad t > 0,$$

for some $B_2 > 0$ *and* $\xi > 0$ *such that* $B_2\tilde{h}(\xi) \le 1/2$.

Then

$$\left|u(t)\right| \le C\left(|u_0| + |u_1|\right)t^{1-\alpha}, \quad t \ge 0,$$

for some positive constant C *and small initial data*.

Proof Let us compare solutions of (11) with those of

$$\begin{cases} D_C^\alpha w(t) = -aw(t) + \int_0^t k(t-s)h(w(s))\,ds, & 1 < \alpha < 2, t > 0, \\ w(0) = w_0 = u_0, \qquad w'(0) = w_1 = u_1. \end{cases} \tag{13}$$

The corresponding variation-of-parameters formula is

$$\begin{aligned} w(t) &= E_\alpha\left(-at^\alpha\right)w_0 + tE_{\alpha,2}\left(-at^\alpha\right)w_1 \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\left(\int_0^s k(s-\sigma)h\left(w(\sigma)\right)d\sigma\right)ds, \quad t \ge 0. \end{aligned} \tag{14}$$

Therefore from (5) and the assumption on h we have

$$\begin{aligned} \left|w(t)\right| &\le \left|E_\alpha\left(-at^\alpha\right)\right||w_0| + M_2(\alpha,a)t^{1-\alpha}|w_1| \\ &\quad + \int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right|\left(\int_0^s k(s-\sigma)\left|w(\sigma)\right|\tilde{h}\left(\left|w(\sigma)\right|\right)d\sigma\right)ds \end{aligned}$$

and

$$\begin{aligned} t^{\alpha-1}\left|w(t)\right| &\le t^{\alpha-1}E_\alpha\left(-at^\alpha\right)|w_0| + M_2(\alpha,a)|w_1| \\ &\quad + t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right| \\ &\quad \times \left(\int_0^s k(s-\sigma)\left|w(\sigma)\right|\tilde{h}\left(\left|w(\sigma)\right|\right)d\sigma\right)ds \end{aligned} \tag{15}$$

for $t \ge 0$. We multiply inside the inner integral in (15) by the expression $\sigma^{\alpha-1}\sigma^{1-\alpha}$:

$$\begin{aligned} t^{\alpha-1}\left|w(t)\right| &\le C_1(\alpha,a)|w_0| + M_2(\alpha,a)|w_1| \\ &\quad + t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right| \\ &\quad \times \left(\int_0^s k(s-\sigma)\sigma^{\alpha-1}\left|w(\sigma)\right|\sigma^{1-\alpha}\tilde{h}\left(\left|w(\sigma)\right|\right)d\sigma\right)ds. \end{aligned}$$

Clearly,

$$\begin{aligned} t^{\alpha-1}\left|w(t)\right| \le{}& C_1(\alpha,a)|w_0| + M_2(\alpha,a)|w_1| \\ &+ t^{\alpha-1}\phi(t)\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right| \\ &\times \left(\int_0^s k(s-\sigma)\sigma^{1-\alpha}\tilde{h}\left(\left|w(\sigma)\right|\right)\,d\sigma\right)ds, \quad t\ge 0, \end{aligned} \tag{16}$$

where

$$\phi(t) = \sup_{0\le\sigma\le t}\sigma^{\alpha-1}\left|w(\sigma)\right|, \quad t\ge 0.$$

If the initial data satisfy

$$C_1(\alpha,a)|w_0| + M_2(\alpha,a)|w_1| < \varsigma/4$$

and $|w(t)|\le\varsigma$ for all $0\le t\le\bar{t}$, then

$$\begin{aligned} &t^{\alpha-1}\left|w(t)\right| \\ &\quad\le C_1(\alpha,a)|w_0| + M_2(\alpha,a)|w_1| \\ &\qquad + t^{\alpha-1}\phi(t)\tilde{h}(\varsigma)\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right|\left(\int_0^s k(s-\sigma)\sigma^{1-\alpha}\,d\sigma\right)ds. \end{aligned} \tag{17}$$

Now if

$$\begin{aligned} &t^{\alpha-1}\tilde{h}(\varsigma)\int_0^t (t-s)^{\alpha-1}\left|E_{\alpha,\alpha}\left(-a(t-s)^\alpha\right)\right|\left(\int_0^s k(s-\sigma)\sigma^{1-\alpha}\,d\sigma\right)ds \\ &\quad\le B_1\tilde{h}(\varsigma)\le 1/2 \end{aligned} \tag{18}$$

for some $B_1>0$, then from (17) we deduce that

$$t^{\alpha-1}\left|w(t)\right| \le C_1(\alpha,a)|w_0| + M_2(\alpha,a)|w_1| + \frac{\phi(t)}{2}, \quad 0\le t\le\bar{t}, \tag{19}$$

and taking the supremum in (19), we get

$$\left|w(t)\right| \le 2\left(C_1(\alpha,a)|w_0| + M_2(\alpha,a)|w_1|\right)t^{1-\alpha}, \quad 0\le t\le\bar{t}. \tag{20}$$

The difficulty here is to make the process continue forever to get this last estimate (20) valid for all t.

If $\bar{t}\ge 1$, then

$$\left|w(\bar{t})\right| \le 2\left(C_1|w_0| + M|w_1|/a\right) < \varsigma/2,$$

and we can continue the process.

If $\bar{t} < 1$, then we go back to (14) and proceed as follows. We get

$$\begin{aligned} |w(t)| &\le M_3(\alpha,a)\big(|w_0| + |w_1|\big) \\ &\quad + \int_0^t (t-s)^{\alpha-1} \big|E_{\alpha,\alpha}\big(-a(t-s)^\alpha\big)\big| \left(\int_0^s k(s-\sigma)\big|w(\sigma)\big|\tilde{h}\big(\big|w(\sigma)\big|\big)\, d\sigma \right) ds. \end{aligned} \quad (21)$$

Next,

$$\begin{aligned} |w(t)| &\le M_3(\alpha,a)\big(|w_0| + |w_1|\big) + \psi(t) \\ &\quad \times \int_0^t (t-s)^{\alpha-1} \big|E_{\alpha,\alpha}\big(-a(t-s)^\alpha\big)\big| \left(\int_0^s k(s-\sigma)\tilde{h}\big(\big|w(\sigma)\big|\big)\, d\sigma \right) ds, \end{aligned} \quad (22)$$

where

$$\psi(t) = \sup_{0\le\sigma\le t} \big|w(\sigma)\big|.$$

If $M_3(\alpha,a)(|w_0| + |w_1|) < \xi/4$ and $|w(t)| \le \xi$ for all $0 \le t \le \bar{t}$, then we get

$$\begin{aligned} |w(t)| &\le M_3(\alpha,a)\big(|w_0| + |w_1|\big) \\ &\quad + \tilde{h}(\xi)\psi(t) \int_0^t (t-s)^{\alpha-1} \big|E_{\alpha,\alpha}\big(-a(t-s)^\alpha\big)\big| \left(\int_0^s k(s-\sigma)\, d\sigma \right) ds. \end{aligned}$$

Notice that the expression

$$\int_0^t (t-s)^{\alpha-1} \big|E_{\alpha,\alpha}\big(-a(t-s)^\alpha\big)\big| \left(\int_0^s k(s-\sigma)\, d\sigma \right) ds$$

is uniformly bounded in view of Lemma 5 and the fact that k is summable.

Now, assuming that

$$\tilde{h}(\xi) \int_0^t (t-s)^{\alpha-1} \big|E_{\alpha,\alpha}\big(-a(t-s)^\alpha\big)\big| \left(\int_0^s k(\sigma)\, d\sigma \right) ds \le B_2\tilde{h}(\xi) < 1/2$$

for some $B_2 > 0$, we find

$$\big|w(t)\big| \le M_3(\alpha,a)\big(|w_0| + |w_1|\big) + \frac{\psi(t)}{2}, \quad 0 \le t \le \bar{t}.$$

Passing to the supremum, we obtain

$$\big|w(t)\big| \le 2M_3(\alpha,a)\big(|w_0| + |w_1|\big), \quad 0 \le t \le \bar{t}, \quad (23)$$

and therefore

$$\big|w(t)\big| \le 2M_3(\alpha,a)\big(|w_0| + |w_1|\big) < \xi/2, \quad 0 \le t \le \bar{t}. \quad (24)$$

Relation (24) shows that the process can be continued. The proof is complete. □

5 Application to neural network theory

In this section, we present an application to neural network theory. For simplicity, we consider the problem

$$\begin{cases} D_C^{\alpha} x_i(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n \int_0^t k_{ij}(s) g_j(x_j(t-s))\,ds + I_i, \\ \quad t > 0, i = 1, \dots, n, \\ x_i(0) = x_{i0}, \qquad x_i'(0) = x_{i1}, \quad i = 1, \dots, n, \end{cases}$$

where $0 < \alpha < 1$, $c_i > 0$, $a_{ij} \in R$, I_i, and x_{i0}, x_{i1}, $i, j = 1, \dots, n$, are given data. From our argument it will be clear that similar results hold for more general problems such as the case of additional discrete delays $\sum_{j=1}^n b_{ij} f_j(x_j(t-\tau))$ and also the case of different activation functions f_j. Notice that we consider the finite distributed delay case.

We start with the following assumptions:

(A1) The functions f_i are Lipschitz continuous on R with Lipschitz constants L_i, $i = 1, 2, \dots, n$, that is,

$$\left|f_i(x) - f_i(y)\right| \le L_i |x - y|, \quad \forall x, y \in R, i = 1, 2, \dots, n.$$

(A2) The functions g_i are Lipschitz continuous on R with Lipschitz constants G_i, $i = 1, 2, \dots, n$, that is,

$$\left|g_i(x) - g_i(y)\right| \le G_i |x - y|, \quad \forall x, y \in R, i = 1, 2, \dots, n.$$

(A3) The delay kernel functions k_{ij} are nonnegative summable functions ($\kappa_{ij} := \int_0^\infty k_{ij}(s)\,ds < \infty$) satisfying (3) or simply (9).

We denote

$$u(t) = x(t) - x^*,$$

where x^* is an equilibrium for problem (13). Then the stability of x^* is shifted to the stability of the 0 state for the system

$$\begin{cases} D_C^{\alpha} u_i(t) = -c_i u_i(t) + \sum_{j=1}^n a_{ij} \bar{f}_j(u_j(t)) + \sum_{j=1}^n \int_0^t k_{ij}(t-s) \bar{g}_j(u_j(s))\,ds, \\ \quad t > 0, i = 1, 2, \dots, n, \\ u_i(0) = \psi_i := x_{i0} - x_i^*, \qquad u_i'(0) = \psi_i' := x_{i1} - x_i^*, \quad i = 1, 2, \dots, n, \end{cases}$$

where

$$\bar{f}_j\big(u_j(t)\big) = f_j\big(u_j(t) + x_j^*\big) - f_j\big(x_j^*\big), \quad j = 1, 2, \dots, n, t \ge 0,$$

and

$$\bar{g}_j\big(u_j(t)\big) = g_j\big(u_j(t) + x_j^*\big) - g_j\big(x_j^*\big), \quad j = 1, 2, \dots, n, t \ge 0,$$

so that, in view of assumptions (A1) and (A2), we obtain

$$\begin{cases} D_C^{\alpha} u_i(t) \le -c_i u_i(t) + \sum_{j=1}^{n} a_{ij} L_i |u_i(t)| + \sum_{j=1}^{n} G_j \int_0^t k_{ij}(t-s)|u_j(s)|\,ds, \\ \quad t > 0, i = 1,2,\ldots,n. \end{cases}$$

We can apply the first theorem to get a global power-type stability result.

For the nonlinear case, we assume:

(A4) The functions g_i are such that

$$\bigl|g_i(x) - g_i(y)\bigr| \le |x-y|\tilde{h}_i\bigl(|x-y|\bigr), \quad \forall x,y \in R, i = 1,2,\ldots,n$$

for some continuous nondecreasing functions $\tilde{h}_i$. The second theorem may be applied to get a local stability of power type.

Example Consider the example

$$\begin{cases} D_C^{\alpha} x_1(t) = -c_1 x_1(t) + a_{11} f_1(x_1(t)) + a_{12} f_2(x_2(t)) \\ \qquad\qquad + \int_0^t k_{11}(s) f_1(x_1(t-s))\,ds + \int_0^t k_{12}(s) f_2(x_2(t-s))\,ds + I_1, \\ D_C^{\alpha} x_2(t) = -c_2 x_2(t) + a_{21} f_1(x_1(t)) + a_{22} f_2(x_2(t)) \\ \qquad\qquad + \int_0^t k_{21}(s) f_1(x_1(t-s))\,ds + \int_0^t k_{22}(s) f_2(x_2(t-s))\,ds + I_2, \\ x_i(0) = x_{i0}, \quad i = 1,2, \end{cases}$$

with $\alpha = 3/2$, $f_i(x) = \tanh x$, $i = 1,2$, $k_{ij}(t) = K_{ij} t^{\mu_{ij}-1} e^{-b_{ij} t}$, $i,j = 1,2$. The initial data may be any values. The rest of the coefficients and parameters are such that the conditions of the first theorem (see also the first example) are satisfied.

The equilibrium solution satisfies

$$\begin{cases} 0 = -c_1 x_1^* + (a_{11} + \int_0^{\infty} k_{11}(s)\,ds) f_1(x_1^*) \\ \quad + (a_{12} + \int_0^{\infty} k_{12}(s)\,ds) f_2(x_2^*) + I_1, \\ 0 = -c_2 x_2^* + (a_{21} + \int_0^{\infty} k_{21}(s)\,ds) f_1(x_1^*) \\ \quad + (a_{22} + \int_0^{\infty} k_{22}(s)\,ds) f_2(x_2^*) + I_2. \end{cases}$$

Having all the conditions in the first theorem satisfied, we conclude the power-type stability.

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