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(ω, c)-Periodic solutions for time varying impulsive differential equations

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Abstract

In this paper, we study a class of (ω, c)-periodic time varying impulsive differential equations and establish the existence and uniqueness results for (ω, c)-periodic solutions of homogeneous problem as well as nonhomogeneous problem.

Keywords: (ω, c)-periodic solutions; Impulsive differential equation; Existence and uniqueness

1 Introduction

It is well known that the concept of (ω, c)-periodic functions is the same of "affine-periodic functions" or "periodic of second kind", which were introduced by Floquet [1] and have been studied in the past decades. Recently, Alvarez et al. [2] introduced a new concept of (ω, c)-periodic function by considering Mathieu's equation $z'' + [\alpha - 2\beta\cos(2t)]z = 0$, and its solution satisfies $z(t+\omega) = cz(t)$, $c \in \mathbb{C}$. Clearly, (ω, c)-periodic functions become the standard ω-periodic functions when $c = 1$ and ω-antiperiodic functions when $c = -1$. For these particular cases, we refer readers to [3–6].

Meanwhile, Alvarez et al. [7] transferred the same idea to study (N, λ)-periodic discrete functions and established the existence and uniqueness of (N, λ)-periodic solutions to a class of Volterra difference equations with infinite delay. Next, Agaoglou et al. [8] applied the concept of (ω, c)-periodic to semilinear evolution equations in complex Banach spaces and studied its existence and uniqueness of (ω, c)-periodic solutions. Li et al. [9] transferred the similar idea to consider (ω, c)-periodic solutions impulsive differential systems.

Although, Floquet [1] studied a homogenous linear periodic system $x'(t) = A(t)x(t)$ with $A(t+\omega) = A(t)$, $t \in \mathbb{R}$, there are quite few analogous results to Floquet's theory for (ω, c)-periodic systems with impulse. Motivated by [1, 2, 8, 9], we consider the following time varying impulsive differential equation:

$$\begin{cases} x'(t) = a(t)x(t) + f(t, x(t)), & t \neq t_i, i \in \mathbb{N} = \{1, 2, \ldots\}, \\ \Delta x|_{t=t_i} = x(t_i^+) - x(t_i^-) = b_i x(t_i^-) + c_i, \end{cases} \tag{1}$$

where $a \in C(\mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $b_i, c_i \in \mathbb{R}$, and $t_i < t_{i+1}$, $i \in \mathbb{N}$. The symbols $x(t_i^+)$ and $x(t_i^-)$ represent the right and left limits of $x(t)$ at $t = t_i$.

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The main purpose of this paper is to derive existence and uniqueness results for (ω,c)-periodic solutions of nonhomogeneous linear problem as well as homogeneous linear problem.

2 Preliminaries

We introduce a Banach space $\mathrm{PC}(\mathbb{R},\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R} : x \in C((t_i,t_{i+1}],\mathbb{R}), \text{and } x(t_i^-) = x(t_i), x(t_i^+) \text{ exists } \forall i \in \mathbb{N}\}$ endowed with the norm $\|x\| = \sup_{t\in\mathbb{R}} |x(t)|$.

Lemma 2.1 (See [10, p.9]) *Suppose that $f \in C(\mathbb{R},\mathbb{R})$. A solution $x \in \mathrm{PC}(\mathbb{R},\mathbb{R})$ of the following nonhomogeneous linear impulsive equation*

$$\begin{cases} x'(t) = a(t)x(t) + f(t), & t \neq t_i, i \in \mathbb{N}, \\ \Delta x|_{t=t_i} = b_i x(t_i^-) + c_i, \\ x(t_0) = x_{t_0}, \end{cases} \tag{2}$$

is given by

$$x(t) = W(t,t_0)x(t_0) + \int_{t_0}^t W(t,s)f(s)\,ds + \sum_{t_0<t_i<t} W(t,t_i)c_i, \quad t \geq t_0, \tag{3}$$

where (*see* [10, p.8])

$$W(t,t_0) = e^{\int_{t_0}^t a(s)\,ds} \prod_{t_0<t_i<t} (1+b_i), \quad t \geq t_0.$$

Lemma 2.2 *For any $t, t_0 \in \mathbb{R}$, $\tau \in \mathbb{R} \setminus \{t_i\}_{i\in\mathbb{N}}$, and $t \geq \tau \geq t_0$, we have*

$$W(t,t_0) = W(t,\tau)W(\tau,t_0). \tag{4}$$

Proof Since $\tau \notin \{t_i\}_{i\in\mathbb{N}}$, we derive

$$\begin{aligned} W(t,t_0) &= e^{\int_{t_0}^t a(s)\,ds} \prod_{t_0<t_i<t} (1+b_i) \\ &= \left(e^{\int_{t_0}^\tau a(s)\,ds} \prod_{t_0<t_i<\tau} (1+b_i)\right) e^{\int_\tau^t a(s)\,ds} \prod_{\tau\leq t_i<t} (1+b_i) \\ &= \left(e^{\int_{t_0}^\tau a(s)\,ds} \prod_{t_0<t_i<\tau} (1+b_i)\right) e^{\int_\tau^t a(s)\,ds} \prod_{\tau<t_i<t} (1+b_i) = W(t,\tau)W(\tau,t_0). \end{aligned}$$

□

Definition 2.3 (See [2]) Let $c \in \mathbb{R} \setminus \{0\}$ and $\omega > 0$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be (ω,c)-periodic if $f(t+\omega) = cf(t)$ for all $t \in \mathbb{R}$.

Lemma 2.4 (See [8, Lemma 2.2]) *Set $\Psi_{\omega,c} := \{x : x \in \mathrm{PC}(\mathbb{R},\mathbb{R}) \text{ and } cx(\cdot) = x(\cdot+\omega)\}$. Let $x \in \Psi_{\omega,c}$, that is, x is a piecewise continuous and (ω,c)-periodic function. Then $x \in \Psi_{\omega,c}$ is equivalent to*

$$x(\omega) = cx(0). \tag{5}$$

Lemma 2.5 *Assume that the following conditions hold*:

(A_1) *$a(\cdot)$ is ω-periodic, i.e., $a(t+\omega)=a(t)$, $\forall t\in\mathbb{R}$.*

(A_2) *Set $t_0=0$ and $t_i<t_{i+1}$, $i\in\mathbb{N}$. There exists $N\in\mathbb{N}$ such that $t_{i+N}=t_i+\omega$, $b_{i+N}=b_i$, and $c_{i+N}=c_i$, $\forall i\in\mathbb{N}$.*

Then the following homogeneous linear impulsive equation

$$\begin{cases} x'(t)=a(t)x(t), & t\neq t_i, i\in\mathbb{N},\\ \Delta x|_{t=t_i}=b_i x(t_i^-), \\ x(0)=x_0, \end{cases} \tag{6}$$

has a solution $x\in\Psi_{\omega,c}$ if and only if $x_0(c-W(\omega,0))=0$.

Proof The solution $x\in PC(\mathbb{R},\mathbb{R})$ of (6) is given by

$$x(t)=x_0W(t,0)=x_0e^{\int_{t_0}^t a(s)\,ds}\prod_{0<t_i<t}(1+b_i),\quad t\geq 0.$$

If there exists $t_i\in(0,t)$ such that $1+b_i=0$, obviously, $x(t+\omega)=cx(t)=0$, and the result holds.

If $1+b_i\neq 0$, $\forall t_i\in(0,t)$ and $t\in[0,\infty)\setminus\{t_i\}_{i\in\mathbb{N}}$, we derive

$$\begin{aligned} x(t+\omega)=cx(t) \quad&\Longleftrightarrow\quad x_0e^{\int_0^{t+\omega} a(s)\,ds}\prod_{0<t_i<t+\omega}(1+b_i)=cx_0e^{\int_0^t a(s)\,ds}\prod_{0<t_i<t}(1+b_i)\\ &\Longleftrightarrow\quad x_0e^{\int_t^{t+\omega} a(s)\,ds}\prod_{t<t_i<t+\omega}(1+b_i)=cx_0\\ &\Longleftrightarrow\quad x_0\biggl(c-e^{\int_t^{t+\omega} a(s)\,ds}\prod_{t<t_i<t+\omega}(1+b_i)\biggr)=0\\ &\Longleftrightarrow\quad x_0\biggl(c-e^{\int_0^{\omega} a(s)\,ds}\prod_{0<t_i<\omega}(1+b_i)\biggr)=0\\ &\Longleftrightarrow\quad x_0\bigl(c-W(\omega,0)\bigr)=0. \end{aligned}$$

In addition, since $x(t_i)=x(t_i^-)$, we obtain $x(t_i+\omega)=cx(t_i)$. □

3 Main results

We consider the (ω,c)-periodic solutions of the following nonhomogeneous linear problem:

$$\begin{cases} x'(t)=a(t)x(t)+f(t), & t\neq t_i, i\in\mathbb{N},\\ \Delta x|_{t=t_i}=b_i x(t_i^-)+c_i, \\ x(0)=x_0, \end{cases} \tag{7}$$

where $f\in C(\mathbb{R},\mathbb{R})$ and f is (ω,c)-periodic. We give the following assumption:

(A_3) $c\neq W(\omega,0)$.

Lemma 3.1 *Assume that (A_1), (A_2), and (A_3) hold. Then the solution $x \in \Upsilon := \mathrm{PC}([0,\omega],\mathbb{R})$ of (7) satisfying (5) is given by*

$$x(t) = \int_0^\omega F(t,s)f(s)\,ds + \sum_{i=1}^N F(t,t_i)c_i, \tag{8}$$

where

$$F(t,s) = \begin{cases} c(c - W(\omega,0))^{-1}W(t,s), & 0 \le s < t, \\ W(t,0)(c - W(\omega,0))^{-1}W(\omega,s), & t \le s < \omega. \end{cases} \tag{9}$$

Proof The solution $x \in \Upsilon$ of (7) is given by

$$x(t) = W(t,0)x_0 + \int_0^t W(t,s)f(s)\,ds + \sum_{0<t_i<t} W(t,t_i)c_i. \tag{10}$$

Thus $x(\omega) = W(\omega,0)x_0 + \int_0^\omega W(\omega,s)f(s)\,ds + \sum_{0<t_i<\omega} W(\omega,t_i)c_i = cx_0$, which is equivalent to $x_0 = (c - W(\omega,0))^{-1}(\int_0^\omega W(\omega,s)f(s)\,ds + \sum_{0<t_i<\omega} W(\omega,t_i)c_i)$ due to $c \neq W(\omega,0)$.

Then we have

$$\begin{aligned} x(t) = {} & W(t,0)\bigl(c - W(\omega,0)\bigr)^{-1}\left(\int_0^\omega W(\omega,s)f(s)\,ds + \sum_{0<t_i<\omega} W(\omega,t_i)c_i\right) \\ & + \int_0^t W(t,s)f(s)\,ds + \sum_{0<t_i<t} W(t,t_i)c_i := I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= W(t,0)\bigl(c - W(\omega,0)\bigr)^{-1}\int_0^\omega W(\omega,s)f(s)\,ds + \int_0^t W(t,s)f(s)\,ds, \\ I_2 &:= W(t,0)\bigl(c - W(\omega,0)\bigr)^{-1}\sum_{0<t_i<\omega} W(\omega,t_i)c_i + \sum_{0<t_i<t} W(t,t_i)c_i. \end{aligned}$$

If $t \in [0,\omega] \setminus \{t_1,\ldots,t_N\}$, by (4) and condition (A_3), we derive

$$\begin{aligned} I_1 = {} & W(t,0)\bigl(c - W(\omega,0)\bigr)^{-1}\int_0^t W(\omega,t)W(t,s)f(s)\,ds + \int_0^t W(t,s)f(s)\,ds \\ & + W(t,0)\bigl(c - W(\omega,0)\bigr)^{-1}\int_t^\omega W(\omega,s)f(s)\,ds \\ = {} & \bigl(W(\omega,0)\bigl(c - W(\omega,0)\bigr)^{-1} + 1\bigr)\int_0^t W(t,s)f(s)\,ds \\ & + \int_t^\omega W(t,0)\bigl(c - W(\omega,0)\bigr)^{-1}W(\omega,s)f(s)\,ds \\ = {} & c\int_0^t \bigl(c - W(\omega,0)\bigr)^{-1}W(t,s)f(s)\,ds + \int_t^\omega W(t,0)\bigl(c - W(\omega,0)\bigr)^{-1}W(\omega,s)f(s)\,ds \\ = {} & \int_0^\omega F(t,s)f(s)\,ds, \end{aligned}$$

and

$$\begin{aligned}
I_2 &= W(t,0)\big(c - W(\omega,0)\big)^{-1} \sum_{0<t_i<t} W(\omega,t)W(t,t_i)c_i + \sum_{0<t_i<t} W(t,t_i)c_i \\
&\quad + W(t,0)\big(c - W(\omega,0)\big)^{-1} \sum_{t<t_i<\omega} W(\omega,t_i)c_i \\
&= \big(W(\omega,0)\big(c - W(\omega,0)\big)^{-1} + 1\big)\big) \sum_{0<t_i<t} W(t,t_i)c_i \\
&\quad + W(t,0)\big(c - W(\omega,0)\big)^{-1} \sum_{t<t_i<\omega} W(\omega,t_i)c_i \\
&= c \sum_{0<t_i<t} \big(c - W(\omega,0)\big)^{-1} W(t,t_i)c_i + \sum_{t<t_i<\omega} W(t,0)\big(c - W(\omega,0)\big)^{-1} W(\omega,t_i)c_i \\
&= \sum_{0<t_i<\omega} F(t,t_i)c_i \\
&= \sum_{i=1}^{N} F(t,t_i)c_i.
\end{aligned}$$

Thus we get (8). Since $x(t_i) = x(t_i^-)$, we can also get the same result for $t \in \{t_1, \ldots, t_N\}$. □

Lemma 3.2 *Let* $\tilde{a} := \max_{t\in[0,\omega]}\{a(t)\}$ *and* $\tilde{b} := \max_{1\le i\le N}\{|1+b_i|\}$. *Then, for any* $t \in [0,\omega]$, *we have*

$$\int_0^\omega \big|F(t,s)\big|\,ds \le P_{\tilde{a}} := \begin{cases} |(c - W(\omega,0))^{-1}| e^{\tilde{a}\omega}\omega\tilde{b}^N(|c|+1), & \tilde{a} > 0, \\ |(c - W(\omega,0))^{-1}|\omega\tilde{b}^N(|c|+1), & \tilde{a} \le 0. \end{cases}$$

Proof Using (9), we derive

$$\begin{aligned}
\int_0^\omega \big|F(t,s)\big|\,ds &\le \big|\big(c - W(\omega,0)\big)^{-1}\big| \left(\int_0^t \big|cW(t,s)\big|\,ds + \int_t^\omega \big|W(t,0)W(\omega,s)\big|\,ds \right) \\
&\le \big|\big(c - W(\omega,0)\big)^{-1}\big| \left(|c| \int_0^t e^{\int_s^t a(\tau)\,d\tau} \prod_{s<t_i<t} |1+b_i|\,ds \right. \\
&\quad \left. + \int_t^\omega e^{(\int_0^t + \int_s^\omega) a(\tau)\,d\tau} \prod_{0<t_i<t \cup s<t_i<\omega} |1+b_i|\,ds \right).
\end{aligned}$$

If $\tilde{a} > 0$, we get

$$\int_0^\omega \big|F(t,s)\big|\,ds \le \big|\big(c - W(\omega,0)\big)^{-1}\big| e^{\tilde{a}\omega}\omega\tilde{b}^N\big(|c|+1\big).$$

If $\tilde{a} \le 0$, we get

$$\int_0^\omega \big|F(t,s)\big|\,ds \le \big|\big(c - W(\omega,0)\big)^{-1}\big| \omega\tilde{b}^N\big(|c|+1\big).$$

The proof is finished. □

Lemma 3.3 *For any $t \in [0,\omega]$, we have*

$$\sum_{i=1}^{N} \bigl|F(t,t_i)c_i\bigr| \le Q_{\tilde{a}} := \begin{cases} |(c - W(\omega,0))^{-1}|(|c|+1)e^{\tilde{a}\omega}\tilde{b}^N \sum_{i=1}^{N} |c_i| & \tilde{a} > 0, \\ |(c - W(\omega,0))^{-1}|(|c|+1)\tilde{b}^N \sum_{i=1}^{N} |c_i| & \tilde{a} \le 0. \end{cases}$$

Proof By (9), we have

$$\begin{aligned} \sum_{i=1}^{N} \bigl|F(t,t_i)c_i\bigr| &\le \bigl|\bigl(c - W(\omega,0)\bigr)^{-1}\bigr| \biggl(\sum_{0<t_i<t} \bigl|cW(t,t_i)c_i\bigr| + \sum_{t\le t_i<\omega} \bigl|W(t,0)W(\omega,t_i)c_i\bigr| \biggr) \\ &\le \bigl|\bigl(c - W(\omega,0)\bigr)^{-1}\bigr| \biggl(\sum_{0<t_i<t} |c_i||c|e^{\int_{t_i}^{t} a(\tau)\,d\tau} \prod_{t_i<t_k<t} |1+b_k| \\ &\quad + \sum_{t\le t_i<\omega} |c_i| e^{(\int_0^t + \int_{t_i}^{\omega}) a(\tau)\,d\tau} \prod_{0<t_k<t \cup t_i<t_k<\omega} |1+b_k| \biggr). \end{aligned}$$

If $\tilde{a} > 0$, we obtain

$$\sum_{i=1}^{N} \bigl|F(t,t_i)c_i\bigr| \le \bigl|\bigl(c - W(\omega,0)\bigr)^{-1}\bigr| \bigl(|c|+1\bigr) e^{\tilde{a}\omega}\tilde{b}^N \sum_{i=1}^{N} |c_i|.$$

If $\tilde{a} \le 0$, we obtain

$$\sum_{i=1}^{N} \bigl|F(t,t_i)c_i\bigr| \le \bigl|\bigl(c - W(\omega,0)\bigr)^{-1}\bigr| \bigl(|c|+1\bigr) \tilde{b}^N \sum_{i=1}^{N} |c_i|.$$

The proof is complete. □

Now we are ready to study the existence of semilinear impulsive problems. We make the following hypotheses:

(A_4) For any $t \in \mathbb{R}$ and $x \in \mathbb{R}$, it holds $f(t+\omega, cx) = cf(t,x)$.

(A_5) There exists $L > 0$ such that $|f(t,x) - f(t,y)| \le L|x-y|$ for any $t \in \mathbb{R}$ and $x,y \in \mathbb{R}$.

(A_6) There exist constants $K, J > 0$ such that $|f(t,x)| \le K|x| + J$ for any $t \in \mathbb{R}$ and $x \in \mathbb{R}$.

Theorem 3.4 *Suppose that (A_1), (A_2), (A_3), (A_4), and (A_5) hold. If $0 < LP_{\tilde{a}} < 1$, then (1) has a unique (ω,c)-periodic solution $x \in \Psi_{\omega,c}$. Moreover, it holds $\|x\| \le \frac{f_0 P_{\tilde{a}} + Q_{\tilde{a}}}{1 - LP_{\tilde{a}}}$, where $f_0 =$* $\max_{t\in[0,\omega]} |f(t,0)|$.

Proof For any $x \in \Psi_{\omega,c}$, i.e., $x(\cdot + \omega) = cx$), we have $f(t+\omega, x(t+\omega)) = f(t, cx(t))$, $t \in \mathbb{R}$. Further, by assumption (A_4), $f(t+\omega, x(t+\omega)) = f(t, cx(t)) = cf(t,x)$, $t \in \mathbb{R}$. Thus, $f(\cdot, x(\cdot)) \in \Psi_{\omega,c}$. For more characterization of the (ω,c)-periodic functions, see [2, Sect. 2].

Let $\mathbb{G} : \Upsilon \to \Upsilon$ be the operator given by

$$(\mathbb{G}x)(t) = \int_0^{\omega} F(t,s) f\bigl(s, x(s)\bigr)\,ds + \sum_{i=1}^{N} F(t,t_i)c_i. \tag{11}$$

By Lemma 2.4 and Lemma 3.1, the existence of (ω,c)-periodic solutions of (1) is equivalent to the existence of the fixed point of (11).

It is easy to show that $\mathbb{G}(\Upsilon) \subseteq \Upsilon$. For any $x, y \in \Upsilon$, we derive

$$\begin{aligned}\bigl|(\mathbb{G}x)(t) - (\mathbb{G}y)(t)\bigr| &\le L \int_0^{\omega} \bigl|F(t,s)\bigr|\bigl|x(s) - y(s)\bigr|\,ds \\ &\le L\|x - y\| \int_0^{\omega} \bigl|F(t,s)\bigr|\,ds \le LP_{\tilde{a}}\|x - y\|,\end{aligned}$$

which implies $\|\mathbb{G}x - \mathbb{G}y\| \le LP_{\tilde{a}}\|x - y\|$. Noticing $0 < LP_{\tilde{a}} < 1$, $\mathbb{G}$ is a contraction mapping. Thus, $\mathbb{G}$ defined in (11) has a unique fixed point satisfying $x(\omega) = cx(0)$ due to Lemma 3.1. Further, by Lemma 2.4, one has $x \in \Psi_{\omega,c}$. From the above, there exists a unique (ω, c)-periodic solution $x \in \Psi_{\omega,c}$ of (1).

Moreover, we have

$$\begin{aligned}\bigl|x(t)\bigr| &\le L \int_0^{\omega} \bigl|F(t,s)\bigr|\bigl|x(s)\bigr|\,ds + \int_0^{\omega} \bigl|F(t,s)\bigr|\bigl|f(s,0)\bigr|\,ds + \sum_{i=1}^{N} \bigl|F(t,t_i)c_i\bigr| \\ &\le LP_{\tilde{a}}\|x\| + f_0 P_{\tilde{a}} + Q_{\tilde{a}},\end{aligned}$$

which implies

$$\|x\| \le \frac{f_0 P_{\tilde{a}} + Q_{\tilde{a}}}{1 - LP_{\tilde{a}}}.$$

The proof is finished. □

Theorem 3.5 *Suppose that* (A_1), (A_2), (A_3), (A_4), *and* (A_6) *hold. If* $KP_{\tilde{a}} < 1$, *then* (1) *has at least one* (ω, c)*-periodic solution* $x \in \Psi_{\omega,c}$.

Proof Let $\mathbb{B}_r = \{x \in \Upsilon : \|x\| \le r\}$, where $r \ge \frac{JP_{\tilde{a}} + Q_{\tilde{a}}}{1 - KP_{\tilde{a}}}$. We consider $\mathbb{G}$ defined in (11) on $\mathbb{B}_r$. For all $x \in \mathbb{B}_r$ and $t \in [0, \omega]$, using Lemmas 3.2 and 3.3, we derive

$$\bigl|(\mathbb{G}x)(t)\bigr| \le K\|x\| \int_0^{\omega} \bigl|F(t,s)\bigr|\,ds + J \int_0^{\omega} \bigl|F(t,s)\bigr|\,ds + Q_{\tilde{a}} \le KP_{\tilde{a}}\|x\| + JP_{\tilde{a}} + Q_{\tilde{a}} \le r,$$

which implies $\|\mathbb{G}x\| \le r$. Thus $\mathbb{G}(B_r) \subset B_r$. In addition, it is easy to see that $\mathbb{G}$ is continuous and $\mathbb{G}(\mathbb{B}_r)$ is pre-compact. By Schauder's fixed point theorem, we obtain that (1) has at least one (ω, c)-periodic solution $x \in \Psi_{\omega,c}$. □

4 Examples

Example 4.1 We consider the following semilinear impulsive equation:

$$\begin{cases} x'(t) = (\cos 2t)x(t) + \rho \sin t \cos x(t), & t \neq t_i, i = 1, 2, \ldots, \\ \Delta x|_{t=t_i} = \frac{1}{2} \sin \frac{(2i-1)\pi}{2} x(t_i^-) + \cos i\pi, \end{cases} \tag{12}$$

where $\rho \in \mathbb{R}$, $t_i = \frac{(3i-1)\pi}{6}$, $\omega = \pi$, $c = -1$, $a(t) = \cos 2t$, $f(t,x) = \rho \sin t \cos x$, $b_i = \frac{1}{2} \sin \frac{(2i-1)\pi}{2}$, and $c_i = \cos i\pi$. Clearly, $t_{i+2} = t_i + \pi$, $b_{i+2} = b_i$, $c_{i+2} = c_i$ for all $i \in \mathbb{N}$, then we obtain $N = 2$, (A_1) and (A_2) hold. Since $W(\omega, 0) = \frac{3}{4} \neq -1 = c$, we get (A_3) holds. Note that $f(\cdot + \omega, cx) = f(\cdot + \pi, -x) = -\rho \sin \cdot \cos x = -f(\cdot, x) = cf(\cdot, x)$, we get (A_4) holds. $|f(t,x) - f(t,y)| \le |\rho||x - y|$, then we get $L = |\rho|$ and (A_5) holds. In addition, $\tilde{a} = 1$, $\tilde{b} = \frac{3}{2}$, $P_{\tilde{a}} = \frac{18\pi e^{\pi}}{7} \doteq 186.939334$, and $Q_{\tilde{a}} = \frac{36e^{\pi}}{7} \doteq 119.009276$.

Letting $0 < |\rho| < \frac{7}{18\pi e^{\pi}} \doteq 0.005349$, we get $0 < LP_{\tilde{a}} < 1$, then all the assumptions of Theorem 3.4 hold. So if $0 < |\rho| < \frac{7}{18\pi e^{\pi}}$, problem (12) has a unique π-antiperiodic solution $x \in \mathrm{PC}([0,\infty)),\mathbb{R})$.

Since $|f(t,x)| \leq |\rho|$, we get $K = 0$, $J = |\rho|$, (A_6) holds, and $KP_{\tilde{a}} = 0 < 1$. Then all the assumptions of Theorem 3.5 hold for any $\rho \in \mathbb{R}$. So (12) has at least one π-antiperiodic solution for any $\rho \in \mathbb{R}$.

Example 4.2 We consider the following semilinear impulsive equation:

$$\begin{cases} x'(t) = (\sin 2\pi t)x(t) + \rho x(t)\cos(2^{-t}x(t)), & t \neq t_i, i = 1,2,\ldots, \\ \Delta x|_{t=t_i} = x(t_i^-) + 1, \end{cases} \tag{13}$$

where $\rho \in \mathbb{R}$, $t_i = \frac{3i-1}{6}$, $\omega = 1$, $c = 2$, $a(t) = \sin 2\pi t$, $f(t,x) = \rho x\cos(2^{-t}x)$, $b_i = 1$ and $c_i = 1$. Clearly, $t_{i+2} = t_i + 1$, $b_{i+2} = b_i$, $c_{i+2} = c_i$ for all $i \in \mathbb{N}$, then we obtain $N = 2$, (A_1) and (A_2) hold. Since $W(\omega,0) = 4 \neq 2 = c$, we get (A_3) holds. Note that $f(\cdot + \omega, cx) = f(\cdot + 1, 2x) = 2\rho x \cdot \cos(2^{-t}x) = 2f(\cdot,x) = cf(\cdot,x)$, we get (A_4) holds. Now $f(\cdot,x)$ does not satisfy the Lipschitz condition. Since $|f(t,x)| \leq |\rho||x|$, we get $K = |\rho|$, $J = 0$, and (A_6) holds. Moreover, $\tilde{a} = 1$, $\tilde{b} = 2$, and $P_{\tilde{a}} = 6e$.

Set $|\rho| < \frac{1}{6e} \doteq 0.061313$. Then $KP_{\tilde{a}} < 1$. Now all the assumptions of Theorem 3.5 hold. Thus,(13) has at least one $(1,2)$-periodic solution $x \in PC([0,\infty)),\mathbb{R})$ if $|\rho| < \frac{1}{6e}$.

5 Conclusion

Existence and uniqueness of (ω,c)-periodic solutions for homogeneous problem and nonhomogeneous as well as semilinear time varying impulsive differential equations are established. In a forthcoming work, we shall extend the study to (ω,c)-periodic solutions for nonlinear impulsive evolution systems in infinite dimensional spaces as follows:

$$\begin{cases} \dot{y} = C(t)y + h(t,y), & t \neq \tau_i, i \in \mathbb{N}, \\ \triangle y\,|_{t=\tau_i} = y(\tau_i^+) - y(\tau_i^-) = Dy(\tau_i^-) + d_i, \end{cases}$$

where the linear operator $\{C(t) : t \geq 0\}$ generates a strongly continuous evolutionary process $\{U(t,s), t \geq s \geq 0\}$ on a Banach space X. D is a bounded linear operator and $d_i \in X$. Motivated by [11–15], we shall also consider (ω,c)-periodic delay differential equations with non-instantaneous impulses.

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