Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On delay-interval-dependent robust stability of LPD discrete-time system with mixed time-varying delays and nonlinear uncertainties

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Abstract

In this work, a delay-interval-dependent robust stability problem for linear parameter dependent (LPD) discrete-time system with discrete and distributed time-varying delays and nonlinear uncertainties is addressed. By adopting combinations of mixed model transformation, various inequality expressions, utilization of zero equations, and new parameter dependent Lyapunov–Krasovskii functional, a new delay-interval-dependent asymptotic stability criterion is derived in the form of linear matrix inequalities. Moreover, an improved delay-interval-dependent asymptotic stability criterion of discrete-time linear system with discrete time-varying delay and nonlinear uncertainties is also achieved. Some numerical examples are also illustrated to exhibit the effectiveness with less conservatism of the proposed stability criterion.

Keywords: Parameter dependence; Lyapunov–Krasovskii functional; Linear matrix inequality (LMI); LPD discrete-time system; Interval time-varying delay

1 Introduction

During the past decades, various stability problems of discrete-time systems with delays have drawn many researchers' attention, for delays in systems may lead to instability or bad performances [1–30]. Discrete-time systems with time-varying delays have been well recognized to be typical examples of applications to engineering [3]. In recent years, various stability analyses of both discrete- and continuous-time systems with respect to uncertainty about time-invariant parameters have been among many challenging problems. An important aspect of uncertainty in a linear system is a linear parameter-dependent (LPD) system in which the uncertain state matrices in the polytope are mathematically expressed by a convex combination of known matrices. Most stability criteria have been obtained via Lyapunov–Krasovskii functional approaches, in which parameter-dependent Lyapunov functions have been employed [1, 8, 13, 14, 25, 28]. Stability analysis for discrete-time systems with interval time-varying delay and nonlinear uncertainties was studied in [21]. The robust stability criteria of LPD continuous-time system with delays were presented in [28]. Some results have been obtained for delay-interval-dependent robust stability of discrete-time systems with discrete and distributed time-varying delays [27]. However, few results have been obtained for robust stability for LPD discrete-time systems with interval time-varying delays and nonlinear perturbations.

Motivated by the above statement, the problem of stability for LPD discrete-time systems with mixed interval time-varying delays and nonlinear uncertainties is studied. Through the construction of a parameter-dependent Lyapunov–Krasovskii functional, model transformation, various forms of inequalities and utilization of zero equations, a delay-interval-dependent robust stability criterion is presented in terms of linear matrix inequalities (LMIs) whose feasibility can be easily checked. Then, based on the derived delay-interval-dependent robust stability criterion, an improved delay-range-dependent stability criterion is proposed for a discrete-time system with interval discrete time-varying delay and nonlinear uncertainties. Moreover, we include some numerical examples to exhibit the effectiveness with less conservatism of the proposed stability criterion.

2 Problem formulation

Notations, definition and lemmas used throughout this paper are given as follows. By Z^+ we denote the set of nonnegative integers; R^n denotes the n-dimensional space with the vector norm $\|\cdot\|$; by $\|\xi\|$ we denote the Euclidean vector norm of $\xi \in R^n$, that is, $\|\xi\|^2 = \xi^T\xi$; $R^{n\times r}$ denotes the space of all $(n \times r)$-dimensional real matrices; A^T denotes the transpose of a matrix A, while I denotes the identity matrix. Matrix A is called semi-positive definite ($A \geq 0$) if $x^TAx \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $x^TAx > 0$ for all $x \neq 0$. Matrix B is called semi-negative definite ($B \leq 0$) if $x^TBx \leq 0$, for all $x \in R^n$; B is negative definite ($B < 0$) if $x^TBx < 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$; $A \geq B$ means $A - B \geq 0$; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\mathrm{Re}\,\lambda : \lambda \in \lambda(A)\}$; $\lambda_{\min}(A) = \min\{\mathrm{Re}\,\lambda : \lambda \in \lambda(A)\}$; $\lambda_{\max}(A(\varphi)) = \max\{\lambda_{\max}(A_i) : i = 1,2,\dots,N\}$; $\lambda_{\min}(A(\varphi)) = \min\{\lambda_{\min}(A_i) : i = 1,2,\dots,N\}$, and $*$ represents the elements below the main diagonal of a symmetric matrix.

Consider the LPD discrete-time system with mixed interval time-varying delays and nonlinear perturbations of the form

$$\begin{aligned} \xi(k+1) = {} & A(\varphi)\xi(k) + B(\varphi)\xi\bigl(k-h(k)\bigr) + C(\varphi)\sum_{i=1}^{\infty}\delta(i)\xi(k-i) \\ & + f_1\bigl(k,\xi(k)\bigr) + f_2\bigl(k,\xi\bigl(k-h(k)\bigr)\bigr), \end{aligned} \tag{1}$$

$$\xi(s) = \phi(s), \quad s \in \{-h_2, -h_2+1, \dots, 0\}, \tag{2}$$

where $\xi(k) \in R^n$ is the state variable for $k \in Z^+$ and $\phi(s)$ is the initial value at s; $A(\varphi)$, $B(\varphi)$, $C(\varphi) \in R^{n\times n}$ stand for uncertain matrices defined by

$$\begin{aligned} & A(\varphi) = \sum_{i=1}^{N}\varphi_iA_i, \qquad B(\varphi) = \sum_{i=1}^{N}\varphi_iB_i, \qquad C(\varphi) = \sum_{i=1}^{N}\varphi_iC_i, \\ & \sum_{i=1}^{N}\varphi_i = 1, \quad \varphi_i \geq 0, A_i, B_i, C_i \in R^{n\times n}, i = 1,\dots,N. \end{aligned}$$

The nonlinear perturbations $f_1(k,\xi(k))$ and $f_2(k,\xi(k-h(k)))$ are bounded in magnitude:

$$f_1^T\bigl(k,\xi(k)\bigr)f_1\bigl(k,\xi(k)\bigr) \leq \gamma^2\xi^T(k)\xi(k), \tag{3}$$

$$f_2^T\big(k,\xi\big(k-h(k)\big)\big)f_2\big(k,\xi\big(k-h(k)\big)\big) \le \beta^2 x^T\big(k-h(k)\big)\xi\big(k-h(k)\big), \tag{4}$$

where γ and β are given positive real constants. The discrete time-varying delay $h(k)$ satisfies

$$0 < h_1 \le h(k) \le h_2,$$

where h_1 and h_2 are known positive integers. Moreover, there exists a constant scalar w such that the function $\delta(\cdot)$ satisfies

$$\sum_{i=1}^{\infty} \delta(i) = w < \infty. \tag{5}$$

Definition 2.1 ([25]) System (1) is said to be robustly stable if there exists a positive-definite function $V(k,\xi(k)) : Z^+ \times R^n \to R$ such that

$$\Delta V\big(k,\xi(k)\big) = V\big(k+1,\xi(k+1)\big) - V\big(k,\xi(k)\big) < 0,$$

along any trajectory of solution of system (1) with (2).

Lemma 2.2 ([5]) *For any constant matrix $W \in R^{n\times n}$, $W = W^T > 0$, two integers r_M and r_m satisfying $r_M \ge r_m$, vector function $\xi : [r_m, r_M] \to R^n$, the following inequality holds*:

$$\left(\sum_{i=r_m}^{r_M} \xi(i)\right)^T W \left(\sum_{i=r_m}^{r_M} \xi(i)\right) \le (r_M - r_m + 1) \sum_{i=r_m}^{r_M} \xi^T(i) W \xi(i).$$

Lemma 2.3 ([26]) *Let $Q \in R^{n\times n}$ be a positive-definite matrix, $\xi_i \in R^n, i = 1, 2, \ldots$. If the sums concerned are well defined, then*

$$\left[\sum_{i=k-M_1}^{k-M_2-1} \sum_{j=i}^{k-M_2-1} \xi_j\right]^T Q \left[\sum_{i=k-M_1}^{k-M_2-1} \sum_{j=i}^{k-M_2-1} \xi_j\right] \le \frac{(M_1 - M_2)^2}{2} \sum_{i=k-M_1}^{k-M_2-1} \sum_{j=i}^{k-M_2-1} \xi_j^T Q \xi_j.$$

Lemma 2.4 ([30]) *Let $M \in R^{n\times n}$ be a positive-definite matrix, $\xi_i \in R^n$ and $a_i \ge 0$, $i = 1, 2, \ldots$. If the sums concerned are well defined, then*

$$\left[\sum_{i=k-M_2}^{k-1} \xi_i\right]^T M \left[\sum_{i=k-M_2}^{k-1} \xi_i\right] \le M_2 \sum_{i=k-M_2}^{k-1} \xi_i^T M \xi_i,$$

$$\left[\sum_{i=-M_2}^{-1} \sum_{i=k+j}^{k-1} \xi_j\right]^T M \left[\sum_{i=-M_2}^{-1} \sum_{i=k+j}^{k-1} \xi_j\right] \le \frac{(M_2)^2}{2} \sum_{i=-M_2}^{-1} \sum_{i=j+k}^{k-1} \xi_j^T M \xi_j,$$

$$\left[\sum_{i=1}^{\infty} a_i \xi_i\right]^T M \left[\sum_{i=1}^{\infty} a_i \xi_i\right] \le \left[\sum_{i=1}^{\infty} a_i\right] \sum_{i=1}^{\infty} a_i \xi_i^T M \xi_i.$$

3 Main results

Referring to system (1), a new delay-range-dependent robust stability approach is presented here. The Lyapunov–Krasovskii functional method is combined with the LMI technique. Let us introduce the following notations to be used later on:

$$
\begin{aligned}
& L_w(\varphi)=\sum_{i=1}^{N}\varphi_i L_i^w, \qquad M(\varphi)=\sum_{i=1}^{N}\varphi_i M_i, \qquad Z_1(\varphi)=\sum_{i=1}^{N}\varphi_i Z_i^1,\\
& Z_2(\varphi)=\sum_{i=1}^{N}\varphi_i Z_i^2, \qquad P_k(\varphi)=\sum_{i=1}^{N}\varphi_i P_i^k, \qquad Q_s(\varphi)=\sum_{i=1}^{N}\varphi_i Q_i^s,\\
& M_r(\varphi)=\sum_{i=1}^{N}\varphi_i M_i^r, \qquad N_r(\varphi)=\sum_{i=1}^{N}\varphi_i N_i^r,\\
& \sum_{i=1}^{N}\varphi_i=1, \quad \varphi_i\geq 0, J, W, L_i^w, M_i, M_i^r, N_i^r, Z_i^1, Z_i^2, P_i^k, Q_i^s\in R^{n\times n},\\
& w=1,2,3, r=1,2,3,4, k=1,2,\ldots,18, s=1,2,\ldots,19, i=1,2,\ldots,N,\\
& \prod_{i,j}=\left[\Sigma_{i,j}^{m,n}\right]_{27\times 27},
\end{aligned}
\tag{6}
$$

where $\Sigma_{i,j}^{m,n}=\Sigma_{i,j}^{n,m\,T}, m,n=1,2,3,\ldots,27, i,j=1,2,3,\ldots,N,$

$$
\begin{aligned}
\Sigma_{i,j}^{1,1} ={}& G_i^1+G_i^2+G_i^{1\,T}+G_i^{2\,T}-L_i^{1\,T}A_j^1+L_i^{1\,T}B_j^1-L_i^{1\,T}+A_i^{1\,T}L_j^1\\
&+B_i^{1\,T}L_j^1-L_i^1+P_i^2+P_i^3+P_i^4+P_i^5+P_i^8+P_i^9+h_1^2Q_i^1\\
&+r^2h_1^2Q_i^4+h_2^2Q_i^7+r^2h_2^2Q_i^{10}+\rho^2Q_i^{13}+r^2\rho^2Q_i^{16}+h_1M_i^1\\
&+N_i^1+N_i^{1\,T}+h_2M_i^2+N_i^2+N_i^{2\,T}+rh_1M_i^3+N_i^3+N_i^{3\,T}\\
&+rh_2M_i^4+N_i^4+N_i^{4\,T}-h_1^2P_i^{17}+\epsilon_1\varphi^2I+wP_i^{19},\\
\Sigma_{i,j}^{1,2} ={}& P_i^1+G_i^{1\,T}+G_i^{2\,T}-L_i^{1\,T}+A_i^{1\,T}L_j^2+B_i^{1\,T}L_j^2-L_i^2+h_1^2Q_i^2\\
&+r^2h_1^2Q_i^5+h_2^2Q_i^8+r^2h_2^2Q_i^{11}+\rho^2Q_i^{14}+r^2\rho^2Q_i^{17},\\
\Sigma_{i,j}^{1,3} ={}& -N_i^1, \qquad \Sigma_{i,j}^{1,4}=-N_i^2, \qquad \Sigma_{i,j}^{1,5}=-G_i^1+L_i^{1\,T}B_j^2, \qquad \Sigma_{i,j}^{1,6}=-N_i^3,\\
\Sigma_{i,j}^{1,7} ={}& -N_i^4, \qquad \Sigma_{i,j}^{1,8}=-G_i^2+L_i^{1\,T}A_j^2+h_1P_i^{17}, \qquad \Sigma_{i,j}^{1,21}=-G_i^1-L_i^{1\,T}B_j^1,\\
\Sigma_{i,j}^{1,22} ={}& -G_i^2+L_i^{1\,T}A_j^2+A_i^{1\,T}L_j^3+B_i^{1\,T}L_j^3-L_i^3,\\
\Sigma_{i,j}^{1,25} ={}& L_i^{1\,T}, \qquad \Sigma_{i,j}^{1,26}=L_i^{1\,T}, \qquad \Sigma_{i,j}^{1,27}=L_i^{1\,T}C_j,\\
\Sigma_{i,j}^{2,2} ={}& P_i^1-L_i^{2\,T}-L_i^2+h_1^2Q_i^3+r^2h_1^2Q_i^6+h_2^2Q_i^9+r^2h_2^2Q_i^{12}+\rho^2Q_i^{15}\\
&+r^2\rho^2Q_i^{18}+h_1P_i^{10}+h_2P_i^{11}+rh_1P_i^{12}+rh_2P_i^{13}+h_1^2P_i^{14}\\
&+h_2^2P_i^{15}+\rho^2P_i^{16}+\frac{h_1^4}{4}P_i^{17}+\frac{(h_2-h_1)^2}{4}P_i^{18}+h_2^2Z_i^1+(rh_2)^2Z_i^2,\\
\Sigma_{i,j}^{2,5} ={}& -G_i^1+L_i^{2\,T}B_j^2, \qquad \Sigma_{i,j}^{2,8}=-G_i^2+L_i^{2\,T}A_j^2, \qquad \Sigma_{i,j}^{2,21}=-G_i^1+L_i^{2\,T}B_j^1,\\
\Sigma_{i,j}^{2,22} ={}& -G_i^2+L_i^{2\,T}A_j^2-L_i^3, \qquad \Sigma_{i,j}^{2,25}=L_i^{2\,T}, \qquad \Sigma_{i,j}^{2,26}=L_i^{2\,T},
\end{aligned}
$$

$$\Sigma_{i,j}^{2,27} = L_i^{2^T} C_j, \qquad \Sigma_{i,j}^{3,3} = -P_i^2 + P_i^6 - P_i^{18}, \qquad \Sigma_{i,j}^{3,23} = P_i^{18},$$
$$\Sigma_{i,j}^{4,4} = -P_i^4 - P_i^6, \qquad \Sigma_{i,j}^{5,5} = -P_i^8 - P_i^{18} + \epsilon_2\beta^2 I, \qquad \Sigma_{i,j}^{5,22} = B_i^{2^T} L_j^3,$$
$$\Sigma_{i,j}^{5,24} = P_i^{18}, \qquad \Sigma_{i,j}^{6,6} = -P_i^3 + P_i^7, \qquad \Sigma_{i,j}^{7,7} = -P_i^5 - P_i^7,$$
$$\Sigma_{i,j}^{8,8} = -P_i^9 - P_i^{17}, \qquad \Sigma_{i,j}^{8,22} = A_i^{2^T} L_j^3, \qquad \Sigma_{i,j}^{9,9} = -Q_i^1, \Sigma_{i,j}^{9,15} = -Q_i^2,$$
$$\Sigma_{i,j}^{10,10} = -Q_i^4, \qquad \Sigma_{i,j}^{10,16} = -Q_i^5, \qquad \Sigma_{i,j}^{11,11} = -Q_i^7, \qquad \Sigma_{i,j}^{11,17} = -Q_i^8,$$
$$\Sigma_{i,j}^{12,12} = -Q_i^{10}, \qquad \Sigma_{i,j}^{12,18} = -Q_i^{11}, \qquad \Sigma_{i,j}^{13,13} = -Q_i^{13}, \qquad \Sigma_{i,j}^{13,19} = -Q_i^{14},$$
$$\Sigma_{i,j}^{14,14} = -Q_i^{16}, \qquad \Sigma_{i,j}^{14,20} = -Q_i^{17}, \qquad \Sigma_{i,j}^{15,15} = -Q_i^3 - P_i^{14}, \qquad \Sigma_{i,j}^{16,16} = -Q_i^6,$$
$$\Sigma_{i,j}^{17,17} = -Q_i^9 - P_i^{15}, \qquad \Sigma_{i,j}^{18,18} = -Q_i^{12}, \qquad \Sigma_{i,j}^{19,19} = -Q_i^{15} - P_i^{16},$$
$$\Sigma_{i,j}^{21,21} = -Z_i^1, \qquad \Sigma_{i,j}^{21,22} = -B_i^{1^T} L_j^3, \qquad \Sigma_{i,j}^{21,25} = L_i^{5^T}, \qquad \Sigma_{i,j}^{21,26} = L_i^{5^T},$$
$$\Sigma_{i,j}^{21,27} = L_i^{5^T} C_j, \qquad \Sigma_{i,j}^{22,22} = L_i^{3^T} A_j^2 + A_i^{2^T} L_j^3 - Z_i^2, \qquad \Sigma_{i,j}^{22,25} = L_i^{3^T},$$
$$\Sigma_{i,j}^{22,26} = L_i^{3^T}, \qquad \Sigma_{i,j}^{22,27} = L_i^{3^T} C_j, \qquad \Sigma_{i,j}^{23,23} = -P_i^{18}, \qquad \Sigma_{i,j}^{24,24} = -P_i^{18},$$
$$\Sigma_{i,j}^{25,25} = -\epsilon_1 I, \qquad \Sigma_{i,j}^{26,26} = -\epsilon_2 I, \qquad \Sigma_{i,j}^{27,27} = -\frac{P_i^{19}}{w}, \qquad G_i^1 = P_i^1 J,$$
$$G_i^2 = P_i^1 W, \qquad \rho = h_2 - h_1, \qquad \varphi(k) = h(k) - h_1, \qquad \beta(k) = h_2 - h(k),$$
$$\psi(k) = \frac{1}{\varphi(k)}\left[\sum_{i=k-h(k)}^{k-h_1-1} \xi(i)\right], \qquad \phi(k) = \frac{1}{\beta(k)}\left[\sum_{i=k-h_2}^{k-h(k)-1} \xi(i)\right],$$

and other terms are 0.

Theorem 3.1 *System* (1) *is robustly stable if there exist positive definite symmetric matrices* $M_i, Z_i^1, Z_i^2, P_i^k, Q_i^s$ ($k = 1,2,\ldots,19, s = 1,2,\ldots,18, i = 1,2,\ldots,N$), *any matrices* $J, W, M_i^r, N_i^r, L_i^l$ ($r = 1,2,3,4$, $l = 1,2,3$, $i = 1,2,\ldots,N$) *of appropriate dimensions, and positive real constants* ϵ_1 *and* ϵ_2 *satisfying the following LMIs*:

$$\prod_{i,i} < -I, \quad i = 1,2,\ldots,N, \tag{7}$$

$$\prod_{i,j} + \prod_{j,i} < \frac{2}{(N-1)} I, \quad i = 1,\ldots,N-1, j = i+1,\ldots,N, \tag{8}$$

$$\begin{bmatrix} Q_i^{3t+1} & Q_i^{3t+2} \\ * & Q_i^{3t+3} \end{bmatrix} \geq 0, \quad t = 0,1,2,\ldots,5, i = 1,2,\ldots,N, \tag{9}$$

$$\begin{bmatrix} M_i^l & N_i^l \\ * & P_i^{9+l} \end{bmatrix} \geq 0, \quad l = 1,2,\ldots,4, i = 1,2,\ldots,N. \tag{10}$$

Proof Applying a model transformation method, system (1) can be replaced by the system

$$\xi(k+1) = \xi(k) + \eta(k), \tag{11}$$

$$\eta(k) = \left[A(\varphi) - I\right]\xi(k) + B(\varphi)\xi\left(k - h(k)\right) + C(\varphi)\sum_{i=1}^{\infty}\delta(i)\xi(k-i)$$

$$+ f_1\bigl(k,\xi(k)\bigr) + f_2\bigl(k,\xi\bigl(k-h(k)\bigr)\bigr). \tag{12}$$

To improve the bound for $h(k)$ in (1), we decompose constant matrices A and B as

$$A(\varphi) = A_1(\varphi) + A_2(\varphi), \tag{13}$$

$$B(\varphi) = B_1(\varphi) + B_2(\varphi), \tag{14}$$

where $A_1(\varphi) = \sum_{i=1}^{N} \varphi_i A_i^1, A_2(\varphi) = \sum_{i=1}^{N} \varphi_i A_i^2, B_1(\varphi) = \sum_{i=1}^{N} \varphi_i B_i^1, B_2(\varphi) = \sum_{i=1}^{N} \varphi_i B_i^2$, and $A_i^1, A_i^2, B_i^1, B_i^2 \in R^{n\times n}$ are constant matrices for all $i = 1,2,3,\ldots,N$. By utilizing the following zero equations, we have

$$\xi(k) - \xi\bigl(k-h(k)\bigr) - \sum_{i=k-h(k)}^{k-1} \eta(i) = 0, \tag{15}$$

$$\xi(k) - \xi\bigl(k-rh(k)\bigr) - \sum_{i=k-rh(k)}^{k-1} \eta(i) = 0. \tag{16}$$

From (15) and (16), we obtain

$$J\xi(k) - J\xi\bigl(k-h(k)\bigr) - J\sum_{i=k-h(k)}^{k-1} \eta(i) = 0, \tag{17}$$

$$W\xi(k) - W\xi\bigl(k-rh(k)\bigr) - W\sum_{i=k-rh(k)}^{k-1} \eta(i) = 0, \tag{18}$$

where $J, W \in R^{n\times n}$ are selected so that the stability criterion of the given system is guaranteed. Substituting (17)–(18) into the system (11)–(12), we obtain

$$\begin{aligned}\xi(k+1) = {} & \xi(k) + \eta(k) + J\xi(k) - J\xi\bigl(k-h(k)\bigr) - J\sum_{i=k-h(k)}^{k-1} \eta(i) \\ & + W\xi(k) - W\xi\bigl(k-rh(k)\bigr) - W\sum_{i=k-rh(k)}^{k-1} \eta(i),\end{aligned} \tag{19}$$

$$\begin{aligned}\eta(k) = {} & \bigl[A_1(\varphi) + B_1(\varphi) - I\bigr]\xi(k) + A_2(\varphi)\xi\bigl(k-rh(k)\bigr) \\ & + B_2(\varphi)\xi\bigl(k-h(k)\bigr) - B_1(\varphi)\sum_{i=k-h(k)}^{k-1} \eta(i) + A_2(\varphi)\sum_{i=k-rh(k)}^{k-1} \eta(i) \\ & + C(\varphi)\sum_{i=1}^{\infty} \delta(i)\xi(k-i) + f_1\bigl(k,\xi(k)\bigr) + f_2\bigl(k,\xi\bigl(k-h(k)\bigr)\bigr).\end{aligned} \tag{20}$$

We define a parameter-dependent Lyapunov–Krasovskii functional for the system (19)–(20) as

$$V\bigl(k,\xi(k)\bigr) = \sum_{i=1}^{8} V_i\bigl(k,\xi(k)\bigr), \tag{21}$$

where

$$\begin{aligned}
V_1\bigl(k,\xi(k)\bigr) &= \xi^T(k)P_1(\varphi)\xi(k),\\
V_2\bigl(k,\xi(k)\bigr) &= \sum_{i=k-h_1}^{k-1}\xi^T(i)P_2(\varphi)\xi(i)+\sum_{i=k-rh_1}^{k-1}\xi^T(i)P_3(\varphi)\xi(i)\\
&\quad+\sum_{i=k-h_2}^{k-1}\xi^T(i)P_4(\varphi)\xi(i)+\sum_{i=k-rh_2}^{k-1}\xi^T(i)P_5(\varphi)\xi(i)\\
&\quad+\sum_{i=k-h_2}^{k-h_1-1}\xi^T(i)P_6(\varphi)\xi(i)+\sum_{i=k-rh_2}^{k-rh_1-1}\xi^T(i)P_7(\varphi)\xi(i)\\
&\quad+\sum_{i=k-h(k)}^{k-1}\xi^T(i)P_8(\varphi)\xi(i)+\sum_{i=k-rh(k)}^{k-1}\xi^T(i)P_9(\varphi)\xi(i),\\
V_3\bigl(k,\xi(k)\bigr) &= h_1\sum_{j=-h_1}^{-1}\sum_{i=k+j}^{k-1}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}^T\begin{bmatrix}Q_1(\varphi) & Q_2(\varphi)\\ * & Q_3(\varphi)\end{bmatrix}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}\\
&\quad+rh_1\sum_{j=-rh_1}^{-1}\sum_{i=k+j}^{k-1}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}^T\begin{bmatrix}Q_4(\varphi) & Q_5(\varphi)\\ * & Q_6(\varphi)\end{bmatrix}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}\\
&\quad+h_2\sum_{j=-h_2}^{-1}\sum_{i=k+j}^{k-1}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}^T\begin{bmatrix}Q_7(\varphi) & Q_8(\varphi)\\ * & Q_9(\varphi)\end{bmatrix}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}\\
&\quad+rh_2\sum_{j=-rh_2}^{-1}\sum_{i=k+j}^{k-1}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}^T\begin{bmatrix}Q_{10}(\varphi) & Q_{11}(\varphi)\\ * & Q_{12}(\varphi)\end{bmatrix}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}\\
&\quad+\rho\sum_{j=-h_2}^{-h_1-1}\sum_{i=k+j}^{k-1}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}^T\begin{bmatrix}Q_{13}(\varphi) & Q_{14}(\varphi)\\ * & Q_{15}(\varphi)\end{bmatrix}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}\\
&\quad+r\rho\sum_{j=-rh_2}^{-rh_1-1}\sum_{i=k+j}^{k-1}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix}^T\begin{bmatrix}Q_{16}(\varphi) & Q_{17}(\varphi)\\ * & Q_{18}(\varphi)\end{bmatrix}\begin{bmatrix}\xi(i)\\ \eta(i)\end{bmatrix},\\
V_4\bigl(k,\xi(k)\bigr) &= \sum_{j=-h_1+1}^{0}\sum_{i=k-1+j}^{k-1}\eta^T(i)P_{10}(\varphi)\eta(i)\\
&\quad+\sum_{j=-h_2+1}^{0}\sum_{i=k-1+j}^{k-1}\eta^T(i)P_{11}(\varphi)\eta(i)\\
&\quad+\sum_{j=-rh_1+1}^{0}\sum_{i=k-1+j}^{k-1}\eta^T(i)P_{12}(\varphi)\eta(i)\\
&\quad+\sum_{j=-rh_2+1}^{0}\sum_{i=k-1+j}^{k-1}\eta^T(i)P_{13}(\varphi)\eta(i)\\
&\quad+\sum_{i=1}^{\infty}\delta(i)\sum_{j=k-i}^{k-1}\xi_j^TM(\varphi)\xi_j,
\end{aligned}$$

$$
\begin{aligned}
V_5\bigl(k,\xi(k)\bigr) &= h_1 \sum_{j=-h_1}^{-1} \sum_{i=k+j}^{k-1} \eta^T(i)P_{14}(\varphi)\eta(i) \\
&\quad + h_2 \sum_{j=-h_2}^{-1} \sum_{i=k+j}^{k-1} \eta^T(i)P_{15}(\varphi)\eta(i) \\
&\quad + \rho \sum_{j=-h_2}^{-h_1-1} \sum_{i=k+j}^{k-1} \eta^T(i)P_{16}(\varphi)\eta(i), \\
V_6\bigl(k,\xi(k)\bigr) &= h_2 \sum_{j=-h_2}^{-1} \sum_{i=k+j}^{k-1} \eta^T(i)Z_1(\varphi)\eta(i) \\
&\quad + rh_2 \sum_{j=-rh_2}^{-1} \sum_{i=k+j}^{k-1} \eta^T(i)Z_2(\varphi)\eta(i), \\
V_7\bigl(k,\xi(k)\bigr) &= \frac{h_1^2}{2} \sum_{i=-h_1}^{-1} \sum_{j=i}^{0} \sum_{l=k+j}^{k-1} \eta^T(l)P_{17}(\varphi)\eta(l) \\
&\quad + \frac{1}{2} \sum_{i=k-h_2}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} \sum_{l=j}^{k-h_1-1} \eta^T(l)P_{18}(\varphi)\eta(l), \\
V_8\bigl(k,\xi(k)\bigr) &= \sum_{i=1}^{\infty} \delta(i) \sum_{j=k-i}^{k-1} \xi^T(j)P_{19}(\varphi)\xi(j).
\end{aligned}
$$

From (21), the forward difference of $V(k,\xi(k))$ is defined by

$$
\Delta V\bigl(k,\xi(k)\bigr) = \sum_{i=1}^{8} \Delta V_i\bigl(k,\xi(k)\bigr), \tag{22}
$$

where

$$
\Delta V_i\bigl(k,\xi(k)\bigr) = V_i\bigl(k+1,\xi(k+1)\bigr) - V_i\bigl(k,\xi(k)\bigr) \tag{23}
$$

for $i = 1,2,\dots,8$. The increments of $V_1(k,\xi(k))$, $V_2(k,\xi(k))$, and $V_3(k,\xi(k))$ are obtained by taking the forward difference of $V_1(k,\xi(k))$, $V_2(k,\xi(k))$, and $V_3(k,\xi(k))$, respectively. Then, we have

$$
\begin{aligned}
\Delta V_1\bigl(k,\xi(k)\bigr) &= \xi^T(k+1)P_1(\varphi)\xi(k+1) - \xi^T(k)P_1(\varphi)\xi(k) \\
&= \bigl[\xi^T(k) + \eta^T(k)\bigr]P_1(\varphi)\Biggl[\xi(k) + \eta(k) + J\xi(k) \\
&\quad - J\xi\bigl(k-h(k)\bigr) - J \sum_{i=k-h(k)}^{k-1} \eta(i) + W\xi(k) \\
&\quad - W\xi\bigl(k-rh(k)\bigr) - W \sum_{i=k-rh(k)}^{k-1} \eta(i)\Biggr]
\end{aligned}
$$

$$
\begin{aligned}
&+ \Biggl[\xi^T(k)J^T - \xi^T\bigl(k-h(k)\bigr)J^T \\
&- \sum_{i=k-h(k)}^{k-1} \eta^T(i)J^T + \xi^T(k)W^T - \xi^T\bigl(k-rh(k)\bigr)W^T \\
&- \sum_{i=k-rh(k)}^{k-1} \eta^T(i)W^T\Biggr] P_1(\varphi)\bigl[\xi(k)+\eta(k)\bigr] \\
&+ 2\xi^T(k)L_1^T(\varphi)\Biggl[-\eta(k) + \bigl[A_1(\varphi)+B_1(\varphi)-I\bigr]\xi(k) \\
&+ A_2(\varphi)\xi\bigl(k-rh(k)\bigr) + B_2(\varphi)\xi\bigl(k-h(k)\bigr) \\
&- B_1(\varphi)\sum_{i=k-h(k)}^{k-1}\eta(i) + A_2(\varphi)\sum_{i=k-rh(k)}^{k-1}\eta(i) \qquad (24) \\
&+ C(\varphi)\sum_{i=1}^{\infty}\delta(i)\xi(k-i) + f_1\bigl(k,\xi(k)\bigr) + f_2\bigl(k,\xi\bigl(k-h(k)\bigr)\bigr)\Biggr] \\
&+ 2\eta^T(k)L_2^T(\varphi)\Biggl[-\eta(k) + \bigl[A_1(\varphi)+B_1(\varphi)-I\bigr]\xi(k) \\
&+ A_2(\varphi)\xi\bigl(k-rh(k)\bigr) + B_2(\varphi)\xi\bigl(k-h(k)\bigr) \\
&- B_1(\varphi)\sum_{i=k-h(k)}^{k-1}\eta(i) + A_2(\varphi)\sum_{i=k-rh(k)}^{k-1}\eta(i) \\
&+ C(\varphi)\sum_{i=1}^{\infty}\delta(i)\xi(k-i) + f_1\bigl(k,\xi(k)\bigr) + f_2\bigl(k,\xi\bigl(k-h(k)\bigr)\bigr)\Biggr] \\
&+ 2\sum_{i=k-rh(k)}^{k-1}\eta^T(i)L_3^T(\varphi)\Biggl[-\eta(k) + \bigl[A_1(\varphi)+B_1(\varphi)-I\bigr]\xi(k) \\
&+ A_2(\varphi)\xi\bigl(k-rh(k)\bigr) + B_2(\varphi)\xi\bigl(k-h(k)\bigr) \\
&- B_1(\varphi)\sum_{i=k-h(k)}^{k-1}\eta(i) + A_2(\varphi)\sum_{i=k-rh(k)}^{k-1}\eta(i) \\
&+ C(\varphi)\sum_{i=1}^{\infty}\delta(i)\xi(k-i) + f_1\bigl(k,\xi(k)\bigr) + f_2\bigl(k,\xi\bigl(k-h(k)\bigr)\bigr)\Biggr] \\
&- \xi^T(k)P_1(\varphi)\xi(k), \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
\Delta V_2\bigl(k,\xi(k)\bigr) &= \xi^T(k)P_2(\varphi)\xi(k) - \xi^T(k-h_1)P_2(\varphi)\xi(k-h_1) \\
&+ \xi^T(k)P_3(\varphi)\xi(k) - \xi^T(k-rh_1)P_3(\varphi)\xi(k-rh_1) \\
&+ \xi^T(k)P_4(\varphi)\xi(k) - \xi^T(k-h_2)P_4(\varphi)\xi(k-h_2) \\
&+ \xi^T(k)P_5(\varphi)\xi(k) - \xi^T(k-rh_2)P_5(\varphi)\xi(k-rh_2) \\
&+ \xi^T(k-h_1)P_6(\varphi)\xi(k-h_1) - \xi^T(k-h_2)P_6(\varphi)\xi(k-h_2) \\
&+ \xi^T(k-rh_1)P_7(\varphi)\xi(k-rh_1)
\end{aligned}
$$

$$
\begin{aligned}
&- \xi^T(k - rh_2)P_7(\varphi)\xi(k - rh_2) \\
&+ \xi^T(k)P_8(\varphi)\xi(k) - \xi^T\bigl(k - h(k)\bigr)P_8(\varphi)\xi\bigl(k - h(k)\bigr) \\
&+ \xi^T(k)P_9(\varphi)\xi(k) - \xi^T\bigl(k - rh(k)\bigr)P_9(\varphi)\xi\bigl(k - rh(k)\bigr),
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\Delta V_3\bigl(k,\xi(k)\bigr) \le{}& \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix}^T \begin{bmatrix} h_1^2Q_1(\varphi) & h_1^2Q_2(\varphi) \\ * & h_1^2Q_3(\varphi) \end{bmatrix} \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix} \\
&- \begin{bmatrix} \sum_{i=k-h_1}^{k-1} \xi(i) \\ \sum_{i=k-h_1}^{k-1} \eta(i) \end{bmatrix}^T \begin{bmatrix} Q_1(\varphi) & Q_2(\varphi) \\ * & Q_3(\varphi) \end{bmatrix} \begin{bmatrix} \sum_{i=k-h_1}^{k-1} \xi(i) \\ \sum_{i=k-h_1}^{k-1} \eta(i) \end{bmatrix} \\
&+ \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix}^T \begin{bmatrix} r^2h_1^2Q_4(\varphi) & r^2h_1^2Q_5(\varphi) \\ * & r^2h_1^2Q_6(\varphi) \end{bmatrix} \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix} \\
&- \begin{bmatrix} \sum_{i=k-rh_1}^{k-1} \xi(i) \\ \sum_{i=k-rh_1}^{k-1} \eta(i) \end{bmatrix}^T \begin{bmatrix} Q_4(\varphi) & Q_5(\varphi) \\ * & Q_6(\varphi) \end{bmatrix} \begin{bmatrix} \sum_{i=k-rh_1}^{k-1} \xi(i) \\ \sum_{i=k-rh_1}^{k-1} \eta(i) \end{bmatrix} \\
&+ \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix}^T \begin{bmatrix} h_2^2Q_7(\varphi) & h_2^2Q_8(\varphi) \\ * & h_2^2Q_9(\varphi) \end{bmatrix} \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix} \\
&- \begin{bmatrix} \sum_{i=k-h_2}^{k-1} \xi(i) \\ \sum_{i=k-h_2}^{k-1} \eta(i) \end{bmatrix}^T \begin{bmatrix} Q_7(\varphi) & Q_8(\varphi) \\ * & Q_9(\varphi) \end{bmatrix} \begin{bmatrix} \sum_{i=k-h_2}^{k-1} \xi(i) \\ \sum_{i=k-h_2}^{k-1} \eta(i) \end{bmatrix} \\
&+ \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix}^T \begin{bmatrix} r^2h_2^2Q_{10}(\varphi) & r^2h_2^2Q_{11}(\varphi) \\ * & r^2h_2^2Q_{12}(\varphi) \end{bmatrix} \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix} \\
&- \begin{bmatrix} \sum_{i=k-rh_2}^{k-1} \xi(i) \\ \sum_{i=k-rh_2}^{k-1} \eta(i) \end{bmatrix}^T \begin{bmatrix} Q_{10}(\varphi) & Q_{11}(\varphi) \\ * & Q_{12}(\varphi) \end{bmatrix} \begin{bmatrix} \sum_{i=k-rh_2}^{k-1} \xi(i) \\ \sum_{i=k-rh_2}^{k-1} \eta(i) \end{bmatrix} \\
&+ \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix}^T \begin{bmatrix} \rho^2Q_{13}(\varphi) & \rho^2Q_{14}(\varphi) \\ * & \rho^2Q_{15}(\varphi) \end{bmatrix} \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix} \\
&- \begin{bmatrix} \sum_{i=k-h_2}^{k-h_1-1} \xi(i) \\ \sum_{i=k-h_2}^{k-h_1-1} \eta(i) \end{bmatrix}^T \begin{bmatrix} Q_{13}(\varphi) & Q_{14}(\varphi) \\ * & Q_{15}(\varphi) \end{bmatrix} \begin{bmatrix} \sum_{i=k-h_2}^{k-h_1-1} \xi(i) \\ \sum_{i=k-h_2}^{k-h_1-1} \eta(i) \end{bmatrix} \\
&+ \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix}^T \begin{bmatrix} r^2\rho^2Q_{16}(\varphi) & r^2\rho^2Q_{17}(\varphi) \\ * & r^2\rho^2Q_{18}(\varphi) \end{bmatrix} \begin{bmatrix} \xi(k) \\ \eta(k) \end{bmatrix} \\
&- \begin{bmatrix} \sum_{i=k-rh_2}^{k-rh_1-1} \xi(i) \\ \sum_{i=k-rh_2}^{k-rh_1-1} \eta(i) \end{bmatrix}^T \begin{bmatrix} Q_{16}(\varphi) & Q_{17}(\varphi) \\ * & Q_{18}(\varphi) \end{bmatrix} \begin{bmatrix} \sum_{i=k-rh_2}^{k-rh_1-1} \xi(i) \\ \sum_{i=k-rh_2}^{k-rh_1-1} \eta(i) \end{bmatrix}.
\end{aligned} \tag{27}
$$

Taking the forward difference of $V_4(k,\xi(k))$ yields

$$
\begin{aligned}
&\Delta V_4\bigl(k,\xi(k)\bigr) \\
&\quad= h_1\eta^T(k)P_{10}(\varphi)\eta(k) - \sum_{i=-h_1+1}^{0} \eta^T(k-1+i)P_{10}(\varphi)\eta(k-1+i) \\
&\qquad+ h_2\eta^T(k)P_{11}(\varphi)\eta(k) - \sum_{i=-h_2+1}^{0} \eta^T(k-1+i)P_{11}(\varphi)
\end{aligned}
$$

$$\begin{aligned}
&\times \eta(k-1+i) + rh_1\eta^T(k)P_{12}(\varphi)\eta(k) - \sum_{i=-rh_1+1}^{0}\eta^T(k-1+i) \\
&\times P_{12}(\varphi)\eta(k-1+i) + rh_2\eta^T(k)P_{13}(\varphi)\eta(k) \\
&- \sum_{i=-rh_2+1}^{0}\eta^T(k-1+i)P_{13}(\varphi)\eta(k-1+i).
\end{aligned}$$

By (10), it is easy to see that

$$\begin{aligned}
&2\xi^T(k)N_1(\varphi)\sum_{i=-h_1+1}^{0}\eta(k-1+i) + \sum_{i=-h_1+1}^{0}\eta^T(k-1+i)P_{10}(\varphi)\eta(k-1+i) \\
&\quad + h_1\xi^T(k)M_1(\varphi)\xi(k) \\
&\quad = \sum_{i=-h_1+1}^{0}\begin{bmatrix}\xi(k) \\ \eta(k-1+i)\end{bmatrix}^T\begin{bmatrix}M_1(\varphi) & N_1(\varphi) \\ * & P_{10}(\varphi)\end{bmatrix}\begin{bmatrix}\xi(k) \\ \eta(k-1+i)\end{bmatrix} \geq 0, \qquad (28)\\
&2\xi^T(k)N_2(\varphi)\sum_{i=-h_2+1}^{0}\eta(k-1+i) + \sum_{i=-h_2+1}^{0}\eta^T(k-1+i)P_{11}(\varphi)\eta(k-1+i) \\
&\quad + h_2\xi^T(k)M_2(\varphi)\xi(k) \\
&\quad = \sum_{i=-h_2+1}^{0}\begin{bmatrix}\xi(k) \\ \eta(k-1+i)\end{bmatrix}^T\begin{bmatrix}M_2(\varphi) & N_2(\varphi) \\ * & P_{11}(\varphi)\end{bmatrix}\begin{bmatrix}\xi(k) \\ \eta(k-1+i)\end{bmatrix} \geq 0, \qquad (29)\\
&2\xi^T(k)N_3(\varphi)\sum_{i=-rh_1+1}^{0}\eta(k-1+i) + \sum_{i=-rh_1+1}^{0}\eta^T(k-1+i)P_{12}(\varphi) \\
&\quad \times \eta(k-1+i) + rh_1\xi^T(k)M_3(\varphi)\xi(k) \\
&\quad = \sum_{i=-rh_1+1}^{0}\begin{bmatrix}\xi(k) \\ \eta(k-1+i)\end{bmatrix}^T\begin{bmatrix}M_3(\varphi) & N_3(\varphi) \\ * & P_{12}(\varphi)\end{bmatrix}\begin{bmatrix}\xi(k) \\ \eta(k-1+i)\end{bmatrix} \geq 0, \qquad (30)\\
&2\xi^T(k)N_4(\varphi)\sum_{i=-rh_2+1}^{0}\eta(k-1+i) + \sum_{i=-rh_2+1}^{0}\eta^T(k-1+i)P_{13}(\varphi) \\
&\quad \times \eta(k-1+i) + rh_2\xi^T(k)M_4(\varphi)\xi(k) \\
&\quad = \sum_{i=-rh_2+1}^{0}\begin{bmatrix}\xi(k) \\ \eta(k-1+i)\end{bmatrix}^T\begin{bmatrix}M_4(\varphi) & N_4(\varphi) \\ * & P_{13}(\varphi)\end{bmatrix}\begin{bmatrix}\xi(k) \\ \eta(k-1+i)\end{bmatrix} \geq 0. \qquad (31)
\end{aligned}$$

From (28), (29), (30), and (31), we obtain

$$\begin{aligned}
&-\sum_{i=-h_1+1}^{0}\eta^T(k-1+i)P_{10}(\varphi)\eta(k-1+i) \\
&\quad \leq h_1\xi^T(k)M_1(\varphi)\xi(k) + 2\xi^T(k)N_1(\varphi)\sum_{i=-h_1+1}^{0}\eta(k-1+i) \\
&\quad = h_1\xi^T(k)M_1(\varphi)\xi(k) + 2\xi^T(k)N_1(\varphi)\xi(k) - 2\xi^T(k)N_1(\varphi)\xi(k-h_1), \qquad (32)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=-h_2+1}^{0} \eta^T(k-1+i)P_{11}(\varphi)\eta(k-1+i) \\
&\quad \le h_2\xi^T(k)M_2(\varphi)\xi(k) + 2\xi^T(k)N_2(\varphi)\sum_{i=-h_2+1}^{0}\eta(k-1+i) \\
&\quad = h_2\xi^T(k)M_2(\varphi)\xi(k) + 2\xi^T(k)N_2(\varphi)\xi(k) - 2\xi^T(k)N_2(\varphi)\xi(k-h_2), \qquad (33) \\
&-\sum_{i=-rh_1+1}^{0} \eta^T(k-1+i)P_{12}(\varphi)\eta(k-1+i) \\
&\quad \le rh_1\xi^T(k)M_3(\varphi)\xi(k) + 2\xi^T(k)N_3(\varphi)\sum_{i=-rh_1+1}^{0}\eta(k-1+i) \\
&\quad = rh_1\xi^T(k)M_3(\varphi)\xi(k) + 2\xi^T(k)N_3(\varphi)\xi(k) \\
&\qquad - 2\xi^T(k)N_3(\varphi)\xi(k-rh_1), \qquad (34) \\
&-\sum_{i=-rh_2+1}^{0} \eta^T(k-1+i)P_{13}(\varphi)\eta(k-1+i) \\
&\quad \le rh_2\xi^T(k)M_4(\varphi)\xi(k) + 2\xi^T(k)N_4(\varphi)\sum_{i=-rh_2+1}^{0}\eta(k-1+i) \\
&\quad = rh_2\xi^T(k)M_4(\varphi)\xi(k) + 2\xi^T(k)N_4(\varphi)\xi(k) \\
&\qquad - 2\xi^T(k)N_4(\varphi)\xi(k-rh_2). \qquad (35)
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
&\Delta V_4\bigl(k,\xi(k)\bigr) \\
&\quad \le h_1\eta^T(k)P_{10}(\varphi)\eta(k) + h_1\xi^T(k)M_1(\varphi)\xi(k) + 2\xi^T(k)N_1(\varphi)\xi(k) \\
&\qquad - 2\xi^T(k)N_1(\varphi)\xi(k-h_1) + h_2\eta^T(k)P_{11}(\varphi)\eta(k) \\
&\qquad + h_2\xi^T(k)M_2(\varphi)\xi(k) + 2\xi^T(k)N_2(\varphi)\xi(k) \\
&\qquad - 2\xi^T(k)N_2(\varphi)\xi(k-h_2) + rh_1\eta^T(k)P_{12}(\varphi)\eta(k) \\
&\qquad + rh_1\xi^T(k)M_3(\varphi)\xi(k) + 2\xi^T(k)N_3(\varphi)\xi(k) \\
&\qquad - 2\xi^T(k)N_3(\varphi)\xi(k-rh_1) + rh_2\eta^T(k)P_{13}(\varphi)\eta(k) \\
&\qquad + rh_2\xi^T(k)M_4(\varphi)\xi(k) + 2\xi^T(k)N_4(\varphi)\xi(k) \\
&\qquad - 2\xi^T(k)N_4(\varphi)\xi(k-rh_2). \qquad (36)
\end{aligned}
$$

By Lemma 2.2, the increments of $V_5(k,\xi(k))$ and $V_6(k,\xi(k))$ can be expressed mathematically as

$$
\begin{aligned}
&\Delta V_5\bigl(k,\xi(k)\bigr) \\
&\quad = h_1^2\eta^T(k)P_{14}(\varphi)\eta(k) - h_1\sum_{i=k-h_1}^{k-1}\eta^T(i)P_{14}(\varphi)\eta(i)
\end{aligned}
$$

$$
\begin{aligned}
&+ h_2^2\eta^T(k)P_{15}(\varphi)\eta(k) - h_2\sum_{i=k-h_2}^{k-1}\eta^T(i)P_{15}(\varphi)\eta(i)\\
&+ \rho^2\eta^T(k)P_{16}(\varphi)\eta(k) - \rho\sum_{i=k-h_2}^{k-h_1-1}\eta^T(i)P_{16}(\varphi)\eta(i)\\
\le{}& h_1^2\eta^T(k)P_{14}(\varphi)\eta(k) - \left(\sum_{i=k-h_1}^{k-1}\eta(i)\right)^T P_{14}(\varphi)\left(\sum_{i=k-h_1}^{k-1}\eta(i)\right)\\
&+ h_2^2\eta^T(k)P_{15}(\varphi)\eta(k) - \left(\sum_{i=k-h_2}^{k-1}\eta(i)\right)^T P_{15}(\varphi)\left(\sum_{i=k-h_2}^{k-1}\eta(i)\right)\\
&+ \rho^2\eta^T(k)P_{16}(\varphi)\eta(k) - \left(\sum_{i=k-h_2}^{k-h_1-1}\eta(i)\right)^T P_{16}(\varphi)\left(\sum_{i=k-h_2}^{k-h_1-1}\eta(i)\right),
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&\Delta V_6\bigl(k,\xi(k)\bigr)\\
&= h_2^2\eta^T(k)Z_1(\varphi)\eta(k) - h_2\sum_{i=k-h_2}^{k-1}\eta^T(i)Z_1(\varphi)\eta(i)\\
&\quad + (rh_2)^2\eta^T(k)Z_2(\varphi)\eta(k) - rh_2\sum_{i=k-rh_2}^{k-1}\eta^T(i)Z_2(\varphi)\eta(i)\\
&\le h_2^2\eta^T(k)Z_1(\varphi)\eta(k) - \left(\sum_{i=k-h(k)}^{k-1}\eta(i)\right)^T Z_1(\varphi)\left(\sum_{i=k-h(k)}^{k-1}\eta(i)\right)\\
&\quad + (rh_2)^2\eta^T(k)Z_2(\varphi)\eta(k) - \left(\sum_{i=k-rh(k)}^{k-1}\eta(i)\right)^T Z_2(\varphi)\\
&\quad \times \left(\sum_{i=k-rh(k)}^{k-1}\eta(i)\right).
\end{aligned}
\tag{38}
$$

The forward difference of $V_7(k,\xi(k))$ can be obtained as

$$
\begin{aligned}
\Delta V_7\bigl(k,\xi(k)\bigr) = \eta^T(k)\left[\frac{h_1^4}{4}P_{17}(\varphi) + \frac{(h_2-h_1)^2}{4}P_{18}(\varphi)\right]\eta(k)\\
- \frac{h_1^2}{2}\sum_{i=-h_1}^{-1}\sum_{j=k+i}^{k-1}\eta^T(j)P_{17}(\varphi)\eta(j)\\
- \frac{1}{2}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1}\eta^T(j)P_{18}(\varphi)\eta(j).
\end{aligned}
\tag{39}
$$

By Lemma 2.3, we obtain

$$
-\frac{h_1^2}{2}\sum_{i=-h_1}^{-1}\sum_{j=k+i}^{k-1}\eta^T(j)P_{17}(\varphi)\eta(j)
$$

$$\leq -\left[h_1\xi(k) - \sum_{i=k-h_1}^{k-1}\xi(i)\right]^T P_{17}(\varphi)\left[h_1\xi(k) - \sum_{i=k-h_1}^{k-1}\xi(i)\right], \tag{40}$$

$$-\frac{1}{2}\sum_{i=k-h_2}^{k-h_1-1}\sum_{j=i}^{k-h_1-1}\eta^T(j)P_{18}(\varphi)\eta(j)$$

$$\begin{aligned}
&= -\frac{1}{2}\sum_{i=k-h(k)}^{k-h_1-1}\sum_{j=i}^{k-h_1-1}\eta^T(j)P_{18}(\varphi)\eta(j) - \frac{1}{2}\sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=i}^{k-h(k)-1}\eta^T(j)P_{18}(\varphi)\eta(j)\\
&\quad -\frac{1}{2}\sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=k-h(k)}^{k-h_1-1}\eta^T(j)P_{18}(\varphi)\eta(j)\\
&\leq -\left[\sum_{i=k-h(k)}^{k-h_1-1}\sum_{j=i}^{k-h_1-1}\eta(j)\right]^T \frac{1}{\varphi^2(k)}P_{18}(\varphi)\left[\sum_{i=k-h(k)}^{k-h_1-1}\sum_{j=i}^{k-h_1-1}\eta(j)\right]\\
&\quad -\left[\sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=i}^{k-h(k)-1}\eta(j)\right]^T \frac{1}{\beta^2(k)}P_{18}(\varphi)\left[\sum_{i=k-h_2}^{k-h(k)-1}\sum_{j=i}^{k-h(k)-1}\eta(j)\right]\\
&= -\left[\xi(k-h_1) - \psi(k)\right]^T P_{18}(\varphi)\left[\xi(k-h_1) - \psi(k)\right]\\
&\quad -\left[\xi\big(k-h(k)\big) - \phi(k)\right]^T P_{18}(\varphi)\left[\xi\big(k-h(k)\big) - \phi(k)\right].
\end{aligned} \tag{41}$$

Substituting (40) and (41) into (39), we obtain

$$\begin{aligned}
\Delta V_7\big(k,\xi(k)\big) &\leq \eta^T(k)\left[\frac{h_1^4}{4}P_{17}(\varphi) + \frac{(h_2-h_1)^2}{4}P_{18}(\varphi)\right]\eta(k)\\
&\quad -\left[h_1\xi(k) - \sum_{i=k-h_1}^{k-1}\xi(i)\right]^T P_{17}(\varphi)\left[h_1\xi(k) - \sum_{i=k-h_1}^{k-1}\xi(i)\right]\\
&\quad -\left[\xi(k-h_1) - \psi(k)\right]^T P_{18}(\varphi)\left[\xi(k-h_1) - \psi(k)\right]\\
&\quad -\left[\xi\big(k-h(k)\big) - \phi(k)\right]^T P_{18}(\varphi)\left[\xi\big(k-h(k)\big) - \phi(k)\right].
\end{aligned} \tag{42}$$

Taking the forward difference of $V_8(k,\xi(k))$, we obtain

$$\begin{aligned}
\Delta V_8\big(k,\xi(k)\big) &= w\xi^T(k)P_{19}(\varphi)\xi(k)\\
&\quad -\frac{1}{w}\left[\sum_{i=1}^{\infty}\delta(i)\xi(k-i)\right]^T P_{19}(\varphi)\left[\sum_{i=1}^{\infty}\delta(i)\xi(k-i)\right].
\end{aligned} \tag{43}$$

From (3) and (4), for any scalars $\epsilon_1, \epsilon_2 > 0$, we obtain

$$\epsilon_1\big(\varphi^2\xi^T(k)\xi(k) - f_1^T\big(k,\xi(k)\big)f_1\big(k,\xi(k)\big)\big) \geq 0, \tag{44}$$

$$\epsilon_2\big(\beta^2\xi^T\big(k-h(k)\big)\xi\big(k-h(k)\big) - f_2^T\big(k,\xi\big(k-h(k)\big)\big)f_2\big(k,\xi\big(k-h(k)\big)\big)\big) \geq 0. \tag{45}$$

It follows from (22)–(45) that

$$\Delta V\big(k,\xi(k)\big) \leq \zeta^T(k)\sum_{i=1}^{N}\sum_{j=1}^{N}\varphi_i\varphi_j\prod_{i,j}\zeta(k), \tag{46}$$

where $\zeta^T(k) = [\xi^T(k)\eta^T(k)\xi^T(k-h_1)\xi^T(k-h_2)\xi^T(k-h(k))\xi^T(k-rh_1) \times \xi^T(k-rh_2)\xi^T(k-rh(k))\sum_{i=k-h_1}^{k-1}\xi^T(i)\sum_{i=k-rh_1}^{k-1}\xi^T(i)\sum_{i=k-h_2}^{k-1}\xi^T(i)\sum_{i=k-rh_2}^{k-1}\xi^T(i) \times \sum_{i=k-h_2}^{k-h_1-1}\xi^T(i)\sum_{i=k-rh_2}^{k-rh_1-1}\xi^T(i)\sum_{i=k-h_1}^{k-1}\eta^T(i)\sum_{i=k-rh_1}^{k-1}\eta^T(i)\sum_{i=k-h_2}^{k-1}\eta^T(i)\sum_{i=k-rh_2}^{k-1}\eta^T(i) \times \sum_{i=k-h_2}^{k-h_1-1}\eta^T(i)\sum_{i=k-rh_2}^{k-rh_1-1}\eta^T(i)\sum_{i=k-h(k)}^{k-1}\eta^T(i)\sum_{i=k-rh(k)}^{k-1}\eta^T(i)\psi(k)\phi(k)f_1(k,\xi(k))f_2(k,\xi(k-h(k)))\sum_{i=1}^{\infty}\delta(i)\xi(k-i)]$ and $\prod_{i,j}$ is defined as in (6). The fact that $\sum_{i=1}^{N}\varphi_i = 1$ leads to the following identities:

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\varphi_i\varphi_j\prod_{i,j} = \sum_{i=1}^{N}\varphi_i^2\prod_{i,i} + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\varphi_i\varphi_j\Big[\prod_{i,j}+\prod_{j,i}\Big], \tag{47}$$

$$(N-1)\sum_{i=1}^{N}\varphi_i^2 - 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\varphi_i\varphi_j = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}[\varphi_i-\varphi_j]^2 \geq 0. \tag{48}$$

By (46)–(48), if (7)–(10) are feasible, then

$$\Delta V\big(k,\xi(k)\big) < -\omega\|\xi\|^2, \tag{49}$$

where $\omega > 0$. Therefore, system (1)–(2) is robustly stable. This completes the proof. □

Remark 1 Theorem 3.1 reduces to LMIs stability condition for discrete-time system with time-varying delays and nonlinear uncertainties [21] if $A(\varphi) = A$, $B(\varphi) = B$, $C(\varphi) = 0$ where $A, B \in R^{n\times n}$. We use the mixed model transformation in this paper, and utilize the various inequalities to reduce the possible conservatism of the obtained condition.

Remark 2 If $A(\varphi) = A$, $B(\varphi) = B$, $C(\varphi) = 0$ and $f_1(k,\xi(k)) = f_2(k,\xi(k-h(k))) = 0$ where $A, B \in R^{n\times n}$, model (1) is simplified to the discrete-time system with time-varying delays studied in [2, 4, 6, 7, 16, 18]. So the investigation in this paper expands and improves the studies in the literature.

4 Numerical examples

Example 4.1 Consider the system

$$\xi(k+1) = A\xi(k) + B\xi\big(k-h(k)\big) + f_1\big(k,\xi(k)\big) + f_2\big(k,\xi\big(k-h(k)\big)\big) \tag{50}$$

with the parameters

$$A = \begin{bmatrix} 0.80 & 0 \\ 0.05 & 0.90 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.10 & 0 \\ -0.20 & -0.10 \end{bmatrix}, \qquad \gamma \geq 0, \beta \geq 0, \tag{51}$$

which were considered in [21].

Decompose the matrices A and B as $A = A_1 + A_2$ and $B = B_1 + B_2$, where

$$A_1 = \begin{bmatrix} -0.5 & -0.1 \\ -0.02 & -0.3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.3 & 0.1 \\ 0.07 & 1.2 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.5 & -0.1 \\ -0.3 & -0.4 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.4 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}.$$

Table 1 The maximum upper bounds of h_2 for Example 4.1

Method	h_1	2	6	10	14
$\gamma = 0, \beta = 0$					
Ramakrishnan and Ray (2013) [21]	h_2	17	18	20	22
Ours	h_2	57	57	57	57
$\gamma = 0, \beta = 0.1$					
Ramakrishnan and Ray (2013) [21]	h_2	11	13	15	17
Ours	h_2	57	57	57	57
$\gamma = 0.1, \beta = 0.1$					
Ramakrishnan and Ray (2013) [21]	h_2	10	11	13	15
Ours	h_2	57	57	57	57

Table 2 The maximum upper bounds of h_2 for Example 4.2

h_1	4	10	20	30
Peng (2012) [18]	18	20	26	35
Feng et al. (2015) [2]	21	22	27	35
Kwon et al. (2013) [7]	22	23	27	36
Kim (2015) [6]	22	23	28	36
Nam et al. (2015) [16]	22	23	29	36
Hien and Trinh (2016) [4]	27	28	35	39
Ours	56	56	56	56

Applying Theorem 3.1 to system (50)–(51), the numerical results of our criterion are dramatically less conservative than those of [21], as shown in Table 1.

Example 4.2 Consider the system

$$\xi(k+1) = A\xi(k) + B\xi\bigl(k - h(k)\bigr) \tag{52}$$

with the parameters

$$A = \begin{bmatrix} 0.80 & 0 \\ 0.05 & 0.90 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.10 & 0 \\ -0.20 & -0.10 \end{bmatrix}, \tag{53}$$

which were considered in [2, 4, 6, 7, 16, 18].

Decompose the matrices A and B as $A = A_1 + A_2$ and $B = B_1 + B_2$, where

$$A_1 = \begin{bmatrix} -0.4 & -0.1 \\ -0.02 & -0.3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.2 & 0.1 \\ 0.07 & 1.2 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.5 & -0.1 \\ -0.3 & -0.4 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.4 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}.$$

Given various values of h_1, the obtained upper bounds for h_2 are shown in Table 2. The numerical results show the asymptotic stability of system (52) with (53). Moreover, by applying Theorem 3.1, less conservative results are achieved.

5 Conclusions

The problem of delay-interval-dependent robust stability for LPD discrete-time system with mixed time-varying delays and nonlinear uncertainties was studied. By utilizing various inequalities, a mixed model transformation, zero equations, and a new parameter-dependent Lyapunov–Krasovskii functional, the new delay-interval-dependent stability criterion is derived and formulated in terms of LMIs for the system. Some numerical examples are also illustrated to exhibit the effectiveness with less conservatism of the proposed stability criterion.

Acknowledgements
The author would like to thank the editors and reviewers for careful reading of the original manuscript and for their valuable suggestions.

Funding
This work was supported by the Faculty of Science at Sriracha, Kasetsart University Sriracha Campus, Thailand (Grant number P003/2018); the Thailand Research Fund (TRF); the Office of the Higher Education Commission (OHEC), Khon Kaen University (Grant number MRG6080042); Research and Academic Affairs Promotion Fund, Faculty of Science, Khon Kaen University, Fiscal Year 2019; and National Research Council of Thailand and Khon Kaen University, Thailand (Grant number 6200069).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 30 January 2019 Accepted: 4 June 2019 Published online: 21 June 2019

References

1. Botmart, T., Niamsup, P.: Robust exponential stability and stabilizability of linear parameter dependent systems with delays. Appl. Math. Comput. **217**(6), 2551–2566 (2010)
2. Feng, Z., Lam, J., Yang, G.H.: Optimal partitioning method for stability analysis of continuous/discrete delay systems. Int. J. Robust Nonlinear Control **25**(4), 559–574 (2015)
3. Gao, H., Chen, T.W.: New results on stability of discrete-time systems with time-varying state delay. IEEE Trans. Autom. Control **52**(2), 328–334 (2007)
4. Hien, L.V., Trinh, H.: New finite-sum inequalities with applications to stability of discrete time-delay systems. Automatica **71**, 197–201 (2016)
5. Jiang, X., Han, Q., Yu, X.: Stability criteria for linear discrete-time systems with interval-like time-varying delay. In: Proceedings of the American Mathematical Society, pp. 2817–2822 (2005)
6. Kim, S.H.: Further results on stability analysis of discrete-time systems with time-varying delays via the use of novel convex combination. Appl. Math. Comput. **261**, 104–113 (2015)
7. Kwon, O.M., Park, M.J., Park, J.H., Lee, S.M., Cha, E.J.: Stability and stabilization for discrete-time systems with time-varying delays via augmented Lyapunov–Krasovskii functional. J. Franklin Inst. **350**(3), 521–540 (2013)
8. la Sen, M.D.: Robust stabilization of a class of polytopic linear time-varying continuous systems under point delays and saturating controls. Appl. Math. Comput. **181**(1), 73–83 (2006)
9. Li, Y.: Impulsive synchronization of stochastic neural networks via controlling partial states. Neural Process. Lett. **46**(1), 56–69 (2017)
10. Li, Y., Li, B., Liu, Y., Lu, J., Wang, Z., Alsaadi, F.E.: Set stability and stabilization of switched Boolean networks with state-based switching. IEEE Access **6** 35624–35630 (2018)
11. Li, Y., Lou, J., Wang, Z., Alsaadi, F.E.: Synchronization of dynamical networks with nonlinearly coupling function under hybrid pinning impulsive controllers. J. Franklin Inst. **355** 6520–6530 (2018)
12. Maharajan, C., Raja, R., Cao, J., Rajchakit, G., Alsaedi, A.: Novel results on passivity and exponential passivity for multiple discrete delayed neutral-type neural networks with leakage and distributed time-delays. Chaos Solitons Fractals **115**, 268–282 (2018)
13. Mukdasai, K.: Robust exponential stability for lpd discrete-time system with interval time-varying delay. J. Appl. Math. **2012**, Article ID 237430 (2012)
14. Mukdasai, K., Niamsup, P.: Robust stability of discrete-time linear parameter dependent system with delay. Thai J. Math. **8**, 11–20 (2010). Special Issue (Annual Meeting in Mathematics)
15. Nam, P.T., Hien, H.M., Phat, V.N.: Asymptotic stability of linear state-delayed neutral systems with polytope type uncertainties. Dyn. Syst. Appl. **19**, 63–72 (2010)

16. Nam, P.T., Pubudu, P.N., Trinh, H.: Discrete Wirtinger-based inequality and its application. J. Franklin Inst. **352**, 1893–1905 (2015)
17. Niamsup, P., Rajchakit, G.: New results on robust stability and stabilization of linear discrete-time stochastic systems with convex polytopic uncertainties. J. Appl. Math. **2013**, Article ID 368259 (2013)
18. Peng, C.: Improved delay-dependent stabilisation criteria for discrete systems with a new finite sum inequality. IET Control Theory Appl. **6**(3), 448–453 (2012)
19. Phat, V.N., Nam, P.T.: Exponential stability and stabilization of uncertain linear time-varying systems using parameter dependent Lyapunov function. Int. J. Control **80**(8), 1333–1341 (2007)
20. Raja, R., Raja, U.K., Samidurai, R., Leelamani, A.: Dissipativity of discrete-time BAM stochastic neural networks with Markovian switching and impulses. J. Franklin Inst. **350**(10), 3217–3247 (2013)
21. Ramakrishnan, K., Ray, G.: Robust stability criteria for a class of uncertain discrete-time systems with time-varying delay. Appl. Math. Model. **37**(3), 1468–1479 (2013)
22. Sowmiya, C., Raja, R., Cao, J., Li, X., Rajchakit, G.: Discrete-time stochastic impulsive BAM neural networks with leakage and mixed time delays: an exponential stability problem. J. Franklin Inst. **355**(10), 4404–4435 (2018)
23. Sowmiya, C., Raja, R., Cao, J., Rajchakit, G., Alsaedi, A.: Enhanced robust finite-time passivity for Markovian jumping discrete-time BAM neural networks with leakage delay. Adv. Differ. Equ. **2017**, 318 (2017)
24. Sowmiya, C., Raja, R., Zhu, Q., Rajchakit, G.: Further mean-square asymptotic stability of impulsive discrete-time stochastic BAM neural networks with Markovian jumping and time-varying delays. J. Franklin Inst. **356**(1), 561–591 (2019)
25. Udpin, S., Niamsup, P.: Robust stability of discrete-time LPD neural networks with time-varying delay. Commun. Nonlinear Sci. Numer. Simul. **14**, 3914–3924 (2009)
26. Wang, T., Xue, M.X., Fei, S.M., Li, T.: Triple Lyapunov functional technique on delay-dependent stability for discrete-time dynamical networks. Neurocomputing **122**, 221–228 (2013)
27. Wang, T., Xue, M.X., Zhang, C., Fei, S.M.: Improved stability criteria on discrete-time systems with time-varying and distributed delays. Int. J. Autom. Comput. **10**(3), 260–266 (2013)
28. Xia, Y., Jia, Y.: Robust stability functionals of state delayed systems with polytopic type uncertain via parameter-dependent Lyapunov functions. Int. J. Control **75**, 1427–1434 (2002)
29. Zhang, W., Su, H.S., Han, Z.Z.: Robust stability for nonlinear discrete-time systems with interval time-varying delay. In: 2011 Third International Conference on Measuring Technology and Mechatronics Automation, pp. 191–194 (2011)
30. Zhang, X., Han, Q.L.: A new finite sum inequality approach to delay-dependent h_∞ control of discrete-time systems with time-varying delay. Int. J. Robust Nonlinear Control **18**(6), 630–647 (2008)