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Bifurcation analysis of an e-SEIARS model with multiple delays for point-to-group worm propagation

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Abstract

In this paper, by taking two important network environment factors (namely point-to-group worm propagation and benign worms) into consideration, a mathematical model with multiple delays to model the worm prevalence is presented. Sufficient conditions for the local stability of the unique endemic equilibrium and the existence of a Hopf bifurcation are demonstrated by choosing the different combinations of the three delays and analyzing the associated characteristic equation. Directly afterward, the stability and direction of the bifurcated periodic solutions are investigated by using center manifold theorem and the normal form theory. Finally, special attention is paid to some numerical simulations in order to verify the obtained theoretical results.

Keywords: Delays; Hopf bifurcation; Stability; Point-to-group propagation; Periodic solutions

1 Introduction

Over the years coupled with the fast development of communication technology and computer network applications, the network security has become an important challenge to the internet [1]. Specially, many computer worms have come into the internet frequently since the first known worm, called Morris, appeared in 1988. Computer worms are self-replicating programs created to carry out activities, which can quickly infect millions of electronic devices (computers, smartphones, etc.) without consent of their owners, and they have brought about huge economic losses and have had high social impact [2, 3]. Therefore, it is urgent to analyze the spreading law and control of computer worms in order to lessen their potential threat. Based on a newfangled observation that the spread of worms among computers is closely similar to the transmission of the infectious disease among a population, many epidemic models, such as SIRS model [4], SEIR model [5], SEIRS [2, 6, 7] model, SEIS-V model [8], SEIQR model [9] and SEIRS-V model [10–12], have been employed to analyze and describe the spread of computer worms in the internet.

As stated in the literature [13], the "point-to-group" (P2G), is extensively exists in the real world, especially in information sharing network. And its typical characteristics is that the group members in such network environment can receive the message or file

Table 1 Parameters and their meanings in this paper

Parameter	Description
μ	The replacement rate of old hosts
N	The total number of hosts
β_1	The infection rate of worms
β_2	The infection rate of benign worms
α	The transition rate from E to I
γ	The transition rate from A to R
ρ	The transition rate from E to R
ε	The transition rate from S to R
η	The transition rate from R to S

from the source simultaneously. Moreover, the information exchange in the same group is more frequent and more trustworthy, which makes it easier for computer viruses to propagate in the same group. Although there are some mathematical models which can depict the spread of computer viruses in the internet, however, the common problem of the above models is that they are not suitable for modeling the spread of computer viruses in point-to-group networks. In view of this fact, and based on the SIRA computer virus propagation model in the literature [14] Wang et al. [15] proposed the following e-SEIAR model with point-to-group worm propagation:

$$\begin{cases} \frac{dS(t)}{dt} = \mu N - \beta_1 S(t)E(t) - \beta_2 S(t)A(t) - (\mu + \varepsilon)S(t), \\ \frac{dE(t)}{dt} = \beta_1 S(t)E(t) - (\mu + \alpha + \rho)E(t), \\ \frac{dI(t)}{dt} = \alpha E(t) - \mu I(t) - \beta_2 I(t)A(t), \\ \frac{dA(t)}{dt} = \beta_2 S(t)A(t) + \beta_2 I(t)A(t) - (\mu + \gamma)A(t), \\ \frac{dR(t)}{dt} = \gamma A(t) + \rho E(t) + \varepsilon S(t) - \mu R(t), \end{cases} \tag{1}$$

where $S(t)$, $E(t)$, $I(t)$, $A(t)$ and $R(t)$ represent the numbers of susceptible, exposed, infective, benign worm and recovered hosts at time t, respectively. Namely, the total hosts are partitioned into five groups: S hosts, E hosts, I hosts, A hosts and R hosts. More parameters are listed in Table 1. Wang et al. [15] studied the local and global stability of system (1).

Obviously, Wang et al. [15] assume that the recovered hosts in system (1) have permanent immunization, which is not consistent with real world. In addition, there is usually a latent period from E hosts to I hosts. Similarly, it also needs a period for anti-virus software to clean the worms in E hosts and A hosts. On the other hand, delay differential equations show more complex dynamics compared with ordinary differential equations [16–18]. For example, there some work about delayed predator–prey models in [19–22], neural network models with delays in [23–26] and delayed epidemic models in [27–30]. Specially, there have been some work about the delayed computer virus models. Zhao and Bi studied the Hopf bifurcation of a delayed SEIR computer virus model with limited anti-virus ability by regarding the latent delay as the bifurcation parameter [31]. In [32], Zhang and Wang investigated the Hopf bifurcation of a delayed SLBQRS model by choosing the time delay due to the period that anti-virus software uses to clean viruses as the bifurcation parameter and derived the explicit formulas determining the direction and stability of the Hopf bifurcation by using the center manifold theorem and the normal form theory. In [33], Ren et al. analyzed the Hopf bifurcation of a delayed SIRS computer virus model by taking the time delay due to the temporary immunization period of the recovered hosts.

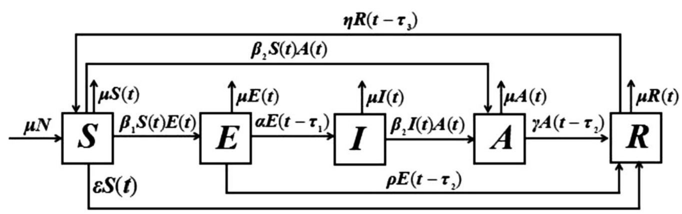

Figure 1 Schematic diagramfor the flowof worms inthe network

Zhao et al. studied the Hopf bifurcation of a delayed SLBS computer virus model by using the different combinations of the two delays as the bifurcation parameter [34]. All the work about the delayed dynamical systems shows that time delays have important effect on the stability of the systems. Therefore, based on the defects in the model proposed by Wang et al. [15] and inspired by the work about the delayed computer virus models in [3, 5, 6, 18, 31–34], we investigate a delayed e-SEIARS model:

$$\begin{cases} \frac{dS(t)}{dt} = \mu N - \beta_1 S(t)E(t) - \beta_2 S(t)A(t) - (\mu + \varepsilon)S(t) + \eta R(t - \tau_3), \\ \frac{dE(t)}{dt} = \beta_1 S(t)E(t) - \mu E(t) - \alpha E(t - \tau_1) - \rho E(t - \tau_2), \\ \frac{dI(t)}{dt} = \alpha E(t - \tau_1) - \mu I(t) - \beta_2 I(t)A(t), \\ \frac{dA(t)}{dt} = \beta_2 S(t)A(t) + \beta_2 I(t)A(t) - \mu A(t) - \gamma A(t - \tau_2), \\ \frac{dR(t)}{dt} = \gamma A(t - \tau_2) + \rho E(t - \tau_2) + \varepsilon S(t) - \mu R(t) - \eta R(t - \tau_3), \end{cases} \tag{2}$$

where τ_1 is the latent period delay of the exposed nodes; τ_2 is the delay due to the period that the anti-virus software uses to clean the worm and τ_3 is the temporary immunization period of the recovered nodes. The dynamical transfer is depicted in Fig. 1.

The rest of paper is organized as follows. In Sect. 2, we analyze local stability of the endemic equilibrium and existence of Hopf bifurcation by taking different combinations of the three delays as bifurcation parameters. In Sect. 3, the properties of the Hopf bifurcation are investigated with aid of the center manifold theory and the normal form method. Numerical simulations are performed in Sect. 4 in order to illustrate the theoretical predictions. Finally, we end our paper with a concluding remark.

2 Local stability and Hopf bifurcation analysis

Straightforward computation shows that if the condition (C_1) $\beta_1(\mu + \gamma) > \beta_2(\mu + \alpha + \rho)$, then system (2) has an endemic equilibrium $D_*(S_*, E_*, I_*, A_*, R_*)$, where

$$S_* = \frac{\mu + \alpha + \rho}{\beta_1},$$

$$E_* = a_1 + a_2 A_*,$$

$$I_* = \frac{\beta_1(\mu + \gamma) - \beta_2(\mu + \alpha + \rho)}{\beta_1 \beta_2},$$

$$A_* = \frac{\mu N - a_1(\mu + \alpha + \rho) - (\mu + \varepsilon)S_* + b_1 \eta}{(a_2 + \beta_2/\beta_1)(\mu + \alpha + \rho) - b_2 \eta},$$

$$R_* = b_1 + b_2 A_*,$$

with

$$
\begin{aligned}
a_1 &= \frac{\mu}{\alpha} I_*, \qquad a_2 = \frac{\beta_2}{\alpha} I_*, \\
b_1 &= \frac{\varepsilon(\mu + \alpha + \rho)}{\beta_1(\mu + \eta)} + \frac{\mu \rho I_*}{\alpha(\mu + \eta)}, \\
b_2 &= \frac{\gamma}{\mu + \eta} + \frac{\beta_2 \rho I_*}{\alpha(\mu + \eta)}.
\end{aligned}
$$

The characteristic equation of the linear section of system (2) at $D_*(S_*, E_*, I_*, A_*, R_*)$ is

$$
\begin{aligned}
&\lambda^5 + M_4\lambda^4 + M_3\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0 \\
&\quad + \left(N_4\lambda^4 + N_3\lambda^3 + N_2\lambda^2 + N_1\lambda + N_0\right)e^{-\lambda\tau_1} \\
&\quad + \left(P_4\lambda^4 + P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0\right)e^{-\lambda\tau_2} \\
&\quad + \left(Q_4\lambda^4 + Q_3\lambda^3 + Q_2\lambda^2 + Q_1\lambda + Q_0\right)e^{-\lambda\tau_3} \\
&\quad + \left(R_3\lambda^3 + R_2\lambda^2 + R_1\lambda + R_0\right)e^{-\lambda(\tau_1+\tau_2)} \\
&\quad + \left(S_3\lambda^3 + S_2\lambda^2 + S_1\lambda + S_0\right)e^{-\lambda(\tau_1+\tau_3)} \\
&\quad + \left(T_3\lambda^3 + T_2\lambda^2 + T_1\lambda + T_0\right)e^{-\lambda(\tau_2+\tau_3)} \\
&\quad + \left(U_3\lambda^3 + U_2\lambda^2 + U_1\lambda + U_0\right)e^{-2\lambda\tau_2} \\
&\quad + \left(V_3\lambda^3 + V_2\lambda^2 + V_1\lambda + V_0\right)e^{-2\lambda\tau_3} \\
&\quad + \left(W_2\lambda^2 + W_1\lambda + W_0\right)e^{-\lambda(\tau_1+\tau_2+\tau_3)} \\
&\quad + \left(X_2\lambda^2 + X_1\lambda + X_0\right)e^{-\lambda(\tau_1+2\tau_3)} \\
&\quad + \left(Y_2\lambda^2 + Y_1\lambda + Y_0\right)e^{-\lambda(\tau_2+2\tau_3)} \\
&\quad + \left(Z_2\lambda^2 + Z_1\lambda + Z_0\right)e^{-\lambda(2\tau_2+\tau_3)} \\
&\quad + (L_1\lambda + L_0)e^{-\lambda(\tau_1+\tau_2+2\tau_3)} + (E_1\lambda + E_0)e^{-2\lambda(\tau_2+\tau_3)} = 0,
\end{aligned}
\tag{3}
$$

where

$$
\begin{aligned}
M_0 &= m_{55}(m_{34}m_{43} - m_{33}m_{44})(m_{11}m_{22} - m_{12}m_{21}), \\
M_1 &= (m_{11}m_{22} - m_{12}m_{21})(m_{33}m_{44} + m_{33}m_{55} + m_{44}m_{55}) \\
&\quad + (m_{11} + m_{22})m_{33}m_{44}m_{55} + m_{12}m_{21}m_{34}m_{43} \\
&\quad - m_{34}m_{43}(m_{11}m_{22} + m_{11}m_{55} + m_{22}m_{55}), \\
M_2 &= (m_{12}m_{21} - m_{11}m_{22})(m_{33} + m_{44} + m_{55}) - m_{33}m_{44}m_{55} \\
&\quad + m_{34}m_{43}(m_{11} + m_{22} + m_{55}) \\
&\quad - (m_{11} + m_{22})(m_{33}m_{44} + m_{33}m_{55} + m_{44}m_{55}), \\
M_3 &= m_{33}m_{44} + m_{33}m_{55} + m_{44}m_{55} + m_{11}m_{22} \\
&\quad - m_{12}m_{21}(m_{11} + m_{22})(m_{33} + m_{44} + m_{55}) - m_{34}m_{43}, \\
M_4 &= -(m_{11} + m_{22} + m_{33} + m_{44} + m_{55}),
\end{aligned}
$$

$$
\begin{aligned}
N_0 &= m_{55}(m_{14}m_{21}m_{43}n_{32} + m_{11}m_{34}m_{43}n_{22} - m_{11}m_{33}m_{44}n_{22}),\\
N_1 &= n_{22}\bigl[m_{55}(m_{11}m_{33} + m_{11}m_{44} + m_{33}m_{44}) + m_{11}m_{33}m_{44}\bigr]\\
&\quad - m_{14}m_{21}m_{43}n_{32} - m_{34}m_{43}n_{22}(m_{11} + m_{55}),\\
N_2 &= n_{22}\bigl[m_{34}m_{43} - m_{55}(m_{11} + m_{33} + m_{44}) - m_{11}m_{33} - m_{11}m_{44} - m_{33}m_{44}\bigr],\\
N_3 &= n_{22}(m_{11} + m_{33} + m_{44} + m_{55}), \qquad N_4 = -n_{22},\\
P_0 &= m_{11}m_{55}p_{22}(m_{34}m_{43} - m_{33}m_{44}) + m_{33}m_{55}p_{44}(m_{12}m_{21} - m_{11}m_{22}),\\
P_1 &= (m_{33} + m_{55})(m_{11}m_{22}p_{44} + m_{11}m_{44}p_{22} - m_{12}m_{21}p_{44})\\
&\quad + m_{33}m_{55}\bigl[p_{22}(m_{11} + m_{44}) + p_{44}(m_{11} + m_{22})\bigr]\\
&\quad - m_{34}m_{43}p_{22}(m_{11} + m_{55}),\\
P_2 &= p_{22}\bigl[m_{34}m_{43} - m_{11}m_{44} - m_{33}m_{55} - (m_{11} + m_{44})(m_{33} + m_{55})\bigr]\\
&\quad + p_{44}\bigl[m_{12}m_{21} - m_{11}m_{22} - m_{33}m_{55} - (m_{11} + m_{22})(m_{33} + m_{55})\bigr],\\
P_3 &= p_{22}(m_{11} + m_{33} + m_{44} + m_{55}) + p_{44}(m_{11} + m_{22} + m_{33} + m_{55}),\\
P_4 &= -(p_{22} + p_{44}),\\
Q_0 &= q_{55}(m_{12}m_{21} - m_{11}m_{22})(m_{33}m_{44} - m_{34}m_{43}) - m_{22}m_{33}m_{44}m_{55}q_{15},\\
Q_1 &= q_{15}\bigl[m_{22}m_{33}(m_{44} + m_{55}) + m_{44}m_{55}(m_{22} + m_{33})\bigr]\\
&\quad + q_{55}\bigl[(m_{11}m_{22} - m_{12}m_{21})(m_{33} + m_{44}) + (m_{33}m_{44} - m_{34}m_{43})(m_{11} + m_{22})\bigr],\\
Q_2 &= q_{55}\bigl[m_{12}m_{21} + m_{34}m_{43} - m_{11}m_{22} - m_{33}m_{44} - (m_{11} + m_{22})(m_{33} + m_{44})\bigr]\\
&\quad - q_{15}\bigl[m_{22}m_{33} + m_{44}m_{55} + (m_{22} + m_{33})(m_{44} + m_{55})\bigr],\\
Q_3 &= q_{15}(m_{22} + m_{33} + m_{44} + m_{55}) + q_{55}(m_{11} + m_{22} + m_{33} + m_{44}),\\
Q_4 &= -(q_{15} + q_{55}), \qquad R_0 = -m_{11}m_{33}m_{55}n_{22}p_{44},\\
R_1 &= p_{44}n_{22}(m_{11}m_{33} + m_{11}m_{55} + m_{33}m_{55}),\\
R_2 &= -p_{44}n_{22}(m_{11} + m_{33} + m_{55}), \qquad R_3 = p_{44}n_{22},\\
S_0 &= q_{55}(m_{14}m_{21}m_{43}n_{32} - m_{11}m_{33}m_{44}n_{22}) - m_{33}m_{44}m_{55}n_{22}q_{15},\\
S_1 &= n_{22}\bigl[q_{15}(m_{33}m_{44} + m_{33}m_{55} + m_{44}m_{55}) + q_{55}(m_{11}m_{33} + m_{11}m_{44} + m_{33}m_{44})\bigr],\\
S_2 &= -n_{22}\bigl[q_{15}(m_{33} + m_{44} + m_{55}) + q_{55}(m_{11} + m_{33} + m_{44})\bigr], \qquad S_3 = n_{22}(q_{15} + q_{55}),\\
T_0 &= m_{33}q_{55}(m_{12}m_{21}p_{44} - m_{11}m_{22}p_{44} - m_{11}m_{44}p_{22})\\
&\quad - m_{33}m_{55}q_{15}(m_{22}p_{44} + m_{44}p_{22}),\\
T_1 &= q_{15}\bigl[p_{22}(m_{33}m_{44} + m_{33}m_{55} + m_{44}m_{55})\\
&\quad + p_{44}(m_{22}m_{33} + m_{22}m_{55} + m_{33}m_{55})\bigr]\\
&\quad + q_{55}\bigl[p_{22}(m_{11}m_{33} + m_{11}m_{44} + m_{33}m_{44})\\
&\quad + p_{44}(m_{11}m_{22} + m_{11}m_{33} + m_{22}m_{33} - m_{12}m_{21})\bigr],
\end{aligned}
$$

$$
\begin{aligned}
T_2 &= -q_{15}\big[p_{22}(m_{33}+m_{44}+m_{55}) + p_{44}(m_{22}+m_{33}+m_{55})\big] \\
&\quad - q_{55}\big[p_{22}(m_{11}+m_{33}+m_{44}) + p_{44}(m_{11}+m_{22}+m_{33})\big], \\
T_3 &= (q_{15}+q_{55})(p_{22}+p_{44}), \\
U_0 &= -m_{11}m_{33}m_{55}p_{22}p_{44}, \\
U_1 &= p_{22}p_{44}(m_{11}m_{33}+m_{11}m_{55}+m_{33}m_{55}), \\
U_2 &= -p_{22}p_{44}(m_{11}+m_{33}+m_{55}), \qquad U_3 = p_{22}p_{44}, \\
V_0 &= -m_{22}m_{33}m_{44}q_{15}q_{55}, \\
V_1 &= q_{15}q_{55}(m_{22}m_{33}+m_{22}m_{44}+m_{33}m_{44}), \\
V_2 &= -q_{15}q_{55}(m_{22}+m_{33}+m_{44}), \qquad V_3 = q_{15}q_{55}, \\
W_0 &= -m_{33}n_{22}p_{44}(m_{55}q_{15}+m_{11}q_{55}), \\
W_1 &= n_{22}p_{44}\big[q_{15}(m_{33}+m_{55}) + q_{55}(m_{11}+m_{33})\big], \\
W_2 &= -n_{22}p_{44}(q_{15}+q_{55}), \\
X_0 &= -m_{33}m_{44}n_{22}q_{15}q_{55}, \qquad X_1 = n_{22}q_{15}q_{55}(m_{33}+m_{44}), \qquad X_2 = -n_{22}q_{15}q_{55}, \\
Y_0 &= -m_{33}q_{15}q_{55}(m_{22}p_{44}+m_{44}p_{22}), \\
Y_1 &= q_{15}q_{55}\big[p_{22}(m_{33}+m_{44}) + p_{44}(m_{22}+m_{33})\big], \\
Y_2 &= -q_{15}q_{55}(p_{22}+p_{44}), \\
Z_0 &= -m_{33}p_{22}p_{44}(m_{11}q_{55}+m_{33}q_{15}), \\
Z_1 &= p_{22}p_{44}\big[q_{55}(m_{11}+m_{33}) + q_{15}(m_{33}+m_{55})\big], \\
Z_2 &= -p_{22}p_{44}(q_{15}+q_{55}), \\
L_0 &= -m_{33}n_{22}p_{44}q_{15}q_{55}, \qquad L_1 = n_{22}p_{44}q_{15}q_{55}, \\
E_0 &= -m_{33}p_{22}p_{44}q_{15}q_{55}, \qquad E_1 = p_{22}p_{44}q_{15}q_{55},
\end{aligned}
$$

and

$$
\begin{aligned}
&m_{11} = -(\beta_1E_* + \beta_2A_* + \mu + \varepsilon), \qquad m_{12} = -\beta_1S_*, \qquad m_{14} = -\beta_2S_*, \qquad q_{15} = \eta, \\
&m_{21} = \beta_1S_*, \qquad m_{22} = -\mu, \qquad n_{22} = -\alpha, \qquad p_{22} = -\rho, \\
&m_{33} = -(\mu + \beta_2A_*), \qquad m_{34} = -\beta_2I_*, \qquad n_{32} = \alpha, \\
&m_{41} = \beta_2A_*, \qquad m_{43} = \beta_2A_*, \qquad m_{44} = \beta_2(S_* + I_*) - \mu, \qquad p_{44} = -\gamma, \\
&m_{51} = \varepsilon, \qquad m_{55} - \mu, \qquad p_{52} = \rho, \qquad p_{54} = \gamma, \qquad q_{55} = -\eta.
\end{aligned}
$$

Case 1. $\tau_1 = \tau_2 = \tau_3 = 0$, Eq. (3) becomes

$$
\lambda^5 + M_{14}\lambda^4 + M_{13}\lambda^3 + M_{12}\lambda^2 + M_{11}\lambda + M_{10} = 0, \tag{4}
$$

where

$$
\begin{aligned}
M_{10} &= M_0 + N_0 + P_0 + Q_0 + R_0 + S_0 + T_0 \\
&\quad + U_0 + V_0 + W_0 + X_0 + Y_0 + Z_0 + L_0 + E_0, \\
M_{11} &= M_1 + N_1 + P_1 + Q_1 + R_1 + S_1 + T_1 \\
&\quad + U_1 + V_1 + W_1 + X_1 + Y_1 + Z_1 + L_1 + E_1, \\
M_{12} &= M_2 + N_2 + P_2 + Q_2 + R_2 + S_2 + T_2 \\
&\quad + U_2 + V_2 + W_2 + X_2 + Y_2 + Z_2, \\
M_{13} &= M_3 + N_3 + P_3 + Q_3 + R_3 + S_3 + T_3 \\
&\quad + U_3 + V_3, \\
M_{14} &= M_4 + N_4 + P_4 + Q_4.
\end{aligned}
$$

Based on the Routh–Hurwitz criteria, it can be concluded that $D_*(S_*, E_*, I_*, A_*, R_*)$ is locally asymptotically stable when $\tau = 0$ if Eqs. (5)–(9) are satisfied, which we refer to as condition (C_2):

$$\det_1 = M_{14} > 0, \tag{5}$$

$$\det_2 = \begin{vmatrix} M_{14} & 1 \\ M_{12} & M_{13} \end{vmatrix} > 0, \tag{6}$$

$$\det_3 = \begin{vmatrix} M_{14} & 1 & 0 \\ M_{12} & M_{13} & M_{14} \\ 0 & M_{11} & M_{12} \end{vmatrix} > 0, \tag{7}$$

$$\det_4 = \begin{vmatrix} M_{14} & 1 & 0 & 0 \\ M_{12} & M_{13} & M_{14} & 1 \\ M_{10} & M_{11} & M_{12} & M_{13} \\ 0 & 0 & M_{10} & M_{11} \end{vmatrix} > 0, \tag{8}$$

$$\det_5 = \begin{vmatrix} M_{14} & 1 & 0 & 0 & 0 \\ M_{12} & M_{13} & M_{14} & 1 & 0 \\ M_{10} & M_{11} & M_{12} & M_{13} & M_{14} \\ 0 & 0 & M_{10} & M_{11} & M_{12} \\ 0 & 0 & 0 & 0 & M_{10} \end{vmatrix} > 0. \tag{9}$$

Case 2. $\tau_1 > 0$, $\tau_2 = \tau_3 = 0$, Eq. (3) becomes

$$
\begin{aligned}
&\lambda^5 + M_{24}\lambda^4 + M_{23}\lambda^3 + M_{22}\lambda^2 + M_{21}\lambda + M_{20} \\
&\quad + \left(N_{24}\lambda^4 + N_{23}\lambda^3 + N_{22}\lambda^2 + N_{21}\lambda + N_{20}\right)e^{-\lambda\tau_1},
\end{aligned}
\tag{10}
$$

with

$$
\begin{aligned}
M_{20} &= M_0 + P_0 + Q_0 + T_0 + U_0 \\
&\quad + V_0 + Y_0 + Z_0 + E_0,
\end{aligned}
$$

$$M_{21} = M_1 + P_1 + Q_1 + T_1 + U_1 + V_1 + Y_1 + Z_1 + E_1,$$
$$M_{22} = M_2 + P_2 + Q_2 + T_2 + U_2 + V_2 + Y_2 + Z_2,$$
$$M_{23} = M_3 + P_3 + Q_3 + T_3 + U_3 + V_3,$$
$$M_{24} = M_4 + P_4 + Q_4,$$
$$N_{20} = N_0 + R_0 + S_0 + W_0 + X_0 + L_0,$$
$$N_{21} = N_1 + R_1 + S_1 + W_1 + X_1 + L_1,$$
$$N_{22} = N_2 + R_2 + S_2 + W_2 + X_2,$$
$$N_{23} = N_3 + R_3 + S_3, \qquad N_{24} = N_4.$$

Let $\lambda = i\omega$ ($\omega > 0$) be the root of Eq. (10), then from Eq. (10) separating real and imaginary part we obtain

$$\begin{cases} f_{21}(\omega)\sin\tau_1\omega + f_{22}(\omega)\cos\tau_1\omega = f_{23}(\omega), \\ f_{21}(\omega)\cos\tau_1\omega - f_{22}(\omega)\sin\tau_1\omega = f_{24}(\omega), \end{cases} \tag{11}$$

where

$$f_{21}(\omega) = N_{21}\omega - N_{23}\omega^3,$$
$$f_{22}(\omega) = N_{24}\omega^4 - N_{22}\omega^2 + N_{20},$$
$$f_{23}(\omega) = M_{22}\omega^2 - M_{24}\omega^4 - M_{20},$$
$$f_{24}(\omega) = M_{23}\omega^3 - \omega^5 - M_{21}\omega.$$

Equation (11) gives the following equation with respect to ω:

$$\omega^{10} + h_{24}\omega^8 + h_{23}\omega^6 + h_{22}\omega^4 + h_{21}\omega^2 + h_{20} = 0, \tag{12}$$

with

$$h_{20} = M_{20}^2 - N_{20}^2,$$
$$h_{21} = M_{21}^2 - 2M_{20}M_{22} + 2N_{20}N_{22} - N_{21}^2,$$
$$h_{22} = M_{22}^2 + 2M_{20}M_{24} - 2M_{21}M_{23} - N_{22}^2 - 2N_{20}N_{24} + 2N_{21}N_{23},$$
$$h_{23} = 2M_{21} + M_{23}^2 - 2M_{22}M_{24} + 2N_{22}N_{24} - N_{23}^2,$$
$$h_{24} = M_{24}^2 - 2M_{23} - N_{24}^2.$$

We suppose that (C_{21}): Eq. (12) has at least one positive root ω_0.

Furthermore, we have

$$\tau_0 = \frac{1}{\omega_0} \times \arccos\left[\frac{f_{21}(\omega_0) \times f_{24}(\omega_0) + f_{22}(\omega_0) \times f_{23}(\omega_0)}{f_{21}^2(\omega_0) + f_{22}^2(\omega_0)}\right]. \tag{13}$$

Now differentiating Eq. (10) with respect to τ_1, we get

$$\begin{aligned}\left[\frac{d\lambda}{d\tau_1}\right]^{-1} &= \frac{5\lambda^4 + 4M_{24}\lambda^3 + 3M_{23}\lambda^2 + 2M_{22}\lambda + M_{21}}{\lambda(\lambda^5 + M_{24}\lambda^4 + M_{23}\lambda^3 + M_{22}\lambda^2 + M_{21}\lambda + M_{20})} \\ &\quad + \frac{4N_{24}\lambda^3 + 3N_{23}\lambda^2 + 2N_{22}\lambda + N_{21}}{\lambda(N_{24}\lambda^4 + N_{23}\lambda^3 + N_{22}\lambda^2 + N_{21}\lambda + N_{20})} - \frac{\tau_1}{\lambda}.\end{aligned} \tag{14}$$

Substituting $\lambda = i\omega_0$ and simplifying we get

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{10}} = \frac{f'_{20}(\nu_0)}{f_{21}^2(\omega_0) + f_{22}^2(\omega_0)},$$

where $f_{20}(\nu) = \nu^5 + h_{24}\nu^4 + h_{23}\nu^3 + h_{22}\nu^2 + h_{21}\nu + h_{20}$ and $\nu = \omega^2$.

Therefore, if (C_{22}): $f'(\nu_0) \neq 0$ is satisfied, then $\mathrm{Re}[\frac{d\lambda}{d\tau_1}]^{-1}_{\tau_1=\tau_{10}} \neq 0$. Thus, we have the following conclusions based on the Hopf bifurcation theorem in [35].

Theorem 1 *For system* (2), *if the conditions* (C_{21})–(C_{22}) *hold, then* $D_*(S_*, E_*, I_*, A_*, R_*)$ *is locally asymptotically stable when* $\tau_1 \in [0, \tau_{10})$; *system* (2) *undergoes a Hopf bifurcation at* $D_*(S_*, E_*, I_*, A_*, R_*)$ *when* $\tau_1 = \tau_{10}$.

Case 3. $\tau_2 > 0$, $\tau_1 = \tau_3 = 0$, Eq. (3) becomes

$$\begin{aligned}&\lambda^5 + M_{34}\lambda^4 + M_{33}\lambda^3 + M_{32}\lambda^2 + M_{31}\lambda + M_{30} \\ &\quad + \left(P_{34}\lambda^4 + P_{33}\lambda^3 + P_{32}\lambda^2 + P_{31}\lambda + P_{30}\right)e^{-\lambda\tau_2} \\ &\quad + \left(U_{33}\lambda^3 + U_{32}\lambda^2 + U_{31}\lambda + U_{30}\right)e^{-2\lambda\tau_2} = 0,\end{aligned} \tag{15}$$

with

$$\begin{aligned}&M_{30} = M_0 + N_0 + Q_0 + S_0 + V_0 + X_0, \\ &M_{31} = M_1 + N_1 + Q_1 + S_1 + V_1 + X_1, \\ &M_{32} = M_2 + N_2 + Q_2 + S_2 + V_2 + X_2, \\ &M_{33} = M_3 + N_3 + Q_3 + S_3 + V_3, \\ &M_{34} = M_4 + N_4 + Q_4, \\ &P_{30} = P_0 + R_0 + T_0 + W_0 + Y_0 + L_0, \\ &P_{31} = P_1 + R_1 + T_1 + W_1 + Y_1 + L_1, \\ &P_{32} = P_2 + R_2 + T_2 + W_2 + Y_2, \\ &P_{33} = P_3 + R_3 + T_3 + W_3, P_{34} = P_4, \\ &U_{30} = U_0 + Z_0 + E_0, \\ &U_{31} = U_1 + Z_1 + E_1, \\ &U_{32} = U_2 + Z_2, \qquad U_{33} = U_3.\end{aligned}$$

Multiplying by $e^{\lambda\tau_2}$, Eq. (15) becomes

$$\begin{aligned} &P_{34}\lambda^4 + P_{33}\lambda^3 + P_{32}\lambda^2 + P_{31}\lambda + P_{30} \\ &\quad + \left(\lambda^5 + M_{34}\lambda^4 + M_{33}\lambda^3 + M_{32}\lambda^2 + M_{31}\lambda + M_{30}\right)e^{\lambda\tau_2} \\ &\quad + \left(U_{33}\lambda^3 + U_{32}\lambda^2 + U_{31}\lambda + U_{30}\right)e^{-\lambda\tau_2} = 0. \end{aligned} \tag{16}$$

Let $\lambda = i\omega(\omega > 0)$ be the root of Eq. (16), then from Eq. (16) separating real and imaginary parts we obtain

$$\begin{cases} f_{31}(\omega)\sin\tau_2\omega + f_{32}(\omega)\cos\tau_2\omega = f_{33}(\omega), \\ f_{34}(\omega)\cos\tau_2\omega - f_{35}(\omega)\sin\tau_2\omega = f_{36}(\omega), \end{cases} \tag{17}$$

where

$$\begin{aligned} &f_{31}(\omega) = (U_{31} - M_{31})\omega - (U_{33} - M_{33})\omega^3 - \omega^5, \\ &f_{32}(\omega) = M_{34}\omega^4 - (M_{32} + U_{32})\omega^2 + M_{30} + U_{30}, \\ &f_{33}(\omega) = P_{32}\omega^2 - P_{34}\omega^4 - P_{30}, \\ &f_{34}(\omega) = (U_{31} + M_{31})\omega - (U_{33} + M_{33})\omega^3 + \omega^5, \\ &f_{35}(\omega) = -M_{34}\omega^4 + (M_{32} - U_{32})\omega^2 - M_{30} + U_{30}, \\ &f_{36}(\omega) = P_{33}\omega^3 - P_{31}\omega. \end{aligned}$$

It follows that

$$\sin\tau_2\omega = \frac{f_{33}(\omega) \times f_{34}(\omega) - f_{32}(\omega) \times f_{36}(\omega)}{f_{31}(\omega) \times f_{34}(\omega) + f_{32}(\omega) \times f_{35}(\omega)} \tag{18}$$

and

$$\cos\tau_2\omega = \frac{f_{31}(\omega) \times f_{36}(\omega) + f_{33}(\omega) \times f_{35}(\omega)}{f_{31}(\omega) \times f_{34}(\omega) + f_{32}(\omega) \times f_{35}(\omega)}. \tag{19}$$

Thus,

$$\begin{aligned} &\left[f_{33}(\omega) \times f_{34}(\omega) - f_{32}(\omega) \times f_{36}(\omega)\right]^2 \\ &\quad + \left[f_{31}(\omega) \times f_{36}(\omega) + f_{33}(\omega) \times f_{35}(\omega)\right]^2 \\ &\quad - \left[f_{31}(\omega) \times f_{34}(\omega) + f_{32}(\omega) \times f_{35}(\omega)\right]^2 = 0. \end{aligned} \tag{20}$$

Similar to Case 2, we assume that (C_{31}) Eq. (20) has at least one positive root ω_0. Then we get

$$\tau_{20} = \frac{1}{\omega_0} \times \arccos\left[\frac{f_{31}(\omega_0) \times f_{36}(\omega_0) + f_{33}(\omega_0) \times f_{35}(\omega_0)}{f_{31}(\omega_0) \times f_{34}(\omega_0) + f_{32}(\omega_0) \times f_{35}(\omega_0)}\right]. \tag{21}$$

Also, differentiating Eq. (16) with respect to τ_2, we get

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{g_{31}(\omega)}{g_{32}(\omega)} - \frac{\tau_2}{\lambda}, \tag{22}$$

where

$$\begin{aligned} g_{31}(\omega) = {} & 4P_{34}\lambda^3 + 3P_{33}\lambda^2 + 2P_{32}\lambda + P_{31} \\ & + \left(5\lambda^4 + 4M_{34}\lambda^3 + 3M_{33}\lambda^2 + 2M_{32}\lambda + M_{31}\right)e^{\lambda\tau_2} \\ & + \left(3U_{33}\lambda^2 + 2U_{32}\lambda + U_{31}\right)e^{-\lambda\tau_2}, \\ g_{32}(\omega) = {} & \left(U_{33}\lambda^4 + U_{32}\lambda^3 + U_{31}\lambda^2 + U_{30}\lambda\right)e^{-\lambda\tau_2} \\ & - \left(\lambda^6 + M_{34}\lambda^5 + M_{33}\lambda^4 + M_{32}\lambda^3 + M_{31}\lambda^2 + M_{30}\lambda\right)e^{\lambda\tau_2}. \end{aligned}$$

Furthermore,

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\tau_2=\tau_{20}} = \frac{U_{3R}V_{3R} + U_{3I}V_{3I}}{V_{3R}^2 + V_{3I}^2}, \tag{23}$$

with

$$\begin{aligned} U_{3R} = {} & \left[5\omega_0^4 - 3(M_{33} + U_{33})\omega_0^2 + M_{31} + U_{31}\right]\cos\tau_{20}\omega_0 \\ & - 2\left[(M_{32} - U_{32})\omega_0 - 2M_{34}\omega_0^3\right]\sin\tau_{20}\omega_0 + P_{31} - 3P_{33}\omega_0^2, \\ U_{3I} = {} & \left[5\omega_0^4 - 3(M_{33} - U_{33})\omega_0^2 + M_{31} - U_{31}\right]\sin\tau_{20}\omega_0 \\ & + 2\left[(M_{32} + U_{32})\omega_0 - 2M_{34}\omega_0^3\right]\cos\tau_{20}\omega_0 + 2P_{31}\omega_0 - 4P_{34}\omega_0^3, \\ V_{3R} = {} & \left[(U_{30} - M_{30})\omega_0 - (U_{32} - M_{32})\omega_0^3 - M_{34}\omega_0^5\right]\sin\tau_{20}\omega_0 \\ & + \left[(U_{33} + M_{33})\omega_0^4 - (U_{31} + M_{31})\omega_0^2 - \omega_0^6\right]\cos\tau_{20}\omega_0, \\ V_{3I} = {} & \left[(U_{30} + M_{30})\omega_0 - (U_{32} + M_{32})\omega_0^3 + M_{34}\omega_0^5\right]\cos\tau_{20}\omega_0 \\ & - \left[(U_{33} - M_{33})\omega_0^4 - (U_{31} - M_{31})\omega_0^2 - \omega_0^6\right]\sin\tau_{20}\omega_0. \end{aligned}$$

Therefore, if (C_{32}): $U_{3R}V_{3R} + U_{3I}V_{3I} \neq 0$ is satisfied, then $\mathrm{Re}[\frac{d\lambda}{d\tau_2}]^{-1}_{\tau_2=\tau_{20}} \neq 0$. Thus, we have the following conclusions based on the Hopf bifurcation theorem in [35].

Theorem 2 *For system* (2), *if the conditions* (C_{31})–(C_{32}) *hold, then* $D_*(S_*, E_*, I_*, A_*, R_*)$ *is locally asymptotically stable when* $\tau_2 \in [0, \tau_{20})$; *system* (2) *undergoes a Hopf bifurcation at* $D_*(S_*, E_*, I_*, A_*, R_*)$ *when* $\tau_2 = \tau_{20}$.

Case 4. $\tau_3 > 0$, $\tau_1 = \tau_2 = 0$, Eq. (3) becomes

$$\begin{aligned} & \lambda^5 + M_{44}\lambda^4 + M_{43}\lambda^3 + M_{42}\lambda^2 + M_{41}\lambda + M_{40} \\ & \quad + \left(Q_{44}\lambda^4 + Q_{43}\lambda^3 + Q_{42}\lambda^2 + Q_{41}\lambda + Q_{40}\right)e^{-\lambda\tau_3} \\ & \quad + \left(V_{43}\lambda^3 + V_{42}\lambda^2 + V_{41}\lambda + V_{40}\right)e^{-2\lambda\tau_3} = 0, \end{aligned} \tag{24}$$

with

$$M_{40} = M_0 + N_0 + P_0 + R_0 + U_0,$$
$$M_{41} = M_1 + N_1 + P_1 + R_1 + U_1,$$
$$M_{42} = M_2 + N_2 + P_2 + R_2 + U_2,$$
$$M_{43} = M_3 + N_3 + P_3 + R_3 + U_3,$$
$$M_{44} = M_4 + N_4 + P_4,$$
$$Q_{40} = Q_0 + S_0 + T_0 + W_0 + Z_0,$$
$$Q_{41} = Q_1 + S_1 + T_1 + W_1 + Z_1,$$
$$Q_{42} = Q_2 + S_2 + T_2 + W_2 + Z_2,$$
$$Q_{43} = Q_3 + S_3 + T_3, \qquad Q_{44} = Q_4,$$
$$V_{40} = V_0 + X_0 + Y_0 + L_0 + E_0,$$
$$V_{41} = V_1 + X_1 + Y_1 + L_1 + E_1,$$
$$V_{42} = V_2 + X_2 + Y_2, \qquad V_{43} = V_3.$$

Multiplying by $e^{\lambda\tau_3}$, Eq. (24) becomes

$$\begin{aligned} &Q_{44}\lambda^4 + Q_{43}\lambda^3 + Q_{42}\lambda^2 + Q_{41}\lambda + Q_{40} \\ &\quad + \left(\lambda^5 + M_{44}\lambda^4 + M_{43}\lambda^3 + M_{42}\lambda^2 + M_{41}\lambda + M_{40}\right)e^{\lambda\tau_3} \\ &\quad + \left(V_{43}\lambda^3 + V_{42}\lambda^2 + V_{41}\lambda + V_{40}\right)e^{-\lambda\tau_3} = 0. \end{aligned} \tag{25}$$

Let $\lambda = i\omega(\omega > 0)$ be the root of Eq. (25), then following the same computation as done in Case 3 we obtain

$$\sin\tau_2\omega = \frac{f_{43}(\omega) \times f_{44}(\omega) - f_{42}(\omega) \times f_{46}(\omega)}{f_{41}(\omega) \times f_{44}(\omega) + f_{42}(\omega) \times f_{45}(\omega)} \tag{26}$$

and

$$\cos\tau_2\omega = \frac{f_{41}(\omega) \times f_{46}(\omega) + f_{43}(\omega) \times f_{45}(\omega)}{f_{41}(\omega) \times f_{44}(\omega) + f_{42}(\omega) \times f_{45}(\omega)}, \tag{27}$$

where

$$f_{41}(\omega) = (V_{41} - M_{41})\omega - (V_{43} - M_{43})\omega^3 - \omega^5,$$
$$f_{42}(\omega) = M_{44}\omega^4 - (M_{42} + V_{42})\omega^2 + M_{40} + V_{40},$$
$$f_{43}(\omega) = Q_{42}\omega^2 - Q_{44}\omega^4 - Q_{40},$$
$$f_{44}(\omega) = (V_{41} + M_{41})\omega - (V_{43} + M_{43})\omega^3 + \omega^5,$$
$$f_{45}(\omega) = -M_{44}\omega^4 + (M_{42} - V_{42})\omega^2 - M_{40} + V_{40},$$
$$f_{46}(\omega) = Q_{43}\omega^3 - Q_{41}\omega.$$

Furthermore,

$$\begin{aligned}&\left[f_{43}(\omega)\times f_{44}(\omega)-f_{42}(\omega)\times f_{46}(\omega)\right]^2\\&\quad+\left[f_{41}(\omega)\times f_{46}(\omega)+f_{43}(\omega)\times f_{45}(\omega)\right]^2\\&\quad-\left[f_{41}(\omega)\times f_{44}(\omega)+f_{42}(\omega)\times f_{45}(\omega)\right]^2=0.\end{aligned}\tag{28}$$

Similar to Case 3, we assume that (H_{41}): Eq. (28) has at least one positive root ω_0. Thus,

$$\tau_{30}=\frac{1}{\omega_0}\times\arccos\left[\frac{f_{41}(\omega_0)\times f_{46}(\omega_0)+f_{43}(\omega_0)\times f_{45}(\omega_0)}{f_{41}(\omega_0)\times f_{44}(\omega_0)+f_{42}(\omega_0)\times f_{45}(\omega_0)}\right]\tag{29}$$

and

$$\left[\frac{d\lambda}{d\tau_3}\right]^{-1}=\frac{g_{41}(\omega)}{g_{42}(\omega)}-\frac{\tau_3}{\lambda},\tag{30}$$

where

$$\begin{aligned}g_{41}(\omega)&=4Q_{44}\lambda^3+3Q_{43}\lambda^2+2Q_{42}\lambda+Q_{41}\\&\quad+\left(5\lambda^4+4M_{44}\lambda^3+3M_{43}\lambda^2+2M_{42}\lambda+M_{41}\right)e^{\lambda\tau_3}\\&\quad+\left(3V_{43}\lambda^2+2V_{42}\lambda+V_{41}\right)e^{-\lambda\tau_3},\\g_{42}(\omega)&=\left(V_{43}\lambda^4+V_{42}\lambda^3+V_{41}\lambda^2+V_{40}\lambda\right)e^{-\lambda\tau_3}\\&\quad-\left(\lambda^6+M_{44}\lambda^5+M_{43}\lambda^4+M_{42}\lambda^3+M_{41}\lambda^2+M_{40}\lambda\right)e^{\lambda\tau_3}.\end{aligned}$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_3}\right]^{-1}_{\tau_3=\tau_{30}}=\frac{U_{4R}V_{4R}+U_{4I}V_{4I}}{V_{4R}^2+V_{4I}^2},\tag{31}$$

with

$$\begin{aligned}U_{4R}&=\left[5\omega_0^4-3(M_{43}+V_{43})\omega_0^2+M_{41}+V_{41}\right]\cos\tau_{30}\omega_0\\&\quad-2\left[(M_{42}-V_{42})\omega_0-2M_{44}\omega_0^3\right]\sin\tau_{30}\omega_0+Q_{41}-3Q_{43}\omega_0^2,\\U_{4I}&=\left[5\omega_0^4-3(M_{43}-V_{43})\omega_0^2+M_{41}-V_{41}\right]\sin\tau_{30}\omega_0\\&\quad+2\left[(M_{42}+V_{42})\omega_0-2M_{44}\omega_0^3\right]\cos\tau_{30}\omega_0+2Q_{42}\omega_0-3Q_{44}\omega_0^3,\\V_{4R}&=\left[(V_{40}-M_{40})\omega_0-(V_{42}-M_{42})\omega_0^3-M_{44}\omega_0^5\right]\sin\tau_{30}\omega_0\\&\quad+\left[(V_{43}+M_{43})\omega_0^4-(V_{41}+M_{41})\omega_0^2-\omega_0^6\right]\cos\tau_{20}\omega_0,\\V_{4I}&=\left[(V_{40}+M_{40})\omega_0-(V_{42}+M_{42})\omega_0^3+M_{44}\omega_0^5\right]\cos\tau_{30}\omega_0\\&\quad-\left[(V_{43}-M_{43})\omega_0^4-(V_{41}-M_{41})\omega_0^2-\omega_0^6\right]\sin\tau_{30}\omega_0.\end{aligned}$$

Therefore, if (C_{42}): $U_{4R}V_{4R}+U_{4I}V_{4I}\neq 0$ is satisfied, then $\operatorname{Re}[\frac{d\lambda}{d\tau_3}]^{-1}_{\tau_3=\tau_{30}}\neq 0$. Thus, we have the following conclusions based on the Hopf bifurcation theorem in [35].

Theorem 3 *For system* (2), *if the conditions* (C_{41})–(C_{42}) *hold, then* $D_*(S_*, E_*, I_*, A_*, R_*)$ *is locally asymptotically stable when* $\tau_3 \in [0, \tau_{30})$; *system* (2) *undergoes a Hopf bifurcation at* $D_*(S_*, E_*, I_*, A_*, R_*)$ *when* $\tau_3 = \tau_{30}$.

Case 5. $\tau_1 > 0$, $\tau_2 \in (0, \tau_{20})$ $\tau_3 \in (0, \tau_{30})$. Let $\lambda = i\omega$ $(\omega > 0)$ be the root of Eq. (3), then we have

$$\begin{cases} f_{51}(\omega)\sin\tau_1\omega + f_{52}(\omega)\cos\tau_1\omega = f_{53}(\omega), \\ f_{51}(\omega)\cos\tau_1\omega - f_{52}(\omega)\sin\tau_1\omega = f_{54}(\omega), \end{cases} \tag{32}$$

with

$$\begin{aligned}
f_{51}(\omega) = {} & N_1\omega - N_3\omega^3 + \left(R_1\omega - R_3\omega^3\right)\cos\tau_2\omega - \left(R_0 - R_2\omega^2\right)\sin\tau_2\omega \\
& + \left(S_1\omega - S_3\omega^3\right)\cos\tau_3\omega - \left(S_0 - S_2\omega^2\right)\sin\tau_3\omega \\
& + W_1\omega\cos(\tau_2 + \tau_3)\omega - \left(W_0 - W_2\omega^2\right)\sin(\tau_2 + \tau_3)\omega \\
& + X_1\omega\cos 2\tau_3\omega - \left(X_0 - X_2\omega^2\right)\sin 2\tau_3\omega \\
& + L_1\omega\cos(\tau_2 + 2\tau_3)\omega - L_0\sin(\tau_2 + 2\tau_3)\omega, \\
f_{52}(\omega) = {} & N_4\omega^4 - N_2\omega^2 + N_0 + \left(R_1\omega - R_3\omega^3\right)\sin\tau_2\omega + \left(R_0 - R_2\omega^2\right)\cos\tau_2\omega \\
& + \left(S_1\omega - S_3\omega^3\right)\sin\tau_3\omega + \left(S_0 - S_2\omega^2\right)\cos\tau_3\omega \\
& + W_1\omega\sin(\tau_2 + \tau_3)\omega + \left(W_0 - W_2\omega^2\right)\cos(\tau_2 + \tau_3)\omega \\
& + X_1\omega\sin 2\tau_3\omega + \left(X_0 - X_2\omega^2\right)\cos 2\tau_3\omega, \\
f_{53}(\omega) = {} & M_2\omega_2 - M_4\omega^4 - M_0 - \left(P_1\omega - P_3\omega^3\right)\sin\tau_2\omega - \left(P_4\omega^4 - P_2\omega^2 + P_0\right)\cos\tau_2\omega \\
& - \left(Q_1\omega - Q_3\omega^3\right)\sin\tau_3\omega - \left(Q_4\omega^4 - Q_2\omega^2 + Q_0\right)\cos\tau_3\omega \\
& - \left(T_1\omega - T_3\omega^3\right)\sin(\tau_2 + \tau_3)\omega - \left(T_0 - T_2\omega^2\right)\cos(\tau_2 + \tau_3)\omega \\
& - \left(U_1\omega - U_3\omega^3\right)\sin 2\tau_2\omega - \left(U_0 - U_2\omega^2\right)\cos 2\tau_2\omega \\
& - \left(V_1\omega - V_3\omega^3\right)\sin 2\tau_3\omega - \left(V_0 - V_2\omega^2\right)\cos 2\tau_3\omega \\
& - Y_1\omega\sin(\tau_2 + 2\tau_3)\omega - \left(Y_0 - Y_2\omega^2\right)\cos(\tau_2 + 2\tau_3)\omega \\
& - Z_1\omega\sin(2\tau_2 + \tau_3)\omega - \left(Z_0 - Z_2\omega^2\right)\cos(2\tau_2 + \tau_3)\omega \\
& - E_1\omega\sin 2(\tau_2 + \tau_3)\omega - E_0\cos 2(\tau_2 + \tau_3)\omega, \\
f_{54}(\omega) = {} & M_3\omega^3 - \omega^5 - M_1\omega - \left(P_1\omega - P_3\omega^3\right)\cos\tau_2\omega + \left(P_4\omega^4 - P_2\omega^2 + P_0\right)\sin\tau_2\omega \\
& - \left(Q_1\omega - Q_3\omega^3\right)\cos\tau_3\omega + \left(Q_4\omega^4 - Q_2\omega^2 + Q_0\right)\sin\tau_3\omega \\
& - \left(T_1\omega - T_3\omega^3\right)\cos(\tau_2 + \tau_3)\omega + \left(T_0 - T_2\omega^2\right)\sin(\tau_2 + \tau_3)\omega \\
& - \left(U_1\omega - U_3\omega^3\right)\cos 2\tau_2\omega + \left(U_0 - U_2\omega^2\right)\sin 2\tau_2\omega \\
& - \left(V_1\omega - V_3\omega^3\right)\cos 2\tau_3\omega + \left(V_0 - V_2\omega^2\right)\sin 2\tau_3\omega \\
& - Y_1\omega\cos(\tau_2 + 2\tau_3)\omega + \left(Y_0 - Y_2\omega^2\right)\sin(\tau_2 + 2\tau_3)\omega
\end{aligned}$$

$$
\begin{aligned}
&- Z_1\omega\cos(2\tau_2+\tau_3)\omega + \left(Z_0 - Z_2\omega^2\right)\sin(2\tau_2+\tau_3)\omega \\
&- E_1\omega\cos 2(\tau_2+\tau_3)\omega + E_0\sin 2(\tau_2+\tau_3)\omega.
\end{aligned}
$$

Thus, we can obtain the following equation with respect to ω:

$$f_{53}^2(\omega) + f_{54}^2(\omega) - f_{51}^2(\omega) - f_{52}^2(\omega) = 0. \tag{33}$$

In order to give the main results in the present paper, we suppose that (C_{51}): Eq. (33) has at least one positive root ω_{1*}.

Thus, we can obtain

$$\tau_{1*} = \frac{1}{\omega_{1*}} \times \arccos\left[\frac{f_{51}(\omega_{1*}) \times f_{54}(\omega_{1*}) + f_{52}(\omega_{1*}) \times f_{53}(\omega_{1*})}{f_{51}^2(\omega_{1*}) + f_{52}^2(\omega_{1*})}\right]. \tag{34}$$

Taking the derivative of λ with respect to τ_1 in Eq. (3), we obtain

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{g_{51}(\omega)}{g_{52}(\omega)} - \frac{\tau_1}{\lambda}, \tag{35}$$

where

$$
\begin{aligned}
g_{51}(\omega) = {} & 5\lambda^4 + 4M_4\lambda^3 + 3M_3\lambda^2 + 2M_2\lambda + M_1 + \left(4N_4\lambda^3 + 3N_3\lambda^2 + 2N_2\lambda + N_1\right)e^{-\lambda\tau_1} \\
& - \left[\tau_2 P_4\lambda^4 + (\tau_2 P_3 - 4P_4)\lambda^3 + (\tau_2 P_2 - 3P_3)\lambda^2\right. \\
& \left. + (\tau_2 P_1 - 2P_2)\lambda + \tau_2 P_0 - P_1\right]e^{-\lambda\tau_2} \\
& - \left[\tau_3 Q_4\lambda^4 + (\tau_3 Q_3 - 4Q_4)\lambda^3 + (\tau_3 Q_2 - 3Q_3)\lambda^2\right. \\
& \left. + (\tau_3 Q_1 - 2Q_2)\lambda + \tau_3 Q_0 - Q_1\right]e^{-\lambda\tau_3} \\
& - \left[\tau_2 R_3\lambda^3 + (\tau_2 R_2 - 3R_3)\lambda^2 + (\tau_2 R_1 - 2R_2)\lambda + \tau_2 R_0 - R_1\right]e^{-(\tau_1+\tau_2)} \\
& - \left[\tau_3 S_3\lambda^3 + (\tau_3 S_2 - 3S_3)\lambda^2 + (\tau_3 S_1 - 2S_2)\lambda + \tau_3 S_0 - S_1\right]e^{-(\tau_1+\tau_3)} \\
& - \left[(\tau_2+\tau_3)T_3\lambda^3 + \left((\tau_2+\tau_3)T_2 - 3T_3\right)\lambda^2\right. \\
& \left. + \left((\tau_2+\tau_3)T_1 - 2T_2\right)\lambda + (\tau_2+\tau_3)T_0 - T_1\right]e^{-(\tau_2+\tau_3)} \\
& - \left[2\tau_2 U_3\lambda^3 + (2\tau_2 U_2 - 3U_3)\lambda^2 + (2\tau_2 U_1 - 2U_2)\lambda + 2\tau_2 U_0 - U_1\right]e^{-2\tau_2} \\
& - \left[2\tau_3 V_3\lambda^3 + (2\tau_3 V_2 - 3V_3)\lambda^2 + (2\tau_3 V_1 - 2V_2)\lambda + 2\tau_3 V_0 - V_1\right]e^{-2\tau_3} \\
& - \left[(\tau_2+\tau_3)W_2\lambda^2 + \left((\tau_2+\tau_3)W_1 - 2W_2\right)\lambda + (\tau_2+\tau_3)W_0 - W_1\right]e^{-\lambda(\tau_1+\tau_2+\tau_3)} \\
& - \left[2\tau_3 X_2\lambda^2 + (2\tau_3 X_1 - 2X_2)\lambda + 2\tau_3 X_0 - X_1\right]e^{-\lambda(\tau_1+2\tau_3)} \\
& - \left[2\tau_3 Y_2\lambda^2 + (2\tau_3 Y_1 - 2Y_2)\lambda + 2\tau_3 Y_0 - Y_1\right]e^{-\lambda(\tau_2+2\tau_3)} \\
& - \left[(2\tau_2+\tau_3)Z_2\lambda^2 + \left((2\tau_2+\tau_3)Z_1 - 2Z_2\right)\lambda + (2\tau_2+\tau_3)Z_0 - Z_1\right]e^{-\lambda(2\tau_2+\tau_3)} \\
& - \left[(\tau_2+2\tau_3)L_1\lambda + (\tau_2+2\tau_3)L_0 - L_1\right]e^{-\lambda(\tau_2+2\tau_3)} \\
& - \left[2(\tau_2+\tau_3)E_1\lambda + 2(\tau_2+\tau_3)E_0 - E_1\right]e^{-2\lambda(\tau_2+\tau_3)},
\end{aligned}
$$

$$\begin{aligned} g_{51}(\omega) &= \left(N_4\lambda^4 + N_3\lambda^3 + N_2\lambda^2 + N_1\lambda + N_0\right)e^{-\lambda\tau_1} + \left(R_3\lambda^3 + R_2\lambda^2 + R_1\lambda + R_0\right)e^{-\lambda(\tau_1+\tau_2)} \\ &\quad + \left(S_3\lambda^3 + S_2\lambda^2 + S_1\lambda + S_0\right)e^{-\lambda(\tau_1+\tau_3)} + \left(W_2\lambda^2 + W_1\lambda + W_0\right)e^{-\lambda(\tau_1+\tau_2+\tau_3)} \\ &\quad + \left(X_2\lambda^2 + X_1\lambda + X_0\right)e^{-\lambda(\tau_1+2\tau_3)} + (L_1\lambda + L_0)e^{-\lambda(\tau_1+\tau_2+2\tau_3)}. \end{aligned}$$

Define

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{1*}} = \frac{U_{5R}V_{5R} + U_{5I}V_{5I}}{V_{5R}^2 + V_{5I}^2}. \tag{36}$$

Thus, if (C_{52}): $U_{5R}V_{5R} + U_{5I}V_{5I} \neq 0$ is satisfied, then $\operatorname{Re}[\frac{d\lambda}{d\tau_1}]^{-1}_{\tau_1=\tau_{1*}} \neq 0$. Furthermore, we have the following conclusions based on the Hopf bifurcation theorem in [35].

Theorem 4 *For system* (2), *if the conditions* (C_{51})–(C_{52}) *hold, then* $D_*(S_*, E_*, I_*, A_*, R_*)$ *is locally asymptotically stable when* $\tau_1 \in [0, \tau_{1*})$; *system* (2) *undergoes a Hopf bifurcation at* $D_*(S_*, E_*, I_*, A_*, R_*)$ *when* $\tau_1 = \tau_{1*}$.

3 Direction and stability of the Hopf bifurcation

In this section, we will investigate the direction and stability of the Hopf bifurcation at the critical value $\tau_1 = \tau_{1*}$ by employing the center manifold theorem and the normal form theory. Let $\tau_1 = \tau_{1*} + \mu$, $\mu \in R$, then μ is the Hopf bifurcation value for system (2). For convenience, we suppose that $\tau_{3*} < \tau_{2*} < \tau_{1*}$ where $\tau_{3*} \in (0, \tau_{30})$ and $\tau_{2*} \in (0, \tau_{20})$. Let $u_1 = S(\tau_1 t)$, $u_2 = E(\tau_1 t)$, $u_3 = I(\tau_1 t)$, $u_4 = A(\tau_1 t)$ and $u_5 = R(\tau_1 t)$. System (2) can be transformed into the following form:

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{37}$$

where $u(t) = (u_1, u_2, u_3, u_4 u_5)^T \in C = C([-1, 0], \mathrm{R}^5)$, $L_\mu : C \to \mathrm{R}^5$ and $F : \mathrm{R} \times C \to \mathrm{R}^5$ can be defined as

$$L_\mu\phi = (\tau_{1*} + \mu)\left(M\phi(0) + P\phi\left(-\frac{\tau_{2*}}{\tau_{1*}}\right) + Q\phi\left(-\frac{\tau_{3*}}{\tau_{1*}}\right) + N\phi(-1)\right)$$

and

$$F(\mu, \phi) = (\tau_{1*} + \mu)\begin{pmatrix} -\beta_1\phi_1(0)\phi_2(0) - \beta_2\phi_1(0)\phi_4(0) \\ \beta_1\phi_1(0)\phi_2(0) \\ -\beta_2\phi_3(0)\phi_4(0) \\ \beta_2\phi_1(0)\phi_4(0) + \beta_2\phi_3(0)\phi_4(0) \\ 0 \end{pmatrix}$$

with

$$M = \begin{pmatrix} m_{11} & m_{12} & 0 & m_{14} & 0 \\ m_{21} & m_{22} & 0 & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} & 0 \\ m_{41} & 0 & m_{43} & m_{44} & 0 \\ m_{51} & 0 & 0 & 0 & m_{55} \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & n_{22} & 0 & 0 & 0 \\ 0 & n_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & p_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 \\ 0 & p_{52} & 0 & p_{54} & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & q_{15} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{55} \end{pmatrix}.$$

Thus, for $\phi \in C$, there exists $\eta(\theta,\mu)$ such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta). \tag{38}$$

In fact, choosing

$$\eta(\theta,\mu) = \begin{cases} (\tau_{1*}+\mu)(M+N+P+Q), & \theta = 0, \\ (\tau_{1*}+\mu)(N+P+Q), & \theta \in [-\frac{\tau_{3*}}{\tau_{1*}}, 0), \\ (\tau_{1*}+\mu)(N+P), & \theta \in [-\frac{\tau_{2*}}{\tau_{1*}}, -\frac{\tau_{3*}}{\tau_{1*}}), \\ (\tau_{1*}+\mu)N, & \theta \in (-1, -\frac{\tau_{2*}}{\tau_{1*}}), \\ 0, & \theta = -1, \end{cases} \tag{39}$$

where $\delta(\theta)$ is the Dirac delta function.

For $\phi \in C([-1,0], R^5)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases}$$

Then system (37) equals

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{40}$$

For $\varphi \in C^1([0,1], (R^5)^*)$, define

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \int_{-1}^{0} d\eta^T(s,0)\varphi(-s), & s = 0, \end{cases}$$

and

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\varphi}(\xi-\theta)\, d\eta(\theta)\phi(\xi)\, d\xi, \tag{41}$$

where $\eta(\theta) = \eta(\theta,0)$.

By the discussion above, we can conclude that $\pm i\omega_{1*}\tau_{1*}$ are eigenvalues of $A(0)$ and A^*. Let $\kappa(\theta) = (1,\kappa_2,\kappa_3,\kappa_4,\kappa_5)^T e^{i\tau_{1*}\omega_{1*}\theta}$ be the eigenvector of $A(0)$ corresponding to

$+i\tau_{1*}\omega_{1*}$ and $\kappa^*(s) = \varrho(1, \kappa_2^*, \kappa_3^*, \kappa_4^*, \kappa_5^*)^T e^{i\tau_{1*}\omega_{1*}s}$ be the eigenvector of $A^*(0)$ corresponding to $-i\tau_2^*\omega_2^*$, where

$$\begin{aligned}
\kappa_2 &= \frac{m_{21}}{i\omega_{1*} - m_{22} - n_{22}e^{-i\tau_{1*}\omega_{1*}} - p_{22}e^{-i\tau_{2*}\omega_{1*}}}, \\
\kappa_3 &= \frac{n_{32}e^{-i\tau_{1*}\omega_{1*}}\kappa_2 + n_{34}\kappa_4}{i\omega_{1*} - m_{33}}, \\
\kappa_4 &= \frac{m_{41}(i\omega_{1*} - m_{33}) + m_{43}n_{32}e^{-i\tau_{1*}\omega_{1*}}\kappa_2}{(i\omega_{1*} - m_{33})(i\omega_{1*} - m_{44} - p_{44}e^{-i\tau_{2*}\omega_{1*}}) - m_{43}n_{34}}, \\
\kappa_5 &= \frac{m_{51} + p_{52}e^{-i\tau_{2*}\omega_{1*}}\kappa_2 + p_{54}e^{-i\tau_{2*}\omega_{1*}}\kappa_4}{i\omega_{1*} - m_{55} - q_{55}e^{-i\tau_{2*}\omega_{1*}}}, \\
\kappa_2^* &= -\frac{i\omega_{1*} + m_{11} + m_{41}\kappa_4 + m_{51}\kappa_5}{m_{21}}, \\
\kappa_3^* &= -\frac{m_{43}\kappa_4}{i\omega_{1*} + m_{33}}, \\
\kappa_4^* &= \frac{m_{12}m_{21} + m_{21}p_{22}e^{i\omega_{1*}\tau_{2*}}\kappa_5^* + \widetilde{p}_{22}(i\omega_{1*} + m_{11} + m_{51}\kappa_5^*)}{(m_{41}\widetilde{m}_{33}\widetilde{p}_{22} + m_{21}m_{43}n_{42}e^{i\tau_{1*}\omega_{1*}})/\widetilde{m}_{33}}, \\
\kappa_5^* &= -\frac{q_{15}e^{i\tau_{3*}\omega_{1*}}}{i\omega_{1*} + m_{55} + q_{55}e^{i\tau_{3*}\omega_{1*}}}, \\
\widetilde{m}_{33} &= i\omega_{1*} + m_{33}, \\
\widetilde{p}_{22} &= i\omega_{1*} + m_{22} + n_{22}e^{i\tau_{1*}\omega_{1*}} + p_{22}e^{i\tau_{2*}\omega_{1*}},
\end{aligned}$$

and

$$\begin{aligned}
\bar{\varrho} = \Big[&1 + \kappa_2\bar{\kappa}_2^* + \kappa_3\bar{\kappa}_3^* + \kappa_4\bar{\kappa}_4^* + \kappa_5\bar{\kappa}_5^* + \tau_1^*e^{-i\tau_1^*\omega_1^*}\kappa_2\big(n_{22}\bar{\kappa}_2^* + n_{32}\bar{\kappa}_3^*\big) \\
&+ \tau_2^*e^{-i\tau_2^*\omega_1^*}\big[\kappa_2\big(p_{22}\bar{\kappa}_2^* + p_{52}\bar{\kappa}_5^*\big) + \kappa_4\big(p_{44}\bar{\kappa}_4^* + p_{54}\bar{\kappa}_5^*\big)\big] \\
&+ \tau_3^*e^{-i\tau_3^*\omega_1^*}\kappa_5\big(q_{15} + q_{55}\bar{\kappa}_5^*\big)\Big]^{-1},
\end{aligned}$$

which ensures that $\langle\kappa^*, \kappa\rangle = 1$ and $\langle\kappa^*, \bar{\kappa}\rangle = 0$.

In what follows, we can obtain the expressions of g_{20}, g_{11}, g_{02} and g_{21} by using the algorithms in [35] and a similar computation process to that in [31, 36–38]:

$$\begin{aligned}
g_{20} &= 2\tau_1^*\bar{\varrho}\big[\beta_1\kappa_2\big(\bar{\kappa}_2^* - 1\big) - \beta_2\kappa_4 - \beta_2\kappa_3\kappa_4\bar{\kappa}_3^* + \beta_2\kappa_4\bar{\kappa}_4^*(1 + \kappa_3)\big], \\
g_{11} &= \tau_1^*\bar{\varrho}\big[\beta_1\big(\kappa_2 + \bar{\kappa}_2^*\big)\big(\bar{\kappa}_2^* - 1\big) - \beta_2\big(\kappa_4 + \bar{\kappa}_4^*\big) - \beta_2\bar{\kappa}_3^*\big(\kappa_3\bar{\kappa}_4^* + \bar{\kappa}_3^*\kappa_4\big) \\
&\quad + \beta_2\bar{\kappa}_4^*\big(\kappa_4 + \bar{\kappa}_4^* + \kappa_3\bar{\kappa}_4^* + \bar{\kappa}_3^*\kappa_4\big)\big], \\
g_{02} &= 2\tau_1^*\bar{\varrho}\big[\beta_1\bar{\kappa}_2\big(\bar{\kappa}_2^* - 1\big) - \beta_2\bar{\kappa}_4 - \beta_2\bar{\kappa}_3\bar{\kappa}_4\bar{\kappa}_3^* + \beta_2\bar{\kappa}_4\bar{\kappa}_4^*(1 + \bar{\kappa}_3)\big], \\
g_{21} &= 2\tau_1^*\bar{\varrho}\bigg[\big(\bar{\kappa}_2^* - 1\big)\bigg(W_{11}^{(1)}(0)\kappa_2 + \frac{1}{2}W_{20}^{(1)}(0)\bar{\kappa}_2 + W_{11}^{(2)}(0) + \frac{1}{2}W_{20}^{(2)}(0)\bigg) \\
&\quad + \beta_2\big(\bar{\kappa}_4^* - 1\big)\bigg(W_{11}^{(1)}(0)\kappa_4 + \frac{1}{2}W_{20}^{(1)}(0)\bar{\kappa}_4 + W_{11}^{(4)}(0) + \frac{1}{2}W_{20}^{(4)}(0)\bigg) \\
&\quad + \beta_2\big(\bar{\kappa}_4^* - \bar{\kappa}_3^*\big)\bigg(W_{11}^{(3)}(0)\kappa_4 + \frac{1}{2}W_{20}^{(3)}(0)\bar{\kappa}_4 + W_{11}^{(4)}(0)\kappa_3 + \frac{1}{2}W_{20}^{(4)}(0)\bar{\kappa}_3^*\bigg)\bigg],
\end{aligned}$$

with

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}\kappa(0)}{\tau_1^*\omega_1^*}e^{i\tau_1^*\omega_1^*\theta} + \frac{i\bar{g}_{02}\bar{\kappa}(0)}{3\tau_1^*\omega_1^*}e^{-i\tau_1^*\omega_1^*\theta} + E_1 e^{2i\tau_1^*\omega_1^*\theta}, \\ W_{11}(\theta) &= -\frac{ig_{11}\kappa(0)}{\tau_1^*\omega_1^*}e^{i\tau_1^*\omega_1^*\theta} + \frac{i\bar{g}_{11}\bar{\kappa}(0)}{\tau_1^*\omega_1^*}e^{-i\tau_1^*\omega_1^*\theta} + E_2, \end{aligned}$$

where

$$\begin{aligned} E_1 = 2 &\begin{pmatrix} m'_{11} & -m_{12} & 0 & -m_{14} & -q_{15}e^{-i\tau_3^*\omega_1^*} \\ -m_{21} & m'_{22} & 0 & 0 & 0 \\ 0 & -n_{32}e^{-2i\tau_1^*\omega_1^*} & m'_{33} & -m_{34} & 0 \\ -m_{41} & 0 & -m_{43} & m'_{44} & 0 \\ -m_{51} & -p_{52}e^{-2i\tau_2^*\omega_1^*} & 0 & -p_{54}e^{-2i\tau_2^*\omega_1^*} & m'_{55} \end{pmatrix}^{-1} \\ &\times \begin{pmatrix} -\beta_1\kappa_2 - \beta_2\kappa_4 \\ \beta_1\kappa_2 \\ -\beta_2\kappa_3\kappa_4 \\ \beta_2\kappa_4(1+\kappa_3) \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} E_2 = 2 &\begin{pmatrix} m_{11} & m_{12} & 0 & m_{14} & q_{15} \\ m_{21} & m_{22}+n_{22}+p_{22} & 0 & 0 & 0 \\ 0 & n_{32} & m_{33} & m_{34} & 0 \\ m_{41} & 0 & m_{43} & m_{44}+p_{44} & 0 \\ m_{51} & p_{52} & 0 & p_{54} & m_{55}+q_{55} \end{pmatrix}^{-1} \\ &\times \begin{pmatrix} -\beta_1(\kappa_2+\bar{\kappa}_2) - \beta_2(\kappa_4+\bar{\kappa}_4) \\ \beta_1(\kappa_2+\bar{\kappa}_2) \\ -\beta_2(\kappa_3\bar{\kappa}_4+\bar{\kappa}_3\kappa_4) \\ \beta_2(\kappa_4+\bar{\kappa}_4)+\beta_2(\kappa_3\bar{\kappa}_4+\bar{\kappa}_3\kappa_4) \\ 0 \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} m'_{11} &= 2i\omega_1^* - m_{11}, \\ m'_{22} &= 2i\omega_1^* - m_{22} - n_{22}e^{-2i\tau_1^*\omega_1^*} - p_{22}e^{-2i\tau_2^*\omega_1^*}, \\ m'_{33} &= 2i\omega_1^* - m_{33}, \\ m'_{44} &= 2i\omega_1^* - m_{44} - p_{44}e^{-2i\tau_2^*\omega_1^*}, \\ m'_{55} &= 2i\omega_1^* - m_{55} - q_{55}e^{-i\tau_3^*\omega_1^*}. \end{aligned}$$

Thus, we can compute the following values:

$$\begin{aligned} C_1(0) &= \frac{i}{2\tau_1^*\omega_1^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\{C_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_1^*)\}}, \\ \beta_2 &= 2\mathrm{Re}\{C_1(0)\}, \\ T_2 &= -\frac{\mathrm{Im}\{C_1(0)\} + \mu_2\,\mathrm{Im}\{\lambda'(\tau_1^*)\}}{\tau_1^*\omega_1^*}. \end{aligned} \tag{42}$$

Thus, we can obtain the following results according to the discussion of the properties of Hopf bifurcating periodic solutions of dynamical system in [23].

Theorem 5 *For system* (2), *if* $\mu_2 > 0$ ($\mu_2 < 0$), *then the Hopf bifurcation is supercritical* (*subcritical*); *if* $\beta_2 < 0$ ($\beta_2 > 0$), *then the bifurcating periodic solutions are stable* (*unstable*); *if* $T_2 > 0$ ($T_2 < 0$), *then the periods of the bifurcating periodic solutions increase* (*decrease*).

4 Numerical simulations

In this section, we shall give some numerical simulations to validate the theoretical results obtained in the previous section. Choosing $\mu = 0.06$, $N = 100$, $\beta_1 = 0.1$, $\beta_2 = 0.015$, $\varepsilon = 0.07$, $\eta = 0.3$, $\alpha = 0.1$, $\rho = 0.02$, $\gamma = 0.08$. Then we get the following specific model:

$$\begin{cases} \frac{dS(t)}{dt} = 6 - 0.1S(t)E(t) - 0.015S(t)A(t) - 0.13S(t) + 0.3R(t-\tau_3), \\ \frac{dE(t)}{dt} = 0.1S(t)E(t) - 0.06E(t) - 0.1E(t-\tau_1) + 0.02E(t-\tau_2), \\ \frac{dI(t)}{dt} = 0.1E(t-\tau_1) - 0.06I(t) - 0.015I(t)A(t), \\ \frac{dA(t)}{dt} = 0.015S(t)A(t) + 0.015I(t)A(t) - 0.06A(t) - 0.08A(t-\tau_2), \\ \frac{dR(t)}{dt} = 0.08A(t-\tau_2) + 0.02E(t-\tau_2) + 0.07S(t) - 0.06R(t) - 0.3R(t-\tau_3), \end{cases} \tag{43}$$

which satisfies the condition (C_1). By computing, we obtain the unique endemic equilibrium $D_*(1.8, 44.5473, 7.5333, 35.4424, 10.6964)$ and the condition (C_2) is satisfied.

First, by taking $\tau_1 > 0$ ($\tau_2 = \tau_3 = 0$), $\tau_2 > 0$ ($\tau_1 = \tau_3 = 0$), $\tau_3 > 0$ ($\tau_1 = \tau_2 = 0$) and $\tau_1 > 0$ ($\tau_2 = 3.75 \in (0, \tau_{20})$ and $\tau_3 = 2.35 \in (0, \tau_{30})$), respectively, we obtain $\omega_1 = 1.9353$ and $\tau_{10} = 25.1722$, $\omega_2 = 0.6372$ and $\tau_{20} = 30.0105$, $\omega_3 = 1.4460$ and $\tau_{30} = 6.1042$, $\omega_1^* = 3.2659$ and $\tau_1^* = 6.6884$. The unique endemic equilibrium $D_*(1.8, 44.5473, 7.5333, 35.4424, 10.6964)$ is seen to be locally asymptotically stable for less values of the delays. With the increased values of delays, a Hopf bifurcation occurs at the corresponding critical values of the delays. The bifurcation phenomena of model (43) are shown in Figs. 2–5. Also, by some complex computations, we obtain $C_1(0) = -29.6058 + 17.2864i$, $\mu_2 = 30.0049 > 0$, $\beta_2 = -59.2116$ and $T_2 = -0.8610 < 0$. Thus, we can conclude that the Hopf bifurcation is supercritical, and the periodic solutions are stable and decrease.

5 Conclusions

In this paper, a delayed e-SEIARS model for point-to-group worm propagation is proposed by incorporating the latent period delay, the delay due to the period that the anti-virus

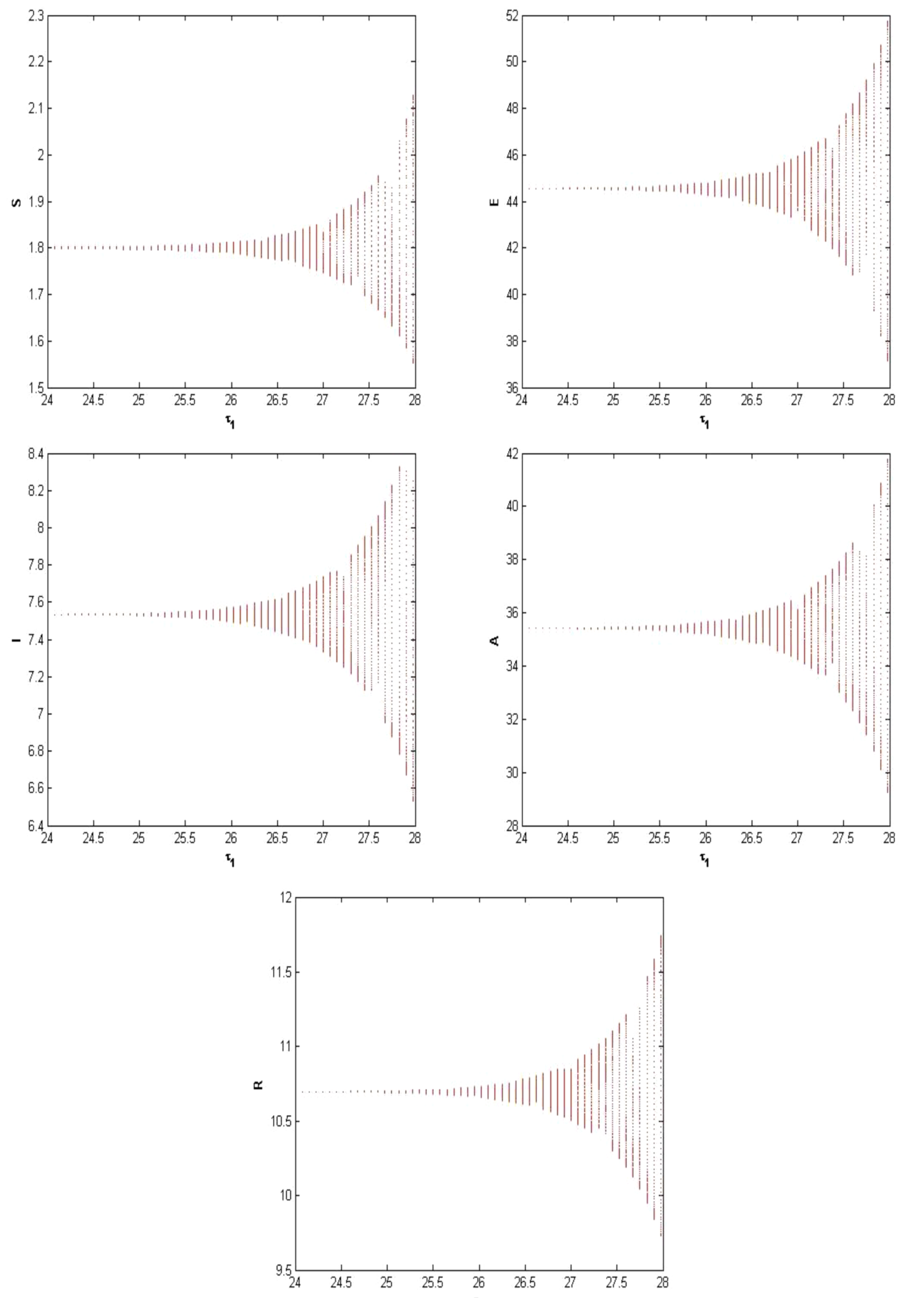

Figure 2 Bifurcation diagram with respectto τ_1

software uses to clean the worm and the temporary immunization delay into the model formulated in the literature [15]. The model not only takes the time delays into account, but also takes two important network environment factors into account, namely point-to-group worm propagation and benign worms.

We mainly focus on effect of the time delays on the proposed model. Local stability and the existence of Hopf bifurcation are discussed by taking different combinations of the time delays as bifurcation parameter. Specially, properties of the Hopf bifurcation are

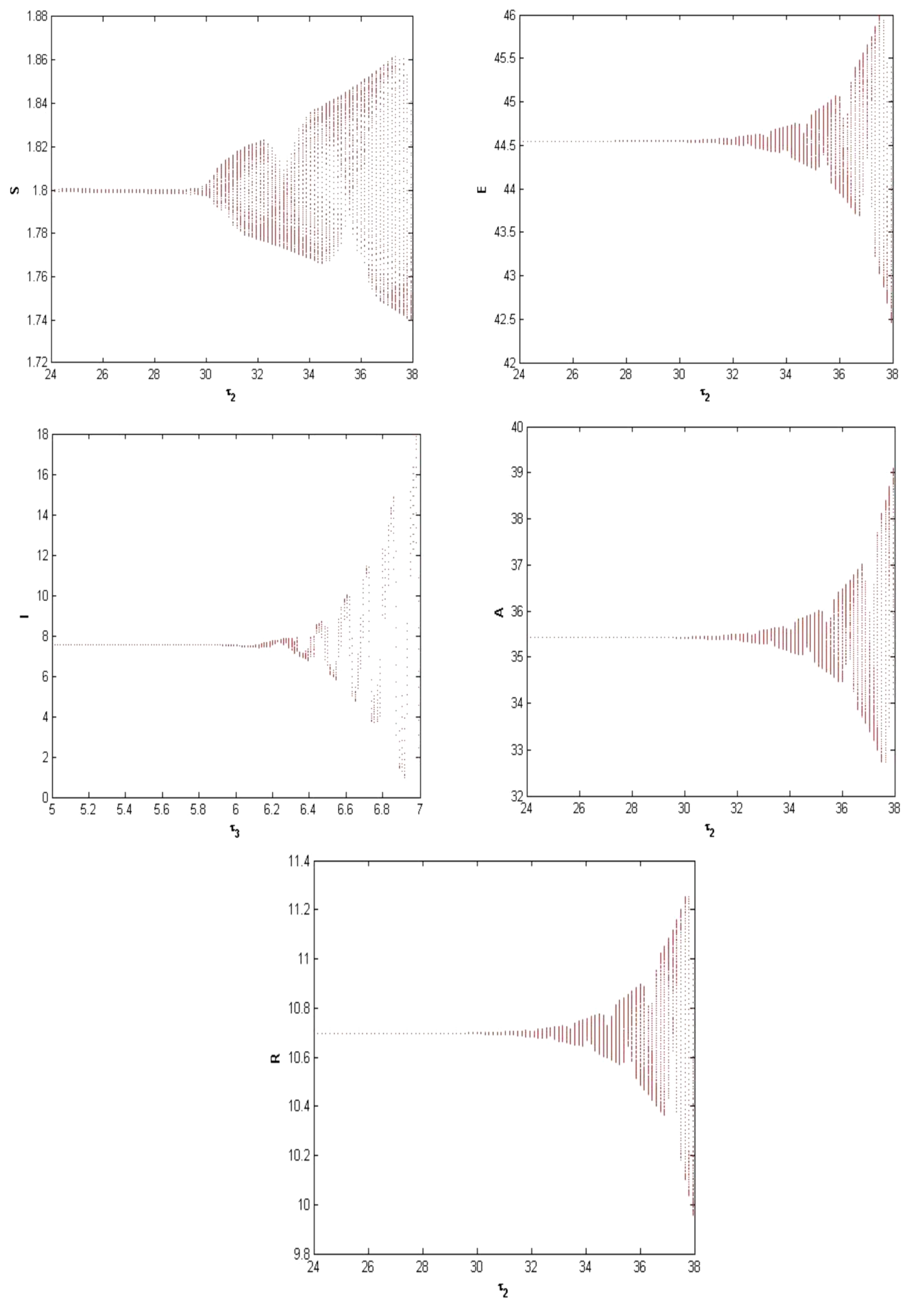

Figure 3 Bifurcation diagram with respect to τ_2

investigated. It is found that when the values of the time delays are below the critical value, the model is locally asymptotically stable. In this case, the worms in model (2) can be predicted and controlled. However, the propagation of the worms will be out of control when the values of the time delays pass through the corresponding critical values. It should be pointed out that our work is only restricted to a theoretical analysis of the model. It may be necessary to make experimental studies in real-world networks in the near future, and this is left for our next study.

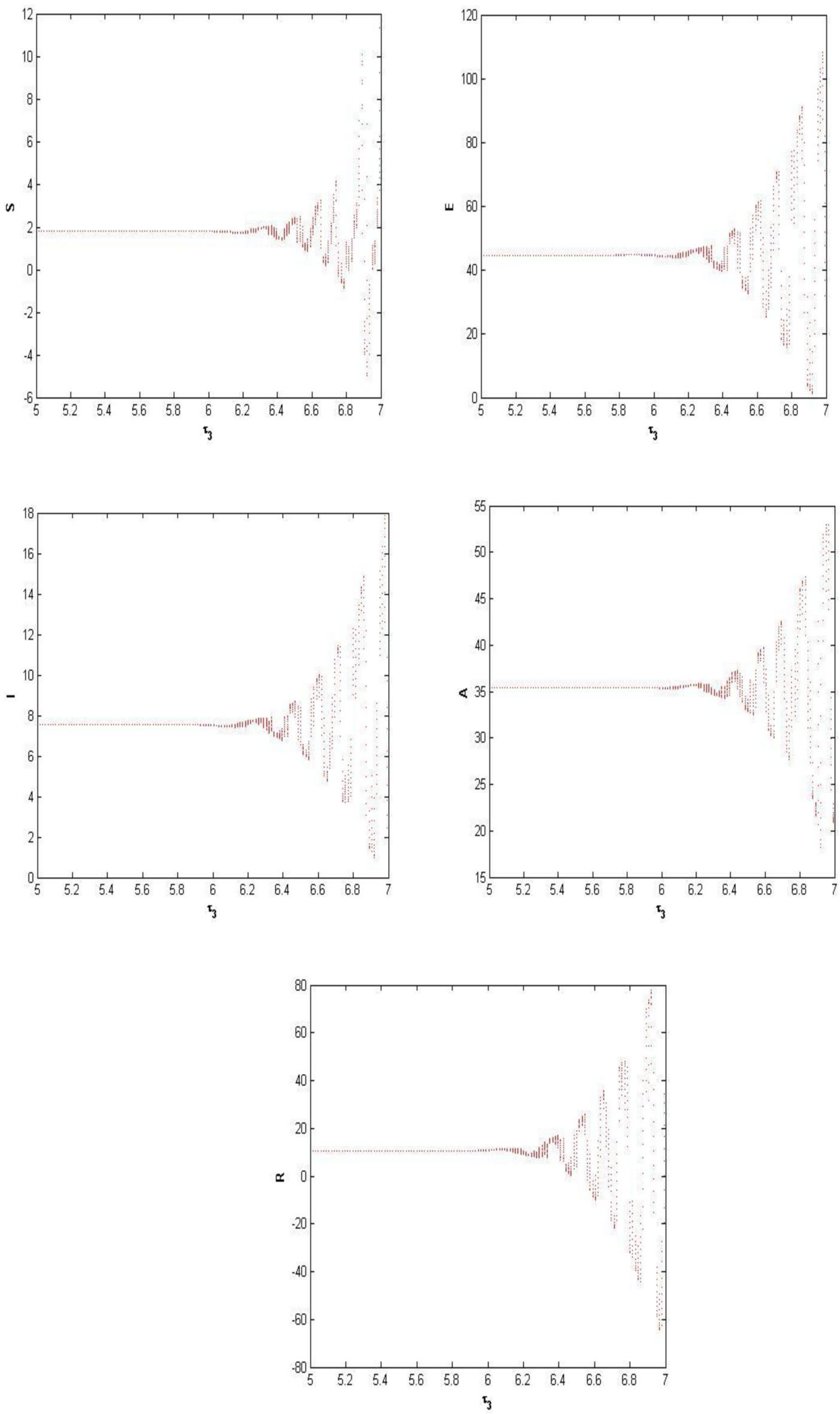

Figure 4 Bifurcation diagram with respect to τ_3

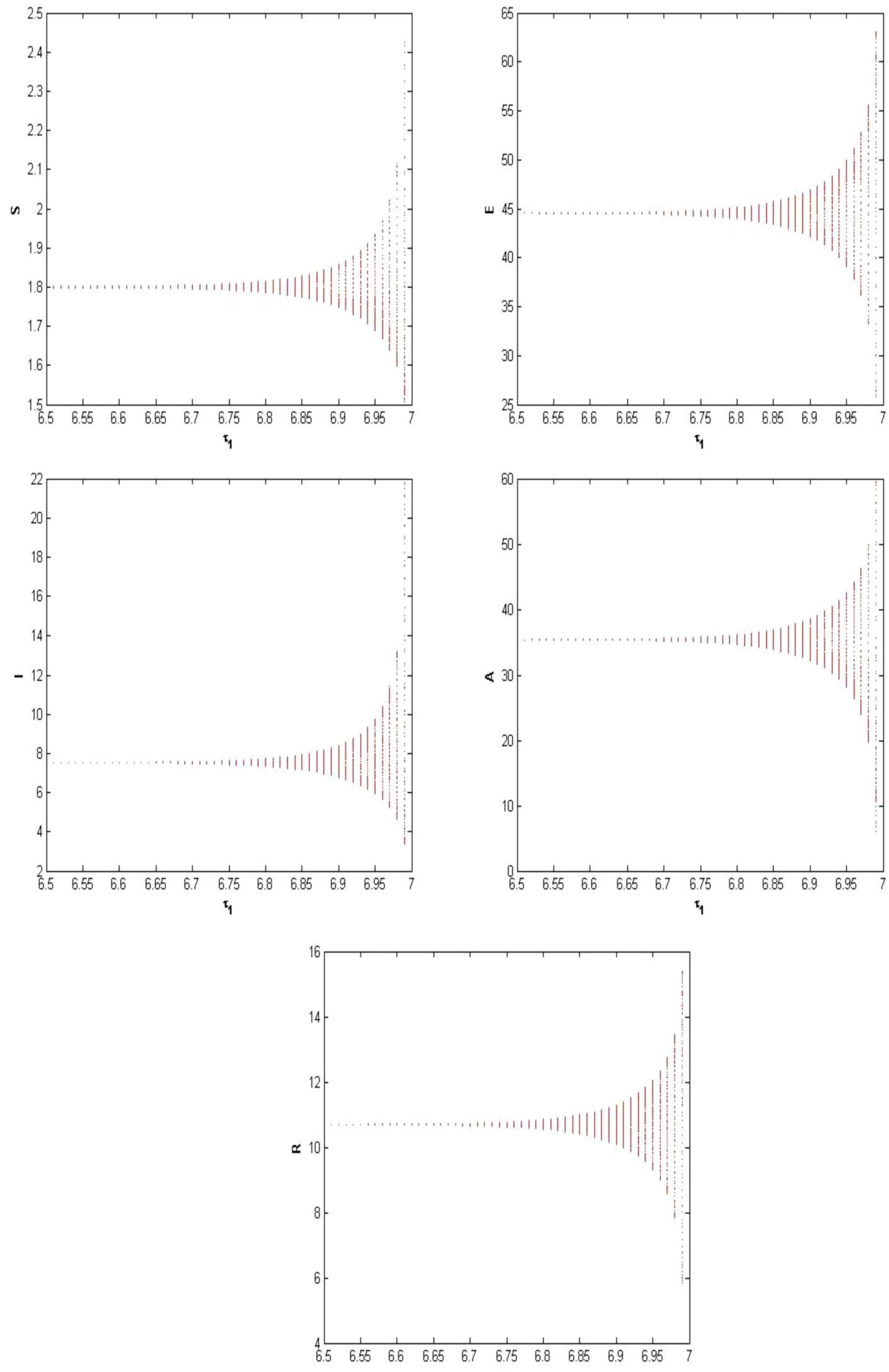

Figure 5 Bifurcation diagram with respect to τ_1 when $\tau_2 = 3.75$ and $\tau_3 = 2.35$

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Availability of data and materials
All of the authors declare that all the data can be accessed in our manuscript in the numerical simulation section.

Competing interests
The authors declare that there is no conflict of interests.

Authors' contributions
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