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Fractional optimal control in transmission dynamics of West Nile virus model with state and control time delay: a numerical approach

N.H. Sweilam^{1*} , O.M. Saad² and D.G. Mohamed²

*Correspondence:

nsweilam@sci.cu.edu.eg

¹Department of Mathematics,
Faculty of Science, Cairo University,
Giza, Egypt

Full list of author information is
available at the end of the article

Abstract

In this paper, an optimal control for a novel fractional West Nile virus model with time delay is presented. The proposed model is governed by a system of fractional delay differential equations, where the fractional derivative is defined in the Grünwald–Letnikov sense. Stability analysis of fixed points is studied. Corresponding fractional optimal control problem, with time delays in both state and control variables, is formulated and studied. Two simple numerical methods are used to study the nonlinear fractional delay optimal control problem. The methods are standard finite difference method and nonstandard finite difference method. Comparative studies are implemented, it is found that the nonstandard finite difference method is better than the standard finite difference method.

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Keywords: West Nile virus mathematical model; Delay differential equations; Nonstandard finite difference method; Grünwald–Letnikov's definition; Fractional optimal control problem

1 Introduction

Time delays play an important role in the modeling of real-life phenomena in different fields of applications, for example, engineering, chemical, and biological systems that are modeled by differential equations. Many methodologies have been proposed to deal with the control and optimal control problems of systems with time delay [1–9]. In [10], an introduction to time-delay control problems is given. In [11], Kharatishvili established a maximum principle for optimal control problems with a constant state delay. Similar results for control problems with pure control delays are presented in [12]. In [13], Halanay proved a maximum principle for optimal control problems with multiple constant delays in state and control variables. Soliman and Ray obtained the similar results in [14]. A simple method for obtaining necessary conditions for control problems with a constant delay in the state variable is introduced in [15]. A maximum principle for control systems with a time-dependent delay in the state variable is derived in [16]. The theory of necessary conditions for optimization problems in functional spaces is applied on a unified approach to control problems with delays in the state variable in [17]. Optimal control for a time-delay

multi-strain tuberculosis fractional model is introduced in [18]. Recently, many interesting studies were presented in nonlinear and time-delay optimal control problems, see [19–21].

Fractional calculus has been the focus of many studies due to its appearance in various sciences [22–30]. The paths of the control variables that maximize or minimize a function of the state and control variables are described by fractional optimal control problems (FOCP). The theory of FOCPs has been under development. The approximation and numerical techniques are used to solve the FOCPs [25, 31]. Since most of the fractional-order differential equations do not have exact analytic solutions, the approximation and numerical techniques must be used [26]. Delayed fractional differential equations (DEDEs) are also used to describe dynamical systems [32], for more details, see [33–35]. Recently, many papers have been devoted to DEDEs (see [36–38], and the references cited therein). Non-standard finite difference method (NSFDM) was firstly developed in 1980s by Mickens and has been applied in many areas of science including biological and epidemic models [39–41]. It is important that the proposed numerical method can preserve the positivity of the solution [42].

In this paper, fractional time-delay optimal control in transmission dynamics of WNV model is presented as an extension of the WNV model given in [26]. Time delays in both state and control variables are formulated and studied. The aim of this paper is to study the fractional time-delay optimal control of this proposed model. We introduced NSFDM and standard finite difference method (SFDM) to study the behavior of this optimal control problem.

West Nile virus (WNV) is an arbovirus and single-standard RNA virus Flavivirus [43]. WNV is transmitted among mosquitoes, birds, humans, and other animals [44]. In 1937, the virus was first isolated in the West Nile district of Uganda [45]. It has spread in Africa, Europe, the Middle East, west and central Asia, Oceania (subtype Kunjin), and North America [46–50]. Owing to the absence of an effective diagnostic test therapeutic treatment against WNV, public health officials have focussed on the use of preventive measures in attempt to halt the spread of WNV in humans [43]. In 2001 [51], the first WNV model was presented by Thomas and Urena. In [52], a single season model with a system of differential equations for WNV transmission in the mosquito-bird population was presented by Wonham et al. They were using local stability results and simulations, showed that while mosquito control decreases WNV outbreak threshold, controlling birds increases it. In [53] Cruz-Pacheco et al. studied a mathematical model for the transmission of WNV infection between mosquito and avian populations, and by using experimental and field data as well as numerical simulations, they found the phenomenon of damped oscillations of the infected bird population. Also, the spatial spread of the virus was established by Lewis et al. in [54]. A comparative study of the discrete-time model in [51] and the continuous-time model in [55] was introduced by Lewis et al. in [56]. Kbenesh et al. determined the cost-effective strategies for combating the spread of WNV in a given population in [57]. Fractional optimal control of WNV model was introduced by using two numerical methods by Sweilam et al. in [26].

This paper is organized as follows. In Sect. 2, preliminaries and notations are presented. In Sect. 3, a mathematical model is formulated. Reproduction number R_0 and stability of the disease-free equilibrium and the endemic equilibrium are studied in Sect. 4 and Sect. 5, respectively. The optimal control problem is formulated in Sect. 6. In Sect. 7, NSFDM

method for fractional time-delay WNV model is presented. A numerical experiment is discussed in Sect. 8. Finally, in Sect. 9, conclusions are presented.

2 Preliminaries and notations

Definition 2.1 ([58]) The left Caputo fractional derivative (LCFD) is defined as follows:

$${}_a^c D_t^\alpha \mathcal{V}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{\mathcal{V}^{(n)}(\tau)}{(t-\tau)^{1-n+\alpha}} d\tau, \quad (1)$$

the right Caputo fractional derivative (RCFD) is defined as follows:

$${}_t^c D_b^\alpha \mathcal{V}(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{\mathcal{V}^{(n)}(\tau)}{(t-\tau)^{1-n+\alpha}} d\tau, \quad (2)$$

where Γ is the Euler gamma function.

Definition 2.2 ([58]) The Grünwald–Letnikov approximation of the fractional derivative of order α of the function $\mathcal{V}(t)$ is defined as follows:

$$D^\alpha \mathcal{V}(t) = \lim_{h \rightarrow 0} (h)^{-\alpha} \sum_{k=0}^{[\frac{t}{h}]} \omega_k^\alpha \mathcal{V}(t - kh), \quad (3)$$

where $[t]$ is the integer part of t , h is the step size, and the Grünwald–Letnikov coefficients are given by

$$\omega_k^\alpha = \left(1 - \frac{\alpha + 1}{k}\right) \omega_{k-1}^\alpha, \quad \text{and} \quad \omega_0^\alpha = (h)^{-\alpha}, \quad k > 1. \quad (4)$$

Definition 2.3 ([59]) The numerical scheme is called NSFD discretization if at least one of the following conditions is satisfied:

- (i) Nonlocal approximation is used [39, 59, 60].
- (ii) The discretization of derivative is not traditional and a nonnegative function $\varphi(\Delta t) = \Delta t + O((\Delta t)^2)$, called a denominator function, is used [41].

The discretization is based on the approximations of temporal derivatives by a generalized forward scheme. If we have $\varphi(\Delta t) = \Delta t$, then the scheme is defined as finite difference or SFDM. Hence, if $\mathcal{M}(t) \in C^1(\mathbb{R})$, the first derivative can be defined as follows:

$$\frac{d\mathcal{M}(t)}{dt} = \frac{\mathcal{M}(t + \Delta t) - \mathcal{M}(t)}{\varphi(\Delta t)}. \quad (5)$$

3 Mathematical model

In this section, a new formulation for the transmission dynamics of WNV model given in [57] is presented. In the following, the transmission dynamics of WNV model is modified to be nonlinear time-delayed fractional differential equations. The WNV introduced in [57] consists of nine nonlinear ODEs which is described by the total mosquito population at time t , denoted by $N_M(t)$, split into the populations of susceptible $M_s(t)$ and infected $M_i(t)$ mosquitoes. Similarly, the total bird population at time t , denoted by $N_B(t)$, is subdivided into susceptible $B_s(t)$ and infected $B_i(t)$ birds. Finally, the total human population

Table 1 All variables in system (6) and their definitions

Variable	Definition
$M_s(t)$	The population of susceptible mosquitoes
$M_i(t)$	The population of infected mosquitoes
$N_M(t)$	The total population of mosquitoes $N_M(t) = M_s(t) + M_i(t)$
$B_s(t)$	The population of susceptible birds
$B_i(t)$	The population of infected birds
$N_B(t)$	The total population of birds $N_B(t) = B_s(t) + B_i(t)$
$S(t)$	The population of susceptible humans
$E(t)$	The population of exposed humans
$I(t)$	The population of infected humans
$H(t)$	The population of hospitalized humans
$R(t)$	The population of recovered humans
$N_H(t)$	The total population of humans $N_H(t) = S(t) + E(t) + I(t) + H(t) + R(t)$

at time t , denoted by $N_H(t)$, is split into susceptible $S(t)$, exposed $E(t)$, infectious $I(t)$, hospitalized $H(t)$, and recovered $R(t)$ humans. Thus, $N_M(t) = M_s(t) + M_i(t)$, $N_B(t) = B_s(t) + B_i(t)$ and $N_H(t) = S(t) + E(t) + I(t) + H(t) + R(t)$ as explained in Table 1. The population of susceptible humans is increased via recruitment of humans (by birth or immigration) into the community at a constant rate λ_H . It is decreased infection (acquired via contact with infected mosquitoes) and by natural death (at a rate λ_H), where b_2 is the per capita rate of biting of humans by mosquitoes and β_3 is the probability of WNV transmission from mosquitoes to humans. Here, it is assumed that new human infections are acquired at a rate that depends on the average number of mosquito bites per unit time and on the transmission probability normalized by total human population [43]. Let us introduce a discrete time delay in the state variables $M_i(t)$, $S(t)$, and $H(t)$, denoted by $\mathbf{d}_M$, $\mathbf{d}_S$, and $\mathbf{d}_H$, respectively (where $\mathbf{d}_M$ represents the time required for the WNV transmission from mosquitoes to humans and $\mathbf{d}_H$ represents the time required for diagnosis and commencement of treatment of the infection of WNV model). The time-delay system of WNV model of fractional-order derivatives is

$$\begin{aligned}
 {}^c D_t^\alpha M_s(t) &= \lambda_M^\alpha - \frac{b_1^\alpha \beta_1^\alpha M_s(t) B_i(t)}{N_B(t)} - \mu_M^\alpha M_s(t), \\
 {}^c D_t^\alpha M_i(t) &= \frac{b_1^\alpha \beta_1^\alpha M_s(t) B_i(t)}{N_B(t)} - \mu_M^\alpha M_i(t), \\
 {}^c D_t^\alpha B_s(t) &= \lambda_B^\alpha - \frac{b_1^\alpha \beta_2^\alpha M_i(t) B_s(t)}{N_B(t)} - (\Psi_B^\alpha + \mu_B^\alpha) B_s(t), \\
 {}^c D_t^\alpha B_i(t) &= \frac{b_1^\alpha \beta_2^\alpha M_i(t) B_s(t)}{N_B(t)} - (d_B^\alpha + \Psi_B^\alpha + \mu_B^\alpha) B_i(t), \\
 {}^c D_t^\alpha S(t) &= \lambda_H^\alpha - \frac{b_2^\alpha \beta_3^\alpha M_i(t - \mathbf{d}_M) S(t - \mathbf{d}_M)}{N_H(t)} - \mu_H^\alpha S(t), \\
 {}^c D_t^\alpha E(t) &= \frac{b_2^\alpha \beta_3^\alpha M_i(t - \mathbf{d}_M) S(t - \mathbf{d}_M)}{N_H(t)} - (\alpha_w^\alpha + \mu_H^\alpha) E(t), \\
 {}^c D_t^\alpha I(t) &= \alpha_w^\alpha E(t) - (\gamma^\alpha + d_I^\alpha + r^\alpha + \mu_H^\alpha) I(t), \\
 {}^c D_t^\alpha H(t) &= \gamma^\alpha I(t) - (d_H^\alpha + \mu_H^\alpha) H(t) - \tau^\alpha H(t - \mathbf{d}_H), \\
 {}^c D_t^\alpha R(t) &= \tau^\alpha H(t - \mathbf{d}_H) + r^\alpha I(t) - \mu_H^\alpha R(t),
 \end{aligned} \tag{6}$$

Table 2 All parameters in system (6) and their interpretation

Parameter	Interpretation
λ_M^α	The recruitment rate of mosquitoes (assumed susceptible)
μ_M^α	The natural death rate of mosquitoes
λ_B^α	The recruitment rate of birds (assumed susceptible)
μ_B^α	The natural death rate of birds
ψ_B^α	The migration rate of birds
d_B^α	The WNV-induced death rate of birds
λ_H^α	The recruitment rate of humans (assumed susceptible)
μ_H^α	The natural death rate of humans
d_H^α	The WNV-induced death rate of humans
d_H^α	The death rate of hospitalized humans
β_1^α	The probability of WNV transmission from an infected bird to a susceptible mosquito
β_2^α	The probability of WNV transmission from an infected mosquito to a susceptible bird
β_3^α	The probability of WNV transmission from mosquitoes to humans
b_1^α	The per capita biting rate of mosquitoes on the primary host (birds) and $b_1^\alpha = \frac{b^\alpha N_B}{N_B + N_H}$
b_2^α	The per capita biting rate of mosquitoes on the humans and $b_2^\alpha = \frac{b^\alpha N_H}{N_B + N_H}$
b^α	The average biting rate of mosquitoes
α_ω^α	The rate of development of clinical symptoms of WNV
r^α	The natural recovery rate
τ^α	The treatment-induced recovery rate
γ^α	The hospitalized rate of infectious humans

Table 3 Parameter values used in system (6)

Parameter	Value	Parameter	Value
μ_H^α	$((3.91 \times 10^{-5})^\alpha - 0.005^\alpha)$	d_H^α	$((5 \times 10^{-5})^\alpha - 0.015^\alpha)$
μ_B^α	$(0.0001^\alpha - 0.0003^\alpha)$	d_B^α	$(0.06^\alpha - 0.2^\alpha)$
μ_M^α	$(0.016^\alpha - 0.07^\alpha)$	λ_M^α	$(51.1 \times 10^{-3})^\alpha$
λ_B^α	2.1^α	λ_H^α	$(5 \times 10^{-2})^\alpha$
d_H^α	$(d_H + 10^{-5})^\alpha$	ψ_B^α	$(5.2 \times 10^{-2})^\alpha$
β_1^α	0.4^α	β_2^α	0.1^α
β_3^α	$(10^{-2})^\alpha$	b^α	3^α
α_ω^α	0.1^α	r^α	$(2 \times 10^{-4})^\alpha$
τ^α	0.05^α	γ^α	$(9 \times 10^{-4})^\alpha$

with the following initial conditions:

$$M_s(\eta) = \zeta_1(\eta), \quad M_i(\eta) = \zeta_2(\eta), \quad B_s(\eta) = \zeta_3(\eta), \quad (7)$$

$$B_i(\eta) = \zeta_4(\eta), \quad S(\eta) = \zeta_5(\eta), \quad E(\eta) = \zeta_6(\eta),$$

$$I(\eta) = \zeta_7(\eta), \quad H(\eta) = \zeta_8(\eta), \quad R(\eta) = \zeta_9(\eta), \quad (8)$$

where ${}_a^c D_t^\alpha$ is the Caputo fractional-order derivative. Let us consider that $\mathbf{d}_d = \mathbf{d}_H = \mathbf{d}$. Since $\zeta \in [-\mathbf{d}, 0]$, where $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_9)^T \in \mathbb{C}$, and $\mathbb{C}$ is the Banach space $\mathbb{C}([0, \mathbf{d}], \mathbb{R}^9)$. The variables in system (6) are defined in Table 1. The list of parameter values and their interpretation are introduced in Tables 2 and 3; for more details, see [57].

4 Reproduction number R_0 and stability of the disease-free equilibrium

In this section, we compute the reproduction number R_0 . The basic reproduction number is the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual [32]. Method in [32] is applied to derive R_0 . The

order of infected variables is

$$\zeta = (M_i, B_i, E, I, H, R)'. \quad (9)$$

The new infections into the infected classes ζ are defined by the vector $\mathbf{F}$, and the other flows within and out of the infected classes ζ are defined by the vector $\mathbf{V}$, where these vectors are given by

$$\mathbf{F} = \begin{pmatrix} \frac{b_1^\alpha \beta_1^\alpha M_s B_i}{N_B} \\ \frac{b_2^\alpha \beta_2^\alpha M_i B_s}{N_B} \\ \frac{b_3^\alpha \beta_3^\alpha M_i S}{N_B} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} -\mu_M^\alpha M_i \\ -(d_B^\alpha + \Psi_B^\alpha + \mu_B^\alpha) B_i \\ -(\alpha_w^\alpha + \mu_H^\alpha) E \\ \alpha_w^\alpha E - (\gamma^\alpha + d_I^\alpha + r^\alpha + \mu_H^\alpha) I \\ \gamma^\alpha I - (d_H^\alpha + \tau^\alpha + \mu_H^\alpha) H \\ \tau^\alpha H + r^\alpha I - \mu_H^\alpha R \end{pmatrix}.$$

Let us differentiate the above matrices $\mathbf{F}$ and $\mathbf{V}$ with respect to the infected variables ζ and evaluate at the disease-free equilibrium

$$\mathbf{F} = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix},$$

where

$$\begin{aligned} f_1 &= \begin{pmatrix} 0 & \frac{b_1^\alpha \beta_1^\alpha M_s}{B_s + S} \\ \frac{b_2^\alpha \beta_2^\alpha B_s}{B_s + S} & 0 \end{pmatrix}, & f_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ f_3 &= \begin{pmatrix} \frac{b_3^\alpha \beta_3^\alpha S}{B_s + S} & 0 & 0 & 0 \end{pmatrix}^T, & f_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ v_1 &= \begin{pmatrix} \mu_M^\alpha & 0 \\ 0 & \mu_B^\alpha + \Psi_B^\alpha + d_B^\alpha \end{pmatrix}, & v_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & v_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^T, \\ v_4 &= \begin{pmatrix} \mu_H^\alpha + \alpha_\omega^\alpha & 0 & 0 & 0 \\ -\alpha_\omega^\alpha & \gamma^\alpha + d_I^\alpha + r^\alpha + \mu_H^\alpha & 0 & 0 \\ 0 & -\gamma^\alpha & \tau^\alpha + \mu_H^\alpha + d_H^\alpha & 0 \\ 0 & -r^\alpha & -\tau^\alpha & \mu_H^\alpha \end{pmatrix}, \end{aligned}$$

then the basic reproduction number R_0 for system (6) with spectral radius ρ is defined by

$$\begin{aligned} R_0 &= \rho(\mathbf{FV}^{-1}) \\ &= \frac{b^\alpha \sqrt{\mu_M^\alpha (\mu_B^\alpha + \Psi_B^\alpha + d_B^\alpha) \beta_1^\alpha \beta_2^\alpha \hat{M}_s \hat{B}_s}}{\mu_M^\alpha (\mu_B^\alpha + \Psi_B^\alpha + d_B^\alpha) (\hat{B}_s + \hat{S})}. \end{aligned} \quad (10)$$

Lemma 4.1 ([61]) *The disease-free equilibrium point ε_{eq} is locally asymptotically stable iff all eigenvalues λ_i satisfy $|\arg \lambda_i| > \frac{\alpha\pi}{2}$.*

Proof To analyze the stability of the disease-free equilibrium (DFE) point, let us have that [62]

$$\begin{aligned} {}^c D_t^\alpha M_s &= 0, & {}^c D_t^\alpha M_i &= 0, & {}^c D_t^\alpha B_s &= 0, \\ {}^c D_t^\alpha B_i &= 0, & {}^c D_t^\alpha S &= 0, & {}^c D_t^\alpha E &= 0, \\ {}^c D_t^\alpha I &= 0, & {}^c D_t^\alpha H &= 0, & {}^c D_t^\alpha R &= 0. \end{aligned} \quad (11)$$

Let us consider the coordinate transformation $m_s(t) = M_s(t) - \check{M}_s$, $m_i(t) = M_i(t) - \check{M}_i$, $b_s(t) = B_s(t) - \check{B}_s$, $b_i(t) = B_i(t) - \check{B}_i$, $s(t) = S(t) - \check{S}$, $e(t) = E(t) - \check{E}$, $i(t) = I(t) - \check{I}$, $h(t) = H(t) - \check{H}$, $r(t) = R(t) - \check{R}$, where any equilibrium point is described by $(\check{M}_s, \check{M}_i, \check{B}_s, \check{B}_i, \check{S}, \check{E}, \check{I}, \check{H}, \check{R})$. When $\check{M}_i = 0$, $\check{B}_i = 0$, and $\check{I} = 0$, then the DFE point is $\varepsilon_{\text{eq}} = (\frac{\lambda_M^\alpha}{\mu_M^\alpha}, 0, \frac{\lambda_B^\alpha}{\psi_B^\alpha + \mu_B^\alpha}, 0, \frac{\lambda_H^\alpha}{\mu_H^\alpha}, 0, 0, 0, 0)$. The characteristic equation can be calculated at the DFE point. It can be written as follows:

$$(J_{\varepsilon_{\text{eq}}} - \lambda \mathcal{I}) = \begin{pmatrix} -\mu_M^\alpha - \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu_M^\alpha - \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_1^\alpha \beta_2^\alpha & -c_1 - \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_1^\alpha \beta_2^\alpha & 0 & -c_2 - \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_2^\alpha \beta_3^\alpha e^{-\lambda \mathbf{d}} & 0 & 0 & -\mu_H^\alpha - \lambda & 0 & 0 & 0 & 0 \\ 0 & b_2^\alpha \beta_3^\alpha e^{-\lambda \mathbf{d}} & 0 & 0 & 0 & -c_3 - \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_\omega^\alpha & -c_4 - \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma^\alpha & -c_5 - \lambda - \tau^\alpha e^{-\lambda \mathbf{d}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r^\alpha & \tau^\alpha e^{-\lambda \mathbf{d}} & -\mu_H^\alpha - \lambda \end{pmatrix},$$

where $\mathcal{I}$ is the unit matrix, $\mathbf{c}_1 = (\psi_B^\alpha + \mu_B^\alpha)$, $\mathbf{c}_2 = (d_B^\alpha + \psi_B^\alpha + \mu_B^\alpha)$, $\mathbf{c}_3 = (\alpha_\omega^\alpha + \mu_H^\alpha)$, $\mathbf{c}_4 = (\gamma^\alpha + d_I^\alpha + r^\alpha + \mu_H^\alpha)$, and $\mathbf{c}_5 = (d_H^\alpha + \mu_H^\alpha)$. Thus the characteristic equation corresponding to the above matrix is [62]

$$\begin{aligned} |J_{\varepsilon_{\text{eq}}} - \lambda \mathcal{I}| &= 0, \\ (\lambda + \mu_H^\alpha)^2 (\lambda + \mu_M^\alpha)^2 (\lambda + \alpha_\omega^\alpha + \mu_H^\alpha) (\lambda + \psi_B^\alpha + \mu_B^\alpha) (\lambda + d_B^\alpha + \psi_B^\alpha + \mu_B^\alpha) \\ &\quad \times (\lambda + d_H^\alpha + \mu_H^\alpha + \tau^\alpha e^{-\lambda \mathbf{d}}) (\lambda + d_I^\alpha + \gamma^\alpha + \mu_H^\alpha + r^\alpha) = 0. \end{aligned} \quad (12)$$

If we have $\mathbf{d} = 0$, then the eigenvalues of the Jacobian matrix J_{ε_0} at the DFE point ε_0 are $\lambda_{1,2} = -\mu_H^\alpha$, $\lambda_{3,4} = -\mu_M^\alpha$, $\lambda_5 = -(\alpha_\omega^\alpha + \mu_H^\alpha)$, $\lambda_6 = -(\psi_B^\alpha + \mu_B^\alpha)$, $\lambda_7 = -(d_B^\alpha + \psi_B^\alpha + \mu_B^\alpha)$, $\lambda_8 = -(d_H^\alpha + \mu_H^\alpha + \tau^\alpha)$, and $\lambda_9 = -(d_I^\alpha + \gamma^\alpha + \mu_H^\alpha + r^\alpha)$. According to the Routh–Hurwitz theorem [35], these roots are negative or have negative real parts and the eigenvalues satisfy Matignon's conditions [63] defined by $|\arg \lambda_i| > \frac{\alpha\pi}{2}$. Thus, the DFE point ε_{eq} is asymptotically stable. If we consider $\mathbf{d} > 0$, then $(\lambda + d_H^\alpha + \mu_H^\alpha + \tau^\alpha e^{-\lambda \mathbf{d}})$ has no pure imaginary roots for any value of the delay $\mathbf{d}$. Hence, all roots of the characteristic equation have negative real parts, and the DFE point ε_{eq} is locally asymptotically stable according to Matignon's conditions [63] defined by $|\arg \lambda_i| > \frac{\alpha\pi}{2}$. $\square$

5 Stability analysis of the endemic equilibrium

In this section, the stability analysis of the endemic equilibrium point is introduced. If at least one of the infected variables is not zero, then system (6) has an endemic equilibrium. The endemic equilibrium point is $\varepsilon_{\text{end}} = (0.5908, 2398.7049, 3.6028, 36.7141, 0.4071, 0.5679, 573.7086, 10.2296, 39.7060)$. The list of parameter values is introduced in Table 3.

The basic reproduction number is $R_0 > 1$. According to the linearized system of (6), we have the following form:

$$\frac{d}{dt}\Phi = \Delta_1\Phi + \Delta_2\Phi_d, \quad (13)$$

where

$$\Phi = (m_s(t), m_i(t), b_s(t), b_i(t), s(t), e(t), i(t), h(t), r(t))^T,$$

$$\Phi_d = (m_s(t-d), m_i(t-d), b_s(t-d), b_i(t-d), s(t-d), e(t-d), i(t-d),$$

$$h(t-d), r(t-d))^T,$$

$$\Delta_1 = \begin{pmatrix} -0.0823 & 0 & 0 & -0.0011 & 0 & 0 & 0 & 0 & 0 \\ 0.0663 & -0.0160 & 0 & 0.0011 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0016 & -1.1343 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0016 & 1.0822 & -0.1121 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0050 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1050 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1000 & -0.0061 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0009 & -0.0050 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0002 & 0 & -0.0050 \end{pmatrix},$$

$$\Delta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{b_2^\alpha \beta_3^\alpha \tilde{S}}{N_H} & 0 & 0 & -\frac{b_2^\alpha \beta_3^\alpha \tilde{M}_i}{N_H} & 0 & 0 & 0 & 0 \\ 0 & \frac{b_2^\alpha \beta_3^\alpha \tilde{S}}{N_H} & 0 & 0 & \frac{b_2^\alpha \beta_3^\alpha \tilde{M}_i}{N_H} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\tau^\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\tau^\alpha & 0 \end{pmatrix}.$$

Corresponding to these matrices, the characteristic equation $|\lambda \mathcal{I} - \Delta_1 - e^{-\lambda d} \Delta_2| = 0$ is written as follows:

$$\begin{aligned} & \lambda^9 + (1.4708 + 0.1582e^{-\lambda d})\lambda^8 + (0.4229 + 0.2139e^{-\lambda d} + 0.0054e^{-2\lambda d})\lambda^7 \\ & + (0.0490 + 0.0658e^{-\lambda d} + 0.0079e^{-2\lambda d})\lambda^6 \\ & + (0.0026 + 0.0074e^{-\lambda d} + 0.0022e^{-2\lambda d})\lambda^5 \\ & + (5.9414 \times 10^{-5} + 3.6988 \times 10^{-4}e^{-\lambda d} + 2.4288 \times 10^{-4}e^{-2\lambda d})\lambda^4 \\ & + (6.7012 \times 10^{-7} + 7.5511 \times 10^{-6}e^{-\lambda d} + 1.1435 \times 10^{-5}e^{-2\lambda d})\lambda^3 \\ & + (3.9229 \times 10^{-9} + 6.8272 \times 10^{-8}e^{-\lambda d} + 2.0107 \times 10^{-7}e^{-2\lambda d})\lambda^2 \\ & + (1.1512 \times 10^{-11} + 2.7933 \times 10^{-10}e^{-\lambda d} + 1.3295 \times 10^{-9}e^{-2\lambda d})\lambda \\ & + (1.3423 \times 10^{-14} + 4.2476 \times 10^{-13}e^{-\lambda d} + 2.9051 \times 10^{-12}e^{-2\lambda d}) = 0. \end{aligned} \quad (14)$$

If we have $\mathbf{d} = 0$, then the characteristic equation is written in the following form:

$$\begin{aligned} &\lambda^9 + 1.6290\lambda^8 + 0.6602\lambda^7 + 0.1227\lambda^6 + 0.0122\lambda^5 + 6.7217 \times 10^{-4}\lambda^4 \\ &+ 1.9656 \times 10^{-5}\lambda^3 + 2.7326 \times 10^{-7}\lambda^2 + 1.6203 \times 10^{-9}\lambda \\ &+ 3.3433 \times 10^{-12} = 0. \end{aligned} \quad (15)$$

The roots of the above equation are $\lambda_1 = -0.0160$, $\lambda_2 = -0.1121$, $\lambda_3 = -0.1132$, $\lambda_4 = -0.1050$, $\lambda_5 = -0.0823$, $\lambda_6 = -1.1343$, $\lambda_7 = -0.0050$, $\lambda_8 = -0.0061$, $\lambda_9 = -0.0550$. These roots are negative or have negative real parts and the eigenvalues satisfy Matignon's conditions [63] defined by $|\arg \lambda_i| > \frac{\alpha\pi}{2}$. Thus, the endemic equilibrium point ε_{end} is asymptotically stable. If we consider $\mathbf{d} > 0$, then these roots of the characteristic equation have no pure imaginary roots for any value of the delay $\mathbf{d}$. Thus, all roots of the characteristic equation have negative real parts according to Matignon's conditions [63] defined by $|\arg \lambda_i| > \frac{\alpha\pi}{2}$. Therefore, the endemic equilibrium is locally asymptotically stable.

6 The optimal control problem

In this section, we present the fractional optimal control in transmission dynamics of WNV model (6) with a time delay in the state variables $M_i(t)$, $S(t)$, and $H(t)$. Two control functions are considered u_1 and u_2 . The control u_1 represents the level of larvacide and adulticide used for mosquito control administered at mosquito breeding sites. The control u_2 measures the level of successful prevention (personal protection) efforts. Hence, we introduce a time delay $\mathbf{d}_u$ in the control u_2 that represents the time required for the WNV prevention. The resulting model of WNV is given by the following system of nonlinear ordinary delay fractional-order differential equations:

$$\begin{aligned} {}^c D_t^\alpha M_s(t) &= \lambda_M^\alpha N_M (1 - u_1(t)) - \frac{b_1^\alpha \beta_1^\alpha M_s(t) B_i(t)}{N_B(t)} - \mu_M^\alpha M_s(t) - r_0^\alpha u_1(t) M_s(t), \\ {}^c D_t^\alpha M_i(t) &= \frac{b_1^\alpha \beta_1^\alpha M_s(t) B_i(t)}{N_B(t)} - \mu_M^\alpha M_i(t) - r_0^\alpha u_1(t) M_i(t), \\ {}^c D_t^\alpha B_s(t) &= \lambda_B^\alpha + \rho^\alpha N_B(t) - \frac{b_1^\alpha \beta_2^\alpha M_i(t) B_s(t)}{N_B(t)} - (\Psi_B^\alpha + \mu_B^\alpha) B_s(t), \\ {}^c D_t^\alpha B_i(t) &= \frac{b_1^\alpha \beta_2^\alpha M_i(t) B_s(t)}{N_B(t)} - (d_B^\alpha + \Psi_B^\alpha + \mu_B^\alpha) B_i(t), \\ {}^c D_t^\alpha S(t) &= \lambda_H^\alpha + \gamma_H^\alpha N_H(t) - \frac{b_2^\alpha \beta_3^\alpha M_i(t - \mathbf{d}) S(t - \mathbf{d}) (1 - u_2(t - \mathbf{d}_u))}{N_H(t)} - \mu_H^\alpha S(t), \\ {}^c D_t^\alpha E(t) &= \frac{b_2^\alpha \beta_3^\alpha M_i(t - \mathbf{d}) S(t - \mathbf{d}) (1 - u_2(t - \mathbf{d}_u))}{N_H(t)} - (\alpha_w^\alpha + \mu_H^\alpha) E(t), \\ {}^c D_t^\alpha I(t) &= \alpha_w^\alpha E(t) - (\gamma^\alpha + d_I^\alpha + r^\alpha + \mu_H^\alpha) I(t), \\ {}^c D_t^\alpha H(t) &= \gamma^\alpha I(t) - (d_H^\alpha + \mu_H^\alpha) H(t) - \tau^\alpha H(t - \mathbf{d}), \\ {}^c D_t^\alpha R(t) &= \tau^\alpha H(t - \mathbf{d}) + r^\alpha I(t) - \mu_H^\alpha R(t). \end{aligned} \quad (16)$$

We consider that $\mathbf{d}_u = \mathbf{d}$. Let us denote the state and control variables of the control system (16) by $\mathfrak{X} = (M_s, M_i, B_s, B_i, S, E, I, H, R) \in \mathbb{R}^9$ and $\mathfrak{U} = (u_1, u_2) \in \mathbb{R}^2$. We consider the

following objective function which is quadratic in the control variable:

$$J^*(\mathfrak{X}, \mathfrak{U}) = \int_0^{T_f} (A_1 E(t) + A_2 I(t) + A_3 N_M(t) + B_1 u_1^2(t) + B_2 u_2^2(t)) dt. \quad (17)$$

The optimal control problem is defined by determining a control function $\mathfrak{U} = (u_1, u_2) \in \mathbb{R}^2$ that minimizes the cost functional $J^*(\mathfrak{X}, \mathfrak{U})$ subject to the constraints (16). To get the necessary optimality conditions of the optimal control problem described by time delay in state and control variables, see [64]. To get the Hamiltonian for the delayed fractional optimal control problem, let us introduce the delayed state variables $\mathfrak{C}_2 = \mathfrak{X}_2(t - \mathbf{d}) = M_i(t - \mathbf{d})$, $\mathfrak{C}_5 = \mathfrak{X}_5(t - \mathbf{d}) = S(t - \mathbf{d})$, $\mathfrak{C}_8 = \mathfrak{X}_8(t - \mathbf{d}) = H(t - \mathbf{d})$ and the delayed control variable $\mathfrak{U}_2 = \mathfrak{U}_2(t - \mathbf{d}) = u_2(t - \mathbf{d})$. The Hamiltonian for the objective functional (17) and the control system (16) is given by

$$\begin{aligned} \mathcal{H}^*(\mathfrak{X}, \mathfrak{C}_2, \mathfrak{C}_5, \mathfrak{C}_8, \lambda^*, u_1, \mathfrak{U}_2) &= -(A_1 E + A_2 I + A_3 N_M + B_1 u_1^2 + B_2 u_2^2) \\ &+ \lambda_1^* \left[\lambda_M^\alpha N_M (1 - u_1) - \frac{b_1^\alpha \beta_1^\alpha M_s B_i}{N_B} - \mu_M^\alpha M_s - r_0^\alpha u_1 M_s \right] \\ &+ \lambda_2^* \left[\frac{b_1^\alpha \beta_1^\alpha M_s B_i}{N_B} - \mu_M^\alpha M_i - r_0^\alpha u_1 M_i \right] \\ &+ \lambda_3^* \left[\lambda_B^\alpha + \rho^\alpha N_B - \frac{b_1^\alpha \beta_2^\alpha M_i B_s}{N_B} - (\Psi_B^\alpha + \mu_B^\alpha) B_s \right] \\ &+ \lambda_4^* \left[\frac{b_1^\alpha \beta_2^\alpha M_i B_s}{N_B} - (d_B^\alpha + \Psi_B^\alpha + \mu_B^\alpha) B_i \right] \\ &+ \lambda_5^* \left[\lambda_H^\alpha + \gamma_H^\alpha N_H - \frac{b_2^\alpha \beta_3^\alpha \mathfrak{C}_2 \mathfrak{C}_5 (1 - \mathfrak{U}_2)}{N_H} - \mu_H^\alpha S \right] \\ &+ \lambda_6^* \left[\frac{b_2^\alpha \beta_3^\alpha \mathfrak{C}_2 \mathfrak{C}_5 (1 - \mathfrak{U}_2)}{N_H} - (\alpha_w^\alpha + \mu_H^\alpha) E \right] \\ &+ \lambda_7^* [\alpha_w^\alpha E - (\gamma^\alpha + d_I^\alpha + r^\alpha + \mu_H^\alpha) I] \\ &+ \lambda_8^* [\gamma^\alpha I - (d_H^\alpha + \mu_H^\alpha) H - \tau^\alpha \mathfrak{C}_8] \\ &+ \lambda_9^* [\tau^\alpha \mathfrak{C}_8 + r^\alpha I - \mu_H^\alpha R], \end{aligned} \quad (18)$$

where the adjoint variable is $\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*, \lambda_5^*, \lambda_6^*, \lambda_7^*, \lambda_8^*, \lambda_9^*) \in \mathbb{R}^9$. The adjoint equations are given by

$$\begin{aligned} {}^c D_{T_f}^\alpha \lambda_1^*(t) &= -\frac{\partial \mathcal{H}^*(t)}{\partial M_s(t)}, \\ {}^c D_{T_f}^\alpha \lambda_2^*(t) &= -\frac{\partial \mathcal{H}^*(t)}{\partial M_i(t)} + \chi_{[0, T_f - \mathbf{d}]} \frac{\partial \mathcal{H}^*(t + \mathbf{d})}{\partial \mathfrak{C}_2}, \\ {}^c D_{T_f}^\alpha \lambda_3^*(t) &= -\frac{\partial \mathcal{H}^*(t)}{\partial B_s(t)}, \quad {}^c D_{T_f}^\alpha \lambda_4^*(t) = -\frac{\partial \mathcal{H}^*(t)}{\partial B_i(t)}, \\ {}^c D_{T_f}^\alpha \lambda_5^*(t) &= -\frac{\partial \mathcal{H}^*(t)}{\partial S(t)} + \chi_{[0, T_f - \mathbf{d}]} \frac{\partial \mathcal{H}^*(t + \mathbf{d})}{\partial \mathfrak{C}_5}, \\ {}^c D_{T_f}^\alpha \lambda_6^*(t) &= -\frac{\partial \mathcal{H}^*(t)}{\partial E(t)}, \quad {}^c D_{T_f}^\alpha \lambda_7^*(t) = -\frac{\partial \mathcal{H}^*(t)}{\partial I(t)}, \end{aligned} \quad (19)$$

$${}_t^c D_{T_f}^\alpha \lambda_8^*(t) = -\frac{\partial \mathcal{H}^*(t)}{\partial H(t)} + \chi_{[0, T_f - \mathbf{d}]} \frac{\partial \mathcal{H}^*(t + \mathbf{d})}{\partial \mathfrak{C}_8},$$

$${}_t^c D_{T_f}^\alpha \lambda_9^*(t) = -\frac{\partial \mathcal{H}^*(t)}{\partial R(t)},$$

where

$$\chi_{[0, T_f - \mathbf{d}]} = \begin{cases} 1, & \text{if } t \in [0, T_f - \mathbf{d}], \\ 0, & \text{otherwise.} \end{cases}$$

The equations for $\lambda_2^*(t)$, $\lambda_5^*(t)$, and $\lambda_8^*(t)$ contain the advanced time $t + \mathbf{d}$. Therefore, the Lagrange multipliers can be written as follows:

$$\begin{aligned} {}_t^c D_{T_f}^\alpha \lambda_1^* &= -A_3 - (\lambda_2^* - \lambda_1^*)b\beta_1 B_i \xi - \lambda_1^*[\lambda_M(1 - u_1) - (\mu_1 + \mu_2 N_M) \\ &\quad - \mu_2 M_s - r_0 u_1] + \lambda_2^* \mu_2 M_i, \\ {}_t^c D_{T_f}^\alpha \lambda_2^* &= -A_3 - \lambda_1^*[\lambda_M(1 - u_1) - \mu_2 M_s] + \lambda_2^*[\mu_1 + \mu_2 N_M + \mu_2 M_i + r_0 u_1] \\ &\quad - (\lambda_4^* - \lambda_3^*)b\beta_2 B_s \xi - b\beta_3 S(1 - u_2)\xi(\lambda_6^* - \lambda_5^*) \\ &\quad + \chi_{[0, T_f - \mathbf{d}]} \frac{b_2^\alpha \beta_3^\alpha \mathfrak{C}_5(1 - \mathfrak{A}_2)}{N_H} (\lambda_6^*(t + \mathbf{d}) - \lambda_5^*(t + \mathbf{d})), \\ {}_t^c D_{T_f}^\alpha \lambda_3^* &= (\lambda_2^* - \lambda_1^*)b\beta_1 B_i M_s \xi^2 - \lambda_3^*[\rho - b\beta_2 M_i(\xi - B_s \xi^2) - \Psi_B - \mu_B] \\ &\quad - \lambda_4^* b\beta_2 M_i(\xi - B_s \xi^2) - (\lambda_5^* - \lambda_6^*)b\beta_3 M_i S(1 - u_2)\xi^2, \\ {}_t^c D_{T_f}^\alpha \lambda_4^* &= -(\lambda_2^* - \lambda_1^*)b\beta_1 M_s(\xi - B_i \xi^2) - \lambda_3^*[\rho + b\beta_2 M_i B_s \xi^2] \\ &\quad + \lambda_4^*[b\beta_2 M_i B_s \xi^2 + (d_B + \Psi_B + \mu_B)] - (\lambda_5^* - \lambda_6^*)b\beta_3 M_i S(1 - u_2)\xi^2, \\ {}_t^c D_{T_f}^\alpha \lambda_5^* &= -(\lambda_1^* - \lambda_2^*)b\beta_1 B_i M_s \xi^2 - (\lambda_3^* - \lambda_4^*)b\beta_2 M_i B_s \xi^2 \\ &\quad - \lambda_5^*[\gamma_H - b\beta_3 M_i(1 - u_2)(\xi - S\xi^2) - \mu_4 S - (\mu_3 + \mu_4 N_H)] \\ &\quad - \lambda_6^*[b\beta_3 M_i(1 - u_2)(\xi - S\xi^2) - \mu_4 E] + \lambda_7^* \mu_4 I + \lambda_8^* \mu_4 H + \lambda_9^* \mu_4 R \\ &\quad + \chi_{[0, T_f - \mathbf{d}]} \frac{b_2^\alpha \beta_3^\alpha \mathfrak{C}_2(1 - \mathfrak{A}_2)}{N_H} (\lambda_6^*(t + \mathbf{d}) - \lambda_5^*(t + \mathbf{d})), \\ {}_t^c D_{T_f}^\alpha \lambda_6^* &= -A_1 - (\lambda_1^* - \lambda_2^*)b\beta_1 B_i M_s \xi^2 - (\lambda_3^* - \lambda_4^*)b\beta_2 M_i B_s \xi^2 \\ &\quad - \lambda_5^*[\gamma_H + b\beta_3 M_i B_s(1 - u_2)\xi^2 - \mu_4 S] \\ &\quad + \lambda_6^*[b\beta_3 M_i S(1 - u_2)\xi^2 + \alpha_w + \mu_4 E + \mu_3 + \mu_4 N_H] \\ &\quad - \lambda_7^*[\alpha_w - \mu_4 I] + \lambda_8^* \mu_4 H + \lambda_9^* \mu_4 R, \\ {}_t^c D_{T_f}^\alpha \lambda_7^* &= -A_2 - (\lambda_1^* - \lambda_2^*)b\beta_1 B_i M_s \xi^2 - (\lambda_3^* - \lambda_4^*)b\beta_2 M_i B_s \xi^2 \\ &\quad - \lambda_5^*[\gamma_H + b\beta_3 M_i S(1 - u_2)\xi^2 - \mu_4 S] + \lambda_6^*[b\beta_3 M_i S(1 - u_2)\xi^2 + \mu_4 E] \\ &\quad + \lambda_7^*[\gamma + d_I + r + \mu_4 I + \mu_3 + \mu_4 N_H] - \lambda_8^*[\gamma - \mu_4 H] - \lambda_9^*[r - \mu_4 R], \\ {}_t^c D_{T_f}^\alpha \lambda_8^* &= -(\lambda_1^* - \lambda_2^*)b\beta_1 M_s B_i \xi^2 - (\lambda_3^* - \lambda_4^*)b\beta_2 M_i B_s \xi^2 \\ &\quad - \lambda_5^*[\gamma_H + b\beta_3 M_i S(1 - u_2)\xi^2 - \mu_4 S] + \lambda_6^*[b\beta_3 M_i S(1 - u_2)\xi^2 + \mu_4 E] \end{aligned} \quad (20)$$

$$\begin{aligned}
& + \lambda_7^* \mu_4 I + \lambda_8^* [d_H + \tau + \mu_4 N_H + \mu_3 + \mu_4 H] - \lambda_9^* [\tau - \mu_4 R] \\
& + \chi_{[0, T_f - d]} \tau^\alpha (\lambda_9^*(t + \mathbf{d}) - \lambda_8^*(t + \mathbf{d})), \\
{}_t^c D_{T_f}^\alpha \lambda_9^* & = -(\lambda_1^* - \lambda_2^*) b \beta_1 M_s B_i \xi^2 - (\lambda_3^* - \lambda_4^*) b \beta_2 M_i B_s \xi^2 \\
& - \lambda_5^* [\gamma_H + b \beta_3 M_i S (1 - u_2) \xi^2 - \mu_4 S] + \lambda_6^* [b \beta_3 M_i S (1 - u_2) \xi^2 + \mu_4 E] \\
& + \lambda_7^* \mu_4 I + \lambda_8^* \mu_4 H + \lambda_9^* [\mu_4 R + \mu_3 + \mu_4 N_H].
\end{aligned}$$

Thus, the terminal state is free, the transversality conditions are

$$\lambda_i^*(T_f) = 0, \quad \forall i = 1, 2, \dots, 9, \quad (21)$$

and the characterization of the control variable u_1 is

$$\frac{\partial \mathcal{H}^*(t)}{\partial u_1(t)} = 0. \quad (22)$$

To characterize the control variable u_2 , we introduce the following switching function:

$$\begin{aligned}
\tilde{\vartheta}_2(t) &= \frac{\partial \mathcal{H}^*(t)}{\partial u_2(t)} + \chi_{[0, T_f - d]} \frac{\partial \mathcal{H}^*(t + \mathbf{d})}{\partial \mathfrak{A}_2} \\
&= \begin{cases} \frac{1}{2} B_2 u_2 + \frac{b_2^\alpha \beta_3^\alpha \mathfrak{C}_2 \mathfrak{C}_5 (\lambda_5^* - \lambda_6^*)}{N_H}, & \text{for } 0 \leq t \leq T_f - \mathbf{d}, \\ \frac{1}{2} B_2 u_2, & \text{for } T_f - \mathbf{d} \leq t \leq T_f. \end{cases} \quad (23)
\end{aligned}$$

Thus, the maximum condition for the control variable $u_2(t)$ is equivalent to the maximization $\tilde{\vartheta}_2(t) u_2(t) = \max_{0 \leq u_2 \leq 1} \tilde{\vartheta}_2(t) u_2(t)$.

7 NSFD for fractional time-delay West Nile virus model

In this section, the discretization of fractional delay WNV model is presented. We apply the NSFD method with Grünwald–Letnikov definition, so the discretization can be written as follows:

$$\begin{aligned}
\sum_{\iota=0}^{n+1} \omega_\iota^\alpha M_s^{n+1-\iota} &= \lambda_M^\alpha N_M^n (1 - u_1^n) - \frac{b_1^\alpha \beta_1^\alpha M_s^n B_i^n}{N_B^n} - \mu_M^\alpha M_s^n - r_0^\alpha u_1^n M_s^n, \\
\sum_{\iota=0}^{n+1} \omega_\iota^\alpha M_i^{n+1-\iota} &= \frac{b_1^\alpha \beta_1^\alpha M_s^n B_i^n}{N_B^n} - \mu_M^\alpha M_i^n - r_0^\alpha u_1^n M_i^n, \\
\sum_{\iota=0}^{n+1} \omega_\iota^\alpha B_s^{n+1-\iota} &= \lambda_B^\alpha + \rho^\alpha N_B^n - \frac{b_1^\alpha \beta_2^\alpha M_i^n B_s^n}{N_B^n} - (\Psi_B^\alpha + \mu_B^\alpha) B_s^n, \\
\sum_{\iota=0}^{n+1} \omega_\iota^\alpha B_i^{n+1-\iota} &= \frac{b_1^\alpha \beta_2^\alpha M_i^n B_s^n}{N_B^n} - (d_B^\alpha + \Psi_B^\alpha + \mu_B^\alpha) B_i^n, \\
\sum_{\iota=0}^{n+1} \omega_\iota^\alpha S^{n+1-\iota} &= \lambda_H^\alpha + \gamma_H^\alpha N_H^n - \frac{b_2^\alpha \beta_3^\alpha M_i^{n-\varrho} S^{n-\varrho} (1 - u_2^{n-\varrho})}{N_H^n} - \mu_H^\alpha S^n, \\
\sum_{\iota=0}^{n+1} \omega_\iota^\alpha E^{n+1-\iota} &= \frac{b_2^\alpha \beta_3^\alpha M_i^{n-\varrho} S^{n-\varrho} (1 - u_2^{n-\varrho})}{N_H^n} - (\alpha_w^\alpha + \mu_H^\alpha) E^n,
\end{aligned} \quad (24)$$

$$\sum_{l=0}^{n+1} \omega_l^\alpha I^{n+1-l} = \alpha_w^\alpha E^n - (\gamma^\alpha + d_l^\alpha + r^\alpha + \mu_H^\alpha) I^n,$$

$$\sum_{l=0}^{n+1} \omega_l^\alpha H^{n+1-l} = \gamma^\alpha I^n - (d_H^\alpha + \mu_H^\alpha) H^n - \tau^\alpha H^{n-\varrho},$$

$$\sum_{l=0}^{n+1} \omega_l^\alpha R^{n+1-l} = \tau^\alpha H^{n-\varrho} + r^\alpha I^n - \mu_H^\alpha R^n,$$

where $\omega_0^\alpha = (\varphi(\Delta t))^{-\alpha}$, $n = -\varrho, -\varrho + 1, \dots, 0, 1$, and the local approximations are used for the nonlinear terms. We use the denominator function of the form $\varphi(\Delta t) = 1 - e^{-\Delta t}$. Thus,

the discretized system (24) can be written as follows:

$$\begin{aligned}
 M_s^{n+1} &= \frac{\lambda_M^\alpha N_M^n (1 - u_1^n) - \frac{b_1^\alpha \beta_1^\alpha M_s^n B_i^n}{N_B^n} - \mu_M^\alpha M_s^n - r_0^\alpha u_1^n M_s^n - \sum_{i=1}^{n+1} \omega_i^\alpha M_s^{n+1-i}}{\omega_0^\alpha}, \\
 M_i^{n+1} &= \frac{\frac{b_1^\alpha \beta_1^\alpha M_s^n B_i^n}{N_B^n} - \mu_M^\alpha M_i^n - r_0^\alpha u_1^n M_i^n - \sum_{i=1}^{n+1} \omega_i^\alpha M_i^{n+1-i}}{\omega_0^\alpha}, \\
 B_s^{n+1} &= \frac{\lambda_B^\alpha + \rho^\alpha N_B^n - \frac{b_1^\alpha \beta_2^\alpha M_i^n B_s^n}{N_B^n} - (\Psi_B^\alpha + \mu_B^\alpha) B_s^n - \sum_{i=1}^{n+1} \omega_i^\alpha B_s^{n+1-i}}{\omega_0^\alpha}, \\
 B_i^{n+1} &= \frac{\frac{b_1^\alpha \beta_2^\alpha M_i^n B_s^n}{N_B^n} - (d_B^\alpha + \Psi_B^\alpha + \mu_B^\alpha) B_i^n - \sum_{i=1}^{n+1} \omega_i^\alpha B_i^{n+1-i}}{\omega_0^\alpha}, \\
 S^{n+1} &= \frac{\lambda_H^\alpha + \gamma_H^\alpha N_H^n - \frac{b_2^\alpha \beta_3^\alpha M_i^{n-\varrho} S^{n-\varrho} (1 - u_2^{n-\varrho})}{N_H^n} - \mu_H^\alpha S^n - \sum_{i=1}^{n+1} \omega_i^\alpha S^{n+1-i}}{\omega_0^\alpha}, \\
 E^{n+1} &= \frac{\frac{b_2^\alpha \beta_3^\alpha M_i^{n-\varrho} S^{n-\varrho} (1 - u_2^{n-\varrho})}{N_H^n} - (\alpha_w^\alpha + \mu_H^\alpha) E^n - \sum_{i=1}^{n+1} \omega_i^\alpha E^{n+1-i}}{\omega_0^\alpha},
 \end{aligned} \tag{25}$$

$$I^{n+1} = \frac{\alpha_w^\alpha E^n - (\gamma^\alpha + d_I^\alpha + r^\alpha + \mu_H^\alpha) I^n - \sum_{i=1}^{n+1} \omega_i^\alpha I^{n+1-i}}{\omega_0^\alpha},$$

$$H^{n+1} = \frac{\gamma^\alpha I^n - (d_H^\alpha + \mu_H^\alpha) H^n - \tau^\alpha H^{n-\ell} - \sum_{i=1}^{n+1} \omega_i^\alpha H^{n+1-i}}{\omega_0^\alpha},$$

$$R^{n+1} = \frac{\tau^\alpha H^{n-\ell} + r^\alpha I^n - \mu_H^\alpha R^n - \sum_{i=1}^{n+1} \omega_i^\alpha R^{n+1-i}}{\omega_0^\alpha},$$

where

$$N_M^n = M_s^n + M_i^n,$$

$$N_B^n = B_s^n + B_i^n, \quad (26)$$

$$N_H^n = S^n + E^n + I^n + H^n + R^n.$$

8 Numerical experiment

Throughout this section for simulations, we use NSFDM and SFDM. These simulations are performed with initial conditions $(M_{s_0}, M_{i_0}, B_{s_0}, B_{i_0}, S_0, E_0, I_0, R_0, H_0) = (10,000, 1000, 1000, 0, 1000, 0, 0, 0, 0)$ and transversality conditions $\lambda_i(T_f) = 0, \forall i = 1, 2, \dots, 9$. For Figs. 1–9, the weights $A_1 = A_2 = 1, A_3 = 10^{-4}$ in the cost functional (17), (i.e., the minimization of the number of exposed and infected humans is given more importance than the reduction of the total number of mosquitoes). Respectively, we use the upper bound of 0.8 and 0.5 on u_1 and u_2 . All parameters are collected and given in Table 3. In Table 4, the sum of numerical values of the state variables $E(T_f) + I(T_f)$ at $T_f = 100$ and the objective function $J^*(\mathfrak{X}, \mathfrak{U})$ in two cases, with and without control at different values of time delay $\mathbf{d}$ and different values of α with $\Delta t = 0.1$, obtained by NSFDM method, are given. According to comparisons that are presented in Table 4, it can be observed that, for example, the cost functional $J^*(\mathfrak{X}, \mathfrak{U})$ with control is 44,507, 56,867 at $\mathbf{d} = 1$ and $\mathbf{d} = 7$, respectively, in case of integer order and $J^*(\mathfrak{X}, \mathfrak{U})$ without control is 68,771, 71,011 at the same values of $\mathbf{d}$. But in case of fractional order $J^*(\mathfrak{X}, \mathfrak{U})$ with control is 38,732, 49,617 and $J^*(\mathfrak{X}, \mathfrak{U})$ without

control is 59,450, 61,304 for $\alpha = 0.98$ and 21,901, 28,350 in control case but in case of without control 32,911, 34,564 for $\alpha = 0.90$ at $\mathbf{d} = 1$ and $\mathbf{d} = 7$, respectively.

The main goal in this study is to reduce the infected and exposed human populations from the increase due to delay in treatment. Through the results obtained in Table 4, it can be noted that, if the time delay $\mathbf{d}$ is increased, the infected and exposed human populations are increased at different values of α . For example, in case of non-delay optimal control problem at $\mathbf{d} = 0$, it can be seen that the infected and exposed human populations are increased if the value of α is increased, and the better results occur at $\alpha = 0.70$. While in the case of delay optimal control problem, it can be noted that at $\mathbf{d} = 1$, the lowest value of the infected and exposed human populations occurs at $\alpha = 0.70$, similarly for $\mathbf{d} = 3, 5, 7$.

In Table 5, comparisons between NSFDM and SFDM at different values of time delay $\mathbf{d}$ and different values of α with $\Delta t = 1$ at $T_f = 100$ are presented. It can be seen that the values of the cost function $J^*(\mathcal{X}, \mathcal{U})$ are 44,560, 62,325 at $\mathbf{d} = 3$ and $\mathbf{d} = 10$, respectively, using NSFDM, in case of integer order. But it takes the values 45,647, 62,325 at the same time delay $\mathbf{d}$ using SFDM. In case of fractional order, the values of $J^*(\mathcal{X}, \mathcal{U})$ are 40,721, 27,976 using NSFDM and 56,813, 38,610 using SFDM for $\alpha = 0.98$, $\alpha = 0.90$ at the same time delay $\mathbf{d}$. Thus, the objective function $J^*(\mathcal{X}, \mathcal{U})$ obtained using NSFDM is better than that obtained using SFDM. It can be concluded that the numerical treatments for the effec-

tiveness of nonstandard finite difference technique for the fractional time-delay optimal control problem are better than the numerical treatments for the time-delay optimal control.

Also, from Table 5 we see that NSFDM helps us to achieve the goal of this research and to obtain the lowest value of the infected and exposed human populations better than SFDM, where the lowest value of these populations occurs at $\alpha = 0.90$ and $\mathbf{d} = 3, 5, 7$.

The numerical simulations of the optimal control problem (16) and (17) using two numerical methods, NSFDM and SFDM, are shown in Figs. 1 and 2. Figure 1 presents the comparison between these proposed methods at $\mathbf{d} = 10$ and $\alpha = 1$, and $\Delta t = 1$. It can be observed that the infected human population $I(t)$ is about 280 and 320 for NSFDM and

SFDM, respectively. It is clear that SFDM is divergent at $\Delta t = 5$, but NSFDM is convergent as explained in Figs. 2 and 3, respectively. The approximate solutions of the proposed optimal control problem are described by Figs. 3–9 at different time delay $\mathbf{d}$ and α . In Fig. 4, the approximate solutions of the fractional delay WNV model (6) without control (i.e., $u_1 = u_2 = 0$) with $\mathbf{d} = 1$ at different α , and $\Delta t = 1$ by using NSFDM are shown. It can be seen that the infected human population $I(t)$ is about 750, 650, and 400 for $\alpha = 1$, $\alpha = 0.98$, and $\alpha = 0.90$, respectively. Numerical comparisons between the approximate solutions of the fractional delay optimal control problem (16) and (17) with and without control (i.e., $u_1 = u_2 = 0$) with $\mathbf{d} = 1$ at $\alpha = 0.90$, and $\Delta t = 1$ by using NSFDM, are introduced in Fig. 5. It can be shown that the infected human population $I(t)$ is about 250 and 450 in two cases

with and without control, respectively. The behavior of the approximate solutions of the fractional delay optimal control problem (16) and (17) at different time delay $\mathbf{d}$ and different α at $\Delta t = 0.1$ is included in Fig. 6. We notice that the delayed infected human population $I(t - \mathbf{d})$ is about 589.0570, 507.9876, and 278.5177 for $\alpha = 1$, $\alpha = 0.98$, and $\alpha = 0.90$, respectively, when $\mathbf{d} = 5$. Figure 7 shows numerical comparisons between the approximate solutions of the optimal control problem (16) and (17) in two cases, the fractional delay optimal control problem at $\mathbf{d} = 5$, and the fraction non-delay optimal control problem at $\mathbf{d} = 0$ when $\alpha = 0.98$. The infected human population $I(t)$ is about 650 and 400 in two cases with and without delay, respectively. Figure 8 shows the relationship between state

variables $M_i(t - \mathbf{d})$ and $M_i(t)$, $S(t - \mathbf{d})$ and $S(t)$, $H(t - \mathbf{d})$ and $H(t)$, and the control variable $u_2(t - \mathbf{d})$ and $u_2(t)$ when $\mathbf{d} = 1$ for $\alpha = 0.98$. Figure 9 describes the approximate solutions of the fractional delay optimal control problem at the same time delay $\mathbf{d} = 1$, $\mathbf{d} = 5$, and $\mathbf{d} = 10$ in case of fractional order at $\alpha = 0.98$. The infected human population $I(t)$ is about 400, 500, and 650 for $\mathbf{d} = 1$, $\mathbf{d} = 5$, and $\mathbf{d} = 10$, respectively.

9 Conclusions

In this paper, fractional time-delay optimal control in transmission dynamics of WNV model is presented. Time delays in both state and control variables are formulated and studied. NSFDM and SFDM are introduced to study the behavior of the proposed model. According to the results implemented in this paper, it can be observed that the numerical

Table 4 Comparisons between the value of objective function $J^*(\mathfrak{X}, \mathfrak{U})$ with and without control cases and $E(T_f) + I(T_f)$ when $0 \leq u_1 \leq 0.8$, $0 \leq u_2 \leq 0.5$, at different time delay $\mathbf{d}$ and different α with $\Delta t = 0.1$, $T_f = 100$ by using NSFDM

$\mathbf{d}$	α	$J^*(\mathfrak{X}, \mathfrak{U})$ with control	$E(T_f) + I(T_f)$	$J^*(\mathfrak{X}, \mathfrak{U})$ without control (i.e., $u_1 = u_2 = 0$)
0	1	43,256	389.4656	68,650
	0.98	37,592	329.6513	59,318
	0.90	21,107	165.8036	32,751
	0.80	9867.9	66.7680	15,225
	0.70	4376.3	24.9857	6859.5
1	1	44,507	403.6675	68,771
	0.98	38,732	341.7494	59,450
	0.90	21,901	172.1452	32,911
	0.80	10,397	69.6851	15,401
	0.70	4759.8	26.4368	7042.2
3	1	47,716	437.1639	69,281
	0.98	41,612	370.0171	59,951
	0.90	23,752	186.2351	33,360
	0.80	11,465	75.5492	15,767
	0.70	5390.6	28.9362	7320.8
5	1	52,063	478.2057	70,195
	0.98	45,446	404.2531	60,795
	0.90	26,022	202.3538	33,950
	0.80	12,616	81.6707	16,120
	0.70	5981.6	31.2947	7517.1
7	1	56,867	522.7261	71,011
	0.98	49,617	440.9938	61,304
	0.90	28,350	218.8961	34,564
	0.80	13,721	87.6141	16,123
	0.70	6521.5	33.4921	8195.7

Table 5 Comparisons between the value of objective function $J^*(\mathfrak{X}, \mathfrak{U})$ using NSFDM and SFDM when $0 \leq u_1 \leq 0.8$, $0 \leq u_2 \leq 0.5$, at different time delay $\mathbf{d}$ and different α with $\Delta t = 1$, $T_f = 100$

$\mathbf{d}$	Method	α	$J^*(\mathfrak{X}, \mathfrak{U})$
3	NSFDM	1	44,560
		0.98	40,721
		0.90	27,976
	SFDM	1	45,647
		0.98	41,616
		0.90	28,328
5	NSFDM	1	46,577
		0.98	42,650
		0.90	29,538
	SFDM	1	49,845
		0.98	45,501
		0.90	31,115
7	NSFDM	1	49,176
		0.98	45,103
		0.90	31,419
	SFDM	1	54,946
		0.98	50,146
		0.90	34,238
10	NSFDM	1	53,826
		0.98	49,412
		0.90	34,499
	SFDM	1	62,325
		0.98	56,813
		0.90	38,610

solutions of the proposed model are different when $\mathbf{d}$ and α take different values. From the comparisons included in Tables 4 and 5, it is found that the values of the objective functional obtained by NSFDM are better than the results obtained by SFDM, it can be applied simply and effectively to solve optimal control of fractional delay.

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Availability of data and materials

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt. ²Department of Mathematics, Faculty of Science, Helwan University, Cairo, Egypt.

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