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Certain generalized fractional calculus formulas and integral transforms involving (p,q)-Mathieu-type series

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Abstract

In this paper, we establish sixteen interesting generalized fractional integral and derivative formulas including their composition formulas by using certain integral transforms involving generalized (p,q)-Mathieu-type series.

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1 Introduction

The generalized fractional calculus operators popularly known as Marichev–Saigo–Maeda operators involving the Appell function $F_3(\cdot)$ or the Horn function in the kernel (see for details [3, 6, 7]) are defined in the following form.

Definition 1 Let $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta \in \mathbb{C}$ and $x > 0$, then, for $\operatorname{Re}(\eta) > 0$,

$$\begin{aligned}\left(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} f\right)(x) = \frac{x^{-\sigma_1}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\sigma_1'} \\ \times F_3\left(\sigma_1, \sigma_1', \nu_1, \nu_1'; \eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\,\mathrm{d}t \end{aligned} \tag{1.1}$$

and

$$\begin{aligned}\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} f\right)(x) = \frac{x^{-\sigma_1'}}{\Gamma(\eta)} \int_x^\infty (t-x)^{\eta-1} t^{-\sigma_1} \\ \times F_3\left(\sigma_1, \sigma_1', \nu_1, \nu_1'; \eta; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\,\mathrm{d}t. \end{aligned} \tag{1.2}$$

Here $F_3(\cdot)$ denotes the Appell hypergeometric function of two variables.

Definition 2 Let $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta \in \mathbb{C}$ and $x > 0$, then, for $\operatorname{Re}(\eta) > 0$,

$$\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} f\right)(x) = \left(I_{0+}^{-\sigma_1',-\sigma_1,-\nu_1',-\nu_1,-\eta} f\right)(x)$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\sigma_1',-\sigma_1,-\nu_1'+n,-\nu_1,-\eta+n} f\right)(x) \quad \left(n = \left[\operatorname{Re}(\eta)\right] + 1\right)$$
$$= \frac{1}{\Gamma(n-\eta)} \left(\frac{d}{dx}\right)^n x^{\sigma_1'} \int_0^x (x-t)^{n-\eta-1} t^{\sigma}$$
$$\times F_3\left(-\sigma_1', -\sigma_1, n-\nu_1', -\nu_1; n-\eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\,\mathrm{d}t \tag{1.3}$$

and

$$\left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} f\right)(x) = \left(I_-^{-\sigma_1',-\sigma_1,-\nu_1',-\nu_1,-\eta} f\right)(x)$$
$$= \left(-\frac{d}{dx}\right)^n \left(I_-^{-\sigma_1',-\sigma_1,-\nu_1',-\nu_1'+n,-\eta+n} f\right)(x) \quad \left(n = \left[\operatorname{Re}(\eta)\right] + 1\right)$$
$$= \frac{1}{\Gamma(n-\eta)} \left(-\frac{d}{dx}\right)^n x^{\sigma_1'} \int_x^\infty (t-x)^{n-\eta-1} t^{\sigma'}$$
$$\times F_3\left(-\sigma_1', -\sigma_1, \nu_1', n-\nu_1; n-\eta; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\,\mathrm{d}t. \tag{1.4}$$

These operators includes Saigo hypergeometric fractional calculus operators, Riemann–Liouville and Erdélyi–Kober fractional calculus operators as special cases for various choices of the parameters (see for details [2, 8, 10] and [12]). In a recent paper, Saxena and Parmar [9] established several interesting Saigo hypergeometric fractional formulas involving the generalized Mathieu series defined by Tomovski and Pogány [14]. More recently, Singh *et al.* [10] established several results by employing Marichev–Saigo–Maeda fractional operators including their composition formulas and using certain integral transforms involving the extended generalized Mathieu series defined by Tomovski and Mehrez [13].

The more generalized form of the so-called (p,q)-Mathieu type series has been considered very recently by Mehrez and Tomovski [4] in the following form:

$$S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}(r,a;p,q;z) = \sum_{n\geq 1} \frac{2a_n^{\beta}(\lambda_1)_n \mathrm{B}_{p,q}(\lambda_2+n,\lambda_3-\lambda_2)}{(a_n^{\alpha}+r^2)^{\mu}\mathrm{B}(\lambda_2,\lambda_3-\lambda_2)} \frac{z^n}{n!}$$
$$\left(\mu,r,\alpha,\beta,\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}^+, \min\left\{\operatorname{Re}(p),\operatorname{Re}(q)\right\} \geq 0; |z| < 1\right), \tag{1.5}$$

where $\mathrm{B}(x,y;p,q)$ is the (p,q)-extended Beta function introduced by Choi *et al.* [1],

$$\mathrm{B}(x,y;p,q) = \mathrm{B}_{p,q}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\,\mathrm{d}t, \tag{1.6}$$

when $\min\{\operatorname{Re}(x),\operatorname{Re}(y)\} > 0; \min\{\operatorname{Re}(p),\operatorname{Re}(q)\} \geq 0$. This (p,q)-Mathieu type series includes various forms of Mathieu-type series as special cases (see for details [4]).

In our present investigation, we require the definition of the Hadamard product (or the convolution) of two analytic functions [9]. If the R_f and R_g are the radii of convergence of the two power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \quad \left(|z| < R_f\right) \quad \text{and} \quad g(z) := \sum_{n=0}^{\infty} b_n z^n \quad \left(|z| < R_g\right),$$

respectively, then the Hadamard product is the newly emerging series defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \quad \bigl(|z| < R\bigr), \tag{1.7}$$

where

$$R = \lim_{n\to\infty}\left|\frac{a_n b_n}{a_{n+1}b_{n+1}}\right| = \left(\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\right)\cdot\left(\lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|\right) = R_f \cdot R_g,$$

so that, in general, we have $R \geqq R_f \cdot R_g$.

In this present note, we aim to develop the compositions of the generalized fractional integral and differential operators (1.1), (1.2), (1.3) and (1.4) for the generalized Mathieu series (1.5) by using the Hadamard product (1.7) in terms of (p,q)-Mathieu type series and Wright hypergeometric function.

2 Fractional formulas of the (p,q)-Mathieu type series

The Wright hypergeometric function ${}_r\Psi_s(z)$ $(r,s \in \mathbb{N}_0)$ having numerator and denominator parameters r and s, respectively, defined for $\alpha_1,\ldots,\alpha_r \in \mathbb{C}$ and $\beta_1,\ldots,\beta_s \in \mathbb{C}\setminus\mathbb{Z}_0^-$ by (see, for example, [2, 8])

$$\begin{aligned} &{}_r\Psi_s\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_r,A_r);\\(\beta_1,B_1),\ldots,(\beta_s,B_s);\end{matrix} z\right] = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+A_1 n)\cdots\Gamma(\alpha_r+A_r n)}{\Gamma(\beta_1+B_1 n)\cdots\Gamma(\beta_s+B_s n)}\frac{z^n}{n!} \\ &\quad\left(A_j\in\mathbb{R}^+ (j=1,\ldots,r); B_j\in\mathbb{R}^+ (j=1,\ldots,s); 1+\sum_{j=1}^{s}B_j-\sum_{j=1}^{r}A_j \geqq 0\right) \end{aligned}\tag{2.1}$$

with

$$|z| < \nabla := \left(\prod_{j=1}^{r} A_j^{-A_j}\right)\cdot\left(\prod_{j=1}^{s} B_j^{B_j}\right).$$

Also, if we take $A_j = B_k = 1$ $(j = 1,\ldots,r; k = 1,\ldots,s)$ in (2.1), this reduces to the generalized hypergeometric function ${}_rF_s$ $(r,s\in\mathbb{N}_0)$ (see, *e.g.*, [2]):

$${}_rF_s\left[\begin{matrix}\alpha_1,\ldots,\alpha_r;\\\beta_1,\ldots,\beta_s;\end{matrix} z\right] = \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_s)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_r)}\,{}_r\Psi_s\left[\begin{matrix}(\alpha_1,1),\ldots,(\alpha_r,1);\\(\beta_1,1),\ldots,(\beta_s,1);\end{matrix} z\right]. \tag{2.2}$$

The following image formulas or power function are useful in our investigation [10].

Lemma 1 *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\varrho \in \mathrm{C}$ *and* $x > 0$. *Then the following relation exists*:

(a) *If* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho) > \max\{0, \operatorname{Re}(\sigma_1+\sigma_1'+\nu_1-\eta), \operatorname{Re}(\sigma_1'-\nu_1')\}$, *then*

$$\begin{aligned}\bigl(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} t^{\varrho-1}\bigr)(x) &= \frac{\Gamma(\varrho)\Gamma(\varrho+\eta-\sigma_1-\sigma_1'-\nu_1)\Gamma(\varrho+\nu_1'-\sigma_1')}{\Gamma(\varrho+\nu_1')\Gamma(\varrho+\eta-\sigma_1-\sigma_1')\Gamma(\varrho+\eta-\sigma_1'-\nu_1)} \\ &\quad\times x^{\varrho+\eta-\sigma_1-\sigma_1'-1}.\end{aligned}\tag{2.3}$$

(b) *If* $\mathrm{Re}(\eta) > 0$ *and* $\mathrm{Re}(\varrho) < 1 + \min\{\mathrm{Re}(-\nu_1), \mathrm{Re}(\sigma_1 + \sigma_1' - \eta), \mathrm{Re}(\sigma_1 + \nu_1' - \eta)\}$, *then*

$$\big(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} t^{\varrho-1}\big)(x) = \frac{\Gamma(1-\varrho-\nu_1)\Gamma(1-\varrho-\eta+\sigma_1+\sigma_1')\Gamma(1-\varrho-\eta+\sigma_1+\nu_1')}{\Gamma(1-\varrho)\Gamma(1-\varrho-\eta+\sigma_1+\sigma_1'+\nu_1')\Gamma(1-\varrho+\sigma_1-\nu_1)} \times x^{\varrho+\eta-\sigma_1-\sigma_1'-1}. \tag{2.4}$$

Lemma 2 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathrm{C}$ *and* $x > 0$. *Then the following relation exists*:

(a) *If* $\mathrm{Re}(\eta) > 0$ *and* $\mathrm{Re}(\varrho) > \max\{0, \mathrm{Re}(\eta - \sigma_1 - \sigma_1' + \nu_1'), \mathrm{Re}(\nu_1 - \sigma_1)\}$, *then*

$$\big(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} t^{\varrho-1}\big)(x) = \frac{\Gamma(\varrho)\Gamma(\varrho-\eta+\sigma_1+\sigma_1'+\nu_1')\Gamma(\varrho-\nu_1+\sigma_1)}{\Gamma(\varrho-\nu_1)\Gamma(\varrho-\eta+\sigma_1+\sigma_1')\Gamma(\varrho-\eta+\sigma_1+\nu_1')} \times x^{\varrho-\eta+\sigma_1+\sigma_1'-1}. \tag{2.5}$$

(b) *If* $\mathrm{Re}(\eta) > 0$ *and* $\mathrm{Re}(\varrho) < 1 + \min\{\mathrm{Re}(\nu_1'), \mathrm{Re}(\eta - \sigma_1 - \sigma_1'), \mathrm{Re}(\eta - \sigma_1' - \nu_1)\}$, *then*

$$\big(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} t^{\varrho-1}\big)(x) = \frac{\Gamma(1-\varrho-\nu_1')\Gamma(1-\varrho+\eta-\sigma_1-\sigma_1')\Gamma(1-\varrho+\eta-\sigma_1'-\nu_1)}{\Gamma(1-\varrho)\Gamma(1-\varrho+\eta-\sigma_1-\sigma_1'-\nu)\Gamma(1-\varrho-\sigma_1'-\nu_1')} \times x^{\varrho-\eta+\sigma_1+\sigma_1'-1}. \tag{2.6}$$

We begin the exposition of the main results with presenting the composition formulas of the generalized fractional operators (1.1), (1.2), (1.3) and (1.4) involving the (p,q)-Mathieu type series by using the Hadamard product (1.7) in terms of the (p,q)-Mathieu type series (1.5) and the Fox–Wright function (2.1).

Theorem 1 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\mathrm{Re}(\eta) > 0$ *and* $\mathrm{Re}(\varrho + \gamma) > \max\{0, \mathrm{Re}(\sigma_1 + \sigma_1' + \nu_1 - \eta), \mathrm{Re}(\sigma_1' - \nu_1')\}$ *with* $|t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following formula for fractional integration holds true*:

$$\begin{aligned}
&\big(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\big\{t^{\varrho-1} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\big(r,a;p,q;t^{\gamma}\big)\big\}\big)(x) \\
&\quad = x^{\varrho+\gamma+\eta-\sigma_1-\sigma_1'-1} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\big(r,a;p,q;x^{\gamma}\big) \\
&\qquad * {}_4\Psi_3\left[\begin{matrix}(1,1),(\varrho+\gamma,\gamma),(\varrho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma,\gamma),(\varrho+\nu_1'-\sigma_1'+\gamma,\gamma);\\ (\varrho+\nu_1'+\gamma,\gamma),(\varrho+\eta-\sigma_1-\sigma_1'+\gamma,\gamma),(\varrho+\eta-\sigma_1'-\nu_1+\gamma,\gamma);\end{matrix}\ x^{\gamma}\right].
\end{aligned}$$

Proof Applying the definitions (1.5), (1.1) and then changing the order of integration and using the relation (2.3), we find for $x > 0$

$$\begin{aligned}
&\big(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\big\{t^{\varrho-1} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\big(r,a;p,q;t^{\gamma}\big)\big\}\big)(x) \\
&\quad = \sum_{k=1}^{\infty} \frac{2a_k^{\beta}(\lambda_1)_k \mathrm{B}(\lambda_2+k,\lambda_3-\lambda_2;p,q)}{(a_k^{\alpha}+r^2)^{\mu}\mathrm{B}(\lambda_2,\lambda_3-\lambda_2)k!}\big(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\big\{t^{\varrho+\gamma k-1}\big\}\big)(x) \\
&\quad = x^{\varrho+\eta-\sigma_1-\sigma_1'-1} \sum_{k=1}^{\infty} \frac{2a_k^{\beta}(\lambda_1)_k \mathrm{B}(\lambda_2+k,\lambda_3-\lambda_2;p,q)}{(a_k^{\alpha}+r^2)^{\mu}\mathrm{B}(\lambda_2,\lambda_3-\lambda_2)k!} \\
&\qquad \times \frac{\Gamma(\varrho+\gamma k)\Gamma(\varrho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma k)\Gamma(\varrho+\nu_1'-\sigma_1'+\gamma k)}{\Gamma(\varrho+\nu_1'+\gamma k)\Gamma(\varrho+\eta-\sigma_1-\sigma_1'+\gamma k)\Gamma(\varrho+\eta-\sigma_1'-\nu_1+\gamma k)} x^{\gamma k}.
\end{aligned} \tag{2.7}$$

Finally, using the Hadamard product (1.7) in (2.7), in view of (1.5) and (2.1), yields the desired formula. □

Theorem 2 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\mathrm{Re}(\eta) > 0$ *and* $\mathrm{Re}(\varrho - \gamma) < 1 + \min\{\mathrm{Re}(-\nu_1), \mathrm{Re}(\sigma_1 + \sigma_1' - \eta), \mathrm{Re}(\sigma_1 + \nu_1' - \eta)\}$ *with* $|1/t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following formula for fractional integration holds true*:

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\frac{1}{t^\gamma}\right)\right\}\right)(x)\\&= x^{\varrho+\eta-\gamma-\sigma_1-\sigma_1'-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\frac{1}{x^\gamma}\right)\\&\quad * {}_4\Psi_3\left[\begin{matrix}(1,1),(1-\varrho-\nu_1+\gamma,\gamma),(1-\varrho-\eta+\sigma_1+\sigma_1'+\gamma,\gamma),(1-\varrho-\eta+\sigma_1+\nu_1'+\gamma,\gamma);\\(1-\varrho+\gamma,\gamma),(1-\varrho-\eta+\sigma_1+\sigma_1'+\nu_1'+\gamma,\gamma),(1-\varrho+\sigma_1-\nu_1+\gamma,\gamma);\end{matrix}\,\frac{1}{x^\gamma}\right].\end{aligned}$$

Theorem 3 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\mathrm{Re}(\eta) > 0$ *and* $\mathrm{Re}(\varrho + \gamma) > \max\{0, \mathrm{Re}(\eta - \sigma_1 - \sigma_1' - \nu_1'), \mathrm{Re}(\nu_1 - \sigma_1)\}$ *with* $|t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following formula for fractional differentiation holds true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;t^\gamma\right)\right\}\right)(x)\\&= x^{\varrho+\gamma-\eta+\sigma_1+\sigma_1'-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;x^\gamma\right)\\&\quad * {}_4\Psi_3\left[\begin{matrix}(1,1),(\varrho+\gamma,\gamma),(\varrho-\eta+\sigma_1+\sigma_1'+\nu_1'+\gamma,\gamma),(\varrho-\nu_1+\sigma_1+\gamma,\gamma);\\(\varrho-\nu_1+\gamma,\gamma),(\varrho-\eta+\sigma_1+\sigma_1'+\gamma,\gamma),(\varrho-\eta+\sigma_1+\nu_1'+\gamma,\gamma);\end{matrix}\,x^\gamma\right].\end{aligned}$$

Proof Applying the definitions (1.5), (1.3) and then changing the order of integration and using the relation (2.5), we find for $x > 0$

$$\begin{aligned}&\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;t^\gamma\right)\right\}\right)(x)\\&= \sum_{k=1}^{\infty}\frac{2a_k^\beta(\lambda_1)_k(\lambda_2)_k}{(a_k^\alpha+r^2)^\mu(\lambda_3)_k k!}\left(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\varrho+\gamma k-1}\right\}\right)(x)\\&= x^{\varrho-\eta+\sigma_1+\sigma_1'-1}\sum_{k=1}^{\infty}\frac{2a_k^\beta(\lambda_1)_k \mathrm{B}(\lambda_2+k,\lambda_3-\lambda_2;p,q)}{(a_k^\alpha+r^2)^\mu \mathrm{B}(\lambda_2+k,\lambda_3-\lambda_2)k!}\\&\quad\times\frac{\Gamma(\varrho+\gamma k)\Gamma(\varrho-\eta+\sigma_1+\sigma_1'+\nu_1'+\gamma k)\Gamma(\varrho-\nu_1+\sigma_1+\gamma k)}{\Gamma(\varrho-\nu_1+\gamma k)\Gamma(\varrho-\eta+\sigma_1+\sigma_1'+\gamma k)\Gamma(\varrho-\eta+\sigma_1+\nu_1'+\gamma k)}x^{\gamma k}.\end{aligned}\tag{2.8}$$

Finally, using the Hadamard product (1.7) in (2.8), in view of (1.5) and (2.1), yields the desired formula (1.3). □

Theorem 4 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\mathrm{Re}(\eta) > 0$ *and* $\mathrm{Re}(\varrho - \gamma) < 1 + \min\{\mathrm{Re}(\nu_1'), \mathrm{Re}(\eta - \sigma_1 - \sigma_1'), \mathrm{Re}(\eta - \sigma_1' - \nu_1)\}$ *with* $|1/t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following fractional differentiation formula holds true*:

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\frac{1}{t^\gamma}\right)\right\}\right)(x)\\&= x^{\varrho-\gamma-\eta+\sigma_1+\sigma_1'-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\frac{1}{x^\gamma}\right)\\&\quad * {}_4\Psi_3\left[\begin{matrix}(1,1),(1-\varrho-\nu_1'+\gamma,\gamma),(1-\varrho+\eta-\sigma_1-\sigma_1'+\gamma,\gamma),(1-\varrho+\eta-\sigma_1'-\nu_1+\gamma,\gamma);\\(1-\varrho+\gamma,\gamma),(1-\varrho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma,\gamma),(1-\varrho-\sigma_1'-\nu_1'+\gamma,\gamma);\end{matrix}\,\frac{1}{x^\gamma}\right].\end{aligned}$$

3 Certain integral transforms

With the help of the results established in the previous section, in this section, we shall present certain very interesting results in the form of several theorems associated with Beta, Laplace and Whittaker transforms. For this purpose, first we would like to define these transforms.

Definition 3 The Euler-Beta transform [11] of the function $f(z)$ is defined, as usual, by

$$\mathcal{B}\{f(z);a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1}f(z)\,\mathrm{d}z. \tag{3.1}$$

Definition 4 The Laplace transform (see, *e.g.*, [11]) of the function $f(z)$ is defined, as usual, by

$$L\{f(z);t\} = \int_0^\infty e^{-tz}f(z)\,\mathrm{d}z \quad \bigl(\mathrm{Re}(t) > 0\bigr). \tag{3.2}$$

The following integral involving the Whittaker function [10]:

$$\int_0^\infty t^{\rho-1}e^{-\frac{1}{2}at}W_{\kappa,\nu}(at)\,\mathrm{d}t = a^{-\rho}\frac{\Gamma(\frac{1}{2}\pm\nu+\rho)}{\Gamma(1-\kappa+\rho)} \quad \left(\mathrm{Re}(a) > 0, \mathrm{Re}(\rho\pm\nu) > -\frac{1}{2}\right), \tag{3.3}$$

is useful in this section, where $W_{\kappa,\nu}$ is the Whittaker function [5, p. 334].

The following interesting results in the form of theorems will be established in this section. As these results are direct consequences of the definitions (3.1), (3.2), (3.3) and Theorems 1 to 4, they are given here without proof.

Theorem 5 *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\varrho \in \mathbb{C}$ *and* $\mu,\alpha,\beta,r,\lambda_1,\lambda_2,\lambda_3,\gamma \in \mathbb{R}^+$ *such that* $\mathrm{Re}(\eta) > 0$ *and* $\mathrm{Re}(\varrho+\gamma) > \max\{0,\mathrm{Re}(\sigma_1+\sigma_1'+\nu_1-\eta),\mathrm{Re}(\sigma_1'-\nu_1')\}$ *with* $|t| < 1$. *Then, for* $\min\{\Re(p),\Re(q)\} \geq 0$, *the following Beta-transform formula holds true*:

$$\begin{aligned}
&B\bigl\{\bigl(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\bigl\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\bigl(r,a;p,q;(tz)^\gamma\bigr)\bigr\}\bigr)(x):l,m\bigr\} \\
&\quad = x^{\varrho+\gamma+\eta-\sigma_1-\sigma_1'-1}\Gamma(m)S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\bigl(r,a;p,q;x^\gamma\bigr) \\
&\qquad * {}_5\Psi_4\left[\begin{matrix}(1,1),(l+\gamma,\gamma),(\varrho+\gamma,\gamma),(\varrho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma,\gamma),(\varrho+\nu_1'-\sigma_1'+\gamma,\gamma); \\ (l+m+\gamma,\gamma),(\varrho+\nu_1'+\gamma,\gamma),(\varrho+\eta-\sigma_1-\sigma_1'+\gamma,\gamma),(\varrho+\eta-\sigma_1'-\nu_1+\gamma,\gamma);\end{matrix} x^\gamma\right].
\end{aligned}$$

Theorem 6 *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\varrho \in \mathbb{C}$ *and* $\mu,\alpha,\beta,r,\lambda_1,\lambda_2,\lambda_3,\gamma \in \mathbb{R}^+$ *such that* $\mathrm{Re}(\eta) > 0$ *and* $\mathrm{Re}(\varrho-\gamma) < 1+\min\{\mathrm{Re}(-\nu_1),\mathrm{Re}(\sigma_1+\sigma_1'-\eta),\mathrm{Re}(\sigma_1+\nu_1'-\eta)\}$ *with* $|1/t| < 1$. *Then, for* $\min\{\Re(p),\Re(q)\} \geq 0$, *the following Beta-transform formula holds true*:

$$\begin{aligned}
&B\left\{\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{z}{t}\right)^\gamma\right)\right\}\right)(x):l,m\right\} \\
&\quad = x^{\varrho-\gamma+\eta-\sigma_1-\sigma_1'-1}\Gamma(m)S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\frac{1}{x^\gamma}\right) \\
&\qquad * {}_5\Psi_4\left[\begin{matrix}(1,1),(l+\gamma,\gamma),(1-\varrho-\nu_1+\gamma,\gamma), \\ (l+m+\gamma,\gamma),(1-\varrho+\gamma,\gamma),\end{matrix}\right. \\
&\qquad\quad \left.\begin{matrix}(1-\varrho-\eta+\sigma_1+\sigma_1'+\gamma,\gamma),(1-\varrho-\eta+\sigma_1+\nu_1'+\gamma,\gamma); \\ (1-\varrho-\eta+\sigma_1+\sigma_1'+\nu_1'+\gamma,\gamma),(1-\varrho+\sigma_1-\nu_1+\gamma,\gamma);\end{matrix}\frac{1}{x^\gamma}\right].
\end{aligned}$$

Theorem 7 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho + \gamma) > \max\{0, \operatorname{Re}(\eta - \sigma_1 - \sigma_1' - \nu_1'), \operatorname{Re}(\nu_1 - \sigma_1)\}$ *with* $|t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following Beta-transform formula holds true*:

$$\begin{aligned} &B\bigl\{\bigl(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\bigl\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\bigl(r,a;p,q;(tz)^{\gamma}\bigr)\bigr\}\bigr)(x):l,m\bigr\} \\ &\quad = x^{\varrho+\gamma-\eta+\sigma_1+\sigma_1'-1}\Gamma(m)S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\bigl(r,a;p,q;x^{\gamma}\bigr) \\ &\qquad * {}_5\Psi_4\left[\begin{matrix}(1,1),(l+\gamma,\gamma),(\varrho+\gamma,\gamma),(\varrho-\eta+\sigma_1+\sigma_1'+\nu_1'+\gamma,\gamma),(\varrho-\nu_1+\sigma_1+\gamma,\gamma); \\ (l+m+\gamma,\gamma),(\varrho-\nu_1+\gamma,\gamma),(\varrho-\eta+\sigma_1+\sigma_1'+\gamma,\gamma),(\varrho-\eta+\sigma_1+\nu_1'+\gamma,\gamma);\end{matrix}x^{\gamma}\right]. \end{aligned}$$

Theorem 8 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho - \gamma) < 1 + \min\{\operatorname{Re}(\nu_1'), \operatorname{Re}(\eta - \sigma_1 - \sigma_1'), \operatorname{Re}(\eta - \sigma_1' - \nu_1)\}$ *with* $|1/t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following Beta-transform formula holds true*:

$$\begin{aligned} &B\left\{\left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{z}{t}\right)^{\gamma}\right)\right\}\right)(x):l,m\right\} \\ &\quad = x^{\varrho-\gamma-\eta+\sigma_1+\sigma_1'-1}\Gamma(m)S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\frac{1}{x^{\gamma}}\right) \\ &\qquad * {}_5\Psi_4\left[\begin{matrix}(1,1),(l+\gamma,\gamma),(1-\varrho-\nu_1'+\gamma,\gamma), \\ (1-\varrho+\gamma,\gamma),(1-\varrho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma,\gamma),\end{matrix}\right. \\ &\qquad \left.\begin{matrix}(1-\varrho+\eta-\sigma_1-\sigma_1'+\gamma,\gamma),(1-\varrho+\eta-\sigma_1'-\nu_1+\gamma,\gamma); \\ (l+m+\gamma,\gamma),(1-\varrho-\sigma_1'-\nu_1'+\gamma,\gamma);\end{matrix}\frac{1}{x^{\gamma}}\right]. \end{aligned}$$

Theorem 9 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho + \gamma) > \max\{0, \operatorname{Re}(\sigma_1 + \sigma_1' + \nu_1 - \eta), \operatorname{Re}(\sigma_1' - \nu_1')\}$ *with* $|t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following Laplace-transform formula holds true*:

$$\begin{aligned} &L\bigl\{z^{l-1}\bigl(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\bigl\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\bigl(r,a;p,q;(tz)^{\gamma}\bigr)\bigr\}\bigr)(x)\bigr\} \\ &\quad = \frac{x^{\varrho+\gamma+\eta-\sigma_1-\sigma_1'-1}}{s^{l}}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{x}{s}\right)^{\gamma}\right) \\ &\qquad * {}_5\Psi_3\left[\begin{matrix}(1,1),(l+\gamma,\gamma),(\varrho+\gamma,\gamma),(\varrho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma,\gamma),(\varrho+\nu_1'-\sigma_1'+\gamma,\gamma); \\ (\varrho+\nu_1'+\gamma,\gamma),(\varrho+\eta-\sigma_1-\sigma_1'+\gamma,\gamma),(\varrho+\eta-\sigma_1'-\nu_1+\gamma,\gamma);\end{matrix}\left(\frac{x}{s}\right)^{\gamma}\right]. \end{aligned}$$

Theorem 10 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho - \gamma) < 1 + \min\{\operatorname{Re}(-\nu_1), \operatorname{Re}(\sigma_1 + \sigma_1' - \eta), \operatorname{Re}(\sigma_1 + \nu_1' - \eta)\}$ *with* $|1/t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following Laplace-transform formula holds true*:

$$\begin{aligned} &L\left\{z^{l-1}\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{z}{t}\right)^{\gamma}\right)\right\}\right)(x)\right\} \\ &\quad = \frac{x^{\varrho-\gamma+\eta-\sigma_1-\sigma_1'-1}}{s^{l}}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{1}{xs}\right)^{\gamma}\right) \\ &\qquad * {}_5\Psi_3\left[\begin{matrix}(1,1),(l+\gamma,\gamma),(1-\varrho-\nu_1+\gamma,\gamma),(1-\varrho-\eta+\sigma_1+\sigma_1'+\gamma,\gamma), \\ (1-\varrho+\gamma,\gamma),(1-\varrho-\eta+\sigma_1+\sigma_1'+\nu_1'+\gamma,\gamma),\end{matrix}\right. \\ &\qquad \left.\begin{matrix}(1-\varrho-\eta+\sigma_1+\nu_1'+\gamma,\gamma); \\ (1-\varrho+\sigma_1-\nu_1+\gamma,\gamma);\end{matrix}\left(\frac{1}{xs}\right)^{\gamma}\right]. \end{aligned}$$

Theorem 11 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho + \gamma) > \max\{0, \operatorname{Re}(\eta - \sigma_1 - \sigma_1' - \nu_1'), \operatorname{Re}(\nu_1 - \sigma_1)\}$ *with* $|t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following formula Laplace-transform holds true*:

$$
\begin{aligned}
&L\bigl\{z^{l-1}\bigl(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\bigl\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\bigl(r,a;p,q;(tz)^{\gamma}\bigr)\bigr\}\bigr)(x):\bigr\} \\
&\quad = \frac{x^{\varrho+\gamma-\eta+\sigma_1+\sigma_1'-1}}{s^l} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{x}{s}\right)^{\gamma}\right) \\
&\qquad * {}_5\Psi_3\left[\begin{matrix}(1,1),(l+\gamma,\gamma),(\varrho+\gamma,\gamma),(\varrho-\eta+\sigma_1+\sigma_1'+\nu_1'+\gamma,\gamma),(\varrho-\nu_1+\sigma_1+\gamma,\gamma); \\ (\varrho-\nu_1+\gamma,\gamma),(\varrho-\eta+\sigma_1+\sigma_1'+\gamma,\gamma),(\varrho-\eta+\sigma_1+\nu_1'+\gamma,\gamma);\end{matrix}\left(\frac{x}{s}\right)^{\gamma}\right].
\end{aligned}
$$

Theorem 12 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho - \gamma) < 1 + \min\{\operatorname{Re}(\nu_1'), \operatorname{Re}(\eta - \sigma_1 - \sigma_1'), \operatorname{Re}(\eta - \sigma_1' - \nu_1)\}$ *with* $|1/t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following Laplace-transform formula holds true*:

$$
\begin{aligned}
&L\left\{z^{l-1}\left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{z}{t}\right)^{\gamma}\right)\right\}\right)(x)\right\} \\
&\quad = \frac{x^{\varrho-\gamma-\eta+\sigma_1+\sigma_1'-1}}{s^l} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{1}{xs}\right)^{\gamma}\right) \\
&\qquad * {}_5\Psi_3\left[\begin{matrix}(1,1),(l+\gamma,\gamma),(1-\varrho-\nu_1'+\gamma,\gamma), \\ (1-\varrho+\gamma,\gamma),(1-\varrho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma,\gamma),\end{matrix}\right. \\
&\qquad \left.\begin{matrix}(1-\varrho+\eta-\sigma_1-\sigma_1'+\gamma,\gamma),(1-\varrho+\eta-\sigma_1'-\nu_1+\gamma,\gamma); \\ (1-\varrho-\sigma_1'-\nu_1'+\gamma,\gamma);\end{matrix}\left(\frac{1}{xs}\right)^{\gamma}\right].
\end{aligned}
$$

Theorem 13 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho + \gamma) > \max\{0, \operatorname{Re}(\sigma_1 + \sigma_1' + \nu_1 - \eta), \operatorname{Re}(\sigma_1' - \nu_1')\}$ *with* $|t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following integral formula holds true*:

$$
\begin{aligned}
&\int_0^{\infty} z^{l-1}e^{-\frac{1}{2}\delta z}W_{\tau,\varsigma}(\delta z)\bigl\{\bigl(I_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\bigl\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\bigl(r,a;p,q;(wtz)^{\gamma}\bigr)\bigr\}\bigr)(x)\bigr\}\,\mathrm{d}z \\
&\quad = \frac{x^{\varrho+\gamma+\eta-\sigma_1-\sigma_1'-1}}{\delta^l} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{wx}{\delta}\right)^{\gamma}\right) \\
&\qquad * {}_6\Psi_4\left[\begin{matrix}(1,1),(\frac{1}{2}+\zeta+l+\gamma,\gamma),(\frac{1}{2}-\zeta+l+\gamma,\gamma), \\ (\frac{1}{2}-\tau+l+\gamma,\gamma),(\varrho+\nu_1'+\gamma,\gamma),\end{matrix}\right. \\
&\qquad \left.\begin{matrix}(\varrho+\gamma,\gamma),(\varrho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma,\gamma),(\varrho+\nu_1'-\sigma_1'+\gamma,\gamma); \\ (\varrho+\eta-\sigma_1-\sigma_1'+\gamma,\gamma),(\varrho+\eta-\sigma_1'-\nu_1+\gamma,\gamma);\end{matrix}\left(\frac{wx}{\delta}\right)^{\gamma}\right].
\end{aligned}
$$

Theorem 14 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho - \gamma) < 1 + \min\{\operatorname{Re}(-\nu_1), \operatorname{Re}(\sigma_1 + \sigma_1' - \eta), \operatorname{Re}(\sigma_1 + \nu_1' - \eta)\}$ *with* $|1/t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following integral formula holds true*:

$$
\begin{aligned}
&\int_0^{\infty} z^{l-1}e^{-\frac{1}{2}\delta z}W_{\tau,\varsigma}(\delta z)\left\{\left(I_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}\left\{t^{\varrho-1}S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{wz}{t}\right)^{\gamma}\right)\right\}\right)(x)\right\}\mathrm{d}z \\
&\quad = \frac{x^{\varrho-\gamma+\eta-\sigma_1-\sigma_1'-1}}{\delta^l} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}\left(r,a;p,q;\left(\frac{w}{x\delta}\right)^{\gamma}\right)
\end{aligned}
$$

$$* {}_6\Psi_4 \left[\begin{matrix} (1,1), (\frac{1}{2}+\zeta+l+\gamma,\gamma), (\frac{1}{2}-\zeta+l+\gamma,\gamma), (1-\varrho-\nu_1+\gamma,\gamma), \\ (\frac{1}{2}-\tau+l+\gamma,\gamma), (1-\varrho+\gamma,\gamma), \end{matrix} \right.$$

$$\left. \begin{matrix} (1-\varrho-\eta+\sigma_1+\sigma_1'+\gamma,\gamma), (1-\varrho-\eta+\sigma_1+\nu_1'+\gamma,\gamma); \\ (1-\varrho-\eta+\sigma_1+\sigma_1'+\nu_1'+\gamma,\gamma), (1-\varrho+\sigma_1-\nu_1+\gamma,\gamma); \end{matrix} \left(\frac{w}{x\delta}\right)^{\gamma} \right].$$

Theorem 15 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho+\gamma) > \max\{0, \operatorname{Re}(\eta-\sigma_1-\sigma_1'-\nu_1'), \operatorname{Re}(\nu_1-\sigma_1)\}$ *with* $|t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following integral formula holds true*:

$$\int_0^\infty z^{l-1} e^{-\frac{1}{2}\delta z} W_{\tau,\varsigma}(\delta z) \bigl\{ \bigl(D_{0,x}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} \bigl\{ t^{\varrho-1} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)} \bigl(r, a; p, q; (wtz)^{\gamma}\bigr) \bigr\} \bigr)(x) \bigr\}$$

$$= \frac{x^{\varrho+\gamma-\eta+\sigma_1+\sigma_1'-1}}{\delta^l} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)} \left(r, a; p, q; \left(\frac{wx}{\delta}\right)^{\gamma} \right)$$

$$* {}_6\Psi_4 \left[\begin{matrix} (1,1), (\frac{1}{2}+\zeta+l+\gamma,\gamma), (\frac{1}{2}-\zeta+l+\gamma,\gamma) \\ (\frac{1}{2}-\tau+l+\gamma,\gamma), (\varrho-\nu_1+\gamma,\gamma), \end{matrix} \right.$$

$$\left. \begin{matrix} (\varrho+\gamma,\gamma), (\varrho-\eta+\sigma_1+\sigma_1'+\nu_1'+\gamma,\gamma), (\varrho-\nu_1+\sigma_1+\gamma,\gamma); \\ (\varrho-\eta+\sigma_1+\sigma_1'+\gamma,\gamma), (\varrho-\eta+\sigma_1+\nu_1'+\gamma,\gamma) \end{matrix} \left(\frac{wx}{\delta}\right)^{\gamma} \right].$$

Theorem 16 *Let* $\sigma_1, \sigma_1', \nu_1, \nu_1', \eta, \varrho \in \mathbb{C}$ *and* $\mu, \alpha, \beta, r, \lambda_1, \lambda_2, \lambda_3, \gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho-\gamma) < 1 + \min\{\operatorname{Re}(\nu_1'), \operatorname{Re}(\eta-\sigma_1-\sigma_1'), \operatorname{Re}(\eta-\sigma_1'-\nu_1)\}$ *with* $|1/t| < 1$. *Then, for* $\min\{\Re(p), \Re(q)\} \geq 0$, *the following integral formula holds true*:

$$\int_0^\infty z^{l-1} e^{-\frac{1}{2}\delta z} W_{\tau,\varsigma}(\delta z) \left\{ \left(D_{x,\infty}^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta} \left\{ t^{\varrho-1} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)} \left(r, a; p, q; \left(\frac{wz}{t}\right)^{\gamma} \right) \right\} \right)(x) \right\}$$

$$= \frac{x^{\varrho-\gamma-\eta+\sigma_1+\sigma_1'-1}}{\delta^l} S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)} \left(r, a; p, q; \left(\frac{w}{x\delta}\right)^{\gamma} \right)$$

$$* {}_6\Psi_4 \left[\begin{matrix} (1,1), (\frac{1}{2}+\zeta+l+\gamma,\gamma), (\frac{1}{2}-\zeta+l+\gamma,\gamma), (1-\varrho-\nu_1'+\gamma,\gamma), \\ (\frac{1}{2}-\tau+l+\gamma,\gamma), (1-\varrho+\gamma,\gamma), \end{matrix} \right.$$

$$\left. \begin{matrix} (1-\varrho+\eta-\sigma_1-\sigma_1'+\gamma,\gamma), (1-\varrho+\eta-\sigma_1'-\nu_1+\gamma,\gamma)p; \\ (1-\varrho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma,\gamma), (1-\varrho-\sigma_1'-\nu_1'+\gamma,\gamma); \end{matrix} \left(\frac{w}{x\delta}\right)^{\gamma} \right].$$

4 Concluding remark and observations

In this paper, we have established 16 interesting generalized fractional integrals and derivative formulas including their composition formulas by using certain integral transforms involving generalized (p,q)-Mathieu type series. The (p,q)-Mathieu type series (1.5) considered by Mehrez and Tomovski contains several special cases as various forms of the Mathieu type series presented in [4]. In particular, if we take $p = q$ in (1.5), we get the p-Mathieu type series defined as

$$S_{\mu,\lambda_1,\lambda_2,\lambda_3}^{(\alpha,\beta)}(r, a; p; z) = \sum_{n\geq 1} \frac{2a_n^{\beta}(\lambda_1)_n \mathrm{B}_p(\lambda_2+n, \lambda_3-\lambda_2)}{(a_n^{\alpha}+r^2)^{\mu} \mathrm{B}(\lambda_2, \lambda_3-\lambda_2)} \frac{z^n}{n!}$$

$$\bigl(\mu, r, \alpha, \beta, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+, \operatorname{Re}(p) \geq 0; |z| < 1\bigr). \tag{4.1}$$

Again, if we take $p = q = 0$ in (1.5), we get the generalized Mathieu type series defined as

$$S^{(\alpha,\beta)}_{\mu,\lambda_1,\lambda_2,\lambda_3}(r,a;z) = \sum_{n\geq 1} \frac{2a_n^{\beta}(\lambda_1)_n(\lambda_2)_n}{(a_n^{\alpha}+r^2)^{\mu}(\lambda_3)_n}\frac{z^n}{n!} \qquad \bigl(\mu,r,\alpha,\beta,\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}^+, |z|<1\bigr). \tag{4.2}$$

Furthermore, if we put $\lambda_2 = \lambda_3$ in (4.2), we get the well-known definition of the Mathieu type series defined earlier by Tomovski and Mehrez [13] as

$$S^{(\alpha,\beta)}_{\mu,\lambda_1}(r,a;z) = \sum_{n\geq 1} \frac{2a_n^{\beta}(\lambda_1)_n}{(a_n^{\alpha}+r^2)^{\mu}}\frac{z^n}{n!} \qquad \bigl(\mu,r,\alpha,\beta,\lambda_1 \in \mathbb{R}^+, |z|<1\bigr). \tag{4.3}$$

The results established in this paper contains various special cases such that, if we take $p = q$ and $p = q = 0$, we can obtain thirty two new results. We left these as an exercise for the interested reader. Furthermore, if we take $p = q = 0$, $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3$, $\varrho = \rho$, $\sigma_1 = \sigma$ and $\sigma_1' = \sigma'$, we recover the 16 known results in corrected form recorded in [10]. Furthermore, all the corollaries obtained earlier by Singh *et al.* [10] in the same paper can also be written correctly by our results. For example, the corrected version of the first result of Singh *et al.* [10] as given in Theorem 1 should read as follows.

Corollary 1 *Let* $\sigma_1,\sigma_1',\nu_1,\nu_1',\eta,\varrho \in \mathbb{C}$ *and* $\mu,\alpha,\beta,r,\lambda_1,\lambda_2,\lambda_3,\gamma \in \mathbb{R}^+$ *such that* $\operatorname{Re}(\eta) > 0$ *and* $\operatorname{Re}(\varrho+\gamma) > \max\{0,\operatorname{Re}(\sigma_1+\sigma_1'+\nu_1-\eta),\operatorname{Re}(\sigma_1'-\nu_1')\}$ *with* $|t|<1$. *Then, for* $x > 0$, *the following formula for fractional integral holds true*:

$$\begin{aligned}
&\bigl(I^{\sigma_1,\sigma_1',\nu_1,\nu_1',\eta}_{0,x}\bigl\{t^{\varrho-1}S^{(\alpha,\beta)}_{\mu,\lambda_1}\bigl(r,a;t^{\gamma}\bigr)\bigr\}\bigr)(x)\\
&\quad = x^{\varrho+\gamma+\eta-\sigma_1-\sigma_1'-1}S^{(\alpha,\beta)}_{\mu,\lambda_1}\bigl(r,a;x^{\gamma}\bigr)\\
&\qquad * {}_4\Psi_3\left[\begin{matrix}(1,1),(\varrho+\gamma,\gamma),(\varrho+\eta-\sigma_1-\sigma_1'-\nu_1+\gamma,\gamma),(\varrho+\nu_1'-\sigma_1'+\gamma,\gamma);\\ (\varrho+\nu_1'+\gamma,\gamma),(\varrho+\eta-\sigma_1-\sigma_1'+\gamma,\gamma),(\varrho+\eta-\sigma_1'-\nu_1+\gamma,\gamma);\end{matrix}\, x^{\gamma}\right].
\end{aligned}$$

Similarly other results can easily be written in corrected forms and we left this as an exercise to the interested reader.

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