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A Caputo–Fabrizio fractional differential equation model for HIV/AIDS with treatment compartment

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Abstract

In recent years, many new definitions of fractional derivatives have been proposed and used to develop mathematical models for a wide variety of real-world systems containing memory, history, or nonlocal effects. The main purpose of the present paper is to develop and analyze a Caputo–Fabrizio fractional derivative model for the HIV/AIDS epidemic which includes an antiretroviral treatment compartment. The existence and uniqueness of the system of solutions of the model are established using a fixed-point theorem and an iterative method. The model is shown to have a disease-free and an endemic equilibrium point. Conditions are derived for the existence of the endemic equilibrium point and for the local asymptotic stability of the disease-free equilibrium point. The results confirm that the disease-free equilibrium point becomes increasingly stable as the fractional order is reduced. Numerical simulations are carried out using a three-step Adams–Bashforth predictor method for a range of fractional orders to illustrate the effects of varying the fractional order and to support the theoretical results.

Keywords: Caputo–Fabrizio fractional derivative; HIV/AIDS epidemic model; Non-singularity; Three-step fractional Adams–Bashforth scheme

1 Introduction

Human immunodeficiency virus (HIV), which leads to acquired immunodeficiency syndrome (AIDS), destroys the human body's ability to fight infections. The disease is very dangerous and can be fatal if untreated. Since the first AIDS case was discovered in 1981 [1], HIV has spread worldwide and over 35 million people have died from AIDS-related illness. At the end of 2016, approximately 36.7 million people were living with HIV, and approximately 1.8 million new infections were occurring globally each year [2]. According to UNAIDS data reported in June 2016 [3], around 18.2 million people living with HIV were receiving antiretroviral therapy (ART). The US Center for Disease control and Prevention (CDC) [4] reported in 2017 that without treatment of HIV/AIDS with antiretroviral medicine, HIV infection advances through several stages and individuals with AIDS typically survive about 3 years. Antiretroviral HIV/AIDS therapy involves the simultaneous management of two or more antiviral drugs which can assist patients to live longer and restore their immune system [5]. In 2016, the regions worst affected by HIV/AIDS were located in eastern and southern Africa with nearly 10.3 million people being treated for

HIV. This number has more than doubled since 2010, while AIDS-related deaths in these regions have decreased by 38% since 2010. Some official reports indicate that between 2000 and 2016 in Africa, HIV-related deaths fell by one-third and the ART helped in saving 13.1 million lives [2]. Although the antiretroviral therapy cannot completely eliminate the virus, it can reduce the level sufficiently to prevent transmission from an HIV+ person to an uninfected person. Effective ART can also help prevent mother-to-child transmission of HIV, and thus significantly decrease the risk of transmission to future generations [6, 7]. Although antiretroviral therapies have been successful in decreasing the mortality rate in some regions, it is still necessary to increase antiretroviral therapies in other regions. Many HIV/AIDS epidemic models have been proposed to predict and control the spread of the disease (see, e.g., [3, 8–11] and the references cited therein).

In recent decades, many physical problems have been modeled using the fractional calculus. The main reasons given for using fractional derivative models are that many systems show memory, history, or nonlocal effects, which can be difficult to model using integer order derivatives. The basic theory and applications of fractional calculus and fractional differential equations can now be found in many studies (see, e.g., [12–16]). Although most of the early studies were based on the use of the Riemann–Liouville fractional order derivative or the Caputo fractional order derivative, it has been pointed out recently that these derivatives have the problem that their kernels have a singularity that occurs at the end point of an interval of definition. As a result, many new definitions of fractional derivatives have now been proposed in the literature (see, e.g., [17–25]). The fundamental differences among the fractional derivatives are their different kernels which can be selected to meet the requirements of different applications. For example, the main differences between the Caputo fractional derivative [13], the Caputo–Fabrizio derivative [19], and the Atangana–Baleanu fractional derivative [26] are that the Caputo derivative is defined using a power law, the Caputo–Fabrizio derivative is defined using an exponential decay law, and the Atangana–Baleanu derivative is defined using a Mittag–Leffler law. Examples of the applications of the new fractional operators to real world problems have been given in a number of recent papers. For example, Tateishi et al. [21] have compared the classical and new fractional time-derivatives in a study of anomalous diffusion. Also, Atangana et al. have compared the Caputo–Fabrizio fractional derivative and the Atangana–Baleanu fractional derivative in modeling fractional delay differential equations [27] and in modeling chaotic systems [26]. They found that the power law derivative of the Riemann–Liouville fractional derivative or the Caputo–Fabrizio fractional derivative provides noisy information due to its specific memory properties. However, the Caputo–Fabrizio fractional derivative gives less noise than the power law one while the Atangana–Baleanu fractional derivative provides an excellent description.

In the present paper, we apply the Caputo–Fabrizio fractional derivative with an exponential decay kernel to a novel HIV/AIDS epidemic model that includes an antiretroviral treatment compartment. The existence and uniqueness of the solution of the fractional model are established using fixed-point theory and an iterative method. The organization of this paper is as follows. The definition of the Caputo–Fabrizio fractional derivative and some of its important properties are given in Sect. 2. The fractional model for HIV/AIDS with treatment compartment is described in Sect. 3. In Sect. 4, the existence and uniqueness of the solutions of the model are discussed. In Sect. 5, we determine the equilibrium points of the model and give conditions for local asymptotic stability. Section 6 describes the numerical method used for solving the model and gives results of numerical simulations. Lastly, some conclusions are presented in Sect. 7.

2 Preliminaries

Because of the singularity in the kernel of the Caputo fractional derivative [28, 29] at the end point of the interval of integration, the Caputo fractional derivative is not always a suitable kernel to accurately describe the memory effect in a real system. Caputo and Fabrizio [19] have recently proposed a new fractional derivative without any singularity in its kernel. The kernel of the new fractional derivative has the form of an exponential function. More recently, Losada and Nieto [20] derived the fractional integral associated with the new fractional Caputo–Fabrizio fractional derivative. In this section, we summarize the definitions and properties for the Caputo–Fabrizio (CF) fractional operators required in this paper.

Let $H^1(a,b) = \{f | f \in L^2(a,b) \text{ and } f' \in L^2(a,b)\}$, where $L^2(a,b)$ is the space of square integrable functions on the interval (a,b).

Definition 1 *Let $f \in H^1(a,b)$ and $\rho \in (0,1)$. Then the Caputo–Fabrizio fractional derivative [19] is defined as*

$$ {}^{CF}D_t^{\rho}\bigl(f(t)\bigr) = \frac{M(\rho)}{1-\rho}\int_a^t f'(x)\exp\left[-\rho\frac{t-x}{1-\rho}\right]dx, \tag{1}$$

where $M(\rho)$ is a normalization function such that $M(0) = M(1) = 1$. However, if $f \notin H^1(a,b)$, then the derivative is defined as

$$ {}^{CF}D_t^{\rho}\bigl(f(t)\bigr) = \frac{\rho M(\rho)}{1-\rho}\int_a^t \bigl(f(t)-f(x)\bigr)\exp\left[-\rho\frac{t-x}{1-\rho}\right]dx. \tag{2}$$

Remark 1 ([19]) If we let $\sigma = \frac{1-\rho}{\rho} \in (0,\infty)$, then $\rho = \frac{1}{1+\sigma} \in (0,1)$. In consequence, Eq. (2) can be reduced to

$$ {}^{CF}D_t^{\rho}\bigl(f(t)\bigr) = \frac{N(\sigma)}{\sigma}\int_a^t f'(x)\exp\left[-\frac{t-x}{\sigma}\right]dx, \tag{3}$$

where $N(\sigma)$ is the normalization term corresponding to $M(\rho)$ such that $N(0) = N(\infty) = 1$.

Remark 2 ([19]) We have the following property:

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t), \quad \text{where } \delta(x-t) \text{ is the Dirac delta function.} \tag{4}$$

The above Caputo–Fabrizio fractional derivative was later modified by Losada and Nieto [20] as

$$ {}^{CF}D_t^{\rho}\bigl(f(t)\bigr) = \frac{(2-\rho)M(\rho)}{2(1-\rho)}\int_a^t f'(x)\exp\left[-\rho\frac{t-x}{1-\rho}\right]dx. \tag{5}$$

The fractional integral corresponding to the derivative in Eq. (5) was defined by Nieto and Losada [20] as follows.

Definition 2 *Let* $0 < \rho < 1$. *The fractional integral of order* ρ *of a function* f *is defined by*

$$ {}^{CF}I_t^{\rho}\big(f(t)\big) = \frac{2(1-\rho)}{(2-\rho)M(\rho)} f(t) + \frac{2\rho}{(2-\rho)M(\rho)} \int_0^t f(x)\,dx, \quad t \geq 0. \tag{6} $$

Remark 3 ([20]) From the definition in Eq. (6), the fractional integral of Caputo–Fabrizio type of a function f of order $0 < \rho < 1$ is a mean between the function f and its integral of order one, i.e.,

$$ \frac{2(1-\rho)}{(2-\rho)M(\rho)} + \frac{2\rho}{(2-\rho)M(\rho)} = 1, \tag{7} $$

and therefore $M(\rho) = \frac{2}{2-\rho}, 0 < \rho < 1$.

Using $M(\rho) = \frac{2}{2-\rho}$, Losada and Nieto proposed the new Caputo derivative and its corresponding integral as follows.

Definition 3 ([20]) *Let* $0 < \rho < 1$. *The fractional Caputo–Fabrizio derivative of order* ρ *of a function* f *is given by*

$$ {}^{CF}D_t^{\rho}\big(f(t)\big) = \frac{1}{1-\rho} \int_a^t f'(x) \exp\left[-\rho \frac{t-x}{1-\rho}\right] dx, \quad t \geq 0, \tag{8} $$

and its fractional integral is defined as

$$ {}^{CF}I_t^{\rho}\big(f(t)\big) = (1-\rho) f(t) + \rho \int_0^t f(x)\,dx, \quad t \geq 0. \tag{9} $$

3 Caputo–Fabrizio fractional model for HIV/AIDS with treatment compartment

In this section, we consider the HIV/AIDS epidemic model with a treatment compartment proposed by Huo et al. [1]. In this model, it is assumed that the total population $N(t)$ at time t is divided into five compartments, namely, $S(t)$ represents the number of susceptible patients, $I(t)$ represents the number of HIV-positive individuals who are infectious (i.e., who are not receiving antiretroviral ARV treatment or for whom the treatment is not effective), $A(t)$ represents the number of individuals with full-blown AIDS who are not receiving ARV treatment or for whom the treatment is not effective, $T(t)$ represents the total number of individuals being treated with ARV and for whom the treatment is effective, and $R(t)$ represents the number of individuals who have changed their sexual habits sufficiently so that they are immune to HIV infection by sexual contact. Yusuf and Benyah [30] determined that the individuals in the $R(t)$ class are people who have safe sexual habits and maintain those habits for the rest of their lives. Hence, the total population is $N(t) = S(t) + I(t) + A(t) + R(t) + T(t)$ and the original integer-order model adopted from

[1] can be written as

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta IS - \mu_1 S - dS, \\
\frac{dI}{dt} &= \beta IS + \alpha_1 T - dI - k_1 I - k_2 I, \\
\frac{dA}{dt} &= k_1 I - (\delta_1 + d)A + \alpha_2 T, \\
\frac{dT}{dt} &= k_2 I - \alpha_1 T - (d + \delta_2 + \alpha_2)T, \\
\frac{dR}{dt} &= \mu_1 S - dR.
\end{aligned} \tag{10}$$

All parameters in the model are assumed to be positive constants and the definitions are as follows. Λ is the recruitment rate of susceptible people into the population, β denotes the contact rate between the susceptible and the infectious people, μ_1 is the rate at which susceptible individuals change their sexual habits per unit time, d is the natural death rate, α_1 is the rate at which treated individuals leave compartment $T(t)$ and return to the infectious compartment, k_1 represents the rate at which individuals leave the infectious class and become individuals with full-blown AIDS, k_2 denotes the rate at which individuals with HIV receive treatment, δ_1 and δ_2 are the disease-induced death rates for individuals in compartments $A(t)$ and $T(t)$, respectively, and α_2 represents the rate at which treated individuals leave the treated class and enter the AIDS compartment $A(t)$.

To obtain our fractional derivative model, we replace the first-order time derivatives of the left-hand side of (10) by the fractional Caputo–Fabrizio derivative defined in Eq. (5). Our new Caputo–Fabrizio fractional model for HIV/AIDS with the treatment compartment can therefore be written as follows:

$$\begin{aligned}
{}^{CF}D_t^{\rho_1} S &= \Lambda - \beta IS - \mu_1 S - dS, \\
{}^{CF}D_t^{\rho_2} I &= \beta IS + \alpha_1 T - dI - k_1 I - k_2 I, \\
{}^{CF}D_t^{\rho_3} A &= k_1 I - (\delta_1 + d)A + \alpha_2 T, \\
{}^{CF}D_t^{\rho_4} T &= k_2 I - \alpha_1 T - (d + \delta_2 + \alpha_2)T, \\
{}^{CF}D_t^{\rho_5} R &= \mu_1 S - dR,
\end{aligned} \tag{11}$$

with initial conditions

$$S(0) = S_0, \qquad I(0) = I_0, \qquad A(0) = A_0, \qquad T(0) = T_0, \qquad R(0) = R_0. \tag{12}$$

In the theoretical treatment, we will assume that the fractional orders $(0 < \rho_i < 1, i = 1, 2, \dots, 5)$ for each of the five populations can be different.

4 Existence and uniqueness of solutions of the model

In this section, we investigate the existence and uniqueness of the solutions of the Caputo–Fabrizio fractional model for HIV/AIDS in Eq. (11) with initial conditions (12). Using fixed point theory (see, e.g., [31, 32]), we can prove existence of solutions for the model as follows.

Applying the Caputo–Fabrizio fractional integral operator in Eq. (6) to both sides of Eq. (11), we obtain

$$\begin{aligned}
&S(t)-S(0)={}^{CF}I_t^{\rho_1}[\Lambda-\beta IS-\mu_1 S-dS],\\
&I(t)-I(0)={}^{CF}I_t^{\rho_2}[\beta IS+\alpha_1 T-dI-k_1 I-k_2 I],\\
&A(t)-A(0)={}^{CF}I_t^{\rho_3}\bigl[k_1 I-(\delta_1+d)A+\alpha_2 T\bigr],\\
&T(t)-T(0)={}^{CF}I_t^{\rho_4}\bigl[k_2 I-\alpha_1 T-(d+\delta_2+\alpha_2)T\bigr],\\
&R(t)-R(0)={}^{CF}I_t^{\rho_5}[\mu_1 S-dR].
\end{aligned} \tag{13}$$

Then, for computational convenience, we define the following kernels:

$$\begin{aligned}
&K_1(t,S)=\Lambda-\beta I(t)S(t)-\mu_1 S(t)-dS(t),\\
&K_2(t,I)=\beta I(t)S(t)+\alpha_1 T(t)-dI(t)-k_1 I(t)-k_2 I(t),\\
&K_3(t,A)=k_1 I(t)-(\delta_1+d)A(t)+\alpha_2 T(t),\\
&K_4(t,T)=k_2 I(t)-\alpha_1 T(t)-(d+\delta_2+\alpha_2)T(t),\\
&K_5(t,R)=\mu_1 S(t)-dR(t),
\end{aligned} \tag{14}$$

and the functions

$$\Omega(\rho)=\frac{2(1-\rho)}{(2-\rho)M(\rho)}\quad\text{and}\quad\omega(\rho)=\frac{2\rho}{(2-\rho)M(\rho)}. \tag{15}$$

In proving the following theorems, we will assume that S, I, A, T, and R are nonnegative bounded functions, i.e., $\|S(t)\|\le\theta_1$, $\|I(t)\|\le\theta_2$, $\|A(t)\|\le\theta_3$, $\|T(t)\|\le\theta_4$, and $\|R(t)\|\le\theta_5$ where $\theta_1,\theta_2,\theta_3,\theta_4$, and θ_5 are some positive constants. Denote

$$\begin{aligned}
&\gamma_1=\beta\theta_2+\mu_1+d,\qquad \gamma_2=\beta\theta_1+d+k_1+k_2,\qquad \gamma_3=\delta_1+d,\\
&\gamma_4=\alpha_1+d+\delta_2+\alpha_2,\qquad \gamma_5=d.
\end{aligned} \tag{16}$$

Applying the definition of the Caputo–Fabrizio fractional integral in Eq. (6) to Eq. (13), we obtain

$$\begin{aligned}
&S(t)-S(0)=\Omega(\rho_1)K_1(t,S)+\omega(\rho_1)\int_0^t K_1(y,S)\,dy,\\
&I(t)-I(0)=\Omega(\rho_2)K_2(t,I)+\omega(\rho_2)\int_0^t K_2(y,I)\,dy,\\
&A(t)-A(0)=\Omega(\rho_3)K_3(t,A)+\omega(\rho_3)\int_0^t K_3(y,A)\,dy,\\
&T(t)-T(0)=\Omega(\rho_4)K_4(t,T)+\omega(\rho_4)\int_0^t K_4(y,T)\,dy,\\
&R(t)-R(0)=\Omega(\rho_5)K_5(t,R)+\omega(\rho_5)\int_0^t K_5(y,R)\,dy.
\end{aligned} \tag{17}$$

Theorem 4 *If the following inequality holds*

$$0 \leq M = \max\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\} < 1, \tag{18}$$

then the kernels K_1, K_2, K_3, K_4, and K_5 satisfy Lipschitz conditions and are contraction mappings.

Proof We consider the kernel K_1. Let S and S_1 be any two functions, then we have

$$\begin{aligned}\left\|K_1(t,S) - K_1(t,S_1)\right\| = {}& \left\|-\beta I(t)\big(S(t) - S_1(t)\big) - \mu_1\big(S(t) - S_1(t)\big)\right. \\ & \left.- d\big(S(t) - S_1(t)\big)\right\|.\end{aligned} \tag{19}$$

Using the triangle inequality for norms on the right-hand side of Eq. (19), we obtain

$$\begin{aligned}\left\|K_1(t,S) - K_1(t,S_1)\right\| &\leq \left\|\beta I(t)\big(S(t) - S_1(t)\big)\right\| + \left\|\mu_1\big(S(t) - S_1(t)\big)\right\| \\ &\quad + \left\|d\big(S(t) - S_1(t)\big)\right\| \\ &\leq \big(\beta\left\|I(t)\right\| + \mu_1 + d\big)\left\|S(t) - S_1(t)\right\| \\ &\leq (\beta\theta_2 + \mu_1 + d)\left\|S(t) - S_1(t)\right\| \\ &= \gamma_1\left\|S(t) - S_1(t)\right\|,\end{aligned} \tag{20}$$

where γ_1 is defined in Eq. (16). Similar results for the kernels K_2, K_3, K_4, and K_5 can be obtained using $\{I, I_1\}, \{A, A_1\}, \{T, T_1\}$, and $\{R, R_1\}$, respectively, as follows:

$$\begin{aligned}&\left\|K_2(t,I) - K_2(t,I_1)\right\| \leq \gamma_2\left\|I(t) - I_1(t)\right\|, \\ &\left\|K_3(t,A) - K_3(t,A_1)\right\| \leq \gamma_3\left\|A(t) - A_1(t)\right\|, \\ &\left\|K_4(t,T) - K_4(t,T_1)\right\| \leq \gamma_4\left\|T(t) - T_1(t)\right\|, \\ &\left\|K_5(t,R) - K_5(t,R_1)\right\| \leq \gamma_5\left\|R(t) - R_1(t)\right\|,\end{aligned}$$

where $\gamma_2, \gamma_3, \gamma_4$, and γ_5 are defined in Eq. (16). Therefore, the Lipschitz conditions are satisfied for K_1, K_2, K_3, K_4, and K_5. In addition, since $0 \leq M = \max\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\} < 1$, the kernels are contractions. □

From Eq. (17), the state variables can be displayed in terms of the kernels as follows:

$$\begin{aligned}&S(t) = S(0) + \Omega(\rho_1)K_1(t,S) + \omega(\rho_1)\int_0^t K_1(y,S)\,dy, \\ &I(t) = I(0) + \Omega(\rho_2)K_2(t,I) + \omega(\rho_2)\int_0^t K_2(y,I)\,dy, \\ &A(t) = A(0) + \Omega(\rho_3)K_3(t,A) + \omega(\rho_3)\int_0^t K_3(y,A)\,dy, \\ &T(t) = T(0) + \Omega(\rho_4)K_4(t,T) + \omega(\rho_4)\int_0^t K_4(y,T)\,dy, \\ &R(t) = R(0) + \Omega(\rho_5)K_5(t,R) + \omega(\rho_5)\int_0^t K_5(y,R)\,dy.\end{aligned} \tag{21}$$

Using Eq. (21), we now introduce the following recursive formulas:

$$\begin{aligned}
S_n(t) &= \Omega(\rho_1)K_1(t,S_{n-1}) + \omega(\rho_1)\int_0^t K_1(y,S_{n-1})\,dy,\\
I_n(t) &= \Omega(\rho_2)K_2(t,I_{n-1}) + \omega(\rho_2)\int_0^t K_2(y,I_{n-1})\,dy,\\
A_n(t) &= \Omega(\rho_3)K_3(t,A_{n-1}) + \omega(\rho_3)\int_0^t K_3(y,A_{n-1})\,dy,\\
T_n(t) &= \Omega(\rho_4)K_4(t,T_{n-1}) + \omega(\rho_4)\int_0^t K_4(y,T_{n-1})\,dy,\\
R_n(t) &= \Omega(\rho_5)K_5(t,R_{n-1}) + \omega(\rho_5)\int_0^t K_5(y,R_{n-1})\,dy.
\end{aligned} \tag{22}$$

The initial components of the above recursive formulas are determined by the given initial conditions as follows:

$$\begin{aligned}
&S_0(t) = S(0), \qquad I_0(t) = I(0), \qquad A_0(t) = A(0),\\
&T_0(t) = T(0), \qquad R_0(t) = R(0).
\end{aligned} \tag{23}$$

The differences between the consecutive terms for the recursive formulas can be written as

$$\begin{aligned}
\phi_n(t) = S_n(t) - S_{n-1}(t) &= \Omega(\rho_1)\bigl(K_1(t,S_{n-1}) - K_1(t,S_{n-2})\bigr)\\
&\quad + \omega(\rho_1)\int_0^t \bigl(K_1(y,S_{n-1}) - K_1(y,S_{n-2})\bigr)\,dy,\\
\psi_n(t) = I_n(t) - I_{n-1}(t) &= \Omega(\rho_2)\bigl(K_2(t,I_{n-1}) - K_2(t,I_{n-2})\bigr)\\
&\quad + \omega(\rho_2)\int_0^t \bigl(K_2(y,I_{n-1}) - K_2(y,I_{n-2})\bigr)\,dy,\\
\xi_n(t) = A_n(t) - A_{n-1}(t) &= \Omega(\rho_3)\bigl(K_3(t,A_{n-1}) - K_3(t,A_{n-2})\bigr)\\
&\quad + \omega(\rho_3)\int_0^t \bigl(K_3(y,A_{n-1}) - K_3(y,A_{n-2})\bigr)\,dy,\\
\chi_n(t) = T_n(t) - T_{n-1}(t) &= \Omega(\rho_4)\bigl(K_4(t,T_{n-1}) - K_4(t,T_{n-2})\bigr)\\
&\quad + \omega(\rho_4)\int_0^t \bigl(K_4(y,T_{n-1}) - K_4(y,T_{n-2})\bigr)\,dy,\\
\eta_n(t) = R_n(t) - R_{n-1}(t) &= \Omega(\rho_5)\bigl(K_5(t,R_{n-1}) - K_5(t,R_{n-2})\bigr)\\
&\quad + \omega(\rho_5)\int_0^t \bigl(K_5(y,R_{n-1}) - K_5(y,R_{n-2})\bigr)\,dy.
\end{aligned} \tag{24}$$

Note that:

$$\begin{aligned}
&S_n(t) = \sum_{i=1}^{n} \phi_i(t), \qquad I_n(t) = \sum_{i=1}^{n} \psi_i(t), \qquad A_n(t) = \sum_{i=1}^{n} \xi_i(t),\\
&T_n(t) = \sum_{i=1}^{n} \chi_i(t), \qquad R_n(t) = \sum_{i=1}^{n} \eta_i(t).
\end{aligned} \tag{25}$$

Next, we formulate the recursive inequalities for the differences $\phi_n(t)$, $\psi_n(t), \xi_n(t), \chi_n(t)$, and $\eta_n(t)$ as follows:

$$\begin{aligned}\|\phi_n(t)\| &= \|S_n(t) - S_{n-1}(t)\| \\ &= \left\| \Omega(\rho_1)\big(K_1(t,S_{n-1}) - K_1(t,S_{n-2})\big) \right. \\ &\quad \left. + \omega(\rho_1)\int_0^t \big(K_1(y,S_{n-1}) - K_1(y,S_{n-2})\big)\,dy \right\|. \end{aligned} \tag{26}$$

Applying the triangle inequality for norms to Eq. (26), we obtain

$$\begin{aligned}\|S_n(t) - S_{n-1}(t)\| &\le \Omega(\rho_1)\|K_1(t,S_{n-1}) - K_1(t,S_{n-2})\| \\ &\quad + \omega(\rho_1)\int_0^t \|K_1(y,S_{n-1}) - K_1(y,S_{n-2})\|\,dy.\end{aligned}$$

Then, since the kernel K_1 satisfies the Lipschitz condition with Lipschitz constant γ_1, we have

$$\begin{aligned}\|S_n(t) - S_{n-1}(t)\| &\le \Omega(\rho_1)\gamma_1\|S_{n-1} - S_{n-2}\| \\ &\quad + \omega(\rho_1)\gamma_1\int_0^t \|S_{n-1} - S_{n-2}\|\,dy.\end{aligned}$$

Thus, we obtain

$$\|\phi_n(t)\| \le \Omega(\rho_1)\gamma_1\|\phi_{n-1}(t)\| + \omega(\rho_1)\gamma_1\int_0^t \|\phi_{n-1}(y)\|\,dy. \tag{27}$$

In a similar manner, we can obtain the following results:

$$\begin{aligned}&\|\psi_n(t)\| \le \Omega(\rho_2)\gamma_2\|\psi_{n-1}(t)\| + \omega(\rho_2)\gamma_2\int_0^t \|\psi_{n-1}(y)\|\,dy, \\ &\|\xi_n(t)\| \le \Omega(\rho_3)\gamma_3\|\xi_{n-1}(t)\| + \omega(\rho_3)\gamma_3\int_0^t \|\xi_{n-1}(y)\|\,dy, \\ &\|\chi_n(t)\| \le \Omega(\rho_4)\gamma_4\|\chi_{n-1}(t)\| + \omega(\rho_4)\gamma_4\int_0^t \|\chi_{n-1}(y)\|\,dy, \\ &\|\eta_n(t)\| \le \Omega(\rho_5)\gamma_5\|\eta_{n-1}(t)\| + \omega(\rho_5)\gamma_5\int_0^t \|\eta_{n-1}(y)\|\,dy.\end{aligned} \tag{28}$$

Theorem 5 *If there exists a time $t_0 > 0$ such that the following inequalities hold*:

$$\Omega(\rho_i)\gamma_i + \omega(\rho_i)\gamma_i t_0 < 1, \quad \textit{for } i = 1,2,\ldots,5, \tag{29}$$

then a system of solutions exists for the fractional HIV model (11)–(12).

Proof Since the functions $S(t), I(t), A(t), T(t)$, and $R(t)$ are assumed to be bounded and each of the kernels satisfies a Lipschitz condition, the following relations can be obtained using Eqs. (27)–(28) recursively:

$$\begin{aligned}
&\|\phi_n(t)\| \le \|S(0)\|\big[\Omega(\rho_1)\gamma_1 + \omega(\rho_1)\gamma_1 t\big]^n,\\
&\|\psi_n(t)\| \le \|I(0)\|\big[\Omega(\rho_2)\gamma_2 + \omega(\rho_2)\gamma_2 t\big]^n,\\
&\|\xi_n(t)\| \le \|A(0)\|\big[\Omega(\rho_3)\gamma_3 + \omega(\rho_3)\gamma_3 t\big]^n,\\
&\|\chi_n(t)\| \le \|T(0)\|\big[\Omega(\rho_4)\gamma_4 + \omega(\rho_4)\gamma_4 t\big]^n,\\
&\|\eta_n(t)\| \le \|R(0)\|\big[\Omega(\rho_5)\gamma_5 + \omega(\rho_5)\gamma_5 t\big]^n.
\end{aligned} \tag{30}$$

Equation (30) shows the existence and smoothness of the functions defined in Eq. (25).

To complete the proof, we prove that the functions $S_n(t)$, $I_n(t)$, $A_n(t)$, $T_n(t)$, $R_n(t)$ converge to a system of solutions of (11)–(12). We define $B_n(t), C_n(t), D_n(t), E_n(t)$, and $F_n(t)$ as the remainder terms after n iterations, i.e.,

$$\begin{aligned}
&S(t) - S(0) = S_n(t) - B_n(t),\\
&I(t) - I(0) = I_n(t) - C_n(t),\\
&A(t) - A(0) = A_n(t) - D_n(t),\\
&T(t) - T(0) = T_n(t) - E_n(t),\\
&R(t) - R(0) = R_n(t) - F_n(t).
\end{aligned} \tag{31}$$

Then, using the triangle inequality and the Lipschitz condition for K_1, we have

$$\begin{aligned}
\|B_n(t)\| &= \bigg\|\Omega(\rho_1)\big(K_1(t,S) - K_1(t,S_{n-1})\big)\\
&\qquad + \omega(\rho_1)\int_0^t \big(K_1(y,S) - K_1(y,S_{n-1})\big)\,dy\bigg\|\\
&\le \Omega(\rho_1)\big\|K_1(t,S) - K_1(t,S_{n-1})\big\|\\
&\qquad + \omega(\rho_1)\int_0^t \big\|K_1(y,S) - K_1(y,S_{n-1})\big\|\,dy,\\
&\le \Omega(\rho_1)\gamma_1\|S - S_{n-1}\| + \omega(\rho_1)\gamma_1\|S - S_{n-1}\|t.
\end{aligned}$$

Applying the above process recursively, we obtain

$$\|B_n(t)\| \le \big[\big(\Omega(\rho_1) + \omega(\rho_1)t\big)\gamma_1\big]^{n+1}\theta_1. \tag{32}$$

Then at t_0, we obtain

$$\|B_n(t)\| \le \big[\big(\Omega(\rho_1) + \omega(\rho_1)t_0\big)\gamma_1\big]^{n+1}\theta_1. \tag{33}$$

Taking the limit on Eq. (33) as $n \to \infty$ and then using condition (29), we obtain $\|B_n(t)\| \to 0$. Using the same process as described above, we have the following relations:

$$\|C_n(t)\| \le \big[\big(\Omega(\rho_2) + \omega(\rho_2)t_0\big)\gamma_2\big]^{n+1}\theta_2, \tag{34}$$

$$\|D_n(t)\| \leq \left[\left(\Omega(\rho_3) + \omega(\rho_3)t_0\right)\gamma_3\right]^{n+1}\theta_3, \tag{35}$$

$$\|E_n(t)\| \leq \left[\left(\Omega(\rho_4) + \omega(\rho_4)t_0\right)\gamma_4\right]^{n+1}\theta_4, \tag{36}$$

$$\|F_n(t)\| \leq \left[\left(\Omega(\rho_5) + \omega(\rho_5)t_0\right)\gamma_5\right]^{n+1}\theta_5. \tag{37}$$

Similarly, taking the limit on Eqs. (34)–(37) as $n \to \infty$ and then using condition (29), we have $\|C_n(t)\| \to 0$, $\|D_n(t)\| \to 0$, $\|E_n(t)\| \to 0$, and $\|F_n(t)\| \to 0$. Therefore, the existence of the system of solutions of system (11)–(12) is proved. □

We now give conditions for the system of solutions to be unique.

Theorem 6 *System* (11) *along with the initial conditions* (12) *has a unique system of solutions if the following conditions hold*:

$$\left(1 - \Omega(\rho_i)\gamma_i - \omega(\rho_i)\gamma_i t\right) > 0 \quad \textit{for } i = 1,2,3,4,5. \tag{38}$$

Proof Assume that $\{S_1(t), I_1(t), A_1(t), T_1(t), R_1(t)\}$ is another set of solutions of model (11)–(12) in addition to the solution set $\{S(t), I(t), A(t), T(t), R(t)\}$ proved to exist in Theorems 4 and 5. Then

$$\begin{aligned} S(t) - S_1(t) = {} & \Omega(\rho_1)\left(K_1(t,S) - K_1(t,S_1)\right) \\ & + \omega(\rho_1)\int_0^t \left(K_1(y,S) - K_1(y,S_1)\right) dy. \end{aligned} \tag{39}$$

Taking the norm on both sides of Eq. (39) and using the triangle inequality, we obtain

$$\begin{aligned} \|S(t) - S_1(t)\| \leq {} & \Omega(\rho_1)\|K_1(t,S) - K_1(t,S_1)\| \\ & + \omega(\rho_1)\int_0^t \|K_1(y,S) - K_1(y,S_1)\| dy. \end{aligned} \tag{40}$$

Using the Lipschitz condition for the kernel K_1, we find

$$\|S(t) - S_1(t)\| \leq \Omega(\rho_1)\gamma_1\|S(t) - S_1(t)\| + \omega(\rho_1)\gamma_1 t\|S(t) - S_1(t)\|. \tag{41}$$

Then, rearranging Eq. (41), we obtain

$$\|S(t) - S_1(t)\|\left(1 - \Omega(\rho_1)\gamma_1 - \omega(\rho_1)\gamma_1 t\right) \leq 0. \tag{42}$$

Finally, applying condition (38) for $i = 1$ to Eq. (42), we obtain

$$\|S(t) - S_1(t)\| = 0, \tag{43}$$

and therefore $S(t) = S_1(t)$.

Applying a similar procedure to each of the following pairs $(I(t), I_1(t))$, $(A(t), A_1(t))$, $(T(t), T_1(t))$, and $(R(t), R_1(t))$ with inequality (38) for $i = 2,3,4,5$, respectively, we obtain

$$I(t) = I_1(t), \qquad A(t) = A_1(t), \qquad T(t) = T_1(t), \qquad R(t) = R_1(t). \tag{44}$$

Thus, the uniqueness of the system of solutions of the fractional order system is proved. □

In summary, the existence of the solutions of the model described in Eqs. (11) and (12) can be obtained by requiring that $0 \leq M = \max\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\} < 1$, where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and γ_5 are the Lipschitz constants of the kernels K_1, K_2, K_3, K_4, and K_5, respectively. Moreover, the uniqueness of the system of solutions of the considered system can be established using the inequalities in Eq. (38).

5 Equilibrium points of the model and asymptotic stability

We can determine the equilibrium points of the fractional order system (11) by equating its right-hand side to zero. Solving the resulting algebraic system, we obtain two equilibrium points, namely, a disease-free and an endemic equilibrium point, which are the same as the equilibrium points given in [1]. Let $E^0 = (S^0, I^0, A^0, T^0, R^0)$ denote the disease-free equilibrium point of the model and $E^* = (S^*, I^*, A^*, T^*, R^*)$ denote the endemic equilibrium point of the model. From [1], we have the disease-free equilibrium point given by

$$E^0 = \left(S^0, I^0, A^0, T^0, R^0\right) = \left(\frac{\Lambda}{\mu_1 + d}, 0, 0, 0, \frac{\Lambda\mu_1}{d(d + \mu_1)}\right), \tag{45}$$

and the endemic equilibrium point given by

$$\begin{aligned} S^* &= \frac{\Lambda}{\beta I^* + \mu_1 + d}, \qquad I^* = \frac{(R_0 - 1)(\mu_1 + d)}{\beta}, \qquad A^* = \frac{k_1 I^* + \alpha_2 T^*}{d + \delta_1}, \\ T^* &= \frac{k_2 I^*}{\alpha_1 + d + \delta_2 + \alpha_2}, \qquad R^* = \frac{\mu_1 \Lambda}{d(\beta I^* + \mu_1 + d)}, \end{aligned} \tag{46}$$

where the basic reproduction number R_0, which can be obtained using the next generation matrix method [33, 34], is written as

$$R_0 = \frac{\beta\Lambda(d + \delta_2 + \alpha_1 + \alpha_2)}{(\mu_1 + d)[(d + k_1 + k_2)(d + \delta_2 + \alpha_1 + \alpha_2) - \alpha_1 k_2]}. \tag{47}$$

It can be noticed that the unique endemic equilibrium point E^* exists if $R_0 > 1$.

Consider the following fractional-order linear system described by the Caputo–Fabrizio derivative:

$$^{CF}D_t^{\rho} x(t) = Ax(t), \tag{48}$$

where $x(t) \in R^n, A \in R^{n\times n}$, and $0 < \rho < 1$.

Definition 7 ([35]) *The characteristic equation of system* (48) *is*

$$\det\bigl(s\bigl(I - (1 - \rho)A\bigr) - \rho A\bigr) = 0. \tag{49}$$

Theorem 8 ([35]) *If $(I - (1 - \rho)A)$ is invertible, then system* (48) *is asymptotically stable if and only if the real parts of the roots to the characteristic equation of system* (48) *are negative.*

The linearization matrix of model (11) evaluated at the disease-free equilibrium point E^0 is

$$J(E^0) = \begin{bmatrix} -d-\mu_1 & -\frac{\beta\Lambda}{\mu_1+d} & 0 & 0 & 0 \\ 0 & \frac{\beta\Lambda}{\mu_1+d}-d-k_1-k_2 & 0 & \alpha_1 & 0 \\ 0 & k_1 & -\delta_1-d & \alpha_2 & 0 \\ 0 & k_2 & 0 & -\alpha_1-d-\delta_2-\alpha_2 & 0 \\ \mu_1 & 0 & 0 & 0 & -d \end{bmatrix}. \tag{50}$$

If model (11) has a commensurate order, i.e., $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_5 = \rho \in (0,1)$, then the characteristic equation of the linearized system of model (11) at E^0 is

$$\det\big(s\big(I-(1-\rho)J\big(E^0\big)\big)-\rho J\big(E^0\big)\big) = 0. \tag{51}$$

Theorem 9 *The disease-free equilibrium point E^0 of model (11) with a commensurate order $\rho \in (0,1)$ is asymptotically stable if and only if real parts of the roots of the characteristic equation (51) are negative.*

Proof Equation (51) is a quintic polynomial equation. Then we denote its five roots by s_1, s_2, s_3, s_4, and s_5. However, it is not difficult to find the first three roots of Eq. (51). They are as follows:

$$s_1 = \frac{\rho d}{(\rho-1)d-1}, \qquad s_2 = \frac{\rho(\delta_1+d)}{(\rho-1)d+(\rho-1)\delta_1-1},$$
$$s_3 = \frac{\rho(\mu_1+d)}{(\rho-1)d+(\rho-1)\mu_1-1}.$$

It is obvious to see that s_1, s_2, and s_3 are negative because $0 < \rho < 1$. The rest two roots of Eq. (51), i.e., s_4, s_5 can be found from the following equation:

$$\det\left(\begin{bmatrix} a & -s(1-\rho)\alpha_1-\rho\alpha_1 \\ -s(1-\rho)k_2-\rho k_2 & b \end{bmatrix}\right) = 0, \tag{52}$$

where

$$\begin{aligned} a &= s\left(1-(1-\rho)\left(\frac{\beta\Lambda}{\mu_1+d}-d-k_1-k_2\right)\right)-\rho\left(\frac{\beta\Lambda}{\mu_1+d}-d-k_1-k_2\right), \\ b &= s\big(1-(1-\rho)(-\alpha_1-d-\delta_2-\alpha_2)\big)-\rho(-\alpha_1-d-\delta_2-\alpha_2). \end{aligned} \tag{53}$$

If real parts of the two roots of Eq. (52) are negative, i.e., $\mathrm{Re}(s_4) < 0$ and $\mathrm{Re}(s_5) < 0$, then the equilibrium point E_0 of model (11) is asymptotically stable by Theorem 8. □

6 Three-step Adams–Bashforth scheme and numerical simulations

In recent years, there have been many new analytical methods developed for solving the wide variety of nonlinear fractional derivative models that have been used as models of real world problems. The new analytical methods include the homotopy analysis Sumudu transform technique (HASTM) [36], the homotopy analysis transform method (HATM)

[37] and the local fractional homotopy perturbation Laplace transform method (LFH-PLTM) [38]. In addition, many numerical methods for obtaining approximate solutions of fractional differential equations have been developed. These methods are typically based on discretization of the independent variable and include modifications of the integer-order methods such as the Adams–Bashforth–Moulton type predictor-corrector methods [39], finite difference methods [40], and finite element methods [41]. In this paper, we will use a three-step fractional Adams–Bashforth scheme to obtain numerical solutions for the Caputo–Fabrizio fractional model (11). In this section, we will first describe the three-step fractional Adams–Bashforth scheme and then apply it to obtain numerical solutions for the fractional HIV/AIDS model in Eqs. (11)–(12) for a range of fractional orders and a range of reasonable parameter values.

In describing the numerical method, we will use the original definition of the Caputo–Fabrizio fractional derivative in Eq. (5) rather than the Losada and Nieto definition in Eq. (1).

Consider the Caputo–Fabrizio fractional differential equation

$$ {}^{CF}D_t^{\rho}\big(u(t)\big) = f\big(t, u(t)\big), \quad 0 < \rho < 1, \tag{54} $$

where ${}^{CF}D_t^{\rho}(\cdot)$ is the Caputo–Fabrizio fractional derivative defined in Eq. (1). Applying the following fractional integral:

$$ {}^{CF}I_t^{\rho}\big(f(t)\big) = \frac{1-\rho}{M(\rho)} f(t) + \frac{\rho}{M(\rho)} \int_0^t f(x)\,dx, \tag{55} $$

to both sides of Eq. (54), we obtain

$$ \begin{aligned} {}^{CF}I_t^{\rho}\,{}^{CF}D_t^{\rho}\big(u(t)\big) &= {}^{CF}I_t^{\rho}\big(f\big(t,u(t)\big)\big), \\ u(t) - u(0) &= {}^{CF}I_t^{\rho}\big(f\big(t,u(t)\big)\big) \\ &= \frac{(1-\rho)}{M(\rho)} f\big(t,u(t)\big) + \frac{\rho}{M(\rho)} \int_0^t f\big(s,u(s)\big)\,ds. \end{aligned} \tag{56} $$

We then discretize the time interval $[0,t]$ in steps of h and obtain the sequence $t_0 = 0, t_{k+1} = t_k + h, k = 0, 1, 2, \dots, n-1, t_n = t$. From Eq. (56), we can construct the following recursive formulas:

$$ u(t_{k+1}) - u(0) = \frac{(1-\rho)}{M(\rho)} f\big(t_k, u(t_k)\big) + \frac{\rho}{M(\rho)} \int_0^{t_{k+1}} f\big(t,u(t)\big)\,dt \tag{57} $$

and

$$ u(t_k) - u(0) = \frac{(1-\rho)}{M(\rho)} f\big(t_{k-1}, u(t_{k-1})\big) + \frac{\rho}{M(\rho)} \int_0^{t_k} f\big(t,u(t)\big)\,dt. \tag{58} $$

Subtracting Eq. (58) from Eq. (57), we obtain

$$ u(t_{k+1}) - u(t_k) = \frac{(1-\rho)}{M(\rho)} \big[f(t_k,u_k) - f(t_{k-1},u_{k-1})\big] + \frac{\rho}{M(\rho)} \int_{t_k}^{t_{k+1}} f\big(t,u(t)\big)\,dt. \tag{59} $$

We now derive a three-step Adams–Bashforth type predictor formula, by approximating the integral $\int_{t_k}^{t_{k+1}} f(t,u(t))\,dt$ in the above equation by the approximation $\int_{t_k}^{t_{k+1}} P_2(t)\,dt$, where $P_2(t)$ is the Lagrange interpolating polynomial of degree two passing through the following three points $(t_{k-2},f(t_{k-2},u(t_{k-2})))$, $(t_{k-1},f(t_{k-1},u(t_{k-1})))$, and $(t_k,f(t_k,u(t_k)))$. That is,

$$P_2(t) = \sum_{i=0}^{2} f(t_{k-i},u_{k-i})L_i(t), \tag{60}$$

where the $L_i(t)$ are the Lagrange basis polynomials on the three points (t_{k-2},t_{k-1},t_k). Using the change of variable $s = \frac{t_{k+1}-t}{h}$, substituting for the Lagrange basis polynomials and integrating, we obtain

$$\begin{aligned}\int_{t_k}^{t_{k+1}} f\bigl(t,u(t)\bigr)\,ds &= h\int_0^1 \biggl[\frac{(s-2)(s-3)}{(1-2)(1-3)} f(t_k,u_k) + \frac{(s-1)(s-3)}{(2-1)(2-3)} f(t_{k-1},u_{k-1}) \\ &\quad + \frac{(s-2)(s-1)}{(3-2)(3-1)} f(t_{k-2},u_{k-2})\biggr]\,ds, \\ &= h\biggl[\frac{23}{12} f(t_k,u_k) - \frac{4}{3} f(t_{k-1},u_{k-1}) + \frac{5}{12} f(t_{k-2},u_{k-2})\biggr],\end{aligned} \tag{61}$$

where $u_{k-2} = u(t_{k-2}), u_{k-1} = u(t_{k-1})$, and $u_k = u(t_k)$. Then, inserting Eq. (61) into Eq. (59), we obtain the iterative formula as follows:

$$\begin{aligned}u_{k+1} &= u_k + \frac{1}{M(\rho)}\biggl[(1-\rho) + \frac{23h\rho}{12}\biggr] f(t_k,u_k) \\ &\quad - \frac{1}{M(\rho)}\biggl[(1-\rho) + \frac{4}{3}h\rho\biggr] f(t_{k-1},u_{k-1}) + \frac{5h\rho}{12M(\rho)} f(t_{k-2},u_{k-2}).\end{aligned} \tag{62}$$

For the special case $\rho = 1$, Eq. (62) reduces to the classical Adams–Bashforth three-step predictor formula.

The truncation error for the three-step formula can be estimated by using the error estimate for the Lagrange interpolating polynomial, namely,

$$\begin{aligned}& f\bigl(t,u(t)\bigr) = P_2(t) + E_2(t), \\ & E_2(t) = \frac{f^{(3)}(\xi_k,u(\xi_k))}{3!}(t-t_k)(t-t_{k-1})(t-t_{k-2}), \quad \xi_k \in (t_{k-2},t_k).\end{aligned} \tag{63}$$

Then we have

$$\begin{aligned}\int_{t_k}^{t_{k+1}} E_2(t)\,dt &= \int_{t_k}^{t_{k+1}} \frac{f^{(3)}(\xi_k,u(\xi_k))}{3!}(t-t_k)(t-t_{k-1})(t-t_{k-2})\,dt \\ &\approx -\frac{h^4 f^{(3)}(\mu_k,u(\mu_k))}{6(3!)}\int_0^1 (s-1)(s-2)(s-3)\,ds \\ &= \frac{3}{8}h^4 f^{(3)}\bigl(\mu_k,u(\mu_k)\bigr),\end{aligned} \tag{64}$$

where $\mu_k \in (t_{k-2},t_{k+1})$, and we have used a mean value theorem to approximate the integral.

Denoting the entire right-hand side of Eq. (62) by $\widetilde{u}_k$, then we have $u_{k+1} = \widetilde{u}_k + \frac{3}{8}h^4 f^{(3)}(\mu_k, u(\mu_k))$. Therefore, the local truncation error of the use of formula (62) is determined by

$$\begin{aligned} \frac{u_{k+1} - \widetilde{u}_k}{h} &= \frac{\rho}{M(\rho)} \cdot \frac{\frac{3}{8}h^4 f^{(3)}(\mu_k, u(\mu_k))}{h}, \\ &= \frac{3}{8M(\rho)} \rho h^3 f^{(3)}\big(\mu_k, u(\mu_k)\big). \end{aligned} \tag{65}$$

Next, we use the three-step fractional Adams–Bashforth scheme in Eq. (62) to obtain numerical solutions of the fractional model (11)–(12). For the numerical simulations, we will assume that all fractional derivatives in system (11) have the same order, i.e., $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_5 = \rho$. We can then write the system in the vector form:

$${}^{CF}D_t^{\rho}\big(\mathbf{u}(t)\big) = \mathbf{f}\big(t, \mathbf{u}(t)\big), \quad 0 < \rho < 1, \tag{66}$$

where

$$\mathbf{u}(t) = \begin{bmatrix} S(t) \\ I(t) \\ A(t) \\ T(t) \\ R(t) \end{bmatrix}, \qquad \mathbf{f}\big(t, \mathbf{u}(t)\big) = \begin{bmatrix} f_1(t, \mathbf{u}(t)) \\ f_2(t, \mathbf{u}(t)) \\ f_3(t, \mathbf{u}(t)) \\ f_4(t, \mathbf{u}(t)) \\ f_5(t, \mathbf{u}(t)) \end{bmatrix}. \tag{67}$$

The scalar functions $f_i, i = 1, 2, \dots, 5$, are defined from the right-hand sides of system (11), i.e., $f_1(t, \mathbf{u}(t)) = \Lambda - \beta IS - \mu_1 S - dS$, $f_2(t, \mathbf{u}(t)) = \beta IS + \alpha_1 T - dI - k_1 I - k_2 I$, $f_3(t, \mathbf{u}(t)) = k_1 I - (\delta_1 + d)A + \alpha_2 T$, $f_4(t, \mathbf{u}(t)) = k_2 I - \alpha_1 T - (d + \delta_2 + \alpha_2)T$, and $f_5(t, \mathbf{u}(t)) = \mu_1 S - dR$. Applying the fractional integral in Eq. (55) to both sides of Eq. (66), we obtain

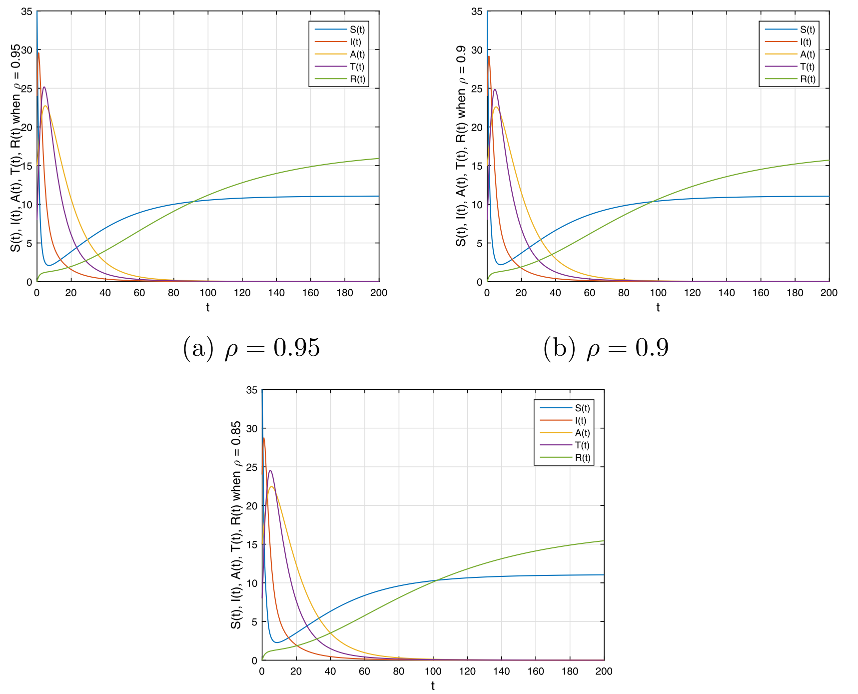
$$\begin{aligned} \mathbf{u}(t) - \mathbf{u}(0) &= {}^{CF}I_t^{\rho}\big(\mathbf{f}\big(t, \mathbf{u}(t)\big) \\ &= \frac{(1-\rho)}{M(\rho)} \mathbf{f}\big(t, \mathbf{u}(t)\big) + \frac{\rho}{M(\rho)} \int_0^t \mathbf{f}\big(s, \mathbf{u}(s)\big)\, ds. \end{aligned} \tag{68}$$

Applying the scheme in Eq. (62) to Eq. (68), we obtain the following iterative formula:

$$\begin{aligned} \mathbf{u}_{k+1} = \mathbf{u}_k &+ \frac{1}{M(\rho)}\left[(1-\rho) + \frac{23h\rho}{12}\right]\mathbf{f}(t_k, \mathbf{u}_k) \\ &- \frac{1}{M(\rho)}\left[(1-\rho) + \frac{4}{3}h\rho\right]\mathbf{f}(t_{k-1}, \mathbf{u}_{k-1}) + \frac{5h\rho}{12M(\rho)}\mathbf{f}(t_{k-2}, \mathbf{u}_{k-2}), \end{aligned} \tag{69}$$

where $\mathbf{u}_{k+1} = \mathbf{u}(t_{k+1})$, $\mathbf{u}_k = \mathbf{u}(t_k)$, $\mathbf{u}_{k-1} = \mathbf{u}(t_{k-1})$, $\mathbf{u}_{k-2} = \mathbf{u}(t_{k-2})$, and $\mathbf{u}_0 = \mathbf{u}(t_0) = [S(t_0), I(t_0), A(t_0), T(t_0), R(t_0)]^T$.

The following parameter values and initial conditions [1] have been used for our simulations: $\Lambda = 0.55$, $\beta = 0.03$, $d = 0.0196$, $k_1 = 0.15$, $k_2 = 0.35$, $\alpha_1 = 0.08$, $\alpha_2 = 0.03$, $\delta_1 = 0.0909$, $\delta_2 = 0.0667$, $\mu_1 = 0.03$, $S(0) = 35$, $I(0) = 24$, $A(0) = 15$, $T(0) = 8$, and $R(0) = 0$. Hence, we have $E^0 = (S^0, I^0, A^0, T^0, R^0) = (11.0887, 0, 0, 0, 16.9725)$ and $R_0 = 0.8825 < 1$. For the computational convenience, we set the fractional orders in system (11) as $\rho_1 = \rho_2 = \rho_3 = \rho_4 = \rho_5 = \rho$ and we choose $M(\rho) = 1$.



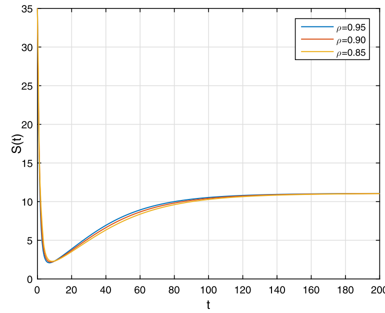
(a) $\rho = 0.95$ (b) $\rho = 0.9$

(c) $\rho = 0.85$

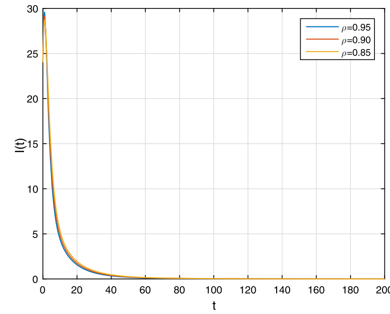
Figure 1 The time series plots for all state variables in model (11) when (**a**) $\rho = 0.95$, (**b**) $\rho = 0.9$, (**c**) $\rho = 0.85$

The numerical simulations, which are generated using scheme (69), are designed to show the behaviors for $\rho = 0.95, 0.9, 0.85$. Using the above parameter values and the selected fractional orders, the roots of the characteristic equation (51) depending on the fractional orders can be solved numerically as follows. For $\rho = 0.95$, the roots of Eq. (51) are $s_1 = -0.4700, s_2 = -0.1860, s_3 = -0.2298, s_4 = -0.1043, s_5 = -0.3350$. For $\rho = 0.9$, the roots of Eq. (51) are $s_1 = -0.4441, s_2 = -0.1760, s_3 = -0.2174, s_4 = -0.9836, s_5 = -0.3119$. The roots of Eq. (51) for $\rho = 0.85$ are $s_1 = -0.4184, s_2 = -0.1661, s_3 = -0.2051, s_4 = -0.9239, s_5 = -0.2895$. Hence, the equilibrium point E_0 of model (11) is asymptotically stable for $\rho = 0.95, 0.9, 0.85$.

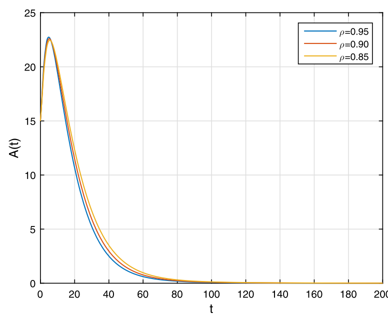
Figures 1(a)–(c) show time series plots for all state variables, $S(t), I(t), A(t), T(t)$, and $R(t)$, in model (11) for the fractional orders $\rho = 0.95, 0.9, 0.85$, respectively. We can observe from these plots that the curves of each state variable have the same trend when ρ is changed. However, their values are slightly different. From Fig. 2, each state solution is plotted with respect to $\rho = 0.95, 0.9, 0.85$. We can observe from Fig. 2(a) that the curves of $S(t)$ are increasing after a certain time and they finally converge to the equilibrium point $S^0 = 11.0887$. Figure 2(b) shows that all graphs of $I(t)$ decrease with time and tend to the equilibrium point $I^0 = 0$. Similar behavior occurs for the curves of $A(t)$ and $T(t)$ which are plotted in Fig. 2(c) and (d), respectively. However, the graphs of $R(t)$ are increasing in time and will tend to the equilibrium point $R^0 = 16.9725$ at longer times. The effect of changing ρ on each state variable can be observed more clearly in Fig. 2 because the trends of the curves are different for different periods of time. In summary, we notice from Fig. 2 that each state solution converges faster to its equilibrium point as ρ is increased. Figures 1–2 show that the obtained numerical solutions are converging to the disease-free equilibrium



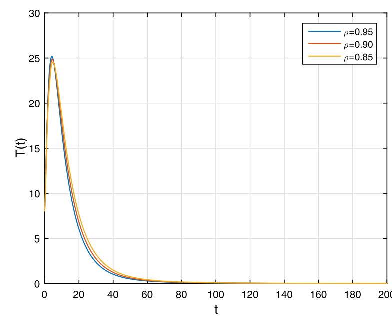
(a) t v.s. $S(t)$



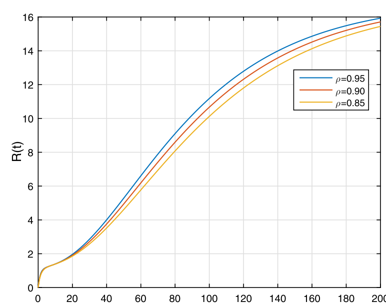
(b) t v.s. $I(t)$



(c) t v.s. $A(t)$



(d) t v.s. $T(t)$



(d) t v.s. $R(t)$

Figure 2 The time series plots for each state variable in model (11) when compared for $\rho = 0.95, 0.9, 0.85$ (**a**) t v.s. $S(t)$, (**b**) t v.s. $I(t)$, (**c**) t v.s. $A(t)$, (**d**) t v.s. $T(t)$, (**e**) t v.s. $R(t)$

point $E^0 = (11.0887, 0, 0, 0, 16.9725)$. This means that the number of HIV-positive individuals who are infectious, the number of individuals with full-blown AIDS, and the total number of individuals being treated with ARV are tending to zero as $t \to \infty$. In other words, HIV-infectious and full-blown AIDS people eventually disappear from the system and there are no HIV/AIDS patients who need the treatment.

7 Conclusions

In this article, a Caputo–Fabrizio fractional differential equation model for HIV/AIDS with an antiretroviral treatment compartment has been investigated. This fractional model is based on the use of the non-singular exponentially decreasing kernels appearing in the Caputo–Fabrizio fractional derivative. Using fixed point theory and an iterative method, the existence and uniqueness of the system of solutions for the model have been demonstrated. We have determined the equilibrium points of the model and the conditions for local asymptotic stability of the disease-free equilibrium point. A three-step fractional Adams–Bashforth scheme has been derived and used to obtain numerical solutions of the fractional system. Finally, we have compared the numerical simulations with respect to different values of the fractional order ρ. The paper gives an example of the use of the Caputo–Fabrizio fractional derivative as a model for real world problems that include history, memory, or nonlocal effects.

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The authors declare that they have no competing interests.

Authors' contributions
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