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Reduction and normal forms for a delayed reaction–diffusion differential system with B–T singularity

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Abstract
In this paper, we expanded our computation to obtain a simpler and detailed reduction and normal form of a delayed reaction–diffusion differential system with Bogdanov–Takens (B–T) singularity. By using the central manifold reduction method, we try to reduce the dimension of phase space without changing the dynamic behavior of the system. Next, by normal form theory, we try to simplify the form of differential equations, and then succeeded in obtaining a simpler and more specific parameterized delayed ordinary differential system on its center manifold. Finally, two examples show that the given algorithm is effective.

Keywords: Ordinary differential equations; Complex dynamical system; B–T singularity; Delayed reaction–diffusion differential system; Central manifold reduction

1 Introduction

Common bifurcations include the Hopf bifurcation and the B–T bifurcation, a B–T bifurcation is a well-studied example of a bifurcation with co-dimension two, meaning that two parameters must be varied for the bifurcation to occur. It is named after Bogdanov and Takens, who independently and simultaneously described this bifurcation [1–3]. At the Bogdanov–Takens bifurcation, for the system there may appear a saddle node bifurcation, Hopf bifurcation or homoclinic bifurcation, and the B–T bifurcation can further provide more information about periodic behavior and global dynamic behavior. We discuss the bifurcation phenomenon, which is to find the universal unfolding of the system, however, due to the diversity of disturbances, finding a universal unfolding is not easy. Over the past 40 years, great progress has been made in the bifurcation analysis of functional differential equations [4–10]. Most of the current bifurcation theory studies are focused on ordinary differential systems or delayed differential systems. In real nature, many phenomena can be more realistic if they are described by partial functional differential equations, so people pay more and more attention to the application of partial functional differential equations. There are two difficulties in bifurcation analysis for reaction–diffusion systems with time delay. If the system contains both time delay and diffusion, it will become an infinite dimensional dynamic system. The characteristic equation of the linearized equation at a certain equilibrium state is a transcendental algebraic equation, and it is difficult to calculate its characteristic root. On the other hand, it is difficult to analyze the eigen-

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values of infinite dimensional operators, especially in the analysis of the existence of the bifurcation and the stability of its periodic solution.

As is well known, the dimension of an ordinary differential system with B–T singularity is at least 2, but this is not true for delayed differential system and reaction–diffusion system. Faria and Magalhães showed that B–T singularity may happen in scalar delayed differential system [4]. Furthermore, in some special cases, reaction–diffusion system can undergo B–T bifurcation only needing one parameter to take its critical value, which suggests that reaction–diffusion system can display more complex dynamical behavior [11]. Xu and Huang gave a necessary and sufficient condition to characterize the B–T singularity in the first place. Meanwhile, Xu and Huang described the bifurcation behavior of system with B–T singularity in detail [12]. In our previous paper [13], we have studied a class of delayed reaction–diffusion systems,

$$\begin{cases} u_t = D\triangle u(x,t) + M(\alpha)u(x,t) + N(\alpha)u(x,t-1) \\ \qquad + f(u(x,t), u(x,t-1), \alpha), & t \geq 0, x \in (0,\pi), \\ \frac{\partial u}{\partial \nu} = 0, & x = 0, \pi, \end{cases} \tag{1}$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ is the bifurcation parameter, $D = \mathrm{diag}(d_1, d_2, \dots, d_n)$, $x \in \Omega = (0,\pi)$, $u(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}^n$. $\triangle$ is the Laplacian operator in $\mathbb{R}$, the homogeneous Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ shows that there is no movement on the boundary.

We have already analyzed the generalized eigenvector associated with zero eigenvalue, an equivalent condition for the determination of B–T singularity is obtained. Next, by using center manifold theory and normal form method, we had a two-dimension ordinary differential system on its center manifold. In this paper, we will expand our computation to obtain a simpler and detailed reduction and normal form of system (1). By using the central manifold reduction method and normal form theory, we try to reduce the dimension of phase space without changing the dynamic behavior of the system, and succeeded in obtaining a simpler and more specific parameterized delayed ordinary differential system on its center manifold. The contribution of this paper and its difference with [13] is that, first, compared with Theorem 2 in [13], Lemma 1 have defined the basis of, $\tilde{P}$ and its conjugate space $\tilde{P}^*$, and determined six conditions for calculating parameters ϕ_1^0, ψ_2^0, coefficients of ϕ_2^0, ψ_1^0 and coefficients of ϕ_1^0, ψ_2^0, we also completed the detailed proof of Lemma 1. Second, we have calculated in detail one basis of $V_2^4(\mathbb{R}^2)$, their images under M_2', one basis of $\mathrm{Im}(M_2^1)^c$, these are the most critical results in the derivation Theorem 1. Third, by comparing with [13], we find that the delayed reaction–diffusion differential system and the delayed differential system have the same normal form of B–T bifurcation, except at $\Phi(\theta)$ and $\Psi(s)$. For the reduced system (26), its local bifurcation behavior is determined by linear and second-order terms, rather than by higher order terms. In (26), we ignore the terms higher than the second-order terms, and give a brief list of the results for sufficiently small μ_1, μ_2 and $\varepsilon = 1$. Thus, Theorem 2 proposed the phase diagram of the system (27) and the boundary lines on parameter plane (μ_1, μ_2). This paper is organized as follows: in Sect. 2, basic assumptions and preliminaries for this paper; by using center manifold Theorem and normal form method, the precise statements and proofs of our main results are shown in Sect. 3; finally, two examples also provided in Sect. 4.

2 Preliminaries

The first of our basic assumptions on system (1) is:

(H1) $M(\alpha)$, $N(\alpha)$ are C^r ($r \geq 2$) smooth matrix-valued functions from $\mathbb{R}^2$ to $\mathbb{R}^{n\times n}$, $f(x,y,\alpha)$ is a C^r ($r \geq 2$) smooth function from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^2$ to $\mathbb{R}^n$, and

$$f(0,0,\alpha) = 0, \qquad \frac{\partial f}{\partial x}(0,0,\alpha) = 0, \qquad \frac{\partial f}{\partial y}(0,0,\alpha) = 0, \quad \forall \alpha \in \mathbb{R}^2, \tag{2}$$

$$\frac{d}{d\alpha} f(0,0,\alpha) = 0, \qquad \frac{d}{d\alpha}\frac{\partial f}{\partial x}(0,0,\alpha) = 0, \qquad \frac{d}{d\alpha}\frac{\partial f}{\partial y}(0,0,\alpha) = 0, \quad \forall \alpha \in \mathbb{R}^2. \tag{3}$$

Let $M = M(0)$, $N = N(0)$, then (1) becomes

$$\begin{aligned} u_t &= D\Delta u + Mu(x,t) + N(x,t-1) + \big(M(\alpha) - M\big)u(x,t) \\ &\quad + \big(N(\alpha) - N\big)u(x,t-1) + f\big(u(x,t), u(x,t-1), \alpha\big). \end{aligned} \tag{4}$$

$C_n = C([-1,0], \mathbb{R}^n)$ is used to represent the space of the continuous mapping from $[-1,0]$ to $\mathbb{R}^n$. Let

$$\eta_\alpha(\theta) = \begin{cases} M(\alpha) + N(\alpha), & \theta = 0, \\ N(\alpha), & -1 < \theta < 0, \\ 0, & \theta = -1. \end{cases}$$

Here $\eta_\alpha(\theta)$ is a matrix-valued function with bounded variation on $[-1,0]$. Note that

$$M(\alpha)u(x,t) + N(\alpha)u(x,t-1) = \int_{-1}^{0} d\eta_\alpha(\theta)u(x,t+\theta).$$

Let $V_1(t) = u_1(\cdot,t), V_2(t) = u_2(\cdot,t), \ldots, V_n(t) = u_n(\cdot,t), V = (V_1, V_2, \ldots, V_n)^T$, then

$$L(\alpha)V_t = \int_{-1}^{0} d\eta_\alpha(\theta)u_t(\theta)$$

can be regarded as a bounded linear operator from C_n to $\mathbb{R}^n$, where $V_t(\theta) = V(t+\theta)$. If $\alpha = 0$, then we have

$$L(0)V_t = \int_{-1}^{0} d\eta_\alpha(\theta)u(t+\theta) = MV(t) + NV(t-1) \triangleq L_0 V_t.$$

According to the definition of L_0, we can get $L_0(\zeta) = (M+N)\zeta$, $L_0(\theta\zeta) = -N\zeta$, $L_0(\theta^2\zeta) = N\zeta$, $\forall \zeta \in \mathbb{R}^n$, and $L_0(e^{\lambda\theta}\zeta) = (M + Ne^{-\lambda})\zeta$, $\forall \zeta \in \mathbb{R}^n$. System (1) becomes

$$\dot{V}(t) = D\Delta V(t) + L(\alpha)V_t + f(V_t, \alpha), \tag{5}$$

(4) becomes

$$\dot{V}(t) = D\Delta V(t) + L_0 V_t + \big[L(\alpha)V_t - L_0 V_t + Ff(V_t, \alpha)\big]. \tag{6}$$

Linearize (6) at $(V_t,\alpha)=(0,0)$, then [14, 15]

$$\dot{V}(t)=D\Delta V(t)+L_0V_t. \tag{7}$$

The solution of (7) defines a C_0 semigroup $\{T_0(t):t\geq 0\}$ on C_n, its infinitesimal generator $\mathcal{A}_0:C_n\to C_n$ can be defined as

$$\begin{aligned}&\mathcal{A}_0\phi=\dot{\phi},\\&\mathcal{D}(\mathcal{A}_0)=\bigl\{\phi\in C_n:\dot{\phi}\in C_n,\phi(0)=\mathcal{D}(\Delta),\dot{\phi}(0)=D\Delta\phi(0)+L_0\phi\bigr\},\end{aligned} \tag{8}$$

(7) is equivalent to

$$\dot{V}=\mathcal{A}_0V.$$

We know that the spectrum of $\mathcal{A}_0$ is only a point spectrum, that is, $\sigma(\mathcal{A}_0)=\sigma_p(\mathcal{A}_0)$, and

$$\lambda\in\sigma_p(\mathcal{A}_0)\quad\Leftrightarrow\quad\exists y\in\operatorname{dom}(\Delta)\backslash\{0\},\text{s.t. }\lambda y-D\Delta y-L_0\bigl(e^{\lambda\cdot}y\bigr)=0. \tag{9}$$

Under the Neumann boundary condition, the characteristic root of Δ is $-k^2$ and the characteristic function is $\gamma_k=\cos(kx)$, $k=0,1,2,\ldots$. Let

$$\eta_\alpha^k(\theta)=\begin{cases}-Dk^2+M(\alpha)+N(\alpha), & \theta=0,\\ N(\alpha), & -1<\theta<0,\\ 0, & \theta=-1.\end{cases}$$

Then $\eta_\alpha^k(\theta)$ is actually a bounded variation matrix-valued function on $[-1,0]$, and

$$-Dk^2\phi(0)+L(\alpha)(\phi)=\int_{-1}^{0}d\bigl[\eta_\alpha^k(\theta)\bigr]\phi(\theta),\quad \phi\in C\bigl([-1,0],\mathbb{R}^n\bigr).$$

The further hypotheses of system (1) are:

(H2) If $\lambda\in\sigma_p(\mathcal{A}_0)\backslash\{0\}$, then $\operatorname{Re}\lambda\neq 0$;

(H3) $\lambda=0$ is an eigenvalue of $\mathcal{A}_0$ with algebraic multiplicity 2 and geometric multiplicity 1.

If conditions (H1)–(H3) can be satisfied, we say that system (1) has a B–T singularity, $(V,\alpha)=(0,0)$ is the B–T point. Theorem 1 in [13] gave an equivalent description for B–T singularity in (1), which can be used as a feasible algorithm for determining the B–T singularity.

3 Reduction and normal forms for system (1)

In this section, we will continue to explore the reduction and normal forms for system (1) with B–T singularity, based on the theory in [4, 16], we find that (1) can be reduced to a simple two-dimensional ordinary differential system on its central manifold. By (6), we can transform the system (1) with parameters into the following system without parameters:

$$\begin{cases}\dot{V}(t)=D\Delta V(t)+L_0V_t+[L(\alpha)V_t-L_0V_t+f(V_t,\alpha)],\\ \dot{\alpha}(t)=0.\end{cases} \tag{10}$$

$\tilde{C}_{n+2}:=C([-1,0],\mathbb{R}^n\times\mathbb{R}^2)$ denotes its phase space.

Let $\tilde{V}(t) = (V(t), \alpha(t)) \in \mathbb{R}^n \times \mathbb{R}^2$ be the solution of (9), then (9) becomes

$$\dot{\tilde{V}}(t) = \tilde{D}\triangle\tilde{V}(t) + \tilde{L}_0\tilde{V}_t + \tilde{f}(\tilde{V}_t), \tag{11}$$

where $\tilde{D} = \operatorname{diag}(D, 0_2)$, $\tilde{L}_0\tilde{V}_t = (L_0V_t, 0)$ is a bounded linear operator from $\tilde{C}_{n+2}$ to $\mathbb{R}^n \times \mathbb{R}^2$. $\tilde{f}(\tilde{V}_t) = [L(\alpha(0)) - L_0]V_t + f(V_t, \alpha(0), 0) := (\hat{f}(V_t, \alpha), 0)$, where $V_t \in C_n$, $\alpha \in C_2 := C([-1, 0], \mathbb{R}^2)$.

Consider the linearized system of (11) at $\tilde{V}_t = 0$

$$\dot{\tilde{V}}(t) = \tilde{D}\triangle\tilde{V}(t) + \tilde{L}_0\tilde{V}_t, \tag{12}$$

using $\tilde{\mathcal{A}}_0$ to represent the infinitesimal generator of C_0-semigroup associated with (12), then $\tilde{\mathcal{A}}_0 = (\mathcal{A}_0, 0)$. The characteristic roots of $\tilde{\mathcal{A}}_0$ include not only all the characteristic roots of $\tilde{\mathcal{A}}_0$, but also two zero eigenvalues when $\dot{\alpha} = 0$. Let $\tilde{\Gamma}$ be the set of all zero eigenvalues (multiplicity computation). Now we consider the decomposition of the phase space C_n of (6). Let $C_n = P \oplus Q$, where P is the invariant space of $\mathcal{A}_0$ associated with zero eigenvalues, Q is the complementary space of P. $C_n^* = C([0, 1], \mathbb{R}^{n*})$ denotes the conjugate space of C_n, where $\mathbb{R}^{n*}$ is a n-dimensional row vector space. The conjugate inner product on $C_n^* \times C_n$ is defined as

$$(\psi, \phi) = \psi(0)\phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi - \theta)\, d\big[\eta_0^{k_0}(\theta)\big]\phi(\xi)\, d\xi. \tag{13}$$

Let $\Phi(\theta) = (\phi_1(\theta), \phi_2(\theta))$, $(-1 \le \theta \le 0)$ and $\Psi(s) = \operatorname{col}(\psi_1(s), \psi_2(s))$, $(0 \le s \le 1)$ denote the basis of P and its conjugate space P^*, respectively, and satisfy $(\Psi, \Phi) = I_2$, where $(\Psi, \Phi) := (\psi_j, \phi_i)$, $i, j = 1, 2$. Since $Q = \{\phi \in C_n \mid (\Psi, \phi) = 0\}$, we can obtain the following lemma.

Lemma 1 *The basis of P and its conjugate space P^* are as follows*:

$$\begin{aligned} &P = \operatorname{span}\Phi, \qquad \Phi(\theta) = \big(\phi_1(\theta), \phi_2(\theta)\big), \quad -1 \le \theta \le 0, \\ &P^* = \operatorname{span}\Psi, \qquad \Psi(s) = \operatorname{col}\big(\psi_1(s), \psi_2(s)\big), \quad 0 \le s \le 1, \end{aligned} \tag{14}$$

where $\phi_1(\theta) = \phi_1^0 \in \mathbb{R}^n \setminus \{0\}$, $\phi_2(\theta) = \phi_2^0 + \phi_1^0\theta$, $\phi_2^0 \in \mathbb{R}^n$, $\psi_2(s) = \psi_2^0 \in \mathbb{R}^{n*} \setminus \{0\}$, $\psi_1(s) = \psi_1^0 - s\psi_2^0$, $\psi_1^0 \in \mathbb{R}^{n*}$, *and there is* $k_0 \in 0, 1, 2, \ldots$ *satisfy*

$$\begin{aligned} &\text{(i)}\quad \big(-Dk_0^2 + M + N\big)\phi_1^0 = 0, \qquad \text{(ii)}\quad \big(-Dk_0^2 + M + N\big)\phi_2^0 = (N + I)\phi_1^0, \\ &\text{(iii)}\quad \psi_2^0\big(-Dk_0^2 + M + N\big) = 0, \qquad \text{(iv)}\quad \psi_1^0\big(-Dk_0^2 + M + N\big) = \psi_2^0(N + I), \\ &\text{(v)}\quad \psi_2^0\phi_2^0 - \frac{1}{2}\psi_2^0N\phi_1^0 + \psi_2^0N\phi_2^0 = 1, \\ &\text{(vi)}\quad \psi_1^0\phi_2^0 - \frac{1}{2}\psi_1^0N\phi_1^0 + \psi_1^0N\phi_2^0 + \frac{1}{6}\psi_2^0N\phi_1^0 - \frac{1}{2}\psi_2^0N\phi_2^0 = 0. \end{aligned} \tag{15}$$

In the case of one constant difference, ϕ_1^0 and ψ_2^0 can be uniquely determined by (i) *and* (iii), *respectively. Then the coefficients of ϕ_2^0 and ψ_1^0, coefficients of ϕ_1^0 and ψ_2^0 are determined by* (ii) *and* (iv), (v) *and* (vi), *respectively.*

Proof By the proof of Theorem 1 in [13], we know that $\phi_1(\theta) = \phi_1^0 \in \mathbb{R}^n \setminus \{0\}$, $\phi_2(\theta) = \phi_2^0 + \phi_1^0\theta$, $\phi_2^0 \in \mathbb{R}^n$, and conditions (i), (ii) in (15) hold. Next, the conjugate operator $\mathcal{A}_0^* : C_n^* \to C_n^*$ of $\mathcal{A}_0$ is written as

$$\begin{aligned} &A_0^*\psi = -\dot{\psi}, \\ &\mathcal{D}(\mathcal{A}_0^*) = \bigl\{\psi \in C^1\bigl([0,r],\mathbb{R}^{n*}\bigr) : \psi(0) \in \mathcal{D}(\Delta), -\dot{\psi}(0) = D\Delta\psi(0) + L_0^*\psi\bigr\}, \end{aligned} \tag{16}$$

where $L_0^* : C_n^* \to \mathbb{R}^{n*}$ is the formally adjoint operator of L_0; we let $\eta_0^*(\theta)$ denote the adjoint of $\eta_0(\theta)$, then

$$L_0^*(\psi) = \int_{-1}^0 \psi(-\theta)\,d\bigl[\eta_0^*(\theta)\bigr].$$

Notice that $\mathcal{A}_0^*\psi_2 = 0$ is equivalent to

$$\begin{cases} -\dot{\psi}_2(s) = 0, & 0 < s \le 1, \\ D\Delta\psi_2(0) + \int_{-1}^0 \psi_2(-\theta)\,d[\eta_0^*(\theta)] = 0, & s = 0. \end{cases} \tag{17}$$

Equation (17) holds if and only if

$$\psi_2(s) = \psi_2^0 \in \mathbb{R}^{n*}\backslash\{0\}.$$

There exists $k_0 \in 0, 1, 2 \cdots$ which satisfies

$$-Dk_0^2\psi_2^0 + \psi_2^0(M+N) = \psi_2^0\bigl(-Dk_0^2 + M + N\bigr) = 0. \tag{18}$$

Notice that $\mathcal{A}_0^*\psi_1 = \psi_2$ is equivalent to

$$\begin{cases} -\dot{\psi}_1(s) = \psi_2^0, & 0 < s \le 1, \\ D\Delta\psi_1(0) + \int_{-1}^0 \psi_1(-\theta)\,d[\eta_0^*(\theta)] = \psi_2^0, & s = 0. \end{cases} \tag{19}$$

So we have $\psi_1(s) = \psi_1^0 - s\psi_2^0$, $\psi_1^0 \in \mathbb{R}^{n*}$, there exists $k_0 \in 0, 1, 2, \ldots$ which satisfies

$$-Dk_0^2\psi_1^0 + \psi_1^0 M + \bigl(\psi_1^0 - \psi_2^0\bigr)N = \psi_2^0,$$

that is,

$$\psi_1^0\bigl(-Dk_0^2 + M + N\bigr) = \psi_2^0(N+I). \tag{20}$$

So (iii) and (iv) in (15) hold. Finally, by the definition of $\Phi(\theta)$ and $\Psi(s)$, we have

$$\begin{aligned} &(\psi_1,\phi_1) = \psi_1^0\phi_1^0 - \frac{1}{2}\psi_2^0 N\phi_1^0 - \psi_1^0 N\phi_1^0 = 1, \\ &(\psi_2,\phi_2) = \psi_2^0\phi_2^0 - \frac{1}{2}\psi_2^0 N\phi_1^0 + \psi_2^0 N\phi_2^0 = 1, \\ &(\psi_1,\phi_2) = \psi_1^0\phi_2^0 - \frac{1}{2}\psi_1^0 N\phi_1^0 + \psi_1^0 N\phi_2^0 + \frac{1}{6}\psi_2^0 N\phi_1^0 - \frac{1}{2}\psi_2^0 N\phi_2^0 = 0, \\ &(\psi_2,\phi_1) = \psi_2^0\phi_1^0 + \psi_2^0 N\phi_1^0 = 0. \end{aligned} \tag{21}$$

In fact, by (i) and (ii) in (15), we know that the fourth equation in (21) hold. The first equation in (21) is equivalent to the second equation, so we can choose an appropriate coefficient for ϕ_1^0, ψ_2^0 so that all equations of (21) hold, thus, we have completed the proof of Lemma 1. □

It is easy to see that $\Phi(\theta)$ satisfy $\dot{\Phi} = \Phi J$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

Now we consider the decomposition $\tilde{C}_{n+2} = \tilde{P} \oplus \tilde{Q}$, where $\tilde{P} = P \times \mathbb{R}^2$ is the invariant space of $\tilde{\mathcal{A}}_0$ associated with $\tilde{\Gamma}$, $\tilde{Q} = Q \times R$, $R = \{v \in C_2 \mid v(0) = 0\}$. The basis of $\tilde{P}$ and its conjugate space $\tilde{P}^*$ are composed of column vectors of $\tilde{\Phi} = \left(\begin{smallmatrix} \Phi & 0 \\ 0 & I_2 \end{smallmatrix}\right)$ and row vectors of $\tilde{\Psi} = \left(\begin{smallmatrix} \Psi & 0 \\ 0 & I_2 \end{smallmatrix}\right)$, and satisfies $(\tilde{\Psi}, \tilde{\Phi}) = I_4$, $\dot{\tilde{\Phi}} = \tilde{\Phi}\tilde{J}$, where $\tilde{J} = \mathrm{diag}(J, 0_2)$.

The central manifold of the delayed reaction–diffusion delayed differential system near the origin can be expressed as $(y(x,\alpha), w(x,\alpha)) : \mathbb{R}^2 \times \mathbb{R}^2 \to \tilde{Q} = Q \times W$, where $y(x,\alpha)$ and $w(x,\alpha)$ satisfy $y(0,0) = w(0,0) = 0$, $Dy(0,0) = Dw(0,0) = 0$, respectively. Based on the theory of [4, 16], for the fixed α, considering the normal form of (5), we can define the expanded phase space of C_n as follows:

$$
\begin{aligned}
BC_n = \bigl\{\phi \mid \phi : [-1,0] \to \mathbb{R}^n,& \\
&\phi \text{ is uniformly continuous on } [-1,0) \text{ and may not be continuous at } 0\bigr\}.
\end{aligned}
$$

BC_n is isomorphic to $C_n \times \mathbb{R}^n$.

Similarly, considering the normal form of (9), we extend $\tilde{C}_{n+2}$ to $B\tilde{C}_{n+2} = BC_n \times BC_2$, and we see that $B\tilde{C}_{n+2}$ is isomorphic to $\tilde{C}_{n+2} \times \mathbb{R}^{n+2}$. Let X_0 and Y_0 denote the matrix-valued functions on $(-1,0]$, where

$$
X_0(\theta) = \begin{cases} 0, & -1 \le \theta < 0, \\ I_n, & \theta = 0, \end{cases} \qquad Y_0(\theta) = \begin{cases} 0, & -1 \le \theta < 0, \\ I_2, & \theta = 0. \end{cases}
$$

Define

$$
\pi : BC_n \to P, \qquad \pi(\phi + X_0\xi) = \Phi\bigl[(\Psi, \phi) + \Psi(0)\xi\bigr],
$$

where $\phi \in C_n$, $\xi \in \mathbb{R}^n$. Define

$$
\begin{aligned}
&\tilde{\pi} : B\tilde{C}_{n+2} \to \tilde{P}, \\
&\tilde{\pi}(\phi + X_0\xi, \psi + Y_0\mu) = \tilde{\Phi}\left\{\left(\tilde{\Psi}, \begin{pmatrix} \phi \\ \psi \end{pmatrix}\right) + \tilde{\Psi}(0)\begin{pmatrix} \xi \\ \mu \end{pmatrix}\right\} = \bigl(\pi(\phi + X_0\xi), \psi(0) + \mu\bigr),
\end{aligned}
$$

where $\phi \in C_n$, $\psi \in C_2$, $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}^2$. Since $B\tilde{C}_{n+2} = \tilde{P} \oplus \mathrm{Ker}\,\tilde{\pi}$, we can decompose

$$
\begin{pmatrix} V_t \\ \alpha_t \end{pmatrix} = \begin{pmatrix} \Phi & 0 \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} x(t) \\ \alpha(t) \end{pmatrix} + \begin{pmatrix} y(t) \\ w(t) \end{pmatrix},
$$

where $(x(t),\alpha(t))\in\mathbb{R}^{n+2}$, $(y,w)\in\operatorname{Ker}\tilde{\pi}$, then

$$\begin{aligned}
&\begin{pmatrix}\dot{x}\\ \dot{\alpha}\end{pmatrix}=\tilde{J}\begin{pmatrix}x\\ \alpha\end{pmatrix}+\tilde{\Psi}(0)\hat{f}\left(\tilde{\Phi}\begin{pmatrix}x\\ \alpha\end{pmatrix}+\begin{pmatrix}y\\ w\end{pmatrix}\right),\\
&\frac{d}{dt}\begin{pmatrix}y\\ w\end{pmatrix}=\tilde{\mathcal{A}}_{\tilde{Q}^1}\begin{pmatrix}y\\ w\end{pmatrix}+(I-\tilde{\pi})[X_0,Y_0]\hat{f}\left(\tilde{\Phi}\begin{pmatrix}x\\ \alpha\end{pmatrix}+\begin{pmatrix}y\\ w\end{pmatrix}\right),
\end{aligned}\tag{22}$$

where $x\in\mathbb{R}^2$, $\alpha\in\mathbb{R}^2$, $y\in Q^1:=Q\cap C_n^1$, $w\in R^1:=R\cap C_2^1$. $\tilde{\mathcal{A}}_{\tilde{Q}^1}$ is an operator from $\tilde{Q}^1:=\tilde{Q}\cap\tilde{C}_{N+2}^1=Q^1\times R^1$ to $\operatorname{Ker}\tilde{\pi}$, which is defined by

$$\tilde{\mathcal{A}}_{\tilde{Q}^1}\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}\dot{\phi}\\ \dot{\psi}\end{pmatrix}+[X_0,Y_0]\left\{\tilde{D}\begin{pmatrix}\Delta\phi\\ \Delta\psi\end{pmatrix}+\tilde{L}_0\begin{pmatrix}\phi\\ \psi\end{pmatrix}-\begin{pmatrix}\dot{\phi}(0)\\ \dot{\psi}(0)\end{pmatrix}\right\}.$$

The Taylor expansion of $\hat{f}(V_t,\alpha)$ with respect to V_t and α is

$$\hat{f}(V_t,\alpha)=\sum_{j\geq 2}\frac{1}{j!}\hat{f}_j(V_t,\alpha).\tag{23}$$

The first item ($j=2$) of (23) can be expressed in the form

$$\begin{aligned}
\frac{1}{2}\hat{f}_2(V_t,\alpha)={}&M_1\alpha_1V(t)+M_2\alpha_2V(t)+N_1\alpha_1V(t-1)+N_2\alpha_2V(t-1)\\
&+\sum_{i=1}^{n}E_iV_i(t)V(t-1)+\sum_{i=1}^{n}F_iV_i(t)V(t)\\
&+\sum_{i=1}^{n}G_iV_i(t-1)V(t-1),
\end{aligned}$$

where $M_i=(M(\alpha)-M)|_{\alpha_i}$, $N_i=(N(\alpha)-N)|_{\alpha_i}$ $(i=1,2)$, E_i, F_i, G_i $(i=1,2,\ldots,n)$ is the coefficient matrices, and there are no terms of $o(\alpha^2)$ in $\hat{f}_2(V_t,\alpha)=0$ since $\forall\alpha\in\mathbb{R}^2,\hat{f}(0,\alpha)=0$. Define

$$f_j^1(x,y,\alpha)=\Psi(0)\hat{f}_j(\Phi x+y,\alpha),\qquad f_j^2(x,y,\alpha)=(I-\pi)X_0\hat{f}_j(\Phi x+y,\alpha).$$

Note that $V(0)=0$, On $BC_n=P\oplus\ker\pi$, (21) can be reduced to the following equation:

$$\begin{aligned}
&\dot{x}=Jx+\sum_{j\geq 2}\frac{1}{j!}f_j^1(x,y,\alpha),\\
&\frac{d}{dt}y=A_{Q^1}y+\sum_{j\geq 2}\frac{1}{j!}f_j^2(x,y,\alpha),
\end{aligned}\tag{24}$$

where $A_{Q^1}:Q^1\subset\ker\pi\to\ker\pi$ is defined by $A_{Q^1}\phi=\dot{\phi}+X_0[L_0\phi-\dot{\phi}(0)]$. $V_2^4(\mathbb{R}^2)$ is a linear space formed by quadratic homogeneous polynomial with respect to $(x,\alpha)=(x_1,x_2,\alpha_1,\alpha_2)$, that is,

$$V_2^4(\mathbb{R}^2)=\left\{\sum_{|(q,l)|=2}c_{(q,l)}x^q\alpha^l:(q,l)\in\mathbb{N}_0^4,c_{(q,l)}\in\mathbb{R}^2\right\}.$$

Define the operator M_2^1 on $V_2^4(\mathbb{R}^2)$

$$\bigl(M_2^1 p\bigr)(x,\alpha) = D_x p(x,\alpha)Jx - Jp(x,\alpha), \quad \forall p \in V_2^4\bigl(\mathbb{R}^2\bigr).$$

We can decompose $V_2^4(\mathbb{R}^2)$ into $\operatorname{Im}(M_2^1) \oplus \operatorname{Im}(M_2^1)^c$ and use $P_{I,2}^1$ to represent projection mapping from $V_2^4(\mathbb{R}^2)$ to $\operatorname{Im}(M_2^1)$.

According to the hypothesis (H2), we can prove that, for any $\mu \in \sigma(\tilde{\mathcal{A}}_0)\backslash\tilde{\Gamma}$ and $q \in \mathbb{N}_0^4$, $(q,\tilde{\lambda}) \neq \mu$ hold, where $\tilde{\lambda} = (0,0,0,0)$ is a vector consisting of elements in $\tilde{\Gamma}$ (calculated multiplicity). That is to say, (11) satisfies the nonresonant condition with respect to $\tilde{\Gamma}$. From (24), we know that the normal form of (11) on the central manifold can be written as

$$\dot{x} = Jx + \frac{1}{2}g_2^1(x,0,\alpha) + h.o.t., \tag{25}$$

where $g_2^1(x,0,\alpha) = (I - P_{I,2}^1)f_2^1(x,0,\alpha) = \operatorname{proj}_{\operatorname{Im}(M_2^1)^c} f_2^1(x,0,\alpha)$. Now we can select one basis of $V_2^4(\mathbb{R}^2)$ as follows:

$$\begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha_1^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha_2^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_1\alpha_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_1\alpha_2 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} x_2\alpha_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_2\alpha_2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha_1\alpha_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \alpha_1^2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \alpha_2^2 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_1\alpha_1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_1\alpha_2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_2\alpha_1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_2\alpha_2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \alpha_1\alpha_2 \end{pmatrix}.$$

Their images under M_2' are

$$\begin{pmatrix} 2x_1x_2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_2\alpha_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_2\alpha_2 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -x_1^2 \\ 2x_1x_2 \end{pmatrix}, \quad \begin{pmatrix} -x_2^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\alpha_1^2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\alpha_2^2 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} -x_1x_2 \\ x_2^2 \end{pmatrix}, \quad \begin{pmatrix} -x_1\alpha_1 \\ \alpha_1x_2 \end{pmatrix}, \quad \begin{pmatrix} -x_1\alpha_2 \\ \alpha_2x_2 \end{pmatrix}, \quad \begin{pmatrix} -x_2\alpha_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -x_2\alpha_2 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} -\alpha_1\alpha_2 \\ 0 \end{pmatrix}.$$

So we can select one basis of $\operatorname{Im}(M_2^1)^c$ as follows:

$$\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \alpha_1^2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \alpha_2^2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_1\alpha_1 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ x_1\alpha_2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_2\alpha_1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x_2\alpha_2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \alpha_1\alpha_2 \end{pmatrix}.$$

Using ϕ_{ji} to represent the ith element of ϕ_j, we obtain

$$\begin{aligned}
\frac{1}{2}\hat{f}_2(\Phi x,\alpha) &= M_1\alpha_1\left[\Phi(0)\begin{pmatrix}x_1\\x_2\end{pmatrix}\right] + M_2\alpha_2\left[\Phi(0)\begin{pmatrix}x_1\\x_2\end{pmatrix}\right]\\
&\quad + N_1\alpha_1\left[\Phi(-1)\begin{pmatrix}x_1\\x_2\end{pmatrix}\right] + N_2\alpha_2\left[\Phi(-1)\begin{pmatrix}x_1\\x_2\end{pmatrix}\right]\\
&\quad + \sum_{i=1}^{n} E_i\big[\phi_{1i}(0),\phi_{2i}(0)\big]\begin{pmatrix}x_1\\x_2\end{pmatrix}\left[\Phi(-1)\begin{pmatrix}x_1\\x_2\end{pmatrix}\right]\\
&\quad + \sum_{i=1}^{n} F_i\big[\phi_{1i}(0),\phi_{2i}(0)\big]\begin{pmatrix}x_1\\x_2\end{pmatrix}\left[\Phi(0)\begin{pmatrix}x_1\\x_2\end{pmatrix}\right]\\
&\quad + \sum_{i=1}^{n} G_i\big[\phi_{1i}(-1),\phi_{2i}(-1)\big]\begin{pmatrix}x_1\\x_2\end{pmatrix}\left[\Phi(-1)\begin{pmatrix}x_1\\x_2\end{pmatrix}\right]\\
&= \big[M_1\phi_1(0) + N_1\phi_1(-1)\big]\alpha_1 x_1 + \big[M_2\phi_1(0) + N_2\phi_1(-1)\big]\alpha_2 x_1\\
&\quad + \big[M_1\phi_2(0) + N_1\phi_1(-1)\big]\alpha_1 x_2 + \big[M_2\phi_2(0) + N_2\phi_2(-1)\big]\alpha_2 x_2\\
&\quad + \sum_{i=1}^{n}\big[E_i\phi_1(-1)\phi_{1i}(0) + F_i\phi_1(0)\phi_{1i}(0) + G_i\phi_1(-1)\phi_{1i}(-1)\big]x_1^2\\
&\quad + \sum_{i=1}^{n}\big\{E_i\big[\phi_2(-1)\phi_{1i}(0) + \phi_1(-1)\phi_{2i}(0)\big] + F_i\big[\phi_2(0)\phi_{1i}(0) + \phi_1(0)\phi_{2i}(0)\big]\\
&\quad + G_i\big[\phi_2(-1)\phi_{1i}(-1) + \phi_1(-1)\phi_{2i}(-1)\big]\big\}x_1x_2\\
&\quad + \sum_{i=1}^{n}\big[E_i\phi_2(-1)\phi_{2i}(0) + F_i\phi_2(0)\phi_{2i}(0) + G_i\phi_2(-1)\phi_{2i}(-1)\big]x_2^2.
\end{aligned}$$

Since $\phi_1(0) = \phi_1(-1) = \phi_1^0$, $\phi_2(0) = \phi_2^0$, $\phi_2(-1) = \phi_2^0 - \phi_1^0$, $\psi_1(0) = \psi_1^0$, $\psi_2(0) = \psi_2^0$, $f_2^1(x,0,\alpha) = \Psi(0)\hat{f}_2(\Phi x,\alpha)$, $g_2^1 = (I - P_{I,2}^1)f_2^1$ can be calculated from the above formulas and the basis of $\mathrm{Im}(M_2^1)^c$, so that we can get the following theorem.

Theorem 1 *Suppose that* (H1)–(H3) *hold, then the delayed reaction–diffusion differential system* (1) *can be reduced to the following two-dimensional ordinary differential system on the central manifold at* $(V_t,\alpha) = (0,0)$:

$$\begin{aligned}
&\dot{x}_1 = x_2,\\
&\dot{x}_2 = \rho_1 x_1 + \rho_1 x_2 + \eta_1 x_1^2 + \eta_2 x_1 x_2 + h.o.t.,
\end{aligned}\tag{26}$$

where

$$\begin{aligned}
\rho_1 &= \psi_2^0(M_1+N_1)\phi_1^0\alpha_1 + \psi_2^0(M_2+N_2)\phi_1^0\alpha_2,\\
\rho_2 &= \big[\psi_1^0(M_1+N_1)\phi_1^0 + \psi_2^0\big((M_1+N_1)\phi_2^0 - N_1\phi_1^0\big)\big]\alpha_1\\
&\quad + \big[\psi_1^0(M_2+N_2)\phi_1^0 + \psi_2^0\big((M_2+N_2)\phi_2^0 - N_2\phi_1^0\big)\big]\alpha_2,\\
\eta_1 &= \psi_2^0\sum_{i=1}^{n}(E_i+F_i+G_i)\phi_1^0\phi_{1i}^0,
\end{aligned}$$

$$\eta_2 = 2\psi_1^0 \sum_{i=1}^{n} (E_i + F_i + G_i)\phi_1^0\phi_{1i}^0 + \psi_2^0 \left[\sum_{i=1}^{n} (E_i + F_i + G_i)\left(\phi_2^0\phi_{1i}^0 + \phi_1^0\phi_{2i}^0\right) - \sum_{i=1}^{n} (E_i + 2G_i)\phi_1^0\phi_{1i}^0 \right].$$

By comparing with [13], we find that the delayed reaction–diffusion differential system and the delayed differential system have the same normal form of B–T bifurcation, except at $\Phi(\theta)$ and $\Psi(s)$. For the reduced system (26), its local bifurcation behavior is determined by linear and second-order terms, rather than by higher order terms [17]. In (26), we ignore the terms higher than the second-order terms, and let $\mu_1 = -\frac{\eta_2^4}{4\eta_1^4}\rho_1^2$, $\mu_2 = |\frac{\eta_2}{\eta_1}|(\rho_2 - \frac{\eta_2}{2\eta_1}\rho_1)$, $\varepsilon = \pm 1$, then (26) becomes

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \mu_1 + \mu_2 x_2 + x_1^2 + \varepsilon x_1 x_2. \end{cases} \tag{27}$$

The bifurcation diagram of system (26) can be found in many thesis, such as [17–19]. Here, we give a brief list of the results for sufficiently small μ_1, μ_2 and $\varepsilon = 1$.

Theorem 2 *The phase diagram of the system* (27) *is shown in Fig.* 1. *On the parameter plane* (μ_1, μ_2), *we have the following boundary lines*:

$$SN^+ = \{(\mu_1, \mu_2) \mid \mu_1 = 0, \mu_2 > 0\},$$
$$SN^- = \{(\mu_1, \mu_2) \mid \mu_1 = 0, \mu_2 < 0\},$$
$$H = \{(\mu_1, \mu_2) \mid \mu_1 = -\mu_2^2, \mu_2 > 0\},$$
$$HL = \left\{(\mu_1, \mu_2) \mid \mu_1 = -\frac{49}{25}\mu_2^2 + o\left(|\mu_2|^{\frac{5}{2}}\right), \mu_2 > 0\right\}.$$

These boundaries divide the plane into several regions. When (μ_1, μ_2) *is inside these regions, the phase diagram of the system* (27) *remains unchanged under small perturbations, and the system structure is stable. When* (μ_1, μ_2) *is on the boundaries, the system structure*

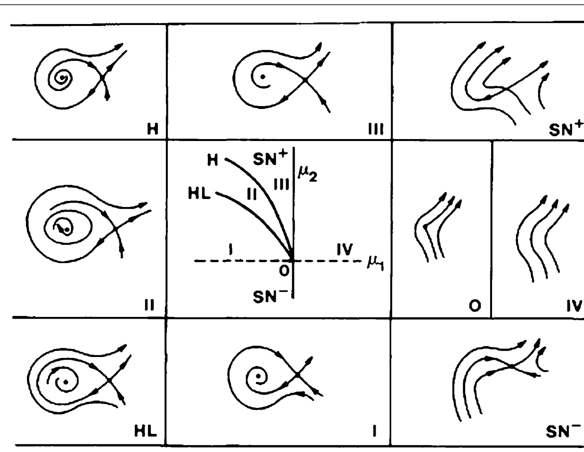

Figure 1 $\varepsilon = 1$ for the B–T bifurcation and phase diagram of system (27)

is unstable. Boundary SN^+ or SN^- corresponds to the saddle bifurcations in equilibrium, boundary H corresponds to the Hopf bifurcations, and boundary HL corresponds to homoclinic orbits bifurcations.

4 Two examples

Example 1 Consider the following two-dimensional delayed reaction–diffusion differential system:

$$\begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = D \begin{pmatrix} \Delta u_1(x,t) \\ \Delta u_2(x,t) \end{pmatrix} + M \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} + N \begin{pmatrix} u_1(x,t-1) \\ u_2(x,t-1) \end{pmatrix}, \tag{28}$$

where $D = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $M = \left(\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}\right)$, $N = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$.

First, in order to verify the condition (H2), we will show that the characteristic equation of system (28) has no pure imaginary roots, that is

$$\det\bigl(\lambda I_2 + Dk^2 I_2 - M - Ne^{-\lambda}\bigr) = 0, \quad k \in \{0, 1, 2, \ldots\},$$

has no pure imaginary roots. Note that $\det(\lambda I_2 - M - Ne^{-\lambda}) = 0 \Leftrightarrow \lambda + k^2 - 3 - e^{-\lambda} = 0$, or $\lambda + k^2 - 2 - e^{-\lambda} = 0$. Assuming that $i\omega$ is a pure imaginary root of the above equation, then

$$\begin{cases} \omega + \sin\omega = 0, \\ k^2 - 3 - \cos\omega = 0, \end{cases}$$

textitor

$$\begin{cases} k^2 - 2 + \cos\omega = 0, \\ \omega - \sin\omega = 0, \end{cases}$$

must hold. Clearly, for $\forall k \in \{0, 1, 2, \ldots\}$ and $\forall\omega \neq 0$, these two formulas are not tenable, thus (H2) is satisfied.

Now we will verify conditions (i)–(iii) of Theorem 1 in [13].

(1) For $k_0 = 1$, $(-Dk_0^2 + M + N) = \left(\begin{smallmatrix} 3 & 2 \\ 0 & 0 \end{smallmatrix}\right)$, that is, $\operatorname{rank}(-Dk_0^2 + M + N) = 1$, condition (i) holds.

(2) Let $\phi_1^0 = \left(\begin{smallmatrix} \frac{1}{3} \\ -\frac{1}{2} \end{smallmatrix}\right)$, then $\mathcal{N}(-Dk_0^2 + M + N) = \operatorname{span}\{\phi_1^0\}$. Let $\phi_2^0 = \left(\begin{smallmatrix} \frac{1}{18} \\ 0 \end{smallmatrix}\right)$, then $(N+I)\phi_1^0 = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \frac{1}{3} \\ \frac{1}{2} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{1}{6} \\ 0 \end{smallmatrix}\right)$, $(-Dk_0^2 + A + B)\phi_2^0 = \left(\begin{smallmatrix} 3 & 2 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \frac{1}{18} \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{1}{6} \\ 0 \end{smallmatrix}\right)$, that is, $(N+I)\phi_1^0 \in \mathcal{R}(-Dk_0^2 + M + N)$, condition (ii) holds.

(3) Since $(N+I)\phi_2^0 - \frac{1}{2}N\phi_1^0 = \left(\begin{smallmatrix} \frac{7}{36} \\ -\frac{1}{4} \end{smallmatrix}\right)$, $\operatorname{rank}((-Dk_0^2 + M + N, N+I)\phi_2^0 - \frac{1}{2}N\phi_1^0) = \operatorname{rank}\left(\begin{smallmatrix} 3 & 2 & \frac{7}{36} \\ 0 & 0 & -\frac{1}{4} \end{smallmatrix}\right) = 2$, we have $(N+I)\phi_2^0 - \frac{1}{2}N\phi_1^0 \notin \mathcal{R}(-Dk_0^2 + M + N)$, condition (iii) holds.

Then we conclude that the delayed reaction–diffusion differential system (28) has a B–T singularity for the linear part and all the nonlinear parts (satisfying (H1)).

Example 2 Consider the following two-dimensional delayed reaction–diffusion differential system:

$$u_t = D\Delta u(x,t) + M(\alpha)u(x,t) + N(\alpha)u(x,t-1) + f\bigl(u(x,t), u(x,t-1), \alpha\bigr), \tag{29}$$

where $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 3 & 1 \\ \alpha_1 & 2+\alpha_2 \end{pmatrix}$, $N = \begin{pmatrix} 1+\alpha_1 & 1 \\ 0 & -1+\alpha_2 \end{pmatrix}$, $f(u(x,t), u(x,t-1), \alpha) = -[u_1^2(x,t), 4u_2^2(x,t-1)]^T$.

From the discussion of Example 1, we know that system (28) has a B–T singularity at $(V, \alpha) = (0,0)$. Let

$$\begin{aligned} \hat{f}\bigl(u(x,t), u(x,t-1), \alpha\bigr) &= \bigl(M(\alpha) - M(0)\bigr)u(x,t) + \bigl(N(\alpha) - N(0)\bigr)u(x,t-1) \\ &\quad + f\bigl(u(x,t), u(x,t-1), \alpha\bigr), \end{aligned}$$

its expansion form is

$$\begin{aligned} &\hat{f}\bigl(u(x,t), u(x,t-1), \alpha\bigr) \\ &\quad = \frac{1}{2}\hat{f}_2\bigl(u(x,t), u(x,t-1), \alpha\bigr) + h.o.t. \\ &\quad = M_1\alpha_1 u(x,t) + M_2\alpha_2 u(x,t) - N_1\alpha_1 u(x,t-1) + N_2\alpha_2 u(x,t-1) \\ &\qquad + F_1 u_1(x,t)u(x,t) + G_2 u_2(x,t-1)u(x,t-1) + h.o.t., \end{aligned} \tag{30}$$

where $M_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $M_1 = N_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $N_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $F_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, $G_2 = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}$.

According to Lemma 1, we choose the basis functions $\Phi(\theta)$ and $\Psi(s)$, where

$$\phi_1^0 = \left(\frac{1}{3}, -\frac{1}{2}\right)^T, \qquad \phi_2^0 = \left(\frac{1}{18}, 0\right)^T, \qquad \psi_2^0 = (0, -4), \qquad \psi_1^0 = \left(0, -\frac{4}{3}\right).$$

By Theorem 1, the system (30) can be reduced to

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = (-\frac{4}{3}\alpha_1 + 2\alpha_2)x_1 + (-\frac{2}{3}\alpha_1 - \frac{4}{3}\alpha_2)x_2 + 4x_1^2 - \frac{16}{3}x_1x_2. \end{cases} \tag{31}$$

Then (31) becomes

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{64}{81}(-\frac{4}{3}\alpha_1 + 2\alpha_2)^2 - \frac{56}{27}\alpha_1 x_2 + x_1^2 - x_1x_2. \end{cases} \tag{32}$$

5 Conclusions

In this paper, we have made an attempt to expand our computation to obtain a simpler and detailed reduction and normal form of system (1). By using the central manifold reduction method and normal form theory, we tried to reduce the dimension of the phase space without changing the dynamic behavior of the system, and succeeded in obtaining a simpler and more specific parameterized delayed ordinary differential system on its center manifold. We showed that system (1) can be reduced to a simple ordinary differential system with dimension two on its center manifold. Although we have theoretically proved that the system (1) can undergo a B–T bifurcation, unfortunately, due to the limited ability of our computer, it is difficult to display the simulation. For future work, we will firstly improve the ability of our computer and try to display the simulation. Secondly we will not only introduce delays in this model, but also we will study its impact on the analysis of dynamic stability, that is, we will consider executing some control by changing the value of the time delays determined by the system parameters to keep the ecological balance.

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Availability of data and materials
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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LL developed the idea for the study and wrote the paper; JC and YM did the analyses; all authors discussed the results and revised the paper.

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