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Riemann–Hilbert approach and N-soliton solution for an eighth-order nonlinear Schrödinger equation in an optical fiber

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Abstract

This paper aims to present an application of the Riemann–Hilbert approach to treat higher-order nonlinear differential equation that is an eighth-order nonlinear Schrödinger equation arising in an optical fiber. Starting from the spectral analysis of the Lax pair, a matrix Riemann–Hilbert problem is formulated strictly. Then, by solving the obtained Riemann–Hilbert problem under the reflectionless case, N-soliton solution is generated for the eighth-order nonlinear Schrödinger equation. Finally, the localized structures and dynamic behaviors of one- and two-soliton solutions are illustrated by some figures.

MSC: 35C08

Keywords: Eighth-order nonlinear Schrödinger equation; Riemann–Hilbert approach; Soliton solutions

1 Introduction

The infinite integrable nonlinear Schrödinger (NLS) equation hierarchy [1] reads as

$$ip_t + A_2K_2\big[p(x,t)\big] - iA_3K_3\big[p(x,t)\big] + A_4K_4\big[p(x,t)\big] - iA_5K_5\big[p(x,t)\big] + \cdots = 0, \tag{1}$$

which is used to investigate the higher-order dispersive effects and nonlinearity. Here $p(x,t)$ denotes a normalized complex amplitude of the optical pulse envelope. The coefficients A_l are arbitrary real constants, and $K_l[p(x,t)]$ are the lth-order operators in the NLS hierarchy

$$\begin{aligned}
K_2\big[p(x,t)\big] &= p_{xx} + 2p|p|^2, \\
K_3\big[p(x,t)\big] &= p_{xxx} + 6|p|^2 p_x, \\
K_4\big[p(x,t)\big] &= p_{xxxx} + 6p^* p_x^2 + 4p|p_x|^2 + 8|p|^2 p_{xx} + 2p^2 p_{xx}^* + 6|p|^4 p, \\
K_5\big[p(x,t)\big] &= p_{xxxxx} + 10|p|^2 p_{xxx} + 30|p|^4 p_x + 10pp_x p_{xx}^* + 10pp_x^* p_{xx} \\
&\quad + 20p^* p_x p_{xx} + 10p_x^2 p_x^*, \\
&\ \ \vdots
\end{aligned}$$

Here the subscripts of $p(x,t)$ mean the partial derivatives with respect to the scaled spatial coordinate x and time coordinate t correspondingly. And the superscript $*$ represents complex conjugate.

As a matter of fact, Equation (1) covers many nonlinear differential equations of important significance, some of which are listed as follows:

(i) For the case of $A_l = 0$, $l \geq 3$, Equation (1) is reduced to the fundamental nonlinear Schrödinger equation describing the propagation of the picosecond pulses in an optical fiber.

(ii) For the case of $A_2 = \frac{1}{2}$ and $A_l = 0$, $l \geq 4$, Equation (1) is reduced to the Hirota equation [2–5] describing the third-order dispersion and time-delay correction to the cubic nonlinearity in ocean waves.

(iii) For the case of $A_2 = \frac{1}{2}$ and $A_l = 0$, $l \geq 5$, Equation (1) becomes a fourth-order dispersive NLS equation [6, 7] describing the ultrashort optical-pulse propagation in a long-distance, high-speed optical fiber transmission system.

(iv) For the case of $A_2 = \frac{1}{2}$ and $A_l = 0$, $l \geq 6$, Equation (1) becomes a fifth-order NLS equation [8] describing the attosecond pulses in an optical fiber.

In recent years, researchers have devoted their attention to many higher-order NLS equations truncating from Equation (1). For instance, an eighth-order NLS equation was under study [9]. The interactions among multiple solitons were discussed, and oscillations in the interaction zones were observed systematically. As a result, it was found that the oscillations in the solitonic interaction zones possess different forms with different spectral parameters and so forth. In a follow-up study [10], the Lax pair and infinitely-many conservation laws were derived via symbolic computation, which verifies the integrability of equation.

All the time, seeking exact solutions of nonlinear models is of an especially important significance in the study of various nonlinear phenomena [11–15]. With this in mind, in this paper, we investigate in detail an eighth-order NLS equation [16]

$$\begin{aligned}
&ip_t + p_{xxxxxxxx} + 16|p|^2 p_{xxxxxx} + 2p^2 p^*_{xxxxxx} + 56p^* p_x p_{xxxxx} + 40pp^*_x p_{xxxxx} \\
&\quad + 12pp_x p^*_{xxxxx} + 98|p|^4 p_{xxxx} + 168|p_x|^2 p_{xxxx} + 112p^* p_{xx} p_{xxxx} + 72pp^*_{xx} p_{xxxx} \\
&\quad + 28p^2|p|^2 p^*_{xxxx} + 42p_x^2 p^*_{xxxx} + 44pp_{xx} p^*_{xxxx} + 68pp_{xxx} p^*_{xxx} + 476|p|^2 p^* p_x p_{xxx} \\
&\quad + 252p_x p^*_{xx} p_{xxx} + 308p|p|^2 p^*_x p_{xxx} + 308p^*_x p_{xx} p_{xxx} + 70p^* p^2_{xxx} + 196p_x p_{xx} p^*_{xxx} \\
&\quad + 168p|p|^2 p_x p^*_{xxx} + 56p^3 p^*_x p^*_{xxx} + 280|p|^6 p_{xx} + 1456|p|^2|p_x|^2 p_{xx} + 490\left(p^*\right)^2 p_x^2 p_{xx} \\
&\quad + 238p^2\left(p^*_x\right)^2 p_{xx} + 588|p|^2 p_x^2 p^*_{xx} + 336p^2|p_x|^2 p^*_{xx} + 140|p|^4 p^2 p^*_{xx} + 42p^3\left(p^*_{xx}\right)^2 \\
&\quad + 392|p|^2 p|p_{xx}|^2 + 322|p|^2 p^* p^2_{xx} + 182p^2_{xx} p^*_{xx} + 560|p|^4 p^* p_x^2 + 560|p|^4 p|p_x|^2 \\
&\quad + 420p^* p_x^2|p_x|^2 + 140p^3|p|^2\left(p^*_x\right)^2 + 378|p_x|^4 p + 70|p|^8 p = 0,
\end{aligned} \tag{2}$$

which works as a model for describing the propagation of ultrashort nonlinear pulses. The same scalar equation can be found from Equation (1) where $K_l(x,t)$ have the same meaning as $H_{l+1}(p,-p^*)$. Here $p(x,t)$ denotes a normalized complex amplitude of the optical pulse envelope.

The principal aim of this paper is to determine multi-soliton solutions for the eighth-order NLS equation (2) with the aid of the Riemann–Hilbert approach [17–29]. This paper is divided into five sections. In the second section, we recall the Lax pair associated with Equation (2) and convert it into a more convenient form. In the third section, we carry out the spectral analysis, from which a matrix Riemann–Hilbert problem is set up on the real axis. In the fourth section, the construction of multi-soliton solutions for Equation (2) is detailedly discussed in the framework of the Riemann–Hilbert problem without reflection. A brief conclusion is given in the final section.

2 Lax pair

Upon the Ablowitz–Kaup–Newell–Segur formalism, Equation (2) is associated with the following Lax pair [16]:

$$\Psi_x = U\Psi, \qquad U = i\varsigma\sigma + iQ, \tag{3a}$$

$$\Psi_{\tilde{t}} = V\Psi, \qquad V = 128i\varsigma^8\sigma + 128i\varsigma^7 Q + \sum_{k=1}^{7}(2\varsigma)^{7-k}V_k^0, \tag{3b}$$

where $\Psi = (\Psi_1, \Psi_2)^{\mathrm{T}}$ is a vector eigenfunction, Ψ_1 and Ψ_2 are the complex functions of x and t, the symbol T signifies transpose of the vector, and ς is an isospectral parameter. Furthermore,

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & p \\ -q & 0 \end{pmatrix}, \qquad V_k^0 = \begin{pmatrix} -i^k F_k(p,q) & i^{k-1}H_k(p,q) \\ i^{k-1}G_k(p,q) & i^k F_k(p,q) \end{pmatrix},$$

and

$$\begin{aligned} &H_1(p,q) = -p_x, \qquad G_1(p,q) = -q_x, \qquad \big(F_k(p,q)\big)_x = -pG_k(p,q) - qH_k(p,q), \\ &H_{k+1}(p,q) = 2pF_k(p,q) + \big(H_k(p,q)\big)_x, \qquad G_{k+1}(p,q) = -2qF_k(p,q) - \big(G_k(p,q)\big)_x. \end{aligned}$$

Particularly,

$$\begin{aligned}
&F_1(p,q) = pq, \qquad H_2(p,q) = 2p^2q - p_{xx}, \\
&G_2(p,q) = -2q^2p + q_{xx}, \qquad F_2(p,q) = p_xq - pq_x, \\
&H_3(p,q) = 6pqp_x - p_{xxx}, \qquad G_3(p,q) = 6pqq_x - q_{xxx}, \\
&F_3(p,q) = pq_{xx} + qp_{xx} - p_xq_x - 3p^2q^2, \\
&H_4(p,q) = -6p^3q^2 + 6qp_x^2 + 4pp_xq_x + 8pqp_{xx} + 2p^2q_{xx} - p_{xxxx}, \\
&G_4(p,q) = 6p^2q^3 - 6pq_x^2 - 4qp_xq_x - 8pqq_{xx} - 2q^2p_{xx} + q_{xxxx}, \\
&F_4(p,q) = -6pq^2p_x + 6p^2qq_x - p_{xx}q_x + p_xq_{xx} + qp_{xxx} - pq_{xxx}, \\
&H_5(p,q) = -30p^2q^2p_x + 10p_x^2q_x + 20qp_xp_{xx} + 10pq_xp_{xx} + 10pp_xq_{xx} + 10pqp_{xxx} - p_{xxxxx}, \\
&G_5(p,q) = -30p^2q^2q_x + 10q_x^2p_x + 10qq_xp_{xx} + 10qp_xq_{xx} + 10pqq_{xxx} + 20pq_xq_{xx} - q_{xxxxx}, \\
&F_5(p,q) = 10p^3q^3 - 5q^2p_x^2 - 5p^2q_x^2 - 10pq^2p_{xx} - 10p^2qq_{xx} + p_{xx}q_{xx} - q_xp_{xxx} - p_xq_{xxx} \\
&\qquad + qp_{xxxx} + pq_{xxxx},
\end{aligned}$$

$$
\begin{aligned}
H_6(p,q) ={}& 20p^4q^3 - 70pq^2p_x^2 - 60p^2qp_xq_x - 10p^3q_x^2 - 50p^2q^2p_{xx} + 50p_xq_xp_{xx} + 20qp_{xx}^2 \\
& - 20p^3qq_{xx} + 20p_x^2q_{xx} + 22pp_{xx}q_{xx} + 30qp_xp_{xxx} + 18pq_xp_{xxx} + 8pp_xq_{xxx} \\
& + 12pqp_{xxxx} + 2p^2q_{xxxx} - p_{xxxxxx}, \\
G_6(p,q) ={}& -20q^4p^3 + 10q^3p_x^2 + 60q^2pp_xq_x + 70qp^2q_x^2 + 20q^3pp_{xx} - 20q_x^2p_{xx} \\
& + 50p^2q^2q_{xx} - 50p_xq_xq_{xx} - 22qp_{xx}q_{xx} - 20pq_{xx}^2 - 8qq_xp_{xxx} - 18qp_xq_{xxx} \\
& - 30pq_xq_{xxx} - 2q^2p_{xxxx} - 12pqq_{xxxx} + q_{xxxxxx}, \\
F_6(p,q) ={}& 30p^2q^3p_x - 30p^3q^2q_x - 10qp_x^2q_x + 10pp_xq_x^2 - 20q^2p_xp_{xx} + 10pqq_xp_{xx} \\
& - 10pqp_xq_{xx} + 20p^2q_xq_{xx} - 10pq^2p_{xxx} + q_{xx}p_{xxx} + 10p^2qq_{xxx} - p_{xx}q_{xxx} \\
& - q_xp_{xxxx} + p_xq_{xxxx} + qp_{xxxxx} - pq_{xxxxx}, \\
H_7(p,q) ={}& 140p^3q^3p_x - 70q^2p_x^3 - 280pqp_x^2q_x - 70p^2q_x^2p_x - 280pq^2p_xp_{xx} - 140p^2qq_xp_{xx} \\
& + 70q_xp_{xx}^2 - 140p^2qp_xq_{xx} + 112p_xp_{xx}q_{xx} - 70p^2q^2p_{xxx} + 98p_xq_xp_{xxx} \\
& + 70qp_{xx}p_{xxx} + 42pq_{xx}p_{xxx} + 28p_x^2q_{xxx} + 28pp_{xx}q_{xxx} + 42qp_xp_{xxxx} \\
& + 28pq_xp_{xxxx} + 14pp_xq_{xxxx} + 14pqp_{xxxxx} - p_{xxxxxxx}, \\
G_7(p,q) ={}& 140p^3q^3q_x - 70q^2p_x^2q_x - 280pqp_xq_x^2 - 70p^2q_x^3 - 140pq^2q_xp_{xx} - 140pq^2p_xq_{xx} \\
& - 280p^2qq_xq_{xx} + 112q_xp_{xx}q_{xx} + 70p_xq_{xx}^2 + 28q_x^2p_{xxx} + 28qq_{xx}p_{xxx} \\
& - 70p^2q^2q_{xxx} + 98p_xq_xq_{xxx} + 42qp_{xx}q_{xxx} + 70pq_{xx}q_{xxx} - 14qq_xp_{xxxx} \\
& + 28qp_xq_{xxxx} + 42pq_xq_{xxxx} + 14pqq_{xxxxx} - q_{xxxxxxx}, \\
F_7(p,q) ={}& -35p^4q^4 + 70pq^3p_x^2 + 70p^2q^2p_xq_x + 70p^3qq_x^2 + 21p_x^2q_x^2 + 70p^2q^3p_{xx} \\
& - 28qp_xq_xp_{xx} - 14pq_x^2p_{xx} - 21q^2p_{xx}^2 + 70p^3q^2q_{xx} - 14qp_x^2q_{xx} - 28pp_xq_xq_{xx} \\
& - 56pqp_{xx}q_{xx} - 21p^2q_{xx}^2 - 28q^2p_xp_{xxx} - 14pqq_xp_{xxx} - 14pqp_xq_{xxx} \\
& - 28p^2q_xq_{xxx} - p_{xxx}q_{xxx} - 14pq^2p_{xxxx} + q_{xx}p_{xxxx} - 14p^2qq_{xxxx} \\
& + p_{xx}q_{xxxx} - q_xp_{xxxxx} - p_xq_{xxxxx} + qp_{xxxxxx} + pq_{xxxxxx}.
\end{aligned}
$$

Then the reductions $q = -p^*$ and $\tilde{t} = -t$ exactly result in Equation (2) based on the zero-curvature equations.

Let us now rewrite the Lax pair (3a)–(3b) in a more convenient form:

$$
\Psi_x = i(\varsigma\sigma + Q)\Psi, \tag{4a}
$$

$$
\Psi_t = -\left(128i\varsigma^8\sigma + Q_1\right)\Psi, \tag{4b}
$$

where

$$
Q = \begin{pmatrix} 0 & p \\ p^* & 0 \end{pmatrix}, \qquad Q_1 = 128i\varsigma^7 Q + \sum_{k=1}^{7} (2\varsigma)^{7-k} V_k^0.
$$

3 Riemann–Hilbert problem

In this section, we focus on putting forward a matrix Riemann–Hilbert problem for Equation (2). Now we assume that the potential function $p(x,t)$ in the Lax pair (4a)–(4b) decays to zero sufficiently fast as $x \to \pm\infty$. It can be known from (4a)–(4b) that when $x \to \pm\infty$,

$$\Psi \propto \mu e^{i\varsigma\sigma x - 128 i\varsigma^8 \sigma t},$$

which motivates us to introduce the variable transformation

$$\Psi = \mu e^{i\varsigma\sigma x - 128 i\varsigma^8 \sigma t}.$$

Under this transformation, the Lax pair (4a)–(4b) can be changed into the form

$$\mu_x = i\varsigma[\sigma,\mu] + U_1\mu, \tag{5a}$$

$$\mu_t = -128 i\varsigma^8[\sigma,\mu] - Q_1\mu, \tag{5b}$$

where $[\cdot,\cdot]$ is the matrix commutator and $U_1 = iQ$. From (5a)–(5b), we find that $\mathrm{tr}(U_1) = \mathrm{tr}(Q_1) = 0$.

In the direct scattering process, we will concentrate on the spectral problem (5a), and the t-dependence will be suppressed. We first introduce two matrix Jost solutions $\mu_\pm$ of (5a) expressed as a collection of columns

$$\mu_- = \big([\mu_-]_1, [\mu_-]_2\big), \qquad \mu_+ = \big([\mu_+]_1, [\mu_+]_2\big), \tag{6}$$

meeting the asymptotic conditions at large distances

$$\mu_- \to \mathbb{I}, \quad x \to -\infty,$$

$$\mu_+ \to \mathbb{I}, \quad x \to +\infty.$$

Here the subscripts of μ indicated refer to which end of the x-axis the boundary conditions are required for, and $\mathbb{I}$ stands for the identity matrix of size 2. Actually, the solutions $\mu_\pm$ are uniquely determined by the integral equations of Volterra type

$$\mu_- = \mathbb{I} + \int_{-\infty}^{x} e^{i\varsigma\sigma(x-y)} U_1(y)\mu_-(y,\varsigma) e^{i\varsigma\sigma(y-x)}\,dy, \tag{7a}$$

$$\mu_+ = \mathbb{I} - \int_{x}^{+\infty} e^{i\varsigma\sigma(x-y)} U_1(y)\mu_+(y,\varsigma) e^{i\varsigma\sigma(y-x)}\,dy. \tag{7b}$$

After direct analysis on Equations (7a)–(7b), we can see that $[\mu_-]_1$, $[\mu_+]_2$ are analytic for $\varsigma \in \mathbb{C}^+$ and continuous for $\varsigma \in \mathbb{C}^+ \cup \mathbb{R}$, while $[\mu_+]_1$, $[\mu_-]_2$ are analytic for $\varsigma \in \mathbb{C}^-$ and continuous for $\varsigma \in \mathbb{C}^- \cup \mathbb{R}$, where $\mathbb{C}^-$ and $\mathbb{C}^+$ are respectively the lower and upper half ς-planes:

$$\mathbb{C}^- = \big\{\varsigma \in \mathbb{C} \,|\, \mathrm{Im}(\varsigma) < 0\big\}, \qquad \mathbb{C}^+ = \big\{\varsigma \in \mathbb{C} \,|\, \mathrm{Im}(\varsigma) > 0\big\}.$$

Next we set out to study the properties of $\mu_\pm$. In fact, it can be shown from Abel's identity and $\mathrm{tr}(U_1) = 0$ that the determinants of $\mu_\pm$ are independent of the variable x. Evaluating

$\det \mu_-$ at $x = -\infty$ and $\det \mu_+$ at $x = +\infty$, we get $\det \mu_\pm = 1$ for $\varsigma \in \mathbb{R}$. In addition, $\mu_- E$ and $\mu_+ E$ are both fundamental solutions of (3a), where $E = \mathrm{e}^{i\varsigma\sigma x}$, they are linearly dependent

$$\mu_- E = \mu_+ ES(\varsigma), \quad \varsigma \in \mathbb{R}. \tag{8}$$

Here $S(\varsigma) = (s_{kj})_{2\times 2}$ is called the scattering matrix and $\det S(\varsigma) = 1$. Furthermore, we find from the properties of $\mu_\pm$ that s_{11} allows analytic extension to $\mathbb{C}^+$ and s_{22} analytically extends to $\mathbb{C}^-$.

A matrix Riemann–Hilbert problem is closely connected with two matrix functions: one is analytic in $\mathbb{C}^+$ and the other is analytic in $\mathbb{C}^-$. In consideration of the analytic properties of $\mu_\pm$, we set

$$P_1(x,\varsigma) = \bigl([\mu_-]_1, [\mu_+]_2\bigr)(x,\varsigma), \tag{9}$$

defining in $\mathbb{C}^+$, be an analytic function of ς. And then, P_1 can be expanded into the asymptotic series at large-ς

$$P_1 = P_1^{(0)} + \frac{P_1^{(1)}}{\varsigma} + \frac{P_1^{(2)}}{\varsigma^2} + O\left(\frac{1}{\varsigma^3}\right), \quad \varsigma \to \infty. \tag{10}$$

Inserting expansion (10) into the spectral problem (5a) and equating terms with the same powers of ς, we obtain

$$i\bigl[\sigma, P_1^{(1)}\bigr] + U_1 P_1^{(0)} = P_{1x}^{(0)}, \qquad i\bigl[\sigma, P_1^{(0)}\bigr] = 0,$$

which yields $P_1^{(0)} = \mathbb{I}$, namely $P_1 \to \mathbb{I}$ as $\varsigma \in \mathbb{C}^+ \to \infty$.

For establishing a matrix Riemann–Hilbert problem, the analytic counterpart of P_1 in $\mathbb{C}^-$ is still needed to be given. Note that the adjoint scattering equation of (5a) reads as

$$H_x = i\varsigma[\sigma, H] - HU_1, \tag{11}$$

and the inverse matrices of $\mu_\pm$ meet this adjoint equation. Then we express the inverse matrices of $\mu_\pm$ as a collection of rows

$$\mu_\pm^{-1} = \begin{pmatrix} [\mu_\pm^{-1}]^1 \\ [\mu_\pm^{-1}]^2 \end{pmatrix}, \tag{12}$$

which obey the boundary conditions $\mu_\pm^{-1} \to \mathbb{I}$ as $x \to \pm\infty$. It is easy to know from (8) that

$$E^{-1}\mu_-^{-1} = R(\varsigma)E^{-1}\mu_+^{-1}, \tag{13}$$

where $R(\varsigma) = (r_{kj})_{2\times 2} = S^{-1}(\varsigma)$. Thus, the matrix function P_2 which is analytic for $\varsigma \in \mathbb{C}^-$ is constructed as

$$P_2(x,\varsigma) = \begin{pmatrix} [\mu_-^{-1}]^1 \\ [\mu_+^{-1}]^2 \end{pmatrix}(x,\varsigma). \tag{14}$$

Analogous to P_1, the very large-ς asymptotic behavior of P_2 turns out to be $P_2 \to \mathbb{I}$ as $\varsigma \in \mathbb{C}^- \to \infty$.

Carrying (6) into Equation (8) gives rise to

$$\big([\mu_-]_1, [\mu_-]_2\big) = \big([\mu_+]_1, [\mu_+]_2\big)\begin{pmatrix} s_{11} & s_{12}\mathrm{e}^{-2i\varsigma x} \\ s_{21}\mathrm{e}^{2i\varsigma x} & s_{22} \end{pmatrix},$$

from which we have

$$[\mu_-]_1 = s_{11}[\mu_+]_1 + s_{21}\mathrm{e}^{2i\varsigma x}[\mu_+]_2.$$

Hence, P_1 is of the form

$$P_1 = \big([\mu_-]_1, [\mu_+]_2\big) = \big([\mu_+]_1, [\mu_+]_2\big)\begin{pmatrix} s_{11} & 0 \\ s_{21}\mathrm{e}^{2i\varsigma x} & 1 \end{pmatrix}.$$

On the other hand, via substituting (12) into Equation (13), we get

$$\begin{pmatrix} [\mu_-^{-1}]^1 \\ [\mu_-^{-1}]^2 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12}\mathrm{e}^{-2i\varsigma x} \\ r_{21}\mathrm{e}^{2i\varsigma x} & r_{22} \end{pmatrix}\begin{pmatrix} [\mu_+^{-1}]^1 \\ [\mu_+^{-1}]^2 \end{pmatrix},$$

from which we can express $[\mu_-^{-1}]^1$ as

$$[\mu_-^{-1}]^1 = r_{11}[\mu_+^{-1}]^1 + r_{12}\mathrm{e}^{-2i\varsigma x}[\mu_+^{-1}]^2.$$

As a consequence, P_2 is written as

$$P_2 = \begin{pmatrix} [\mu_-^{-1}]^1 \\ [\mu_+^{-1}]^2 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12}\mathrm{e}^{-2i\varsigma x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} [\mu_+^{-1}]^1 \\ [\mu_+^{-1}]^2 \end{pmatrix}.$$

With two matrix functions P_1 and P_2 which are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively in hand, we are in a position to deduce a matrix Riemann–Hilbert problem for Equation (2). After denoting that the limit of P_1 is P^+ as $\varsigma \in \mathbb{C}^+ \to \mathbb{R}$ and the limit of P_2 is P^- as $\varsigma \in \mathbb{C}^- \to \mathbb{R}$, a matrix Riemann–Hilbert problem can be given as follows:

$$P^-(x,\varsigma)P^+(x,\varsigma) = \begin{pmatrix} 1 & r_{12}\mathrm{e}^{-2i\varsigma x} \\ s_{21}\mathrm{e}^{2i\varsigma x} & 1 \end{pmatrix}, \tag{15}$$

with its canonical normalization conditions as

$$P_1(x,\varsigma) \to \mathbb{I}, \quad \varsigma \in \mathbb{C}^+ \to \infty,$$

$$P_2(x,\varsigma) \to \mathbb{I}, \quad \varsigma \in \mathbb{C}^- \to \infty,$$

and $r_{11}s_{11} + r_{12}s_{21} = 1$.

4 *N*-Soliton solution

Having described a matrix Riemann–Hilbert problem for Equation (2), we now turn to seeking its multi-soliton solutions. To achieve the goal, we first need to solve the Riemann–Hilbert problem (15) under the assumption of irregularity, which signifies that both $\det P_1$ and $\det P_2$ possess some zeros in the analytic domains of their own. From the definitions of P_1 and P_2 as well as Equation (8), we have

$$\det P_1(\varsigma) = s_{11}(\varsigma), \quad \varsigma \in \mathbb{C}^+,$$
$$\det P_2(\varsigma) = r_{11}(\varsigma), \quad \varsigma \in \mathbb{C}^-,$$

which means that $\det P_1$ and $\det P_2$ have the same zeros as s_{11} and r_{11} respectively, and $r_{11} = (S^{-1})_{11} = s_{22}$.

With the above analysis, it is now necessary to reveal the characteristic feature of zeros. Manifestly, the potential matrix Q possesses the symmetry relation $Q^\dagger = Q$, upon which we deduce

$$\mu_\pm^\dagger(\varsigma^*) = \mu_\pm^{-1}(\varsigma). \tag{16}$$

Here the superscript † stands for the Hermitian of a matrix. For facilitating discussion, we introduce two special matrices $J_1 = \operatorname{diag}(1,0)$ and $J_2 = \operatorname{diag}(0,1)$, and express (9) and (14) in terms of

$$P_1 = \mu_- J_1 + \mu_+ J_2, \tag{17a}$$
$$P_2 = J_1 \mu_-^{-1} + J_2 \mu_+^{-1}. \tag{17b}$$

A direct computation of the Hermitian of expression (17a), using relation (16), generates that

$$P_1^\dagger(\varsigma^*) = P_2(\varsigma), \quad \varsigma \in \mathbb{C}^-, \tag{18}$$

and $S^\dagger(\varsigma^*) = S^{-1}(\varsigma)$, which leads to

$$s_{11}^*(\varsigma^*) = r_{11}(\varsigma), \quad \varsigma \in \mathbb{C}^-. \tag{19}$$

This equality implies that each zero $\pm\varsigma_k$ of s_{11} results in each zero $\pm\varsigma_k^*$ of r_{11} correspondingly. Therefore, our assumption is that $\det P_1$ has simple zeros $\{\varsigma_j \in \mathbb{C}^+, 1 \le j \le N\}$ and $\det P_2$ has simple zeros $\{\hat{\varsigma}_j \in \mathbb{C}^-, 1 \le j \le N\}$, where $\hat{\varsigma}_j = \varsigma_j^*$. The full set of the discrete scattering data is composed of these zeros and the nonzero column vectors υ_j and row vectors $\hat{\upsilon}_j$, which satisfy the following equations:

$$P_1(\varsigma_j)\upsilon_j = 0, \tag{20a}$$
$$\hat{\upsilon}_j P_2(\hat{\varsigma}_j) = 0. \tag{20b}$$

Taking the Hermitian of Equation (20a) and using (18) as well as comparing with Equation (20b), we find that the eigenvectors fulfill the relation

$$\hat{\upsilon}_j = \upsilon_j^\dagger, \quad 1 \le j \le N. \tag{21}$$

Differentiating Equation (20a) in x and t and taking advantage of Lax pair (5a)–(5b), we arrive at

$$P_1(\varsigma_j)\left(\frac{\partial \upsilon_j}{\partial x} - i\varsigma_j \sigma \upsilon_j\right) = 0,$$
$$P_1(\varsigma_j)\left(\frac{\partial \upsilon_j}{\partial t} + 128 i\varsigma_j^8 \sigma \upsilon_j\right) = 0,$$

which yields

$$\upsilon_j = \mathrm{e}^{(i\varsigma_j x - 128 i\varsigma_j^8 t)\sigma} \upsilon_{j,0}, \quad 1 \le j \le N.$$

Here $\upsilon_{j,0}$, $1 \le j \le N$, are complex constant vectors. Making use of relation (21), we have

$$\hat{\upsilon}_j = \upsilon_{j,0}^{\dagger} \mathrm{e}^{(-i\varsigma_j^* x + 128 i\varsigma_j^{*8} t)\sigma}, \quad 1 \le j \le N.$$

However, in order to derive soliton solutions of Equation (2), we investigate the Riemann–Hilbert problem (15) corresponding to the reflectionless case, i.e., $s_{21} = 0$. We introduce an $N \times N$ matrix M defined as

$$M = (M_{kj})_{N\times N} = \left(\frac{\hat{\upsilon}_k \upsilon_j}{\varsigma_j - \hat{\varsigma}_k}\right)_{N\times N}, \quad 1 \le k, j \le N.$$

Thus the solutions [30] to problem (15) can be determined by

$$P_1(\varsigma) = \mathbb{I} - \sum_{k=1}^{N}\sum_{j=1}^{N} \frac{\upsilon_k \hat{\upsilon}_j (M^{-1})_{kj}}{\varsigma - \hat{\varsigma}_j}, \tag{22a}$$

$$P_2(\varsigma) = \mathbb{I} + \sum_{k=1}^{N}\sum_{j=1}^{N} \frac{\upsilon_k \hat{\upsilon}_j (M^{-1})_{kj}}{\varsigma - \varsigma_k}, \tag{22b}$$

where $(M^{-1})_{kj}$ denotes the (k,j)-entry of M^{-1}. From expression (22a), it can be seen that

$$P_1^{(1)} = -\sum_{k=1}^{N}\sum_{j=1}^{N} \upsilon_k \hat{\upsilon}_j \left(M^{-1}\right)_{kj}.$$

In what follows, we shall retrieve the potential function $p(x,t)$ based on the scattering data. Expanding $P_1(\varsigma)$ at large-ς as

$$P_1(\varsigma) = \mathbb{I} + \frac{P_1^{(1)}}{\varsigma} + \frac{P_1^{(2)}}{\varsigma^2} + O\left(\frac{1}{\varsigma^3}\right), \quad \varsigma \to \infty,$$

and carrying this expansion into (5a) gives rise to

$$Q = -\left[\sigma, P_1^{(1)}\right].$$

Consequently, the potential function is reconstructed as

$$p(x,t) = 2\left(P_1^{(1)}\right)_{12},$$

with $(P_1^{(1)})_{12}$ being the $(1,2)$-entry of $P_1^{(1)}$.

To conclude, setting the nonzero vectors $\upsilon_{k,0} = (\alpha_k, \beta_k)^{\mathrm{T}}$ and $\theta_k = i\varsigma_k x - 128i\varsigma_k^8 t$, the general N-soliton solution for the eighth-order NLS equation (2) is written as

$$p(x,t) = -2\sum_{k=1}^{N}\sum_{j=1}^{N}\alpha_k\beta_j^* \mathrm{e}^{\theta_j^*-\theta_k}\left(M^{-1}\right)_{kj}, \tag{23}$$

where

$$M_{kj} = \frac{\alpha_k^*\alpha_j \mathrm{e}^{-\theta_k^*-\theta_j} + \beta_k^*\beta_j \mathrm{e}^{\theta_k^*+\theta_j}}{\varsigma_j - \varsigma_k^*}, \quad 1 \le k,j \le N.$$

The bright one- and two-soliton solutions will be our main concern in the rest of this section. For the simplest case of $N = 1$, the bright one-soliton solution can be readily derived as

$$p(x,t) = -2\alpha_1\beta_1^* \mathrm{e}^{\theta_1^*-\theta_1}\frac{\varsigma_1 - \varsigma_1^*}{|\alpha_1|^2\mathrm{e}^{-\theta_1^*-\theta_1} + |\beta_1|^2\mathrm{e}^{\theta_1^*+\theta_1}}, \tag{24}$$

where $\theta_1 = i\varsigma_1 x - 128i\varsigma_1^8 t$. Furthermore, via fixing $\alpha_1 = 1$ and setting $\varsigma_1 = \tilde{a}_1 + i\tilde{b}_1$ as well as $|\beta_1|^2 = \mathrm{e}^{2\xi_1}$, the solution (24) is then turned into the following form:

$$p(x,t) = -2i\tilde{b}_1\beta_1^* \mathrm{e}^{\theta_1^*-\theta_1}\mathrm{e}^{-\xi_1}\operatorname{sech}\left(\theta_1^* + \theta_1 + \xi_1\right), \tag{25}$$

where

$$\begin{aligned}
\theta_1^* + \theta_1 &= -2\tilde{b}_1 x + 2048\tilde{a}_1^7\tilde{b}_1 t - 14336\tilde{a}_1^5\tilde{b}_1^3 t + 14336\tilde{a}_1^3\tilde{b}_1^5 t - 2048\tilde{a}_1\tilde{b}_1^7 t,\\
\theta_1^* - \theta_1 &= -2i\tilde{a}_1 x - 7168i\tilde{a}_1^2\tilde{b}_1^6 t - 7168i\tilde{a}_1^6\tilde{b}_1^2 t + 17920i\tilde{a}_1^4\tilde{b}_1^4 t + 256i\tilde{b}_1^8 t + 256i\tilde{a}_1^8 t.
\end{aligned}$$

Hence we can further write the bright one-soliton solution (25) as

$$\begin{aligned}
p(x,t) &= -2i\tilde{b}_1\beta_1^* \mathrm{e}^{\theta_1^*-\theta_1}\mathrm{e}^{-\xi_1}\\
&\quad\times \operatorname{sech}\left\{-2\tilde{b}_1\left[x - \left(1024\tilde{a}_1^7 - 7168\tilde{a}_1^5\tilde{b}_1^2 + 7168\tilde{a}_1^3\tilde{b}_1^4 - 1024\tilde{a}_1\tilde{b}_1^6\right)t\right] + \xi_1\right\},
\end{aligned} \tag{26}$$

from which it is indicated that the solution (26) takes the shape of hyperbolic secant function with peak amplitude

$$\mathcal{H} = 2\left|\beta_1^*\right|\tilde{b}_1\mathrm{e}^{-\xi_1}$$

and velocity

$$\mathcal{V} = 1024\tilde{a}_1^7 - 7168\tilde{a}_1^5\tilde{b}_1^2 + 7168\tilde{a}_1^3\tilde{b}_1^4 - 1024\tilde{a}_1\tilde{b}_1^6.$$

To show the localized structures and dynamic behaviors of one-soliton solution (26), we select the involved parameters as $\tilde{a}_1 = 0.3$, $\tilde{b}_1 = 0.25$, $\alpha_1 = \beta_1 = 1$, $\xi_1 = 0$. The plots are depicted in Figs. 1–3.

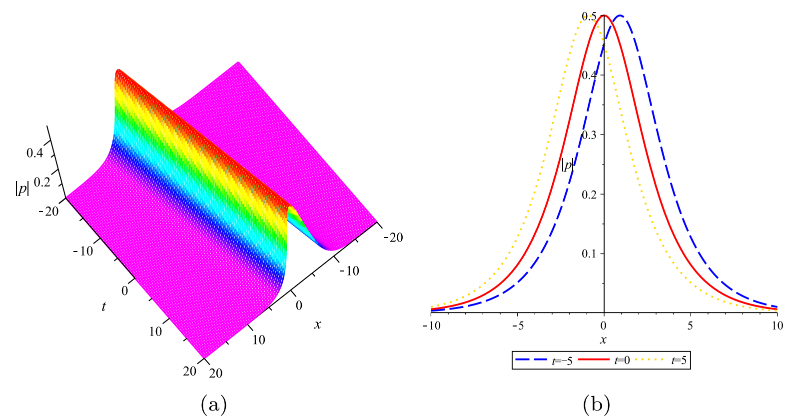

Figure 1 One-soliton solution (26) with $\tilde{a}_1 = 0.3$, $\tilde{b}_1 = 0.25$, $\alpha_1 = \beta_1 = 1$, $\xi_1 = 0$. (**a**) Perspective view of modulus of p; (**b**) The soliton along the x-axis with different time in Fig. 1(a)

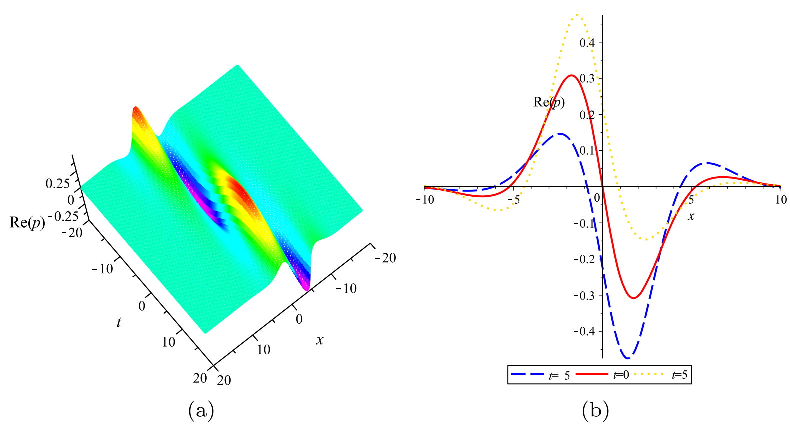

Figure 2 One-soliton solution (26) with $\tilde{a}_1 = 0.3$, $\tilde{b}_1 = 0.25$, $\alpha_1 = \beta_1 = 1$, $\xi_1 = 0$. (**a**) Perspective view of the real part of p; (**b**) The soliton along the x-axis with different time in Fig. 2(a)

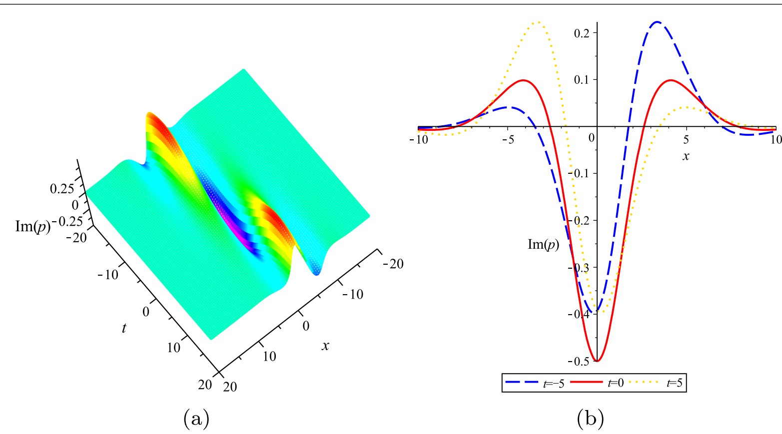

Figure 3 One-soliton solution (26) with $\tilde{a}_1 = 0.3$, $\tilde{b}_1 = 0.25$, $\alpha_1 = \beta_1 = 1$, $\xi_1 = 0$. (**a**) Perspective view of the imaginary part of p; (**b**) The soliton along the x-axis with different time in Fig. 3(a)

Then, for the case of $N = 2$, the bright two-soliton solution for Equation (2) is generated as

$$p(x,t) = \frac{2}{M_{12}M_{21} - M_{11}M_{22}}\big(\alpha_1\beta_1^* e^{\theta_1^*-\theta_1}M_{22} - \alpha_1\beta_2^* e^{\theta_2^*-\theta_1}M_{12} - \alpha_2\beta_1^* e^{\theta_1^*-\theta_2}M_{21} + \alpha_2\beta_2^* e^{\theta_2^*-\theta_2}M_{11}\big), \tag{27}$$

where

$$M_{11} = \frac{|\alpha_1|^2 e^{-\theta_1^*-\theta_1} + |\beta_1|^2 e^{\theta_1^*+\theta_1}}{\varsigma_1 - \varsigma_1^*}, \qquad M_{12} = \frac{\alpha_1^*\alpha_2 e^{-\theta_1^*-\theta_2} + \beta_1^*\beta_2 e^{\theta_1^*+\theta_2}}{\varsigma_2 - \varsigma_1^*},$$
$$M_{21} = \frac{\alpha_2^*\alpha_1 e^{-\theta_2^*-\theta_1} + \beta_2^*\beta_1 e^{\theta_2^*+\theta_1}}{\varsigma_1 - \varsigma_2^*}, \qquad M_{22} = \frac{|\alpha_2|^2 e^{-\theta_2^*-\theta_2} + |\beta_2|^2 e^{\theta_2^*+\theta_2}}{\varsigma_2 - \varsigma_2^*},$$

and $\theta_1 = i\varsigma_1 x - 128i\varsigma_1^8 t$, $\theta_2 = i\varsigma_2 x - 128i\varsigma_2^8 t$, $\varsigma_1 = \tilde{a}_1 + i\tilde{b}_1$, $\varsigma_2 = \tilde{a}_2 + i\tilde{b}_2$.

After assuming that $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2$ as well as $|\beta_1|^2 = e^{2\xi_1}$, the bright two-soliton solution (27) becomes

$$p(x,t) = \frac{2}{M_{12}M_{21} - M_{11}M_{22}} \times \big(\beta_1^* e^{\theta_1^*-\theta_1}M_{22} - \beta_2^* e^{\theta_2^*-\theta_1}M_{12} - \beta_1^* e^{\theta_1^*-\theta_2}M_{21} + \beta_2^* e^{\theta_2^*-\theta_2}M_{11}\big), \tag{28}$$

where

$$M_{11} = -\frac{i}{\tilde{b}_1}e^{\xi_1}\cosh\big(\theta_1^* + \theta_1 + \xi_1\big),$$
$$M_{12} = \frac{2e^{\xi_1}}{(\tilde{a}_2 - \tilde{a}_1) + i(\tilde{b}_1 + \tilde{b}_2)}\cosh\big(\theta_1^* + \theta_2 + \xi_1\big),$$
$$M_{21} = \frac{2e^{\xi_1}}{(\tilde{a}_1 - \tilde{a}_2) + i(\tilde{b}_1 + \tilde{b}_2)}\cosh\big(\theta_2^* + \theta_1 + \xi_1\big),$$
$$M_{22} = -\frac{i}{\tilde{b}_2}e^{\xi_1}\cosh\big(\theta_2^* + \theta_2 + \xi_1\big).$$

The localized structure and dynamic behaviors of two-soliton solution (28) are depicted in Fig. 4 via a selection of the parameters as follows: $\tilde{a}_1 = 0.3$, $\tilde{b}_1 = \tilde{b}_2 = 0.2$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $\tilde{a}_2 = \xi_1 = \xi_2 = 0$.

5 Conclusion

In this investigation, the aim was to explore multi-soliton solutions for an eighth-order nonlinear Schrödinger equation arising in an optical fiber. The method we resort to was the Riemann–Hilbert approach which is based on a matrix Riemann–Hilbert problem. Therefore, we first described a related Riemann–Hilbert problem via analyzing the spectral problem. After solving the resulting Riemann–Hilbert problem without reflection, we finally derived the expression of general N-soliton solution explicitly. We remark that this work mainly emphasizes the effectiveness of the Riemann–Hilbert method in dealing with higher-order nonlinear differential equation. Specifically, an eighth-order nonlinear Schrödinger equation is considered, which can also be generated from the AKNS hierarchy [29].

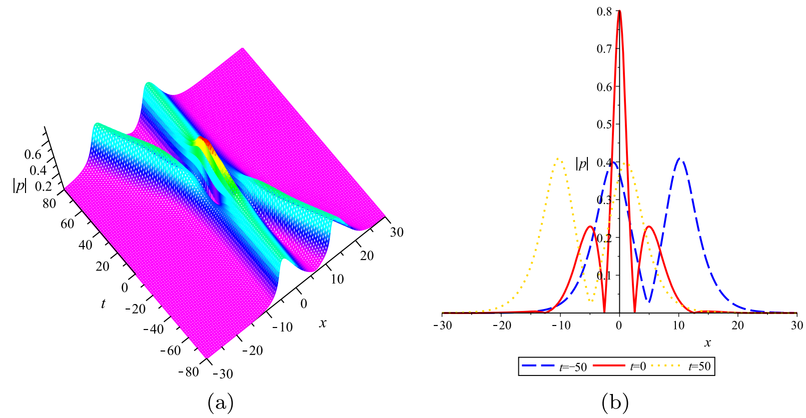

Figure 4 Two-soliton solution (28) with $\tilde{a}_1 = 0.3$, $\tilde{b}_1 = \tilde{b}_2 = 0.2$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $\tilde{a}_2 = \xi_1 = \xi_2 = 0$. (**a**) Perspective view of modulus of p; (**b**) The soliton along the x-axis with different time in Fig. 4(a)

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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