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Existence and finite-time stability of solutions for a class of nonlinear fractional differential equations with time-varying delays and non-instantaneous impulses

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Abstract

In this paper, we mainly consider the existence and finite-time stability of solutions for a kind of ψ-Hilfer fractional differential equations involving time-varying delays and non-instantaneous impulses. By Schauder's fixed point theorem, the contraction mapping principle and the Lagrange mean-value theorem, we present new constructive results as regards existence and uniqueness of solutions. In addition, under some new criteria and by applying the generalized Gronwall inequality, we deduce that the solutions of the addressed equation have finite-time stability. Some results in the literature can be generalized and improved. As an application, three typical examples are delineated to demonstrate the effectiveness of our theoretical results.

Keywords: ψ-Hilfer fractional differential equation; Existence; Finite-time stability; Time-varying delays; Non-instantaneous impulses

1 Introduction

Fractional differential equations play an important role in many fields, especially in biological medicine, dynamics mechanic, population dynamics and communication engineering. There is a growing tendency nowadays for many experts to show their great enthusiasm for this aspect, and a lot of achievements have been made; see the monographs [1–4]. Recently, we found that many researchers have set out to study a new type of impulsive (called non-instantaneous impulsive) fractional differential equations, where the impulsive action starts at an arbitrary fixed point and remains active on a finite time interval, which is very different from the classical instantaneous impulsive case that changes are relatively short compared to the overall duration of the process. For an extensive collection of non-instantaneous impulsive results, we refer the reader to the related literature, such as the monograph [5] and the papers [6–17].

The study of the existence of solutions for fractional differential equations is one of the most interesting and valuable topics [8, 11, 18–22]. In recent contributions [18, 19] one studied equations of the following form:

$$\begin{cases} {}^cD^{\alpha}_{0,t}x(t) = f(t,x(t)), & t \in [0,T] \setminus \{\tau_1,\tau_2,\ldots,\tau_m\}, \\ \Delta x(\tau_k) = I_k(x(\tau_k^-)), & k = 1,2,\ldots,m, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha}_{0,t}$ is the Caputo fractional derivative of the order $\alpha \in (n-1,n)$, $n \in \mathbb{N}$, $f : J \times \mathbb{R} \to \mathbb{R}$, $J = [0,T]$ and $I_k : \mathbb{R} \to \mathbb{R}$ and $0 = \tau_0 < \tau_1 < \cdots < \tau_m < \tau_{m+1} = T$, and we let $x(\tau_k^+) = \lim_{\varepsilon\to 0^+} x(\tau_k + \varepsilon)$ and $x(\tau_k^-) = \lim_{\varepsilon\to 0^-} x(\tau_k + \varepsilon)$ represent the right and left limits of $x(t)$ at $t = \tau_k$, respectively. Here, I_k is a sequence of instantaneous impulse operators, $k = 1,2,\ldots,m$. The authors got the existence results by applying fixed point methods. In general, the classical instantaneous impulses cannot describe some certain dynamics of evolution processes. For example, when we consider the hemodynamic equilibrium of a person, the introduction of drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes. In fact, the above situation can be characterized by the new non-instantaneous impulsive model.

We can see that Yang and Wang studied the following integral boundary value problems for fractional order nonlinear differential equations with non-instantaneous impulses in [11]:

$$\begin{cases} {}^cD^{q}_{0,t}u(t) = f(t,u(t)), & t \in (s_i,t_{i+1}], i = 0,1,2,\ldots,m, q \in (0,1), \\ u(t) = g_i(t,u(t)), & t \in (t_i,s_i], i = 1,2,\ldots,m, \\ u(0) = \int_0^1 u(s)\,ds, \end{cases} \tag{1.2}$$

where ${}^cD^{q}_{0,t}$ denotes the Caputo fractional derivative of the order q with the lower limit zero, $0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_m \le s_m < t_{m+1} = 1$ are pre-fixed numbers, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g_i : [t_i,s_i] \times \mathbb{R} \to \mathbb{R}$ are continuous for all $i = 1,2,\ldots,m$. By using standard fixed point approach, a series of existence results were presented under some conditions.

Yu [8] investigated the following new non-instantaneous impulsive differential equations:

$$\begin{cases} {}^cD^{\alpha}_{s_i,t}x(t) = -\lambda x(t) + f(t,x(t)), & t \in (s_i,t_{i+1}], i = 0,1,2,\ldots,m, \\ x(t) = q + I^{\gamma}_{t_i,t}g_i(t,x(t)) - I^{\alpha}_{0,s_i}f(s_i,x(s_i)), & t \in (t_i,s_i], i = 1,2,\ldots,m, \end{cases} \tag{1.3}$$

where γ, $\alpha \in (0,1)$, $\gamma \ne \alpha$, $\lambda \ge 0$, ${}^cD^{\alpha}_{s_i,t}$ is the Caputo fractional derivative of the order α with the lower limit s_i, $0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_m \le s_m \le t_{m+1} = T$ are pre-fixed numbers, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g_i : [t_i,s_i] \times \mathbb{R} \to \mathbb{R}$ are continuous for all $i = 1,2,\ldots,m$ and $q \in \mathbb{R}$. $I^{\gamma}_{t_i,t}g_i$ and $I^{\alpha}_{0,s_i}f$ are given by

$$I^{\gamma}_{t_i,t}g_i\bigl(t,x(t)\bigr) = \frac{1}{\Gamma(\gamma)}\int_{t_i}^{t}(t-s)^{\gamma-1}g_i\bigl(s,x(s)\bigr)\,ds,$$
$$I^{\alpha}_{0,s_i}f\bigl(s_i,x(s_i)\bigr) = \frac{1}{\Gamma(\alpha)}\int_0^{s_i}(s_i-s)^{\alpha-1}f\bigl(s,x(s)\bigr)\,ds.$$

The authors gave a suitable formula of piecewise continuous solutions, and they presented existence results on a compact interval by the contraction mapping principle.

What will happen if the nonlinear equations (1.2) and (1.3) extend into fractional differential equation with delay? We are particularly interested in fractional differential equation involving time-varying delays.

On the other hand, finite-time stability analysis is also one of the most crucial themes for fractional systems, such as [23–27]. In detail, in [23, 24], the authors investigated finite-time stability of Caputo delta fractional difference equations, and a finite-time stability criterion was proposed for the addressed equations. In [25], the authors presented finite-time stability results of nonlinear fractional delay differential equations under mild conditions on the nonlinear term. Li and Wang introduced the concept of a delayed Mittag-Leffler type matrix function, and then they presented the finite-time stability results by virtue of a delayed Mittag-Leffler type matrix in [26]. In [27], the authors firstly established an interesting impulsive Gronwall inequality with maxima involving a Hadamard type singular kernel, which could be applied to making a prior estimation. Secondly, they applied the above inequality and fixed point approach to show two existence results. Finally, they showed the finite-time stability results. In [21, 22], the authors investigated existence and uniqueness theorems for Caputo fractional differential equations. In [20], Ameen et al. studied the Ulam stability and existence theorems for Caputo generalized fractional differential equations where the kernel of the fractional derivative was function dependent so that the result generalized many existing results in history. Further, for more details about some other properties of the solutions, we can see [28–43].

However, when we add non-instantaneous impulsive effects into the fractional systems [24–27], what can we get? Heavily inspired by the papers mentioned, in this paper, we mainly plan to research the existence and finite-time stability results under non-instantaneous impulsive conditions. We first assume two increasing finite sequences of points $\{t_i\}_{i=1}^{p+1}$ and $\{s_i\}_{i=0}^{p}$ are given such that $s_0 = 0 < t_i \leq s_i < t_{i+1}$, $i = 1, 2, \ldots, p$, and points t_0, $T \in \mathbb{R}_+$ are given such that $0 < t_0 < t_1$, $t_p < T \leq t_{p+1}$, p being a natural number. We are concerned with the following ψ-Hilfer fractional differential equation with time-varying delays and non-instantaneous impulses:

$$\begin{cases} {}^{H}D_t^{\alpha,\beta;\psi} x(t) = A(t)x(t) + B(t)x(t-h(t)) + f(t, x(t), x(t-h(t))), \\ \quad t \in J_1 = (s_k, t_{k+1}] \cap [t_0, T], k = 0, 1, \ldots, p, \\ x(t) = \frac{\phi_k(t, x(t))}{\Gamma(\gamma)\Gamma(2-\gamma)}, \quad t \in J_2 = (t_k, s_k] \cap [t_0, T], k = 1, 2, \ldots, p, \\ x(s_k) = x(s_k^+) = x(s_k^-), \quad k = 1, 2, \ldots, p, \\ x(t) = \frac{\phi(t)}{\Gamma(\gamma)\Gamma(2-\gamma)}, \quad t \in J_3 = [-h, t_0], \end{cases} \tag{1.4}$$

where $J = J_1 \cup J_2 \cup J_3$, ${}^{H}D_t^{\alpha,\beta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative of order $\alpha \in (0, 1]$ and type $0 \leq \beta \leq 1$, with respect to function ψ (see [44]), $\gamma = \alpha + \beta(1 - \alpha)$. $x(t)$ is the quantity of state mapping the interval J to $\mathbb{R}$, $A(t)$, $B(t) : \mathbb{R} \to \mathbb{R}$ are bounded operators, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\phi_k : (t_k, s_k] \cap [t_0, T] \times \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, p$, and $h(t)$ is a continuous delay function satisfying $0 \leq h(t) \leq h$, $t \in J$, and $\phi(t)$ denotes an element of the state space $C([-h, t_0], \mathbb{R})$ that is a Banach space of all continuous functions with the norm defined in the following manner as $\|\phi\| := \sup_{-h \leq t \leq t_0} |\phi(t)|$.

Remark 1.1 ([6]) If $t_k = s_k$, $k = 1, 2, \dots, p$, then system (1.4) reduces to an impulsive differential equation. In this case at any point of instantaneous impulse t_k the amount of jump of the solution $x(t)$ is given by $I_k = \frac{\phi_k(t,x(t))}{\Gamma(\gamma)\Gamma(2-\gamma)}$.

Compared with some recent results in the literature, such as [6, 8–13] and some others, the chief contributions of our study contain at least the following four issues:

(1) In [6, 8–10], authors discussed several types of stability except the finite-time stability, and we first introduce the definition of finite-time stability into the ψ-Hilfer fractional differential equation with non-instantaneous impulses.

(2) Compare with [6, 8–13], in system (1.4), we study the equation with time-varying delays, which is a significant breakthrough in dealing with a non-instantaneous impulsive ψ-Hilfer fractional differential system.

(3) The model we are concerned with is more generalized, and some ones in the references are the special cases of it. Thus, the generalized models are originally discussed in the present paper. Furthermore, our conclusions can also be applied to the addressed equation with disturbance term, and we can see it by Remark 3.3.

(4) An innovative method based on the Lagrange mean-value theorem and the generalized Gronwall inequality is exploited to discuss the existence and finite-time stable of the solutions for the ψ-Hilfer fractional order differential equation with time-varying delays and non-instantaneous impulses, and the results established are essentially new.

This article is organized as follows: In Sect. 2, we will recall some well-known results for our consideration. Some lemmas and definitions are useful to our work. Section 3 is devoted to researching the existence and uniqueness of solutions for Eq. (1.4). In Sect. 4, we will investigate the finite-time stability of this ψ-Hilfer fractional order differential equation, and then we will come up with the main theorems. To explain the results clearly, we finally provide three examples in Sect. 5.

2 Preliminaries

In this section, we plan to introduce some basic definitions and lemmas which are used throughout this paper.

Definition 2.1 ([45], One parameter Mittag-Leffler function) The Mittag-Leffler function is given by the series

$$E_\mu(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + 1)},$$

where $\mu \in \mathbb{C}$, $\Re(\mu) > 0$ and $\Gamma(z)$ is the gamma function given by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt,$$

$\Re(z) > 0$. In particular, if $\mu = 1$, we have $E_1(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j+1)} = \sum_{j=0}^{\infty} \frac{z^j}{j!} = e^z$.

Lemma 2.2 ([45], Generalized Gronwall inequality) *Let u, v be two integrable functions and g a continuous function, with domain $[a,b]$. Let $\psi \in C^1[a,b]$ an increasing function such that $\psi'(t) \neq 0$, $\forall t \in [a,b]$. Assume that*:

(1) *u and v are nonnegative;*
(2) *g is nonnegative and nondecreasing.*

If

$$u(t) \le v(t) + g(t) \int_a^t \psi'(s)\big(\psi(t) - \psi(s)\big)^{\alpha-1} u(s)\,ds,$$

then, for all $t \in [a,b]$, we have

$$u(t) \le v(t) + \int_a^t \left[\sum_{k=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^k}{\Gamma(\alpha k)} \psi'(s)\big(\psi(t) - \psi(s)\big)^{\alpha k-1} v(s)\right] ds.$$

Moreover, if $v(t)$ is a nondecreasing function on $[a,b]$, then

$$u(t) \le v(t) E_\alpha\big(g(t)\Gamma(\alpha)\big[\psi(t) - \psi(s)\big]^\alpha\big),$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function defined by Definition 2.1.

Now we introduce a few spaces: let $C(J,\mathbb{R}) = \{x : J \to \mathbb{R} \text{ is continuous }\}$, and consider the piecewise continuous function space $\mathrm{PC}(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x \in C((t_k, t_{k+1}], \mathbb{R})$, and there exist $x(t_k^-)$ and $x(t_k^+)$ with $x(t_k^-) = x(t_k), k = 0, 1, 2, \dots, p\}$ endowed with the norm $\|x\|_{\mathrm{PC}} := \sup\{|x(t)| : t \in J\}$. The weighted space $\mathrm{PC}_{\gamma;\psi}([a,b],\mathbb{R})$ of the functions x on $(a,b]$ is defined by $\mathrm{PC}_{\gamma;\psi}([a,b],\mathbb{R}) = \{x : (a,b] \to \mathbb{R}, (\psi(t) - \psi(a))^\gamma x(t) \in PC([a,b],\mathbb{R})\}$, where $0 \le \gamma \le 1$, with the norm $\|x\|_{\mathrm{PC}_{\gamma;\psi}} := \sup\{|(\psi(t) - \psi(a))^\gamma x(t)|, t \in [a,b]\}$.

Definition 2.3 ([44, 46]) The ψ-Riemann–Liouville fractional integral and fractional derivative of order α $(n-1 < \alpha < n)$ for an integrable function $\Phi : [a,b] \to \mathbb{R}$ with respect to another function $\psi : [a,b] \to \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [a,b]$ $(-\infty \le a < b \le +\infty)$ are defined as follows:

$$I_t^{\alpha;\psi} \Phi(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\xi)\big(\psi(t) - \psi(\xi)\big)^{\alpha-1} \Phi(\xi)\,d\xi$$

for each $t \in I$.

Definition 2.4 ([44, 46]) Let $n-1 < \alpha < n$ with $n \in \mathbb{N}^+$, $I = [a,b]$ be the interval such that $-\infty \le a < b \le +\infty$ and $\psi \in C^n([a,b],\mathbb{R})$ be a function such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in I$. The ψ-Hilfer functional derivative of the function $\Phi \in C^n([a,b],\mathbb{R})$ of order α and the type $0 \le \beta \le 1$ is defined by

$${}^H D_t^{\alpha,\beta;\psi} \Phi(t) = I_t^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n I_t^{(1-\beta)(n-\alpha);\psi} \Phi(t).$$

The right-sided ψ-Hilfer functional derivative is defined in an analogous form [44].

Lemma 2.5 ([44]) *If $f \in \mathrm{PC}_{\gamma;\psi}[a,b]$, $0 < \alpha < 1$, $0 \le \beta \le 1$, $\gamma = \alpha + \beta(1-\alpha)$, then*

$$I_t^{\alpha;\psi}\, {}^H D_t^{\alpha,\beta;\psi} f(t) = f(t) - \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} I_t^{(1-\beta)(1-\alpha);\psi} f(a).$$

Lemma 2.6 *$\mathrm{PC}(J,\mathbb{R})$ is a Banach space in the field $\mathbb{R}$.*

Lemma 2.7 *$\mathrm{PC}_{\gamma;\psi}([a,b],\mathbb{R})$ is a Banach space in the field $\mathbb{R}$.*

Definition 2.8 A function $x \in \mathrm{PC}_{\gamma;\psi}(J,\mathbb{R})$ is called a mild solution of system (1.4) if $x(t) = \frac{\phi(t)}{\Gamma(\gamma)\Gamma(2-\gamma)}$ for all $t \in [-h,t_0]$, and $x(s_k) = x(s_k^+) = x(s_k^-)$, and $x(t) = \frac{\phi_k(t,x(t))}{\Gamma(\gamma)\Gamma(2-\gamma)}$ for all $t \in (t_k,s_k] \cap [t_0,T]$, each $k = 1,2,\dots,p$, and

$$\begin{aligned} x(t) ={}& \frac{\phi(t_0)}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha-1} \\ &\times \bigl[A(\xi)x(\xi) + B(\xi)x\bigl(\xi-h(\xi)\bigr) + f\bigl(\xi,x(\xi),x\bigl(\xi-h(\xi)\bigr)\bigr)\bigr]\,d\xi \end{aligned}$$

for all $t \in [t_0,t_1] \cap [t_0,T]$ and

$$\begin{aligned} x(t) ={}& \frac{\phi_k(s_k,x(s_k))}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_{s_k}^{t} \psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha-1} \\ &\times \bigl[A(\xi)x(\xi) + B(\xi)x\bigl(\xi-h(\xi)\bigr) + f\bigl(\xi,x(\xi),x\bigl(\xi-h(\xi)\bigr)\bigr)\bigr]\,d\xi \end{aligned}$$

for all $t \in [s_k,t_{k+1}] \cap [t_0,T]$ and every $k = 1,2,\dots,p$.

Lemma 2.9 *Let $0 < \alpha \le 1$, $0 \le \beta \le 1$, and $\gamma = \alpha + \beta(1-\alpha)$. If $x \in \mathrm{PC}_{\gamma;\psi}(J,\mathbb{R})$, then x satisfies the problem (1.4) if and only if x satisfies the Volterra integral equation*

$$x(t) = \begin{cases} \frac{\phi(t)}{\Gamma(\gamma)\Gamma(2-\gamma)}, & t \in [-h,t_0], \\ \frac{\phi(t_0)}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}[A(\xi)x(\xi) + B(\xi)x(\xi-h(\xi)) & \\ \quad + f(\xi,x(\xi),x(\xi-h(\xi)))]\,d\xi, & t \in [t_0,t_1] \cap [t_0,T], \\ \frac{\phi_k(t,x(t))}{\Gamma(\gamma)\Gamma(2-\gamma)}, & t \in (t_k,s_k] \cap [t_0,T], k = 1,2,\dots,p, \\ \frac{\phi_k(s_k,x(s_k))}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_{s_k}^{t} \psi'(\xi)(\psi(t)-\psi(\xi))^{\alpha-1}[A(\xi)x(\xi) + B(\xi)x(\xi-h(\xi)) & \\ \quad + f(\xi,x(\xi),x(\xi-h(\xi)))]\,d\xi, & t \in [s_k,t_{k+1}] \cap [t_0,T], k = 1,2,\dots,p. \end{cases} \tag{2.1}$$

Proof ($\Rightarrow$) Let $x \in \mathrm{PC}_{\gamma;\psi}(J,\mathbb{R})$ be a solution of system (1.4), and we show that x is also a solution of (2.1). We can easily obtain $x(t) = \frac{\phi(t)}{\Gamma(\gamma)\Gamma(2-\gamma)}$ for all $t \in [-h,t_0]$. Let $g(t) = A(t)x(t) + B(t)x(t-h(t)) + f(t,x(t),x(t-h(t)))$. For any $t \in [t_0,t_1] \cap [t_0,T]$, applying the fractional integral operator $I_{t_0^+}^{\alpha;\psi}$ on both sides of the first equation in system (1.4), and using Lemma 2.5, we have

$$\begin{aligned} x(t) ={}& \frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)} \frac{1}{\Gamma(1-\gamma)}\int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{-\gamma} x(t_0)\,d\xi \\ &+ I_{t_0^+}^{\alpha;\psi} g(t) \\ ={}& \frac{\phi(t_0)}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t)-\psi(\xi)\bigr)^{\alpha-1} g(\xi)\,d\xi. \end{aligned} \tag{2.2}$$

On the interval $(t_1,s_1] \cap [t_0,T]$, we can obtain

$$x(t) = \frac{\phi_1(t,x(t))}{\Gamma(\gamma)\Gamma(2-\gamma)}. \tag{2.3}$$

For each $t \in (s_1, t_2] \cap [t_0, T]$, by the same approach, we have

$$x(t) = \frac{\phi_1(s_1, x(s_1))}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)} \int_{s_1}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1} g(\xi)\, d\xi. \tag{2.4}$$

The rest functions on intervals can be done in the same manner. Hence, we can obtain (2.1).

($\Leftarrow$) Assume that $x \in \mathrm{PC}_{\gamma;\psi}(J, \mathbb{R})$ satisfying the Volterra integral equation, Eq. (2.1), and we prove that x also satisfies the fractional system (1.4). The following proof process is similar to the relevant conclusion, and we can refer to Lemma 3.1 in [46]. The proof of this lemma is completed. □

Definition 2.10 ([24]) System (1.4) is finite-time stable w.r.t. $\{\delta, \sigma, J\}$, $\delta < \sigma$, if and only if

$$\frac{\|\phi\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \le \delta \quad \Rightarrow \quad \|x\|_{\mathrm{PC}_{\gamma;\psi}} \le \sigma, \quad \forall t \in J.$$

3 Existence and uniqueness of solutions

In this section, we will consider the existence and uniqueness of solutions for the ψ-Hilfer fractional differential equation with time-varying delays and non-instantaneous impulsive effects. $B^+(J)$ denotes the set of all nonnegative bounded functions on interval J.

Define the operator $H : \mathrm{PC}_{\gamma;\psi}(J, \mathbb{R}) \to \mathrm{PC}_{\gamma;\psi}(J, \mathbb{R})$ by

$$(Hx)(t) = \begin{cases} \frac{\phi(t)}{\Gamma(\gamma)\Gamma(2-\gamma)}, & t \in [-h, t_0], \\ \frac{\phi(t_0)}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1} [A(\xi)x(\xi) + B(\xi)x(\xi - h(\xi)) \\ \quad + f(\xi, x(\xi), x(\xi - h(\xi)))]\, d\xi, & t \in [t_0, t_1] \cap [t_0, T], \\ \frac{\phi_k(t, x(t))}{\Gamma(\gamma)\Gamma(2-\gamma)}, & t \in (t_k, s_k] \cap [t_0, T], k = 1, 2, \ldots, p, \\ \frac{\phi_k(s_k, x(s_k))}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \psi'(\xi)(\psi(t) - \psi(\xi))^{\alpha-1} [A(\xi)x(\xi) + B(\xi)x(\xi - h(\xi)) \\ \quad + f(\xi, x(\xi), x(\xi - h(\xi)))]\, d\xi, & t \in [s_k, t_{k+1}] \cap [t_0, T], k = 1, 2, \ldots, p. \end{cases} \tag{3.1}$$

Before stating and proving the results, we introduce the following hypotheses:

(H_0) There exist constants $M_A, M_B > 0$ such that, for any $t \in J$,

$$\bigl|A(t)x(t)\bigr| \le M_A \cdot \bigl|x(t)\bigr| \quad \text{and} \quad \bigl|B(t)x(t)\bigr| \le M_B \cdot \bigl|x(t)\bigr|.$$

(H_1) Assume that the nonlinear function f satisfies: there exists a positive function $l(t) \in B^+(J)$ such that

$$\bigl|f(t, x_1, y_1) - f(t, x_2, y_2)\bigr| \le l(t) \cdot \bigl(|x_1 - x_2| + |y_1 - y_2|\bigr),$$

where $t \in J$, $x_1, x_2, y_1, y_2 \in \mathrm{PC}_{\gamma;\psi}(J, \mathbb{R})$, and $f(t, 0, 0) = 0$, $k = 1, 2, \ldots, m$.

(H_2) For each $k = 1, 2, \ldots, p$, and $x_1, x_2 \in \mathrm{PC}_{\gamma;\psi}(J, \mathbb{R})$, there exist constants λ_{k0}, λ_{k1} such that

$$\bigl|\phi_k(t_1, x_1) - \phi_k(t_2, x_2)\bigr| \le \lambda_{k0}|t_1 - t_2| + \lambda_{k1}|x_1 - x_2|.$$

Furthermore, for any $t \in J$, we have $|(\psi(T) - \psi(t_0))^{\gamma} \phi_k(t, x(t))| \le d_k$, where d_k is a positive constant.

Throughout this paper, we always assume that

$$\sup_{\xi \in [t_0, t]} l(\xi) = \bar{l}$$

and

$$\lambda = \max\{\lambda_{k0}, \lambda_{k1}\}, \quad k = 1, 2, \ldots, p.$$

Theorem 3.1 *Suppose the validity of conditions* (H_0)–(H_2), *then system* (1.4) *has at least one solution in* $\Omega_\rho = \{x \in \mathrm{PC}_{\gamma;\psi}(J, \mathbb{R}) : \|x\|_{\mathrm{PC}_{\gamma;\psi}} \le \rho\}$ *if*

$$\frac{\overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma) \cdot \rho} \le 1 - \frac{(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^{\alpha}}{\Gamma(\alpha+1)} \tag{3.2}$$

holds, where $\overline{M}_1 = \max\{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}, d_k\}$, $k = 1, 2, \ldots, p$.

Proof We define $\Omega_\rho = \{x \in \mathrm{PC}_{\gamma;\psi}(J, \mathbb{R}) : \|x\|_{\mathrm{PC}_{\gamma;\psi}} \le \rho\}$, which is a closed, bounded and convex subset of $\mathrm{PC}_{\gamma;\psi}(J, \mathbb{R})$. We shall use Schauder's fixed point theorem to prove that operator H has a fixed point. The proof will be given in the following steps.

Step 1: H maps bounded sets into bounded sets.

It is enough to show that there exists a positive constant ρ, we have $\|Hx\|_{\mathrm{PC}_{\gamma;\psi}} \le \rho$ for each $x \in \Omega_\rho$.

For all $t \in [-h, t_0]$, we can easily get $\|Hx\|_{\mathrm{PC}_{\gamma;\psi}} = \frac{\|\phi\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \le \rho$.

As for $t \in [t_0, t_1] \cap [t_0, T]$ and by the condition (3.1) and (3.2), we have

$$\begin{aligned}
&\bigl|\bigl(\psi(t) - \psi(t_0)\bigr)^{\gamma}(Hx)(t)\bigr| \\
&\quad\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1}\bigl[M_A \cdot \bigl|\bigl(\psi(t) - \psi(t_0)\bigr)^{\gamma} x(\xi)\bigr| \\
&\qquad + M_B \cdot \bigl|\bigl(\psi(t) - \psi(t_0)\bigr)^{\gamma} x\bigl(\xi - h(\xi)\bigr)\bigr| \\
&\qquad + l(\xi) \cdot \bigl(\bigl(\psi(t) - \psi(t_0)\bigr)^{\gamma}\bigl|x(\xi)\bigr| + \bigl(\psi(t) - \psi(t_0)\bigr)^{\gamma}\bigl|x\bigl(\xi - h(\xi)\bigr)\bigr|\bigr)\bigr]\,d\xi \\
&\qquad + \frac{|(\psi(t) - \psi(t_0))^{\gamma}\phi(t_0)|}{\Gamma(\gamma)\Gamma(2-\gamma)} \\
&\quad\le \frac{(M_A + M_B + 2\bar{l}) \cdot \|x\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1}\,d\xi + \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \\
&\quad\le \frac{\rho(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^{\alpha}}{\Gamma(\alpha+1)} + \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \\
&\quad\le \rho.
\end{aligned} \tag{3.3}$$

According to the definition of the norm in the weighted space, one can obtain

$$\|Hx\|_{\mathrm{PC}_{\gamma;\psi}} \le \rho. \tag{3.4}$$

For each $t \in (t_k, s_k] \cap [t_0, T]$, $k = 1, 2, \ldots, p$, and according to (3.2), we have

$$\begin{aligned} \|Hx\|_{\mathrm{PC}_{\gamma;\psi}} &= \left|\left(\psi(t) - \psi(t_k)\right)^{\gamma} (Hx)(t)\right| \\ &\leq \frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)} \leq \frac{\overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)} \leq \rho. \end{aligned} \tag{3.5}$$

As for any $t \in [s_k, t_{k+1}] \cap [t_0, T]$, we have

$$\begin{aligned} &\left|\left(\psi(t) - \psi(s_k)\right)^{\gamma} (Hx)(t)\right| \\ &\quad \leq \frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{\rho(M_A + M_B + 2\bar{l})}{\Gamma(\alpha)} \int_{s_k}^{t} \psi'(\xi)\left(\psi(t) - \psi(\xi)\right)^{\alpha-1} d\xi \\ &\quad \leq \frac{\overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{\rho(M_A + M_B + 2\bar{l})(\psi(t) - \psi(s_k))^{\alpha}}{\Gamma(\alpha+1)} \\ &\quad \leq \frac{\overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{\rho(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^{\alpha}}{\Gamma(\alpha+1)} \leq \rho. \end{aligned} \tag{3.6}$$

Hence, we can get $\|Hx\|_{\mathrm{PC}_{\gamma;\psi}} \leq \rho$, for $t \in [s_k, t_{k+1}] \cap [t_0, T]$.

Step 2: H is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ ($n \to \infty$) in Ω_ρ. As for $t \in [-h, t_0]$, we have

$$\|Hx_n - Hx\|_{\mathrm{PC}_{\gamma;\psi}} = 0, \tag{3.7}$$

which implies that H is continuous.

For $\forall t \in [t_0, t_1] \cap [t_0, T]$, we have

$$\begin{aligned} &\left|\left(\psi(t) - \psi(t_0)\right)^{\gamma} (Hx_n - Hx)(t)\right| \\ &\quad \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\left(\psi(t) - \psi(\xi)\right)^{\alpha-1} \Big[M_A \cdot \left|\left(\psi(t) - \psi(t_0)\right)^{\gamma} \left(x_n(\xi) - x(\xi)\right)\right| \\ &\qquad + M_B \cdot \left|\left(\psi(t) - \psi(t_0)\right)^{\gamma} \left(x_n\left(\xi - h(\xi)\right) - x\left(\xi - h(\xi)\right)\right)\right| \\ &\qquad + l(\xi) \cdot \Big(\left(\psi(t) - \psi(t_0)\right)^{\gamma} \left|x_n(\xi) - x(\xi)\right| \\ &\qquad + \left(\psi(t) - \psi(t_0)\right)^{\gamma} \left|x_n\left(\xi - h(\xi)\right) - x\left(\xi - h(\xi)\right)\right|\Big)\Big] d\xi \\ &\quad \leq \frac{(M_A + M_B + 2\bar{l}) \cdot \|x_n - x\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\left(\psi(t) - \psi(\xi)\right)^{\alpha-1} d\xi \\ &\quad \leq \frac{(\psi(T) - \psi(t_0))^{\alpha}}{\Gamma(\alpha+1)} (M_A + M_B + 2\bar{l}) \cdot \|x_n - x\|_{\mathrm{PC}_{\gamma;\psi}}, \end{aligned} \tag{3.8}$$

then we get

$$\|Hx_n - Hx\|_{\mathrm{PC}_{\gamma;\psi}} \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

For all $t \in (t_k, s_k] \cap [t_0, T]$, $k = 1, 2, \dots, p$, we have

$$\begin{aligned}
&\big|\big(\psi(t) - \psi(t_k)\big)^{\gamma}(Hx_n - Hx)(t)\big| \\
&\quad \leq \frac{1}{\Gamma(\gamma)\Gamma(2-\gamma)}\big[\lambda_{k1}\big(\psi(t) - \psi(t_k)\big)^{\gamma} \cdot \big|x_n(t) - x(t)\big|\big] \\
&\quad \leq \frac{\lambda}{\Gamma(\gamma)\Gamma(2-\gamma)} \cdot \|x_n - x\|_{\mathrm{PC}_{\gamma;\psi}} \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.10}$$

For each $t \in [s_k, t_{k+1}] \cap [t_0, T]$, $k = 1, 2, \dots, p$, we have

$$\begin{aligned}
&\big|\big(\psi(t) - \psi(s_k)\big)^{\gamma}(Hx_n - Hx)(t)\big| \\
&\quad \leq \frac{\lambda_{k1} \cdot (\psi(t) - \psi(s_k))^{\gamma}|x_n(s_k) - x(s_k)|}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{1}{\Gamma(\alpha)}\int_{s_k}^{t} \psi'(\xi)\big(\psi(t) - \psi(\xi)\big)^{\alpha-1} \\
&\quad\quad \times \big[M_A \cdot \big|\big(\psi(t) - \psi(s_k)\big)^{\gamma}\big(x_n(\xi) - x(\xi)\big)\big| \\
&\quad\quad + M_B \cdot \big|\big(\psi(t) - \psi(s_k)\big)^{\gamma}\big(x_n\big(\xi - h(\xi)\big) - x\big(\xi - h(\xi)\big)\big)\big| \\
&\quad\quad + l(\xi) \cdot \big(\big(\psi(t) - \psi(s_k)\big)^{\gamma}\big|x_n(\xi) - x(\xi)\big| \\
&\quad\quad + \big(\psi(t) - \psi(s_k)\big)^{\gamma}\big|x_n\big(\xi - h(\xi)\big) - x\big(\xi - h(\xi)\big)\big|\big)\big]\,d\xi \\
&\quad \leq \frac{\lambda\|x_n - x\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{(M_A + M_B + 2\bar{l}) \cdot \|x_n - x\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\alpha)} \\
&\quad\quad \times \int_{s_k}^{t} \psi'(\xi)\big(\psi(t) - \psi(\xi)\big)^{\alpha-1}\,d\xi \\
&\quad \leq \left(\frac{\lambda}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{(M_A + M_B + 2\bar{l})(\psi(T) - \psi(s_k))^{\alpha}}{\Gamma(\alpha+1)}\right) \cdot \|x_n - x\|_{\mathrm{PC}_{\gamma;\psi}} \\
&\quad \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.11}$$

Step 3: H maps bounded sets into equicontinuous sets.
It is obvious that Hx is equicontinuous on the time interval $[-h, t_0]$.
As for $r_1, r_2 \in [t_0, t_1] \cap [t_0, T]$, $r_1 < r_2$, we have

$$\begin{aligned}
&\big|\big[\big(\psi(t) - \psi(t_0)\big)^{\gamma}\big]\big[(Hx)(r_2) - (Hx)(r_1)\big]\big| \\
&\quad \leq \frac{1}{\Gamma(\alpha)}\int_{t_0}^{r_1} \psi'(\xi)\big[\big(\psi(r_2) - \psi(\xi)\big)^{\alpha-1} - \big(\psi(r_1) - \psi(\xi)\big)^{\alpha-1}\big] \\
&\quad\quad \times \big[M_A \cdot \big(\psi(t) - \psi(t_0)\big)^{\gamma}\big|x(\xi)\big| + M_B \cdot \big(\psi(t) - \psi(t_0)\big)^{\gamma}\big|x\big(\xi - h(\xi)\big)\big| \\
&\quad\quad + \big(\psi(t) - \psi(t_0)\big)^{\gamma}\big|f\big(\xi, x(\xi), x\big(\xi - h(\xi)\big)\big)\big|\big]\,d\xi \\
&\quad\quad + \frac{1}{\Gamma(\alpha)}\int_{r_1}^{r_2} \psi'(\xi)\big(\psi(r_2) - \psi(\xi)\big)^{\alpha-1}\big[M_A \cdot \big(\psi(t) - \psi(t_0)\big)^{\gamma}\big|x(\xi)\big| \\
&\quad\quad + M_B \cdot \big(\psi(t) - \psi(t_0)\big)^{\gamma}\big|x\big(\xi - h(\xi)\big)\big| \\
&\quad\quad + \big(\psi(t) - \psi(t_0)\big)^{\gamma}\big|f\big(\xi, x(\xi), x\big(\xi - h(\xi)\big)\big)\big|\big]\,d\xi
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\rho(M_A + M_B + 2\bar{l})}{\Gamma(\alpha)} \int_{t_0}^{r_1} \psi'(\xi)\bigl[\bigl(\psi(r_2) - \psi(\xi)\bigr)^{\alpha-1} - \bigl(\psi(r_1) - \psi(\xi)\bigr)^{\alpha-1}\bigr]\,d\xi \\
&\quad + \frac{\rho(M_A + M_B + 2\bar{l})}{\Gamma(\alpha)} \int_{r_1}^{r_2} \psi'(\xi)\bigl(\psi(r_2) - \psi(\xi)\bigr)^{\alpha-1}\,d\xi. \qquad (3.12)
\end{aligned}$$

By the Lagrange mean-value theorem, we have

$$\begin{aligned}
&\bigl|\bigl[\bigl(\psi(t) - \psi(t_0)\bigr)^{\gamma}\bigr]\bigl[(Hx)(r_2) - (Hx)(r_1)\bigr]\bigr| \\
&\quad\leq \frac{(\alpha-1)\rho\psi'(r)(M_A + M_B + 2\bar{l})\cdot(r_2 - r_1)}{\Gamma(\alpha)} \int_{t_0}^{r_1} \psi'(\xi)\bigl(\psi(r) - \psi(\xi)\bigr)^{\alpha-2}\,d\xi \\
&\qquad + \frac{\rho(M_A + M_B + 2\bar{l})}{\Gamma(\alpha)} \int_{r_1}^{r_2} \psi'(\xi)\bigl(\psi(r_2) - \psi(\xi)\bigr)^{\alpha-1}\,d\xi, \quad r_1 \leq r \leq r_2, \qquad (3.13)
\end{aligned}$$

as $r_1 \to r_2$, the right-hand side of the above inequality tends to zero. Then one can obtain $\|(Hx)(r_2) - (Hx)(r_1)\|_{\mathrm{PC}_{\gamma;\psi}} \to 0$, and Hx is equicontinuous on interval $[t_0, t_1] \cap [t_0, T]$.

In general, for the time interval $(t_k, s_k] \cap [t_0, T]$, $k = 1, 2, \ldots, p$, we can obtain the following inequality:

$$\begin{aligned}
&\bigl\|(Hx)(r_2) - (Hx)(r_1)\bigr\|_{\mathrm{PC}_{\gamma;\psi}} \\
&\quad\leq \frac{\lambda_{k0}\cdot\|r_2 - r_1\|_{\mathrm{PC}_{\gamma;\psi}} + \lambda_{k1}\cdot\|x(r_2) - x(r_1)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \\
&\quad\to 0, \quad \text{as } r_1 \to r_2. \qquad (3.14)
\end{aligned}$$

Hence, Hx is equicontinuous.

For all $t \in [s_k, t_{k+1}] \cap [t_0, T]$, similarly, we use the Lagrange mean-value theorem and get

$$\bigl\|(Hx)(r_2) - (Hx)(r_1)\bigr\|_{\mathrm{PC}_{\gamma;\psi}} \to 0, \quad \text{as } r_1 \to r_2. \qquad (3.15)$$

Therefore, Hx is equicontinuous. Applying the PC-type Ascoli–Arzela theorem, we can conclude that $H : \Omega_\rho \to \Omega_\rho$ is completely continuous. As a consequence of the Schauder's fixed point theorem, we deduce that H has a fixed point in Ω_ρ which is a solution of system (1.4). The proof is completed. □

Theorem 3.2 *Assume that the conditions* (H_0)–(H_2) *hold, then system* (1.4) *has one solution in* $\mathrm{PC}_{\gamma;\psi}(J, \mathbb{R})$ *if*

$$0 \leq \frac{\lambda}{\Gamma(\gamma)\Gamma(2-\gamma)} + \frac{(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^{\alpha}}{\Gamma(\alpha+1)} < 1 \qquad (3.16)$$

holds.

Proof For $\forall t \in [t_0, t_1] \cap [t_0, T]$, according to the operator expression (3.1), we have

$$\begin{aligned}
&\|Hy - Hx\|_{\mathrm{PC}_{\gamma;\psi}} \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1}\bigl[\bigl(M_A + l(\xi)\bigr) \cdot \bigl\|y(\xi) - x(\xi)\bigr\|_{\mathrm{PC}_{\gamma;\psi}} \\
&\qquad + \bigl(M_B + l(\xi)\bigr) \cdot \bigl\|y\bigl(\xi - h(\xi)\bigr) - x\bigl(\xi - h(\xi)\bigr)\bigr\|_{\mathrm{PC}_{\gamma;\psi}}\bigr]\,d\xi \\
&\quad \le \frac{(M_A + M_B + 2\bar{l}) \cdot \|y - x\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1}\,d\xi \\
&\quad = \frac{M_A + M_B + 2\bar{l}}{\Gamma(\alpha)} \cdot \frac{(\psi(t) - \psi(t_0))^{\alpha}}{\alpha} \cdot \|y - x\|_{\mathrm{PC}_{\gamma;\psi}} \\
&\quad \le \frac{(M_A + M_B + 2\bar{l})(\psi(t_1) - \psi(t_0))^{\alpha}}{\Gamma(\alpha + 1)} \cdot \|y - x\|_{\mathrm{PC}_{\gamma;\psi}}.
\end{aligned} \tag{3.17}$$

By (3.16) and (3.17), we get

$$\begin{aligned}
\|Hy - Hx\|_{\mathrm{PC}_{\gamma;\psi}} &\le \frac{(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^{\alpha}}{\Gamma(\alpha + 1)} \cdot \|y - x\|_{\mathrm{PC}_{\gamma;\psi}} \\
&< \|y - x\|_{\mathrm{PC}_{\gamma;\psi}},
\end{aligned} \tag{3.18}$$

which implies that the operator H is a contractive mapping.

As for each $t \in (t_k, s_k] \cap [t_0, T]$, $k = 1, 2, \ldots, p$, by (3.1) and (3.16), we have

$$\|Hy - Hx\|_{\mathrm{PC}_{\gamma;\psi}} \le \frac{\lambda \cdot \|y - x\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2 - \gamma)} < \|y - x\|_{\mathrm{PC}_{\gamma;\psi}}, \tag{3.19}$$

and one can see that the operator H is a contractive mapping.

As for all $t \in [s_k, t_{k+1}] \cap [t_0, T]$, similarly, we have

$$\begin{aligned}
\|Hy - Hx\|_{\mathrm{PC}_{\gamma;\psi}} &\le \left(\frac{\lambda}{\Gamma(\gamma)\Gamma(2 - \gamma)} + \frac{(M_A + M_B + 2\bar{l})(\psi(t) - \psi(s_k))^{\alpha}}{\Gamma(\alpha + 1)}\right) \\
&\quad \cdot \|y - x\|_{\mathrm{PC}_{\gamma;\psi}} \\
&\le \left(\frac{\lambda}{\Gamma(\gamma)\Gamma(2 - \gamma)} + \frac{(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^{\alpha}}{\Gamma(\alpha + 1)}\right) \\
&\quad \cdot \|y - x\|_{\mathrm{PC}_{\gamma;\psi}} \\
&< \|y - x\|_{\mathrm{PC}_{\gamma;\psi}},
\end{aligned} \tag{3.20}$$

and we can also see H is a contractive mapping. As a consequence of the Banach fixed point theorem, we conclude that the operator H has a unique fixed point $x \in \mathrm{PC}_{\gamma;\psi}(J, \mathbb{R})$, which is the solution of system (1.4). This completes the proof of Theorem 3.2. □

If we add the disturbance term $G(t)$ into (1.4), we get the following system:

$$\begin{cases} {}^{H}D_t^{\alpha,\beta;\psi}x(t) = A(t)x(t) + B(t)x(t-h(t)) + G(t) + f(t,x(t),x(t-h(t))), \\ \quad t \in J_1 = (s_k, t_{k+1}] \cap [t_0, T], k = 0, 1, \ldots, p, \\ x(t) = \frac{\phi_k(t,x(t))}{\Gamma(\gamma)\Gamma(2-\gamma)}, \quad t \in J_2 = (t_k, s_k] \cap [t_0, T], k = 1, 2, \ldots, p, \\ x(s_k) = x(s_k^+) = x(s_k^-), \quad k = 1, 2, \ldots, p, \\ x(t) = \frac{\phi(t)}{\Gamma(\gamma)\Gamma(2-\gamma)}, \quad t \in J_3 = [-h, t_0], \end{cases} \tag{3.21}$$

where $G(t) \in C(J, R)$ is bounded satisfying $G(t) \leq \zeta$ for all $t \in J$.

Remark 3.3 According to the proof of Theorem 3.1 and Theorem 3.2, we find that the disturbance term $G(t)$ will not affect the unique result, but will affect the existence result in system (3.21). Therefore, the condition (3.2) in Theorem 3.1 should be modified into

$$\frac{\overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)\cdot\rho} \leq 1 - \frac{[(M_A + M_B + 2\bar{l}) + \frac{\zeta}{\rho}](\psi(T) - \psi(t_0))^{\alpha}}{\Gamma(\alpha+1)},$$

where $\overline{M}_1$ is defined as Theorem 3.1.

4 Finite-time stability results

In this section, we mainly investigate the finite-time stability of Eq. (1.4).

Theorem 4.1 *Suppose the validity of conditions* (H_0), (H_1) *and* (H_2), *and there exist two positive real numbers* δ *and* σ *such that* $\delta < \sigma$, *and* $\frac{\|\phi\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \leq \delta$, *then system* (1.4) *is finite-time stable on* J *provided that*

$$\int_{t_0}^{T} \sum_{n=1}^{\infty} \frac{(M_A + M_B + 2\bar{l})^n}{\Gamma(n\alpha)} \psi'(\xi)\big(\psi(T) - \psi(\xi)\big)^{n\alpha-1}\,d\xi \leq \frac{\sigma\Gamma(\gamma)\Gamma(2-\gamma)}{\overline{M}_1} - 1 \tag{4.1}$$

holds, where $\overline{M}_1 = \max\{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}, d_k\}$.

Proof In view of the expression of the solution (2.1), for all $t \in [-h, t_0]$, it is obvious that system (1.4) is finite-time stable.

For $\forall t \in [t_0, t_1] \cap [t_0, T]$, we have

$$\begin{aligned} \|x\|_{\mathrm{PC}_{\gamma;\psi}} \leq{}& \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\big(\psi(t) - \psi(\xi)\big)^{\alpha-1}\big[M_A \cdot \big\|x(\xi)\big\|_{\mathrm{PC}_{\gamma;\psi}} + M_B \\ &\cdot \big\|x\big(\xi - h(\xi)\big)\big\|_{\mathrm{PC}_{\gamma;\psi}} + l(\xi)\cdot\big(\big\|x(\xi)\big\|_{\mathrm{PC}_{\gamma;\psi}} + \big\|x\big(\xi - h(\xi)\big)\big\|_{\mathrm{PC}_{\gamma;\psi}}\big)\big]\,d\xi \\ &+ \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ ={}& \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\big(\psi(t) - \psi(\xi)\big)^{\alpha-1}\big[\big(M_A + l(\xi)\big)\cdot\big\|x(\xi)\big\|_{\mathrm{PC}_{\gamma;\psi}} \\ &+ \big(M_B + l(\xi)\big)\cdot\big\|x\big(\xi - h(\xi)\big)\big\|_{\mathrm{PC}_{\gamma;\psi}}\big]\,d\xi + \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}. \end{aligned} \tag{4.2}$$

Let us set $u(t) = \sup_{\theta\in[-h,t]} \|x(\theta)\|_{\mathrm{PC}_{\gamma;\psi}}$, $\forall t \in [t_0, t_1] \cap [t_0, T]$, and $\|x(\xi)\|_{\mathrm{PC}_{\gamma;\psi}} \le u(\xi)$, $\|x(\xi - h(\xi))\|_{\mathrm{PC}_{\gamma;\psi}} \le u(\xi)$, $\forall \xi \in [t_0, t]$, from (4.2) it follows that

$$\begin{aligned}\|x(t)\|_{\mathrm{PC}_{\gamma;\psi}} \le{}& \frac{M_A + M_B + 2\bar{l}}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1} u(\xi)\,d\xi \\ &+ \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}.\end{aligned}\tag{4.3}$$

Note that for all $\theta \in [t_0, t]$ we can obtain

$$\begin{aligned}\|x(\theta)\|_{\mathrm{PC}_{\gamma;\psi}} \le{}& \frac{M_A + M_B + 2\bar{l}}{\Gamma(\alpha)} \int_{t_0}^{\theta} \psi'(\xi)\bigl(\psi(\theta) - \psi(\xi)\bigr)^{\alpha-1} u(\xi)\,d\xi \\ &+ \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}.\end{aligned}\tag{4.4}$$

Let

$$a(\theta) = \int_{t_0}^{\theta} \psi'(\xi)\bigl(\psi(\theta) - \psi(\xi)\bigr)^{\alpha-1} u(\xi)\,d\xi, \quad 0 < \alpha \le 1,\tag{4.5}$$

then

$$\begin{aligned}a'(\theta) ={}& \lim_{\xi\to\theta^-} \psi'(\xi)\bigl(\psi(\theta) - \psi(\xi)\bigr)^{\alpha-1} u(\xi) \\ &+ (\alpha - 1) \int_{t_0}^{\theta} \psi'(\xi)\bigl(\psi(\theta) - \psi(\xi)\bigr)^{\alpha-2} \psi'(\theta) u(\xi)\,d\xi \\ \ge{}& 0.\end{aligned}\tag{4.6}$$

Therefore, $a(\theta)$ is a nondecreasing function, and we have

$$\begin{aligned}\|x(\theta)\|_{\mathrm{PC}_{\gamma;\psi}} \le{}& \frac{M_A + M_B + 2\bar{l}}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1} u(\xi)\,d\xi \\ &+ \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}.\end{aligned}\tag{4.7}$$

Hence, we can get

$$\begin{aligned}u(t) ={}& \sup_{\theta\in[-h,t]} \|x(\theta)\|_{\mathrm{PC}_{\gamma;\psi}} \\ \le{}& \max\Bigl\{ \sup_{\theta\in[-h,t_0]} \|x(\theta)\|_{\mathrm{PC}_{\gamma;\psi}}, \sup_{\theta\in[t_0,t]} \|x(\theta)\|_{\mathrm{PC}_{\gamma;\psi}} \Bigr\} \\ \le{}& \max\Bigl\{ \delta, \frac{M_A + M_B + 2\bar{l}}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1} u(\xi)\,d\xi \\ &+ \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \Bigr\}.\end{aligned}\tag{4.8}$$

By using the generalized Gronwall inequality (see Lemma 2.2) and (4.1), we have

$$\begin{aligned}
&\frac{M_A + M_B + 2\bar{l}}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1} u(\xi)\,d\xi + \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \\
&\quad \le \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \\
&\quad\quad \times \left(1 + \int_{t_0}^{T} \sum_{n=1}^{\infty} \frac{(M_A + M_B + 2\bar{l})^n}{\Gamma(n\alpha)} \psi'(\xi)\bigl(\psi(T) - \psi(\xi)\bigr)^{n\alpha-1}\,d\xi\right) \\
&\quad \le \sigma,
\end{aligned} \tag{4.9}$$

and one can obtain $\|x(t)\|_{\mathrm{PC}_{\gamma;\psi}} \le u(t) \le \sigma$ from (4.8) and (4.9).

For each $t \in (t_k, s_k] \cap [t_0, T]$, $k = 1, 2, \ldots, p$, and we can derive $\frac{\overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)} \le \sigma$ from (4.1). Therefore, we have

$$\bigl\|x(t)\bigr\|_{\mathrm{PC}_{\gamma;\psi}} \le \frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)} \le \frac{\overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)} \le \sigma. \tag{4.10}$$

As for $t \in [s_k, t_{k+1}] \cap [t_0, T]$, $k = 1, 2, \ldots, p$, we set $u(t) = \sup_{\theta \in [-h,t]} \|x(\theta)\|_{\mathrm{PC}_{\gamma;\psi}}$, $\forall t \in [s_k, t_{k+1}] \cap [t_0, T]$, similarly, we have

$$\begin{aligned}
u(t) \le \max\biggl\{&\delta, \frac{M_A + M_B + 2\bar{l}}{\Gamma(\alpha)} \int_{s_k}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1} u(\xi)\,d\xi \\
&+ \frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)}\biggr\}.
\end{aligned} \tag{4.11}$$

By the generalized Gronwall inequality (see Lemma 2.2) and (4.1), we have

$$\begin{aligned}
&\frac{M_A + M_B + 2\bar{l}}{\Gamma(\alpha)} \int_{s_k}^{t} \psi'(\xi)\bigl(\psi(t) - \psi(\xi)\bigr)^{\alpha-1} u(\xi)\,d\xi + \frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)} \\
&\quad \le \frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)} \\
&\quad\quad \times \left(1 + \int_{s_k}^{t} \sum_{n=1}^{\infty} \frac{(M_A + M_B + 2\bar{l})^n}{\Gamma(n\alpha)} \psi'(\xi)\bigl(\psi(T) - \psi(\xi)\bigr)^{n\alpha-1}\,d\xi\right) \\
&\quad \le \frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)} \\
&\quad\quad \times \left(1 + \int_{t_0}^{T} \sum_{n=1}^{\infty} \frac{(M_A + M_B + 2\bar{l})^n}{\Gamma(n\alpha)} \psi'(\xi)\bigl(\psi(T) - \psi(\xi)\bigr)^{n\alpha-1}\,d\xi\right) \\
&\quad \le \sigma,
\end{aligned} \tag{4.12}$$

and it follows that $\|x(t)\|_{\mathrm{PC}_{\gamma;\psi}} \le u(t) \le \sigma$ from (4.11) and (4.12). By Definition 2.10, we see that system (1.4) is finite-time stable on J. The proof of the theorem is completed. □

Theorem 4.2 *Assume that* (H_0), (H_1) *and* (H_2) *are satisfied, and* $x(t)$ *is the solution of system* (1.4). *If* $(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^\alpha < \Gamma(\alpha + 1)$, *then we have the following*:

$$M \le \max\left\{ \frac{\|\phi\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}, \frac{\Gamma(\alpha+1)\cdot \overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)[\Gamma(\alpha+1) - (M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^\alpha]} \right\}, \tag{4.13}$$

where $M = \sup_{t\in J} \|x(t)\|_{\mathrm{PC}_{\gamma;\psi}}$, *and* $\overline{M}_1 = \max\{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}, d_k\}$, $k = 1, 2, \ldots, p$.

Proof For all $t \in [-h, t_0]$, by (2.1), we have

$$\|x(t)\|_{\mathrm{PC}_{\gamma;\psi}} \le \frac{\|\phi\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}. \tag{4.14}$$

For each $t \in [t_0, t_1] \cap [t_0, T]$, and by (2.1), we have

$$\begin{aligned} &\|x(t)\|_{\mathrm{PC}_{\gamma;\psi}} \\ &\quad\le \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \psi'(\xi)\big(\psi(t) - \psi(\xi)\big)^{\alpha-1}\big[M_A \cdot \|x(\xi)\|_{\mathrm{PC}_{\gamma;\psi}} + M_B \\ &\qquad \cdot \|x(\xi - h(\xi))\|_{\mathrm{PC}_{\gamma;\psi}} + l(\xi)\cdot\big(\|x(\xi)\|_{\mathrm{PC}_{\gamma;\psi}} + \|x(\xi - h(\xi))\|_{\mathrm{PC}_{\gamma;\psi}}\big)\big]\,d\xi \\ &\qquad + \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ &\quad\le \frac{M(M_A + M_B + 2\bar{l})}{\Gamma(\alpha)} \int_{t_0}^t \psi'(\xi)\big(\psi(t) - \psi(\xi)\big)^{\alpha-1}\,d\xi + \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ &\quad\le \frac{M(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^\alpha}{\Gamma(\alpha+1)} + \frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}. \end{aligned} \tag{4.15}$$

For $t \in (t_k, s_k] \cap [t_0, T]$, $k = 1, 2, \ldots, p$, we have

$$\|x(t)\|_{\mathrm{PC}_{\gamma;\psi}} \le \frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)}. \tag{4.16}$$

As for all $t \in [s_k, t_{k+1}] \cap [t_0, T]$, similarly to (4.15), $k = 1, 2, \ldots, p$, we have

$$\begin{aligned} \|x(t)\|_{\mathrm{PC}_{\gamma;\psi}} &\le \frac{M(M_A + M_B + 2\bar{l})(\psi(T) - \psi(s_k))^\alpha}{\Gamma(\alpha+1)} + \frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)} \\ &\le \frac{M(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^\alpha}{\Gamma(\alpha+1)} + \frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)}, \end{aligned} \tag{4.17}$$

By inequalities (4.15)–(4.17), it follows that

$$M \le \frac{M(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^\alpha}{\Gamma(\alpha+1)} + \frac{\overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)}, \tag{4.18}$$

which implies that

$$M \leq \frac{\Gamma(\alpha+1)\cdot \overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)[\Gamma(\alpha+1)-(M_A+M_B+2\bar{l})(\psi(T)-\psi(t_0))^{\alpha}]}. \tag{4.19}$$

Hence, by (4.14) and (4.19), we can easily get

$$\begin{aligned} M &= \sup_{t\in J}\|x(t)\|_{\mathrm{PC}_{\gamma;\psi}} \\ &\leq \max\Biggl\{\frac{\|\phi\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}, \\ &\quad \frac{\Gamma(\alpha+1)\cdot \overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)[\Gamma(\alpha+1)-(M_A+M_B+2\bar{l})(\psi(T)-\psi(0))^{\alpha}]}\Biggr\}. \end{aligned} \tag{4.20}$$

The proof is completed. □

Theorem 4.3 *Under the hypotheses* (H_0), (H_1) *and* (H_2), *the system* (1.4) *is finite-time stable w.r.t.* $\{\delta,\sigma,J\}$ *with* $\delta<\sigma$ *and* $\frac{\|\phi\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}\leq\delta$, *if the following conditions are satisfied*:

- (H_3) *for all* $k=1,2,\ldots,p$, *we have* $d_k\leq\frac{\|\phi\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}$.
- (H_4) $(M_A+M_B+2\bar{l})(\psi(T)-\psi(t_0))^{\alpha}<\Gamma(\alpha+1)$.
- (H_5) $\frac{\delta}{\sigma}\leq 1-\frac{(M_A+M_B+2\bar{l})(\psi(T)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)}$.

Proof We know $\gamma=\alpha+\beta(1-\alpha)\in(0,1]$, $\Gamma(\gamma)\Gamma(2-\gamma)\geq 1$, and by (H_3), $\frac{d_k}{\Gamma(\gamma)\Gamma(2-\gamma)}\leq d_k\leq\frac{\|\phi\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}\leq\delta$, and $\frac{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}\leq\frac{\|\phi\|_{\mathrm{PC}_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)}\leq\delta$. Therefore, by (H_4) and (H_5), one can easily conclude that

$$\frac{\Gamma(\alpha+1)\cdot \overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)[\Gamma(\alpha+1)-(M_A+M_B+2\bar{l})(\psi(T)-\psi(t_0))^{\alpha}]}\leq\delta\cdot\frac{\sigma}{\delta}=\sigma, \tag{4.21}$$

here $\overline{M}_1=\max\{\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}},d_k\}$. From Theorem 4.2, we get $\|x\|_{\mathrm{PC}_{\gamma;\psi}}\leq M\leq\sigma$, and we conclude that the solution in (1.4) has finite-time stability. □

5 Example

In this section, we will present the following three examples to illustrate our main results.

Example 5.1 Assume that $\psi(t)=\frac{1}{20}\ln(1+t)$, $\alpha=0.5$, $\beta\to 1$, then $\gamma\to 1$. Let $h(t)=e^{\cos t}$, $[t_0,T]=[1,10]$, then $h(t)\in[\frac{1}{e},e]$, $[-h,t_0]=[-e,1]$, $e\approx 2.718$ is a natural constant. If $f=0.1\sin x(t)+0.1\cos x(t-2)$, $\phi_k(t,x(t),x(t_k-0))=\frac{1}{10}e^{-|x(t)|}-\frac{1}{10}\cos x(t_k-0)$, $\phi(t)=t$, then $\bar{l}=0.1$, $\lambda=0.1$, $d_k=\frac{2}{5}$, $\|\phi(t_0)\|_{\mathrm{PC}_{\gamma;\psi}}=1$, $\overline{M}_1=1$, where $k=1,2,\ldots,p$. We consider the bounded operators

$$A(t)x(t)=\sin x(t)\cdot x(t)\quad\text{and}\quad B(t)x(t)=e^{-t}\cdot x(t),$$

then $M_A=1$, $M_B=\frac{1}{e}$. By Mathematica software, we know

$$\frac{\lambda}{\Gamma(\gamma)\Gamma(2-\gamma)}+\frac{(M_A+M_B+2\bar{l})(\psi(T)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)}\approx 0.61<1,$$

which implies that all the conditions in Theorem 3.2 are satisfied. Therefore, the specific system (1.4) has a unique solution in $\mathrm{PC}_{\gamma;\psi}(J,\mathbb{R})$. When we discuss the existence results of system (1.4), for all $\rho \geq 2.05$, one have

$$\frac{\overline{M}_1}{\Gamma(\gamma)\Gamma(2-\gamma)\cdot\rho} < 0.49,$$

and

$$1 - \frac{(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^{\alpha}}{\Gamma(\alpha+1)} \approx 0.49.$$

Obviously, we can easily verify all conditions in Theorem 3.1, and we conclude that system (1.4) has at least one solution in Ω_ρ, where $\Omega_\rho = \{x \in \mathrm{PC}_{\gamma;\psi}(J,\mathbb{R}) : \|x\|_{\mathrm{PC}_{\gamma;\psi}} \leq \rho\}$.

Remark 5.2 Since there are few papers research the existence and uniqueness of solutions for the fractional order nonlinear differential equation involving time-varying delays and non-instantaneous impulses, one can see that all the results in [7–12] cannot directly be applicable to the Example 5.1 to obtain the results. This implies that the results in this paper are essentially new.

Example 5.3 Assumed that all data are the same as in the above Example 5.1, for $1 \leq x < +\infty$, and we let

$$F(x) = \sum_{n=1}^{x} \frac{(M_A + M_B + 2\bar{l})^n}{\Gamma(n\alpha)} \big(\psi(T) - \psi(t_0)\big)^{n\alpha-1}.$$

By Fig. 1, we have $\sum_{n=1}^{\infty} \frac{(M_A+M_B+2\bar{l})^n}{\Gamma(n\alpha)}(\psi(T) - \psi(t_0))^{n\alpha-1} \approx 7.55$, then

$$\int_{t_0}^{T} \sum_{n=1}^{\infty} \frac{(M_A + M_B + 2\bar{l})^n}{\Gamma(n\alpha)} \psi'(\xi)\big(\psi(T) - \psi(t_0)\big)^{n\alpha-1}\, d\xi \approx 0.64.$$

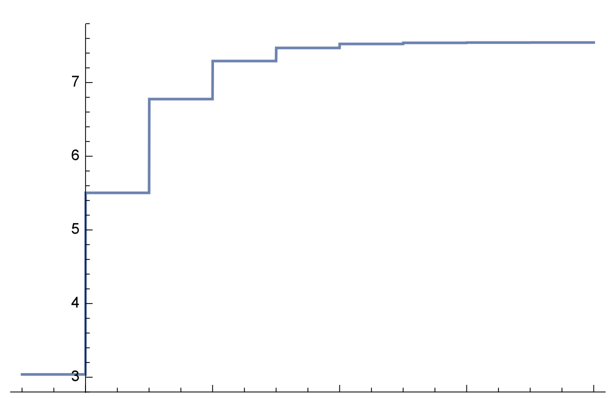

Figure 1 The graph of the function $F(x)$

Meanwhile, let $\sigma = 30$, $\delta = 11$, and we have

$$\frac{\sigma\Gamma(\gamma)\Gamma(2-\gamma)}{\overline{M}_1} - 1 \approx 29 > 0.64,$$

which implies (4.1) holds. Therefore, by Theorem 4.1, system (1.4) is finite-time stable on $[-e, 10]$.

Example 5.4 All the conditions are the same as above, one can easily conclude that $d_k = \frac{2}{5} < \frac{\|\phi\|_{PC_{\gamma;\psi}}}{\Gamma(\gamma)\Gamma(2-\gamma)} = 10$, $(M_A + M_B + 2\bar{l})(\psi(T) - \psi(t_0))^\alpha \approx 0.46$, and $\Gamma(\alpha+1) \approx 0.89$, and $\frac{\delta}{\sigma} \approx 0.37$. We can easily demonstrate that conditions (H_3), (H_4), (H_5) in Theorem 4.3 hold. Therefore, system (1.4) has finite-time stability on $[-e, 10]$.

6 Conclusion

In this paper, we mainly consider a kind of ψ-Hilfer fractional order differential equation. The addressed equation has time-varying delay terms and non-instantaneous impulsive effects, which are quite different from the related references discussed in the literature [18, 19, 21, 22, 38–42]. The nonlinear fractional order differential system studied in the present paper is more generalized and more practical. By applying Schauder's fixed point theorem, contraction mapping principle, and the definition of finite-time stability, we employ a novel argument, and the easily verifiable sufficient conditions have been provided to determine the existence, uniqueness and finite-time stability of the solutions for the considered equations. Finally, three typical numerical examples have been presented at the end of this paper to illustrate the effectiveness and feasibility of the proposed criterion. Consequently, this paper shows theoretically and numerically that some related references known in the literature can be enriched and complemented.

An interesting extension of our study would be to discuss the stability with unknown parameters [37] and Ulam stability [20, 43] for the ψ-Hilfer fractional differential equations with time-varying delay terms. This topic will be the subject of a forthcoming paper.

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