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Representing by several orthogonal polynomials for sums of finite products of Chebyshev polynomials of the first kind and Lucas polynomials

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Abstract

In this paper, we investigate sums of finite products of Chebyshev polynomials of the first kind and those of Lucas polynomials. We express each of them as linear combinations of Hermite, extended Laguerre, Legendre, Gegenbauer, and Jacobi polynomials whose coefficients involve some terminating hypergeometric functions ${}_1F_1$ and ${}_2F_1$. These are obtained by means of explicit computations.

Keywords: Lucas polynomials; Chebyshev polynomials of the first kind; Sums of finite products; Orthogonal polynomials

1 Introduction and preliminaries

In this section, we will first fix some notations that will be used throughout this paper and then recall the necessary basic facts about orthogonal polynomials. As we will limit the facts to the minimum, the interested reader is advised to refer to general books on orthogonal polynomials, for example [2, 4].

For any nonnegative integer n, the falling factorial polynomials $(x)_n$ and the rising factorial polynomials $\langle x\rangle_n$ are respectively given by

$$(x)_n = x(x-1)\cdots(x-n+1) \quad (n \geq 1), \qquad (x)_0 = 1, \tag{1.1}$$

$$\langle x\rangle_n = x(x+1)\cdots(x+n-1) \quad (n \geq 1), \qquad \langle x\rangle_0 = 1. \tag{1.2}$$

The two factorial polynomials are related by

$$(-1)^n(x)_n = \langle -x\rangle_n, \qquad (-1)^n\langle x\rangle_n = (-x)_n, \tag{1.3}$$

$$\frac{(2n-2j)!}{(n-j)!} = \frac{2^{2n-2j}(-1)^j\langle\frac{1}{2}\rangle_n}{\langle\frac{1}{2}-n\rangle_j} \quad (n \geq j \geq 0), \tag{1.4}$$

$$\frac{(2n+2j)!}{(n+j)!} = 2^{2n+2j}\left\langle\frac{1}{2}\right\rangle_n\left\langle n+\frac{1}{2}\right\rangle_j \quad (n,j \geq 0), \tag{1.5}$$

$$\Gamma(\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!} \quad (n \geq 0), \tag{1.6}$$

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$$\frac{\Gamma(x+1)}{\Gamma(x+1-n)} = (x)_n, \qquad \frac{\Gamma(x+n)}{\Gamma(x)} = \langle x\rangle_n \quad (n \geq 0), \tag{1.7}$$

where $\Gamma(x)$ is the gamma function. The hypergeometric function is defined by

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{n=0}^{\infty} \frac{\langle a_1\rangle_n \cdots \langle a_p\rangle_n}{\langle b_1\rangle_n \cdots \langle b_q\rangle_n} \frac{x^n}{n!}. \tag{1.8}$$

Next, we need to recall some basic facts about Chebyshev polynomials of the first kind $T_n(x)$, Hermite polynomials $H_n(x)$, extended Laguerre polynomials $L_n^{\alpha}(x)$, Legendre polynomials $P_n(x)$, Gegenbauer polynomials $C_n^{(\lambda)}(x)$, and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. All of these facts can be also found in [7–10, 13, 14]. Also, we will mention some necessary facts on Lucas polynomials $L_n(x)$. Here we note that when $\alpha = 0$, the extended Laguerre polynomials $L_n^0(x)$ are usually denoted by $L_n(x)$ and called Laguerre polynomials. However, in this paper $L_n(x)$ always indicates the Lucas polynomials and never means the Laguerre polynomials.

In terms of generating functions, the above mentioned polynomials are given as in the following:

$$F(t,x) = \frac{2-xt}{1-xt-t^2} = \sum_{n=0}^{\infty} L_n(x)t^n, \tag{1.9}$$

$$G(t,x) = \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{1.10}$$

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \tag{1.11}$$

$$(1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n \quad (\alpha > -1), \tag{1.12}$$

$$\left(1-2xt+t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{1.13}$$

$$\frac{1}{(1-2xt+t^2)^{\lambda}} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x)t^n \quad \left(\lambda > -\frac{1}{2}, \lambda \neq 0, |t| < 1, |x| \leq 1\right), \tag{1.14}$$

$$\frac{\alpha+\beta}{R(1-t+R)^{\alpha}(1+t+R)^{\beta}} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n \quad \left(R = \sqrt{1-2xt+t^2}, \alpha, \beta > -1\right). \tag{1.15}$$

Those special polynomials are also explicitly given as follows:

$$L_n(x) = n\sum_{l=0}^{[\frac{n}{2}]} \frac{1}{n-l}\binom{n-l}{l} x^{n-2l} \quad (n \geq 1), \tag{1.16}$$

$$\begin{aligned} T_n(x) &= {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right) \\ &= \frac{n}{2}\sum_{l=0}^{[\frac{n}{2}]} (-1)^l \frac{1}{n-l}\binom{n-l}{l}(2x)^{n-2l} \quad (n \geq 1), \end{aligned} \tag{1.17}$$

$$H_n(x) = n! \sum_{l=0}^{[\frac{n}{2}]} \frac{(-1)^l}{l!(n-2l)!} (2x)^{n-2l} \quad (n \geq 0), \tag{1.18}$$

$$\begin{aligned} L_n^{\alpha}(x) &= \frac{\langle \alpha + 1 \rangle_n}{n!} {}_1F_1(-n; \alpha + 1; x) \\ &= \sum_{l=0}^{n} \frac{(-1)^l \binom{n+\alpha}{n-l}}{l!} x^l \quad (n \geq 0), \end{aligned} \tag{1.19}$$

$$\begin{aligned} P_n(x) &= {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right) \\ &= \frac{1}{2^n} \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n}{l} \binom{2n-2l}{n} x^{n-2l} \quad (n \geq 0), \end{aligned} \tag{1.20}$$

$$\begin{aligned} C_n^{(\lambda)}(x) &= \binom{n+2\lambda-1}{n} {}_2F_1\left(-n, n+2\lambda; \lambda + \frac{1}{2}; \frac{1-x}{2}\right) \\ &= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda) k!(n-2k)!} (2x)^{n-2k} \quad (n \geq 0), \end{aligned} \tag{1.21}$$

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{\langle \alpha + 1 \rangle_n}{n!} {}_2F_1\left(-n, 1 + \alpha + \beta + n; \alpha + 1; \frac{1-x}{2}\right) \\ &= \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k} \quad (n \geq 0). \end{aligned} \tag{1.22}$$

Next, we would like to mention Rodrigues-type formulas for Hermite and extended Laguerre polynomials and Rodrigues' formulas for Legendre, Gegenbauer, and Jacobi polynomials.

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \tag{1.23}$$

$$L_n^{\alpha}(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n} \left(e^{-x} x^{n+\alpha}\right), \tag{1.24}$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n, \tag{1.25}$$

$$\left(1 - x^2\right)^{\lambda - \frac{1}{2}} C_n^{(\lambda)}(x) = \frac{(-2)^n}{n!} \frac{\langle \lambda \rangle_n}{\langle n + 2\lambda \rangle_n} \frac{d^n}{dx^n} \left(1 - x^2\right)^{n + \lambda - \frac{1}{2}}, \tag{1.26}$$

$$(1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1-x)^{n+\alpha} (1+x)^{n+\beta}. \tag{1.27}$$

The most important properties of the special polynomials in (1.23)–(1.27) are their orthogonalities with respect to various weight functions which are as follows:

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n! \sqrt{\pi} \delta_{m,n}, \tag{1.28}$$

$$\int_0^{\infty} x^{\alpha} e^{-x} L_n^{\alpha}(x) L_m^{\alpha}(x)\, dx = \frac{1}{n!} \Gamma(\alpha + n + 1) \delta_{m,n}, \tag{1.29}$$

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1} \delta_{m,n}, \tag{1.30}$$

$$\int_{-1}^{1}\left(1-x^2\right)^{\lambda-\frac{1}{2}} C_n^{(\lambda)}(x)C_m^{(\lambda)}(x)\,dx = \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}\delta_{m,n}, \tag{1.31}$$

$$\begin{aligned}&\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\,dx\\ &\quad= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}\delta_{m,n}.\end{aligned} \tag{1.32}$$

The sums of finite products of Chebyshev polynomials of the first kind in (1.33) and those of Lucas polynomials in (1.34) are the two main objects of study in this paper which are respectively denoted by $\alpha_{m,r}(x)$ and $\beta_{m,r}(x)$.

$$\begin{aligned}\alpha_{m,r}(x) &= \sum_{l=0}^{m}\sum_{i_1+\cdots+i_{r+1}=m-l}\binom{r+l}{r}x^l T_{i_1}(x)\cdots T_{i_{r+1}}(x)\\ &\quad-\sum_{l=0}^{m-2}\sum_{i_1+\cdots+i_{r+1}=m-l-2}\binom{r+l}{r}x^l T_{i_1}(x)\cdots T_{i_{r+1}}(x) \quad (m\geq 2, r\geq 1),\end{aligned} \tag{1.33}$$

$$\begin{aligned}\beta_{m,r}(x) &= \sum_{l=0}^{m}\sum_{i_1+\cdots+i_{r+1}=m-l}\binom{r+l}{r}\left(\frac{x}{2}\right)^l L_{i_1}(x)\cdots L_{i_{r+1}}(x)\\ &\quad+\sum_{l=0}^{m-2}\sum_{i_1+\cdots+i_{r+1}=m-l-2}\binom{r+l}{r}\left(\frac{x}{2}\right)^l L_{i_1}(x)\cdots L_{i_{r+1}}(x) \quad (m\geq 2, r\geq 1).\end{aligned} \tag{1.34}$$

Note here that $\alpha_{m,r}(x)$ and $\beta_{m,r}(x)$ are polynomials of degree m.

The purpose of this paper is to study the sums of finite products of Chebyshev polynomials of the first kind in (1.33) and those of Lucas polynomials in (1.34), and to express each of them as linear combinations of Hermite, extended Laguerre, Legendre, Gegenbauer, and Jacobi polynomials. These will be done by explicit computations with the help of Propositions 2.1 and 2.2 in the next section.

Now, we state our main results of this paper, namely Theorems 1.1 and 1.2.

Theorem 1.1 *Let m, r be any integers with $m\geq 2$, $r\geq 1$. Then we have the following identities*:

$$\begin{aligned}&\sum_{l=0}^{m}\sum_{i_1+\cdots+i_{r+1}=m-l}\binom{r+l}{r}x^l T_{i_1}(x)\cdots T_{i_{r+1}}(x)\\ &\qquad-\sum_{l=0}^{m-2}\sum_{i_1+\cdots+i_{r+1}=m-l-2}\binom{r+l}{r}x^l T_{i_1}(x)\cdots T_{i_{r+1}}(x)\\ &\quad=\frac{(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\frac{{}_1F_1(-j;1-m-r;-1)}{j!(m-2j)!}H_{m-2j}(x) \qquad (1.35)\\ &\quad=\frac{(m+r)2^m}{r!}\sum_{k=0}^{m}\frac{(-1)^k}{\Gamma(\alpha+k+1)}\\ &\qquad\times\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-\frac{1}{4})^l(m+r-1-l)!\Gamma(m+\alpha+1-2l)}{l!(m-k-2l)!}L_k^{\alpha}(x) \qquad (1.36)\end{aligned}$$

$$= \frac{(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{(2m-4j+1)}{j!(m-j+\frac{1}{2})_{m-j}}$$

$$\times {}_2F_1\left(-j, j-m-\frac{1}{2}; 1-m-r; 1\right) P_{m-2j}(x) \tag{1.37}$$

$$= \frac{\Gamma(\lambda)(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{(m+\lambda-2j)}{j!\Gamma(m+\lambda+1-j)}$$

$$\times {}_2F_1(-j, j-m-\lambda; 1-m-r; 1) C_{m-2j}^{(\lambda)}(x) \tag{1.38}$$

$$= \frac{(m+r)(-2)^m}{r!} \sum_{k=0}^{m} \frac{(-2)^k \Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)}$$

$$\times \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(-\frac{1}{4})^l (m+r-1-l)!}{l!(m-k-2l)!}$$

$$\times {}_2F_1(k+2l-m, k+\beta+1; 2k+\alpha+\beta+2; 2) P_k^{(\alpha,\beta)}(x). \tag{1.39}$$

Theorem 1.2 *Let m, r be any integers with $m \geq 2$, $r \geq 1$. Then we have the following identities*:

$$\sum_{l=0}^{m} \sum_{i_1+\cdots+i_{r+1}=m-l} \binom{r+l}{r} \left(\frac{x}{2}\right)^l L_{i_1}(x) \cdots L_{i_{r+1}}(x)$$

$$+ \sum_{l=0}^{m-2} \sum_{i_1+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} \left(\frac{x}{2}\right)^l L_{i_1}(x) \cdots L_{i_{r+1}}(x)$$

$$= \frac{2^{r+1-m}(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{{}_1F_1(-j, 1-m-r; 4)}{j!(m-2j)!} H_{m-2j}(x) \tag{1.40}$$

$$= \frac{2^{r+1}(m+r)}{r!} \sum_{k=0}^{m} \frac{(-1)^k}{\Gamma(\alpha+k+1)}$$

$$\times \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(m+r-1-l)!\Gamma(m+\alpha+1-2l)}{l!(m-k-2l)!} L_k^{\alpha}(x) \tag{1.41}$$

$$= \frac{2^{r-m}(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{(2m-4j+1)}{j!(m-j+\frac{1}{2})_{m-j+1}}$$

$$\times {}_2F_1\left(-j, j-m-\frac{1}{2}; 1-m-r; -4\right) P_{m-2j}(x) \tag{1.42}$$

$$= 2^{r+1-m}\Gamma(\lambda) \frac{(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{(m+\lambda-2j)}{j!\Gamma(m+\lambda+1-j)}$$

$$\times {}_2F_1(-j, j-m-\lambda; 1-m-r; -4) C_{m-2j}^{(\lambda)}(x) \tag{1.43}$$

$$= \frac{(-1)^m 2^{r+1}(m+r)}{r!} \sum_{k=0}^{m} \frac{(-2)^k \Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)}$$

$$\times \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(m+r-1-l)!}{l!(m-k-2l)!}$$
$$\times {}_2F_1(k+2l-m, k+\beta+1; 2k+\alpha+\beta+2; 2) P_k^{(\alpha,\beta)}(x). \tag{1.44}$$

Before moving on to the next section, we want to mention some of the previous works that are related to the present results. Along the same line as this paper, certain sums of finite products of Chebyshev polynomials of the first, second, third, and fourth kinds, and of Legendre, Laguerre, Fibonacci, and Lucas polynomials are expressed in terms of all four kinds of Chebyshev polynomials in [11, 16, 19, 24, 26]. Also, certain sums of finite products of Chebyshev polynomials of the second, third, and fourth kinds, and of Fibonacci, Legendre, and Laguerre polynomials are expressed in terms of Hermite, extended Laguerre, Legendre, Gegenbauer, and Jacobi polynomials in [5, 12, 23, 27]. Also, we would like to remark here that some Appell and non-Appell polynomials are also expressed as linear combinations of Bernoulli polynomials. Indeed, for Appell polynomials, some sums of finite products of Bernoulli and Euler polynomials are expressed in terms of Bernoulli polynomials in [1, 20]. As for non-Appell polynomials, some sums of finite products of Chebyshev polynomials of the first, second, third, and fourth kinds, and of Legendre, Laguerre, Genocchi, Fibonacci, and Lucas polynomials are expressed in terms of Bernoulli polynomials in [15, 17, 18, 21, 22, 25]. Actually, all of these were obtained by deriving Fourier series expansions for the functions closely related to such sums of finite products of special polynomials.

Finally, we let the reader look at the papers [3, 6] for some related works.

2 Proof of Theorem 1.1

In this section, we will show (1.35)–(1.37) of Theorem 1.1, leaving (1.38) and (1.39) as exercises to the reader. For this, we will first state the next two results that will be needed in showing Theorems 1.1 and 1.2. Here we note that facts (a), (b), (c), (d), and (e) of Proposition 2.1 are respectively from (3.7) of [9], (2.3) of [13], (2,3) of [10], (2,3) of [7], and (2,7) of [14]. We also observe here that the formulas in Proposition 2.1 are obtained from the orthogonalities in (1.28)–(1.32), Redrigues' and Rodrigues-type formulas in (1.23)–(1.27), and integration by parts.

Proposition 2.1 *Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then the following hold.*

(a) $q(x) = \sum_{k=0}^{n} C_{k,1} H_k(x)$, *where*

$$C_{k,1} = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} q(x) \frac{d^k}{dx^k} \left(e^{-x^2}\right) dx.$$

(b) $q(x) = \sum_{k=0}^{n} C_{k,2} L_k(x)$, *where*

$$C_{k,2} = \frac{1}{\Gamma(\alpha+k+1)} \int_{0}^{\infty} q(x) \frac{d^k}{dx^k} \left(e^{-x^2} x^{k+\alpha}\right) dx.$$

(c) $q(x) = \sum_{k=0}^{n} C_{k,3} P_k(x)$, *where*

$$C_{k,3} = \frac{2k+1}{2^{k+1} k!} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} \left(x^2-1\right)^k dx.$$

(d) $q(x) = \sum_{k=0}^{n} C_{k,4} C_k^{(\lambda)}(x)$, *where*

$$C_{k,4} = \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k \sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} \left(1-x^2\right)^{k+\lambda-\frac{1}{2}} dx.$$

(e) $q(x) = \sum_{k=0}^{n} C_{k,5} P_k^{(\alpha,\beta)}(x)$, *where*

$$C_{k,5} = \frac{(-1)^k (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k+\alpha}(1+x)^{k+\beta} dx.$$

The next proposition is stated in [23].

Proposition 2.2 *Let m, k be any nonnegative integers. Then the following hold.*

(a)
$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!\sqrt{\pi}}{(\frac{m}{2})! 2^m}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(b)
$$\int_{-1}^{1} x^m \left(1-x^2\right)^k dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{2^{2k+2} k! m! (k+\frac{m}{2}+1)!}{(\frac{m}{2})!(2k+m+2)!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(c)
$$\int_{-1}^{1} x^m \left(1-x^2\right)^{k+\lambda-\frac{1}{2}} dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{m}{2}+\frac{1}{2})}{\Gamma(k+\lambda+\frac{m}{2}+1)}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(d)
$$\int_{-1}^{1} x^m (1-x)^{k+\alpha}(1+x)^{k+\beta} dx = 2^{2k+\alpha+\beta+1} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} 2^j \times \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+j+1)}{\Gamma(2k+\alpha+\beta+j+2)}.$$

The following lemma was proved in [22] which follows by differentiating (1.10).

Lemma 2.3 *Let m, r be any integers with $m \geq 2$, $r \geq 1$. Then we have the following identity*:

$$\sum_{l=0}^{m} \sum_{i_1+\cdots+i_{r+1}=m-l} \binom{r+l}{r} x^l T_{i_1}(x) \cdots T_{i_{r+1}}(x) - \sum_{l=0}^{m-2} \sum_{i_1+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} x^l T_{i_1}(x) \cdots T_{i_{r+1}}(x) = \frac{1}{2^{r-1} r!} T_{m+r}^{(r)}(x), \tag{2.1}$$

where the first and second inner sums on the left-hand side are respectively over all nonnegative integers $i_1, \ldots, i_{r+1}$, with $i_1 + \cdots + i_{r+1} = m-l$ and those with $i_1 + \cdots + i_{r+1} = m-l-2$.

From (1.17), the rth derivative of $T_n(x)$ is given by

$$T_n^{(r)}(x) = \frac{n}{2} \sum_{l=0}^{[\frac{n-r}{2}]} (-1)^l \frac{1}{n-l} \binom{n-l}{l} 2^{n-2l} (n-2l)_r x^{n-2l-r}. \tag{2.2}$$

Actually, we need the following particular case:

$$T_{m+r}^{(r+k)}(x)=\frac{m+r}{2}\sum_{l=0}^{[\frac{m-k}{2}]}(-1)^l\frac{1}{m+r-l}\binom{m+r-l}{l}2^{m+r-2l}(m+r-2l)_{r+k}x^{m-k-2l}. \tag{2.3}$$

With $\alpha_{m,r}(x)$ as in (1.33), we let

$$\alpha_{m,r}(x)=\sum_{k=0}^{m}C_{k,1}H_k(x). \tag{2.4}$$

Then, from (a) of Proposition 2.1, (2.1), (2.3), and integration by parts k times, we have

$$\begin{aligned}
C_{k,1}&=\frac{(-1)^k}{2^k k!\sqrt{\pi}}\int_{-\infty}^{\infty}\alpha_{m,r}(x)\frac{d^k}{dx^k}e^{-x^2}\,dx\\
&=\frac{(-1)^k}{2^k k!\sqrt{\pi}2^{r-1}r!}\int_{-\infty}^{\infty}T_{m+r}^{(r)}(x)\frac{d^k}{dx^k}e^{-x^2}\,dx\\
&=\frac{1}{2^k k!\sqrt{\pi}2^{r-1}r!}\int_{-\infty}^{\infty}T_{m+r}^{(r+k)}(x)e^{-x^2}\,dx\\
&=\frac{1}{2^k k!\sqrt{\pi}2^{r-1}r!}\frac{m+r}{2}\sum_{l=0}^{[\frac{m-k}{2}]}(-1)^l\frac{1}{m+r-l}\binom{m+r-l}{l}2^{m+r-2l}\\
&\quad\times(m+r-2l)_{r+k}\int_{-\infty}^{\infty}x^{m-k-2l}e^{-x^2}\,dx,
\end{aligned} \tag{2.5}$$

where we note from (a) of Proposition 2.2 that

$$\int_{-\infty}^{\infty}x^{m-k-2l}e^{-x^2}\,dx=\begin{cases}0, & \text{if } k\not\equiv m \pmod 2,\\ \frac{(m-k-2l)!\sqrt{\pi}}{(\frac{m-k}{2}-l)!2^{m-k-2l}}, & \text{if } k\equiv m \pmod 2.\end{cases} \tag{2.6}$$

Now, from (2.4)–(2.6) and after some simplifications, we get

$$\begin{aligned}
\alpha_{m,r}(x)&=\frac{m+r}{r!}\sum_{0\le k\le m, k\equiv m \pmod 2}\frac{1}{k!}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-1)^l(m+r-l-1)!}{l!(\frac{m-k}{2}-l)!}H_k(x)\\
&=\frac{(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\frac{1}{j!(m-2j)!}\sum_{l=0}^{j}\frac{(-1)^l\langle -j\rangle_l}{l!\langle 1-m-r\rangle_l}H_{m-2j}(x)\\
&=\frac{(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\frac{{}_1F_1(-j;1-m-r;-1)}{j!(m-2j)!}H_{m-2j}(x).
\end{aligned} \tag{2.7}$$

This shows (1.35) of Theorem 1.1.

Next, let us put

$$\alpha_{m,r}(x)=\sum_{k=0}^{m}C_{k,2}L_k^{\alpha}(x). \tag{2.8}$$

Then, from (b) of Proposition 2.1, (2.1), (2.3), and integration by parts k times, we have

$$\begin{aligned} C_{k,2} &= \frac{1}{\Gamma(\alpha+k+1)2^{r-1}r!}\int_0^\infty T_{m+r}^{(r)}(x)\frac{d^k}{dx^k}\left(e^{-x}x^{k+\alpha}\right)dx \\ &= \frac{(-1)^k}{\Gamma(\alpha+k+1)2^{r-1}r!}\frac{m+r}{2}\sum_{l=0}^{[\frac{m-k}{2}]}(-1)^l\frac{1}{m+r-l}\binom{m+r-l}{l}2^{m+r-2l} \\ &\quad\times (m+r-2l)_{r+k}\Gamma(m+\alpha+1-2l). \end{aligned} \tag{2.9}$$

By (2.8) and (2.9) and after simplifications, we have

$$\begin{aligned} \alpha_{m,r}(x) &= \frac{2^m(m+r)}{r!}\sum_{k=0}^{m}\frac{(-1)^k}{\Gamma(\alpha+k+1)} \\ &\quad\times\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(-\frac{1}{4})^l(m+r-1-l)!\Gamma(m+\alpha+1-2l)}{l!(m-k-2l)!}L_k^\alpha(x). \end{aligned} \tag{2.10}$$

This completes the proof for (1.36) of Theorem 1.1.

Lastly, we set

$$\alpha_{m,r}(x) = \sum_{k=0}^{m}C_{k,3}P_k(x). \tag{2.11}$$

Then, by (c) of Proposition 2.1, (2.1), (2.3), and integration by parts k times, we get

$$\begin{aligned} C_{k,3} &= \frac{(-1)^k(2k+1)}{2^{k+1}k!2^{r-1}r!}\frac{m+r}{2}\sum_{l=0}^{[\frac{m-k}{2}]}(-1)^l\frac{1}{m+r-l}\binom{m+r-l}{l} \\ &\quad\times 2^{m+r-2l}(m+r-2l)_{r+k}\int_{-1}^{1}x^{m-k-2l}\left(x^2-1\right)^k dx. \end{aligned} \tag{2.12}$$

Here, from (b) of Proposition 2.2, we note that

$$\begin{aligned} &\int_{-1}^{1}x^{m-k-2l}\left(1-x^2\right)^k dx \\ &\quad= \begin{cases} 0, & \text{if } k\not\equiv m \pmod 2, \\ \frac{2^{2k+2}k!(m-k-2l)!(\frac{m+k}{2}-l+1)!}{(\frac{m-k}{2}-l)!(m+k-2l+2)!}, & \text{if } k\equiv m \pmod 2. \end{cases} \end{aligned} \tag{2.13}$$

From (2.11)–(2.13), and after some simplifications, we obtain

$$\begin{aligned} \alpha_{m,r}(x) &= \frac{2^m(m+r)}{r!}\sum_{0\le k\le m, k\equiv m \pmod 2}(2k+1)2^{k+1} \\ &\quad\times\frac{(-\frac{1}{4})^l(m+r-1-l)!(\frac{m+k}{2}-l+1)!}{l!(\frac{m-k}{2}-l)!(m+k-2l+2)!} \\ &= \frac{2^{2m+1}(m+r)}{r!}\sum_{j=0}^{[\frac{m}{2}]}2^{-2j}(2m-4j+1) \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{l=0}^{j} \frac{(-\frac{1}{4})^l (m+r-1-l)!(m-j-l+1)!}{l!(j-l)!(2m-2j-2l+2)!} P_{m-2j}(x) \\
&= \frac{(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{(2m-4j+1)}{j!(m-j+\frac{1}{2})_{m-j}} \\
&\times \sum_{l=0}^{j} \frac{\langle -j\rangle_l \langle j-m-\frac{1}{2}\rangle_l}{l!\langle 1-m-r\rangle_l} P_{m-2j}(x) \\
&= \frac{(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{(2m-4j+1)}{j!(m-j+\frac{1}{2})_{m-j}} \\
&\times {}_2F_1\left(-j, j-m-\frac{1}{2}; 1-m-r; 1\right) P_{m-2j}(x).
\end{aligned} \tag{2.14}
$$

This finishes up the proof for (1.37) of Theorem 1.1.

3 Proof of Theorem 1.2

Here we will show (1.43) and (1.44) for Theorem 1.2, leaving (1.40)–(1.42) as exercises to the reader. The following lemma can be derived by differentiating (1.9), as was shown in [16].

Lemma 3.1 *Let m, r be any integers with $m \geq 2$, $r \geq 1$. Then we have the following identities*:

$$
\begin{aligned}
&\sum_{l=0}^{m} \sum_{i_1+\cdots+i_{r+1}=m-l} \binom{r+l}{r} \left(\frac{x}{2}\right)^l L_{i_1}(x)\cdots L_{i_{r+1}}(x) \\
&\quad + \sum_{l=0}^{m-2} \sum_{i_1+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} \left(\frac{x}{2}\right)^l L_{i_1}(x)\cdots L_{i_{r+1}}(x) \\
&= \frac{2^{r+1}}{r!} L_{m+r}^{(r)}(x),
\end{aligned} \tag{3.1}
$$

where the first and second inner sums on the left-hand side are respectively over all nonnegative integers $i_1, \ldots, i_{r+1}$ with $i_1 + \cdots + i_{r+1} = m-l$ and those with $i_1 + \cdots + i_{r+1} = m-l-2$.

From (1.16), the rth derivative of $L_n(x)$ is given by

$$
L_n^{(r)}(x) = n \sum_{l=0}^{[\frac{n-r}{2}]} \frac{1}{n-l} \binom{n-l}{l} (n-2l)_r x^{n-2l-r}. \tag{3.2}
$$

Especially, we have

$$
L_{m+r}^{(r+k)}(x) = (m+r) \sum_{l=0}^{[\frac{m-k}{2}]} \frac{1}{m+r-l} \binom{m+r-l}{l} (m+r-2l)_{r+k} x^{m-k-2l}. \tag{3.3}
$$

With $\beta_{m,r}(x)$ as in (1.34), we let

$$\beta_{m,r}(x) = \sum_{k=0}^{m} C_{k,4} C_k^{(\lambda)}(x). \tag{3.4}$$

Then, by (d) of Proposition 2.1, (3.1), (3.3), and integration by parts k times, we have

$$\begin{aligned}
C_{k,4} &= \frac{(k+\lambda)\Gamma(\lambda)2^{r+1}}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})r!} \int_{-1}^{1} L_{m+r}^{(r)}(x) \frac{d^k}{dx^k}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}\,dx \\
&= \frac{(k+\lambda)\Gamma(\lambda)2^{r+1}}{2^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})r!} \int_{-1}^{1} L_{m+r}^{(r+k)}(x) \frac{d^k}{dx^k}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}\,dx \\
&= \frac{(k+\lambda)\Gamma(\lambda)2^{r+1}}{2^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})r!}(m+r) \sum_{l=0}^{[\frac{m-k}{2}]} \frac{1}{m+r-l}\binom{m+r-l}{l} \\
&\quad\times (m+r-2l)_{r+k} \int_{-1}^{1} x^{m-k-2l}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}\,dx.
\end{aligned} \tag{3.5}$$

From (c) of Proposition 2.2, we note that

$$\int_{-1}^{1} x^{m-k-2l}\left(1-x^2\right)^{k+\lambda-\frac{1}{2}}\,dx = \begin{cases} 0, & \text{if } k \not\equiv m \pmod 2, \\ \frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{m-k}{2}-l+\frac{1}{2})}{\Gamma(k+\lambda+\frac{m-k}{2}-l+1)}, & \text{if } k \equiv m \pmod 2. \end{cases} \tag{3.6}$$

Combining (3.4)–(3.6), and after some simplifications, we obtain

$$\begin{aligned}
\beta_{m,r}(x) &= \frac{\Gamma(\lambda)2^{r+1}(m+r)}{\sqrt{\pi}r!} \sum_{0\le k\le m, k\equiv m \pmod 2} \frac{(k+\lambda)}{2^k} \\
&\quad\times \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(m+r-1-l)!\Gamma(\frac{m-k}{2}-l+\frac{1}{2})}{l!(m-k-2l)!\Gamma(k+\lambda+\frac{m-k}{2}-l+1)} C_k^{(\lambda)}(x) \\
&= 2^{r+1-m}\Gamma(\lambda)\frac{(m+r)}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{(m-2j+\lambda)}{\Gamma(m+\lambda-j+1)} \\
&\quad\times \sum_{l=0}^{j} \frac{4^l(m+r-1-l)!(m+\lambda-l)_l}{l!(j-l)!} \\
&= 2^{r+1-m}\Gamma(\lambda)\frac{(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{(m-2j+\lambda)}{j!\Gamma(m+\lambda-j+1)} \\
&\quad\times \sum_{l=0}^{j} \frac{4^l(j)_l(m+\lambda-j)_l}{l!(m+r-1)_l} C_{m-2j}^{(\lambda)}(x) \\
&= 2^{r+1-m}\Gamma(\lambda)\frac{(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{(m-2j+\lambda)}{j!\Gamma(m+\lambda-j+1)} \\
&\quad\times \sum_{l=0}^{j} \frac{(-4)^l\langle -j\rangle_l\langle j-m-\lambda\rangle_l}{l!\langle 1-m-r\rangle_l} C_{m-2j}^{(\lambda)}(x)
\end{aligned}$$

$$= 2^{r+1-m}\Gamma(\lambda)\frac{(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\frac{(m-2j+\lambda)}{j!\Gamma(m+\lambda-j+1)}$$
$$\times {}_2F_1(-j,j-m-\lambda;1-m-r;-4)C_{m-2j}^{(\lambda)}(x). \tag{3.7}$$

This shows (1.43) of Theorem 1.2. Next, we put

$$\beta_{m,r}(x) = \sum_{k=0}^{m} C_{k,5}P_k^{(\alpha,\beta)}(x). \tag{3.8}$$

Then, from (e) of Proposition 2.1, (3.1), (3.3), and integration by parts k times, we get

$$C_{k,5} = \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)2^{r+1}(m+r)}{2^{\alpha+\beta+k+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)r!}$$
$$\times \sum_{l=0}^{[\frac{m-k}{2}]}\frac{1}{m+r-l}\binom{m+r-l}{l}(m+r-2l)_{r+k}$$
$$\times \int_{-1}^{1} x^{m-k-2l}(1-x)^{k+\alpha}(1+x)^{k+\beta}\,dx, \tag{3.9}$$

where we note from (d) of Proposition 2.2 that

$$\int_{-1}^{1} x^{m-k-2l}(1-x)^{k+\alpha}(1+x)^{k+\beta}\,dx$$
$$= 2^{2k+\alpha+\beta+1}\sum_{j=0}^{m-k-2l}\binom{m-k-2l}{j}(-1)^{m-k-j}2^j$$
$$\times \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+j+1)}{\Gamma(2k+\alpha+\beta+j+2)}. \tag{3.10}$$

Combining (3.8)–(3.10), and after some simplifications, we have

$$\beta_{m,r}(x) = -\frac{(-1)^m 2^{r+1}(m+r)}{r!}\sum_{k=0}^{m}\frac{(-2)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(\beta+k+1)}$$
$$\times \sum_{l=0}^{[\frac{m-k}{2}]}\frac{(m+r-1-l)!}{l!}\sum_{j=0}^{m-k-2l}\frac{(-2)^j\Gamma(k+\beta+j+1)}{j!(m-k-2l-j)!\Gamma(2k+\alpha+\beta+j+2)}, \tag{3.11}$$

where the inner most sum is

$$\sum_{j=0}^{m-k-2l}\frac{(-2)^j\Gamma(k+\beta+j+1)}{j!(m-k-2l-j)!\Gamma(2k+\alpha+\beta+j+2)}$$
$$= \frac{\Gamma(k+\beta+1)}{\Gamma(2k+\alpha+\beta+2)(m-k-2l)!}$$
$$\times {}_2F_1(k+2l-m,k+\beta+1;2k+\alpha+\beta+2;2). \tag{3.12}$$

Finally, from (3.11) and (3.12), we obtain

$$\begin{aligned}\beta_{m,r}(x) = {} & \frac{(-1)^m 2^{r+1}(m+r)}{r!} \sum_{k=0}^{m} \frac{(-2)^k \Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \\ & \times \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(m+r-1-l)!}{l!(m-k-2l)!} \\ & \times {}_2F_1(k+2l-m, k+\beta+1; 2k+\alpha+\beta+2; 2) P_k^{(\alpha,\beta)}(x). \end{aligned} \tag{3.13}$$

Thus we have shown the desired result (1.44) of Theorem 1.2.

4 Conclusion

As is well known, the Chebyshev polynomials of the first kind $T_n(x)$ and the Lucas polynomials $L_n(x)$ are related by

$$L_n(x) = 2i^{-n} T_n\left(\frac{ix}{2}\right). \tag{4.1}$$

From (1.33), (1.34), and (4.1), we have the following relation:

$$i^{-m} 2^{r+1} \alpha_{m,r}\left(\frac{ix}{2}\right) = \beta_{m,r}(x). \tag{4.2}$$

We now have our last result from Theorems 1.1 and 1.2, and (4.2).

Theorem 4.1 *Let m, r be any integers with $m \geq 2$, $r \geq 1$. Then the following identities hold*:

$$\begin{aligned} & \frac{i^{-m} 2^{r+1}(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{{}_1F_1(-j; 1-m-r; -1)}{j!(m-2j)!} H_{m-2j}\left(\frac{ix}{2}\right) \\ & = \frac{i^{-m} 2^{m+r+1}(m+r)}{r!} \sum_{k=0}^{m} \frac{(-1)^k}{\Gamma(\alpha+k+1)} \\ & \quad \times \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(-\frac{1}{4})^l (m+r-1-l)! \Gamma(m+\alpha+1-2l)}{l!(m-k-2l)!} L_k^{\alpha}\left(\frac{ix}{2}\right) \\ & = \frac{i^{-m} 2^{r+1}(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{(2m-4j+1)}{j!(m-j+\frac{1}{2})_{m-j}} \\ & \quad \times {}_2F_1\left(-j; j-m-\frac{1}{2}; 1-m-r; 1\right) P_{m-2j}\left(\frac{ix}{2}\right) \\ & = \frac{i^{-m} 2^{r+1} \Gamma(\lambda)(m+r)!}{r!} \sum_{j=0}^{[\frac{m-k}{2}]} \frac{(m+\lambda-2j)}{j! \Gamma(m+\lambda+1-j)} \\ & \quad \times {}_2F_1(-j; j-m-\lambda; 1-m-r; 1) C_{m-2j}^{(\lambda)}\left(\frac{ix}{2}\right) \\ & = \frac{(-i)^{-m} 2^{m+r+1}(m+r)}{r!} \sum_{k=0}^{m} \frac{(-2)^k \Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{(-\frac{1}{4})^l (m+r-1-l)!}{l!(m-k-2l)!} \end{aligned}$$

$$
\begin{aligned}
&\times {}_2F_1(k+2l-m,k+\beta+1;2k+\alpha+\beta+2;2)P_k^{(\alpha,\beta)}\left(\frac{ix}{2}\right)\\
&=\frac{2^{r+1-m}(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\frac{{}_1F_1(-j;1-m-r;4)}{j!(m-2j)!}H_{m-2j}(x)\\
&=\frac{2^{r+1}(m+r)}{r!}\sum_{k=0}^{m}\frac{(-1)^k}{\Gamma(\alpha+\beta+1)}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(m+r-1-l)!\Gamma(m+\alpha+1-2l)}{l!(m-k-2l)!}L_k^{\alpha}(x)\\
&=\frac{2^{r-m}(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\frac{(2m-4j+1)}{j!(m-j+\frac{1}{2})_{m-j+1}}{}_2F_1\left(-j;j-m-\frac{1}{2};1-m-r;-4\right)P_{m-2j}(x)\\
&=2^{r+1-m}\Gamma(\lambda)\frac{(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\frac{(m+\lambda-2j)}{j!\Gamma(m+\lambda+1-j)}\\
&\quad\times {}_2F_1(-j;j-m-\lambda;1-m-r;-4)C_{m-2j}^{(\lambda)}(x)\\
&=\frac{(-1)^m2^{r+1}(m+r)}{r!}\sum_{k=0}^{m}\frac{(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(m+r-1-l)!}{l!(m-k-2l)!}\\
&\quad\times {}_2F_1(k+2l-m,k+\beta+1;2k+\alpha+\beta+2;2)P_k^{(\alpha,\beta)}(x). \qquad (4.3)
\end{aligned}
$$

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Authors' contributions
TK and DSK conceived of the framework and structured the whole paper; TK wrote the paper; LCJ and DVD checked the results of the paper; DSK and TK completed the revision of the article. All authors read and approved the final manuscript.

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