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Existence results for a fraction hybrid differential inclusion with Caputo–Hadamard type fractional derivative

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Abstract

In this manuscript, we talk over the existence of solutions of a class of hybrid Caputo–Hadamard fractional differential inclusions with Dirichlet boundary conditions. Our results are based on the Arzelá–Ascoli theorem and some suitable theorems of fixed point theory. As well, to illustrate our results, we confront the exceptional case of the fractional differential inclusions with examples.

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1 Introduction

During the last decade, the subject of fractional differential equations and inclusions has been developed intensively (for example, see [1–8] and the references therein). An excellent account on the study of fractional differential equations can be found in [9–12]. Hybrid fractional differential equations and inclusions, certain classes of equations and inclusions involve the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Also, they have been examined by several researchers (for example, see [13–15]). Most of the workplace on the fractional differential equations and inclusions is based on Riemann–Liouville, Caputo, and Hadamard type fractional derivatives. In 2012, Jarad et al. modified the Hadamard fractional derivative into a more suitable one having physical interpretable initial conditions similar to the singles in the Caputo setting and called it Caputo–Hadamard fractional derivative. To determine the properties of the modified derivative, refer to [16]. In 2011, Zhao et al. investigated the existence results for the initial value problems of hybrid fractional integro-differential equation

$$D^q\left[\frac{\theta(t)}{u(t,\theta(t))}\right] = v\big(t,\theta(t)\big),$$

with real-valued initial condition $\theta(0) = \theta_0$, where D^q denotes the Riemann–Liouville fractional derivative of order $0 < q < 1$, $t \in J = [0, T)$, $u \in C(J \times \mathbb{R}, \mathbb{R} - \{0\})$, and v in $C(J \times \mathbb{R}, \mathbb{R})$ (for more details, see [13]). In 2016, Ahmad et al. studied the existence of solutions for a nonlocal boundary value problem of hybrid fractional integro-differential inclusions

$$\begin{cases} {}^CD^\alpha[\frac{x(t)-\sum_{i=1}^m I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}] \in G(t,x(t)), \\ x(0) = \mu(x), \qquad x(1) = A, \end{cases}$$

where ${}^CD^\alpha$ and I^α denote the Caputo fractional derivative and Riemann–Liouville integral of order α, respectively, $t \in J = [0,1]$, $m \in \mathbb{N}$, $A \in \mathbb{R}$, $1 < \alpha \le 2$, $\beta_i > 0$ for $i = 1,2,\dots,m$, the functions $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$, $h_i : J \times \mathbb{R} \to \mathbb{R}$ for $i = 1,2,\dots,m$, $\mu : C(J,\mathbb{R}) \to \mathbb{R}$, and the multifunction $G : J \times \mathbb{R} \to P(\mathbb{R})$ satisfies certain conditions (for more details, see [14]).

Motivated by these articles, we look into the existence of solutions for the following hybrid Caputo–Hadamard fractional differential inclusion:

$$\begin{cases} {}^C_HD^\alpha[\frac{x(t)-f(t,x(t),I^{\beta_1}h_1(t,x(t)),I^{\beta_2}h_2(t,x(t)),\dots,I^{\beta_n}h_n(t,x(t)))}{g(t,x(t),I^{\gamma_1}x(t),I^{\gamma_2}x(t),\dots,I^{\gamma_m}x(t))}] \in K(t,x(t)), \\ x(1) = \mu(x), \qquad x(e) = \eta(x), \end{cases} \tag{1}$$

where ${}^C_HD^\alpha$ and I^α denote the Caputo–Hadamard fractional derivative and Hadamard integral of order α, respectively, $t \in J = [1,e]$, $n,m \in \mathbb{N}$, $1 < \alpha \le 2$, $\beta_i > 0$ for $i = 1,2,\dots,n$, $\gamma_i > 0$ for $i = 1,2,\dots,m$, the functions $f : J \times \mathbb{R}^{n+1} \to \mathbb{R}$, $g : J \times \mathbb{R}^{m+1} \to \mathbb{R} - \{0\}$, $h_i : J \times \mathbb{R} \to \mathbb{R}$ for $i = 1,2,\dots,n$, functions μ, η map $C(J,\mathbb{R})$ into $\mathbb{R}$, and the multifunction $K : J \times \mathbb{R} \to P(\mathbb{R})$ satisfies certain conditions.

The Hadamard fractional integral of order $\alpha > 0$ of a function y is defined by

$$I_a^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{y(s)}{s}\, ds,$$

where $a > 0$, $t > a$. In particular, $I_1^\alpha y(t) := I^\alpha y(t)$.

Definition 1 Let $n \in \mathbb{N}$, $n-1 < \alpha \le n$, $y \in AC_\delta^n[a,b]$, where $0 < a < b < \infty$ and

$$AC_\delta^n[a,b] = \left\{ y : [a,b] \to \mathbb{R} : \delta^{n-1}y(t) \in AC[a,b], \delta = t\frac{d}{dt} \right\}.$$

(i) If $\alpha \neq n$, then the Caputo–Hadamard fractional derivative of order α is defined by

$${}^C_HD_a^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} \delta^n \frac{y(s)}{s}\, ds = I_a^{n-\alpha}\delta^n y(t).$$

(ii) If $\alpha = n$, then the Caputo–Hadamard fractional derivative of order n is defined by

$${}^C_HD_a^n y(t) = \delta^n y(t).$$

In particular, ${}^C_HD_1^0 y(t) := y(t)$ and ${}^C_HD_1^\alpha y(t) := {}^C_HD^\alpha y(t)$.

Lemma 2 ([16]) *Let $\alpha > 0$, $n = [\alpha] + 1$, and $\beta > 0$. Then*

$${}^C_HD_a^\alpha \left(\ln \frac{t}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{t}{a}\right)^{\beta-\alpha-1} \quad \text{and} \quad {}^C_HD_a^\alpha \left(\ln \frac{t}{a}\right)^k = 0,$$

where $\beta > n$, $k = 0,1,\dots,n-1$, and ${}^C_HD_a^\alpha c = 0$ for all $c \in \mathbb{R}$.

Lemma 3 ([16]) *Let $n \in \mathbb{N}$, $n-1 < \alpha \leq n$, and $x \in AC_\delta^n[a,b]$. Then one has*

$$I_a^\alpha \left({}_H^C D_a^\alpha\right) x(t) = x(t) + \sum_{i=0}^{n-1} c_i \left(\ln \frac{t}{a}\right)^i,$$

where $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$.

Lemma 4 ([16]) *Let $n \in \mathbb{N}$, $n-1 < \alpha \leq n$, and $x \in C[a,b]$. Then one has ${}_H^C D_a^\alpha (I_a^\alpha) x(t) = x(t)$.*

Let $(X, \|\cdot\|)$ be a normed space. Denote the set of compact and convex subsets of X, the set of closed subsets of X, and the set of bounded subsets of X by $P_{cp,cv}(X)$, $P_{cl}(X)$, and $P_{bd}(X)$, respectively. An element $x \in X$ is called a fixed point of $T : X \to 2^X$ whenever $x \in T(x)$. A multifunction $T : J \to 2^{\mathbb{R}}$ is said to be measurable whenever the function $t \mapsto d(x, T(t))$ is measurable for all $x \in \mathbb{R}$. A multifunction $T : J \times \mathbb{R} \to 2^{\mathbb{R}}$ is a Caratheodory multifunction whenever the map $t \mapsto T(t,x)$ is measurable for all $x \in \mathbb{R}$ and the map $x \mapsto T(t,x)$ is upper semi-continuous for almost all $t \in J$. Also, a Caratheodory multifunction $T : J \times \mathbb{R} \to 2^{\mathbb{R}}$ is called L^1-Caratheodory whenever, for each $\rho > 0$, there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\left\| T(t,x) \right\| = \sup\left\{ |v| : v \in F(t,x) \right\} \leq \phi_\rho(t)$$

for all $x \in \mathbb{R}$ with $|x| \leq \rho$ and almost all $t \in [1,e]$.

Suppose that x belongs to $C(J, \mathbb{R})$. We define the set of the selections of a multifunction $T : J \times \mathbb{R} \to 2^{\mathbb{R}}$ at x by

$$S_{T,x} = \left\{ v \in L^1[1,e] \,|\, v(t) \in T\big(t, x(t)\big) : \forall t \in J \right\}.$$

Lemma 5 ([17]) *The selections $S_{T,x} \neq \emptyset$ for all x belong to $C(J, \mathbb{R})$ whenever a multifunction $T : J \times \mathbb{R} \to 2^{\mathbb{R}}$ is an L^1-Caratheodory multifunction.*

Lemma 6 ([17]) *Let $T : J \times X \to P_{cp,cv}(X)$ be an L^1-Caratheodory multifunction and linear mapping $\Theta : L^1(J, X) \to C(J, X)$ be continuous. Then the operator*

$$\begin{cases} \Theta \circ S_T : C(J, X) \to P_{cp,cv}(C(J), X), \\ (\Theta \circ S_T)(x) = \Theta(S_{T,x}), \end{cases}$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 7 ([17]) *If a multifunction $T : J \times X \to P_{cl}(X)$ is compact and has a closed graph, then it is upper semi-continuous.*

Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls centered at the origin 0 of radius r in a Banach algebra X, respectively.

Theorem 8 ([18]) *Suppose that A, B, C from X to X are three operators such that*

(a) $\frac{I}{A}$ *is well-defined and one-to-one,*
(b) *A and C are Lipschitzians with the Lipschitz constants q_1 and q_2, respectively,*

(c) *B is completely continuous,*
(d) $Mq_1 + q_2 < \frac{1}{2}$, *where*

$$M = \left\| B\left(\overline{B_r(0)}\right) \right\| = \sup\left\{ |Bx| : x \in \overline{B_r(0)} \right\}. \tag{2}$$

Then either
(i) *the operator equation $AxBx + Cx = x$ has a solution in $\overline{B_r(0)}$, or*
(ii) *there exists $u \in X$ with $\|u\| = r$ such that $\mu u = AuBu + Cu$ for some $\mu > 1$.*

Theorem 9 ([19]) *Let $A, C : \overline{B_r(0)} \to X$ and $B : \overline{B_r(0)} \to P_{cp,cv}(X)$ be three operators such that*
(a) *A and C are single-valued Lipschitz with the Lipschitz constants q_1 and q_2, respectively*;
(b) *B is upper-semi continuous and compact*;
(c) $Mq_1 + q_2 < \frac{1}{2}$, *where M is defined by* (2).
Then either
(i) *the operator inclusion $x \in AxBx + Cx$ has a solution, or*
(ii) *there exists $u \in X$ with $\|u\| = r$ such that $\mu u \in AuBu + Cu$ for some $\mu > 1$.*

2 Main results

In this section, by using fixed point theorems, we look into the existence of solution for the boundary value problem (1). In the first step, by applying the following lemma, we break down the solution kind of hybrid fractional differential inclusion (1).

Lemma 10 *Let $y \in C(J, \mathbb{R})$ and*

$$\frac{x(\cdot) - f(\cdot, x(\cdot), I^{\beta_1} h_1(\cdot, x(\cdot)), I^{\beta_2} h_2(\cdot, x(\cdot)), \ldots, I^{\beta_n} h_n(\cdot, x(\cdot)))}{g(\cdot, x(\cdot), I^{\gamma_1} x(\cdot), I^{\gamma_2} x(\cdot), \ldots, I^{\gamma_m} x(\cdot))} \in AC_\delta^2[1, e].$$

Then the unique solution of the hybrid fractional differential equation

$$\begin{aligned} &{}_H^C D^\alpha \left[\frac{x(t) - f(t, x(t), I^{\beta_1} h_1(t, x(t)), I^{\beta_2} h_2(t, x(t)), \ldots, I^{\beta_n} h_n(t, x(t)))}{g(t, x(t), I^{\gamma_1} x(t), I^{\gamma_2} x(t), \ldots, I^{\gamma_m} x(t))} \right] \\ &\quad = y(t), \end{aligned} \tag{3}$$

via the boundary conditions $x(1) = \mu(x)$ and $x(e) = \eta(x)$, is

$$\begin{aligned} x(t) &= g\left(t, x(t), I^{\gamma_1} x(t), I^{\gamma_2} x(t), \ldots, I^{\gamma_m} x(t)\right) \\ &\quad \times \left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s} \right)^{\alpha - 1} \frac{y(s)}{s} \, ds + a_0 \right. \\ &\quad \left. + \left(a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{1}{s} \right)^{\alpha - 1} \frac{y(s)}{s} \, ds - a_0 \right) \ln t \right] \\ &\quad + f\left(t, x(t), I^{\beta_1} h_1\left(t, x(t)\right), I^{\beta_2} h_2\left(t, x(t)\right), \ldots, I^{\beta_n} h_n\left(t, x(t)\right)\right) \end{aligned} \tag{4}$$

for all $t \in J$, where

$$a_0 = \frac{\mu(x) - f(1, \mu(x), \overbrace{0, \ldots, 0}^{n})}{g(1, \mu(x), \underbrace{0, \ldots, 0}_{m})},$$

$$a_1 = \frac{\eta(x) - f(t, x(t), I^{\beta_1} h_1(t, x(t)), I^{\beta_2} h_2(t, x(t)), \ldots, I^{\beta_n} h_n(t, x(t)))|_{t=e}}{g(t, x(t), I^{\gamma_1} x(t), I^{\gamma_2} x(t), \ldots, I^{\gamma_m} x(t))|_{t=e}}.$$

Proof Let $x(t)$ be a solution of equation (3). By using the operator I^α on both sides of equation (3) and applying Lemma 3, we have

$$\begin{aligned} &\frac{x(t) - f(t, x(t), I^{\beta_1} h_1(t, x(t)), I^{\beta_2} h_2(t, x(t)), \ldots, I^{\beta_n} h_n(t, x(t)))}{g(t, x(t), I^{\gamma_1} x(t), I^{\gamma_2} x(t), \ldots, I^{\gamma_m} x(t))} \\ &\quad = I^\alpha y(t) + c_0 + c_1 \ln t, \end{aligned}$$

where $c_0, c_1 \in \mathbb{R}$ denote arbitrary constants. Thus

$$\begin{aligned} x(t) &= g\bigl(t, x(t), I^{\gamma_1} x(t), I^{\gamma_2} x(t), \ldots, I^{\gamma_m} x(t)\bigr) \\ &\quad \times \left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{y(s)}{s}\, ds + c_0 + c_1 \ln t\right] \\ &\quad + f\bigl(t, x(t), I^{\beta_1} h_1\bigl(t, x(t)\bigr), I^{\beta_2} h_2\bigl(t, x(t)\bigr), \ldots, I^{\beta_n} h_n\bigl(t, x(t)\bigr)\bigr). \end{aligned}$$

Now, by utilizing the boundary conditions, we conclude $c_0 = a_0$ and

$$c_1 = a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{1}{s}\right)^{\alpha-1} \frac{y(s)}{s}\, ds - a_0.$$

Conversely, it is easy to check that equation (4) satisfies the boundary conditions $x(1) = \mu(x)$ and $x(e) = \eta(x)$. On the other hand, by deformation equation (4) to

$$\begin{aligned} &\frac{x(t) - f(t, x(t), I^{\beta_1} h_1(t, x(t)), I^{\beta_2} h_2(t, x(t)), \ldots, I^{\beta_n} h_n(t, x(t)))}{g(t, x(t), I^{\gamma_1} x(t), I^{\gamma_2} x(t), \ldots, I^{\gamma_m} x(t))} \\ &\quad = I^\alpha y(t) + a_0 + \left(a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{1}{s}\right)^{\alpha-1} \frac{y(s)}{s}\, ds - a_0\right) \ln t, \end{aligned}$$

and by applying Lemmas 2 and 4, we have

$${}^{C}_{H}D^\alpha \left[\frac{x(t) - f(t, x(t), I^{\beta_1} h_1(t, x(t)), I^{\beta_2} h_2(t, x(t)), \ldots, I^{\beta_n} h_n(t, x(t)))}{g(t, x(t), I^{\gamma_1} x(t), I^{\gamma_2} x(t), \ldots, I^{\gamma_m} x(t))}\right] = y(t),$$

where a_0, a_1 are defined in Lemma 10. This finishes our proof. □

A function x belonging to $C(J,\mathbb{R})$ is a solution of problem (1) whenever it satisfies the boundary conditions, and there exists a function $y \in S_{K,x}$ such that

$$\begin{aligned} x(t) &= g\bigl(t,x(t),I^{\gamma_1}x(t),I^{\gamma_2}x(t),\dots,I^{\gamma_m}x(t)\bigr) \\ &\quad\times\left[\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}\,ds + a_0\right. \\ &\quad\left.+\left(a_1\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{1}{s}\right)^{\alpha-1}\frac{y(s)}{s}\,ds - a_0\right)\ln t\right] \\ &\quad+f\bigl(t,x(t),I^{\beta_1}h_1\bigl(t,x(t)\bigr),I^{\beta_2}h_2\bigl(t,x(t)\bigr),\dots,I^{\beta_n}h_n\bigl(t,x(t)\bigr)\bigr) \end{aligned}$$

for $t \in J$, where

$$\begin{aligned} a_0 &= \frac{\mu(x) - f(1,\mu(x),\overbrace{0,\dots,0}^{n})}{g(1,\mu(x),\underbrace{0,\dots,0}_{m})}, \\ a_1 &= \frac{\eta(x) - f(t,x(t),I^{\beta_1}h_1(t,x(t)),I^{\beta_2}h_2(t,x(t)),\dots,I^{\beta_n}h_n(t,x(t)))|_{t=e}}{g(t,x(t),I^{\gamma_1}x(t),I^{\gamma_2}x(t),\dots,I^{\gamma_m}x(t))|_{t=e}}. \end{aligned}$$

Let $\mathcal{X} = C(J,\mathbb{R})$ denote the Banach algebra of all continuous functions from J to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)| : t \in J\}$.

Theorem 11 *Suppose that*

(H_1) *The multifunction $K : J \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ is an L^1-Caratheodory such that*

$$\bigl\|K(t,x)\bigr\| = \sup\bigl\{|y| : y \in K(t,x)\bigr\} \le p(t)\varphi\bigl(|x|\bigr)$$

for $(t,x) \in J \times \mathbb{R}$, where $p \in C(J,\mathbb{R}^+)$ and $\varphi : [0,\infty) \to (0,\infty)$ is a continuous and increasing function;

(H_2) *Function $f : J \times \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous, and there exists $\phi \in C(J,\mathbb{R}^+)$ such that*

$$\bigl|f(t,x_1,x_2,\dots,x_{n+1}) - f\bigl(t,x'_1,x'_2,\dots,x'_{n+1}\bigr)\bigr| \le \phi(t)\sum_{i=1}^{n+1}\bigl|x_i - x'_i\bigr|,$$

where $x_i, x'_i \in \mathbb{R}$ for $i = 1,2,\dots,n+1$, $t \in J$, and $n \in \mathbb{N}$;

(H_3) *Function $g : J \times \mathbb{R}^{m+1} \to \mathbb{R} - \{0\}$ is continuous, and there exists $\psi \in C(J,\mathbb{R}^+)$ such that*

$$\bigl|g(t,x_1,x_2,\dots,x_{m+1}) - f\bigl(t,x'_1,x'_2,\dots,x'_{m+1}\bigr)\bigr| \le \psi(t)\sum_{i=1}^{m+1}\bigl|x_i - x'_i\bigr|,$$

where $x_i, x'_i \in \mathbb{R}$ for $i = 1,2,\dots,m+1$, $t \in J$, and $m \in \mathbb{N}$;

(H_4) *Functions $h_i : J \times \mathbb{R} \to \mathbb{R}$ are continuous for $i = 1,2,\dots,n$, and there exists $p_i \in C(J,\mathbb{R}^+)$ for $i = 1,2,\dots,n$ such that $|h_i(t,x) - h_i(t,x')| \le p_i(t)|x - x'|$ for $i = 1,2,\dots,n$, where $x, x' \in \mathbb{R}$, $n \in \mathbb{N}$, $t \in J$;*

(H_5) *There exist constants* $M_0, M_1 > 0$ *such that*

$$\left|\frac{\eta(x) - f(t, x(t), I^{\beta_1} h_1(t, x(t)), I^{\beta_2} h_2(t, x(t)), \ldots, I^{\beta_n} h_n(t, x(t)))|_{t=e}}{g(t, x(t), I^{\gamma_1} x(t), I^{\gamma_2} x(t), \ldots, I^{\gamma_m} x(t))|_{t=e}}\right| < M_1$$

and

$$\left|\frac{\mu(x) - f(1, \mu(x), \overbrace{0, \ldots, 0}^{n})}{g(1, \mu(x), \underbrace{0, \ldots, 0}_{m})}\right| < M_0$$

for all $x \in C(J, \mathbb{R})$.

The fractional differential inclusion (1) *admits a solution whenever, for* $i = 1, 2, \ldots, n$,

$$\begin{aligned}&\left(\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)} + 2M_0 + M_1\right)\|\psi\|\left(1 + \sum_{i=1}^{m} \frac{1}{\Gamma(\gamma_i + 1)}\right)\\&\quad + \|\phi\|\left(1 + \sum_{i=1}^{n} \frac{\|p_i\|}{\Gamma(\beta_i + 1)}\right) < \frac{1}{2},\end{aligned}$$

where $G_0 = \sup_{t\in J} |g(t, \underbrace{0, \ldots, 0}_{m+1})|$, $F_0 = \sup_{t\in J} |f(t, \underbrace{0, \ldots, 0}_{n+1})|$, $H_{i,0} = \sup_{t\in J} |h_i(t, 0)|$, *and*

$$r > \frac{G_0[\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)} + 2M_0 + M_1] + \|\phi\| \sum_{i=1}^{n} \frac{H_{i,0}}{\Gamma(\beta_i+1)} + F_0}{1 - \|\psi\|(1 + \sum_{i=1}^{m} \frac{1}{\Gamma(\gamma_i+1)})(\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)} + 2M_0 + M_1) - \|\phi\|(1 + \sum_{i=1}^{n} \frac{\|p_i\|}{\Gamma(\beta_i+1)})}.$$

Proof Let $\overline{B_r(0)} = \{x \in \mathcal{X} : \|x\| \le r\}$. Define operators $A : \overline{B_r(0)} \to \mathcal{X}$, $B : \overline{B_r(0)} \to P(\mathcal{X})$ and $C : \overline{B_r(0)} \to \mathcal{X}$ by

$$\begin{aligned}(Ax)(t) &= g\bigl(t, x(t), I^{\gamma_1} x(t), I^{\gamma_2} x(t), \ldots, I^{\gamma_m} x(t)\bigr),\\(Bx)(t) &= \biggl\{v(t) : v(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{y(s)}{s}\,ds + a_0\\&\quad + \left(a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln\frac{1}{s}\right)^{\alpha-1} \frac{y(s)}{s}\,ds - a_0\right)\ln t, y \in S_{K,x}\biggr\},\\(Cx)(t) &= f\bigl(t, x(t), I^{\beta_1} h_1\bigl(t, x(t)\bigr), I^{\beta_2} h_2\bigl(t, x(t)\bigr), \ldots, I^{\beta_n} h_n\bigl(t, x(t)\bigr)\bigr),\end{aligned}$$

respectively. We establish that the operators A, B, and C satisfy all the conditions of Theorem 9. Since the operator K is L^1-Caratheodory, $S_{K,x} \neq \emptyset$ for all $x \in \mathcal{X}$. First, we show that Bx is a convex subset of $\mathcal{X}$ for all $x \in \overline{B_r(0)}$. Consider $x \in \overline{B_r(0)}$ and $v_1, v_2 \in Bx$. Choose $y_1, y_2 \in S_{K,x}$ such that

$$\begin{aligned}v_1(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{y_1(s)}{s}\,ds + a_0\\&\quad + \left(a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln\frac{1}{s}\right)^{\alpha-1} \frac{y_1(s)}{s}\,ds - a_0\right)\ln t,\end{aligned}$$

$$\begin{aligned} v_2(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{y_2(s)}{s}\, ds + a_0 \\ &\quad + \left(a_1 \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln\frac{1}{s}\right)^{\alpha-1} \frac{y_2(s)}{s}\, ds - a_0\right) \ln t. \end{aligned}$$

At present, for any $0 \le \lambda \le 1$, we have

$$\begin{aligned} \lambda v_1(t) + (1-\lambda) v_2(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{\lambda y_1(s) + (1-\lambda) y_2(s)}{s}\, ds + a_0 \\ &\quad + \left[a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln\frac{1}{s}\right)^{\alpha-1} \frac{\lambda y_1(s) + (1-\lambda) y_2(s)}{s}\, ds - a_0\right] \ln t. \end{aligned}$$

Since K has convex values, hence $S_{K,x}$ is convex and $\lambda y_1 + (1-\lambda) y_2 \in S_{K,x}$. So $\lambda v_1 + (1-\lambda) v_2 \in Bx$, which results in that Bx is a convex subset of $\mathcal{X}$. Now, we show that Bx is a compact subset of $\mathcal{X}$ for all $x \in \overline{B_r(0)}$. Let $x \in \overline{B_r(0)}$ and $\{v_n\}$ be a sequence in Bx. For each n, choose $y_n \in S_{K,x}$ such that

$$\begin{aligned} v_n(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{y_n(s)}{s}\, ds + a_0 \\ &\quad + \left(a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln\frac{1}{s}\right)^{\alpha-1} \frac{y_n(s)}{s}\, ds - a_0\right) \ln t. \end{aligned}$$

Since K is compact-valued, $\{y_n\}$ has a subsequence which pointwise converges to y. We denote this subsequence again by $\{y_n\}$. Since $y(t) \in K(t, x(t))$ and $\|K(t, x(t))\| \le p(t)\varphi(|x(t)|)$, by using Lebesgue dominated convergence theorem $y \in S_{K,x}$, we have

$$\begin{aligned} v_n(t) \to v(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{y(s)}{s}\, ds + a_0 \\ &\quad + \left(a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln\frac{1}{s}\right)^{\alpha-1} \frac{y(s)}{s}\, ds - a_0\right) \ln t \end{aligned}$$

pointwise on J. Let $t_1 < t_2 \in J$. Then we have

$$\begin{aligned} \left|v_n(t_2) - v_n(t_1)\right| &\le \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_2} \left(\ln\frac{t_2}{s}\right)^{\alpha-1} \frac{y_n(s)}{s}\, ds \right. \\ &\quad \left. - \int_1^{t_1} \left(\ln\frac{t_1}{s}\right)^{\alpha-1} \frac{y_n(s)}{s}\, ds \right| \\ &\quad + \left| a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln\frac{1}{s}\right)^{\alpha-1} \frac{y_n(s)}{s}\, ds - a_0 \right| \\ &\quad \times (\ln t_2 - \ln t_1) \\ &\le \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left| \left(\ln\frac{t_2}{s}\right)^{\alpha-1} - \left(\ln\frac{t_2}{s}\right)^{\alpha-1} \right| \frac{\|p\|\varphi(r)}{s}\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left| \left(\ln\frac{t_2}{s}\right)^{\alpha-1} \right| \frac{\|p\|\varphi(r)}{s}\, ds \end{aligned}$$

$$+\left(|a_1|+\frac{1}{\Gamma(\alpha)}\int_1^e\left|\left(\ln\frac{1}{s}\right)^{\alpha-1}\right|\frac{\|p\|\varphi(r)}{s}\,ds+|a_0|\right)$$
$$\times(\ln t_2-\ln t_1).$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Hence $\{v_n\}$ is an equicontinuous sequence. Therefore $\{v_n\}$ has a subsequence that uniformly converges to $v \in Bx$, which proves that Bx is a compact subset of $\mathcal{X}$. In this section, we shall show that the operator B is compact on $\overline{B_r(0)}$. Let $S \subset \overline{B_r(0)}$. We prove that $\overline{B(S)}$ is a compact set in $\mathcal{X}$. To do this, we apply Arzelá–Ascoli theorem and show that $B(S)$ is a uniformly bounded and equicontinuous set. Let $x \in S$, $v \in Bx$, and $t_1 < t_2 \in J$. Choose $y \in S_{K,x}$ such that

$$\begin{aligned} v(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}\,ds+a_0 \\ &\quad+\left(a_1-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{1}{s}\right)^{\alpha-1}\frac{y(s)}{s}\,ds-a_0\right)\ln t. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \bigl|v(t)\bigr| &\le \frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{|y(s)|}{s}\,ds+|a_0| \\ &\quad+\left(|a_1|+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{1}{s}\right)^{\alpha-1}\frac{|y(s)|}{s}\,ds+|a_0|\right)\ln t \\ &\le \frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{\|p\|\varphi(r)}{s}\,ds \\ &\quad+M_0+\left(M_1+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{1}{s}\right)^{\alpha-1}\frac{\|p\|\varphi(r)}{s}\,ds+M_0\right) \\ &\le \frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)}+2M_0+M_1. \end{aligned}$$

Thus, $\|v\| \le \frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)}+2M_0+M_1$. Also, we have

$$\begin{aligned} \bigl|v(t_2)-v(t_1)\bigr| &\le \frac{1}{\Gamma(\alpha)}\left|\int_1^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{y(s)}{s}\,ds-\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\frac{y(s)}{s}\,ds\right| \\ &\quad+\left|a_0-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{1}{s}\right)^{\alpha-1}\frac{y(s)}{s}\,ds-a_0\right|(\ln t_2-\ln t_1) \\ &\le \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left|\left(\ln\frac{t_2}{s}\right)^{\alpha-1}-\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\right|\frac{\|p\|\varphi(r)}{s}\,ds \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\right|\frac{\|p\|\varphi(r)}{s}\,ds \\ &\quad+\left(|a_1|+\frac{1}{\Gamma(\alpha)}\int_1^e\left|\left(\ln\frac{1}{s}\right)^{\alpha-1}\right|\frac{\|p\|\varphi(r)}{s}\,ds+|a_0|\right) \\ &\quad\times(\ln t_2-\ln t_1). \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Thus, $B(S)$ is a uniformly bounded and equicontinuous set in $\mathcal{X}$ and B is a compact operator. In the following, we shall show that the operator B is upper semi-continuous on $\overline{B_r(0)}$. To do this, we apply Lemma 7 and show that B has a closed graph. Let $x_n \to x_0$ and $v_n \in Bx_n$ $(n \geq 1)$ with $v_n \to v_0$. We prove that $v_0 \in Bx_0$. For each natural number n, choose $y_n \in S_{K,x_n}$ such that

$$\begin{aligned} v_n(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{y_n(s)}{s}\, ds + a_0 \\ &\quad + \left(a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{1}{s}\right)^{\alpha-1} \frac{y_n(s)}{s}\, ds - a_0\right) \ln t. \end{aligned}$$

Define the continuous linear operator $\theta : L^1(J,\mathbb{R}) \to \mathcal{X}$ by

$$\begin{aligned} \theta(y)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{y(s)}{s}\, ds + a_0 \\ &\quad + \left(a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{1}{s}\right)^{\alpha-1} \frac{y(s)}{s}\, ds - a_0\right) \ln t. \end{aligned}$$

By using Lemma 6, $\theta o S_K$ is a closed graph operator. Since $v_n \in \theta(S_{K,x_n})$ for all n and $x_n \to x_0$, there exists $y_0 \in S_{K,x_0}$ such that

$$\begin{aligned} v_0(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{y_0(s)}{s}\, ds + a_0 \\ &\quad + \left(a_1 - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\ln \frac{1}{s}\right)^{\alpha-1} \frac{y_0(s)}{s}\, ds - a_0\right) \ln t. \end{aligned}$$

Hence, $v_0 \in Bx_0$. This implies that B has a closed graph, and as a result, B is upper semi-continuous. Now, we show that the operators A, C are Lipschitz on $\overline{B_r(0)}$. Let $x, y \in \overline{B_r(0)}$. Then we have

$$\begin{aligned} \bigl|Ax(t) - Ay(t)\bigr| &= \bigl|g\bigl(t, x(t), I^{\gamma_1}x(t), I^{\gamma_2}x(t), \ldots, I^{\gamma_m}x(t)\bigr) \\ &\quad - g\bigl(t, y(t), I^{\gamma_1}y(t), I^{\gamma_2}y(t), \ldots, I^{\gamma_m}y(t)\bigr)\bigr| \\ &\leq \psi(t)\left[\bigl|x(t) - y(t)\bigr| + \sum_{i=1}^m \bigl|I^{\gamma_i}x(t) - I^{\gamma_i}y(t)\bigr|\right] \\ &\leq \psi(t)\left[\bigl|x(t) - y(t)\bigr| \right.\\ &\quad \left. + \sum_{i=1}^m \frac{1}{\Gamma(\gamma_i)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma_i - 1} \frac{|x(s) - y(s)|}{s}\, ds\right] \\ &\leq \|\psi\|\left[\|x - y\| + \|x - y\| \sum_{i=1}^m \frac{1}{\Gamma(\gamma_i + 1)}\right], \end{aligned}$$

which implies that

$$\|Ax - Ay\| \leq \underbrace{\|\psi\|\left(1 + \sum_{i=1}^m \frac{1}{\Gamma(\gamma_i + 1)}\right)}_{q_1} \|x - y\| = q_1\|x - y\|.$$

Also,

$$\begin{aligned}
\left|Cx(t)-Cy(t)\right| &= \big|f\big(t,x(t),I^{\beta_1}h_1\big(t,x(t)\big),I^{\beta_2}h_2\big(t,x(t)\big),\ldots,I^{\beta_n}h_n\big(t,x(t)\big)\big) \\
&\quad -f\big(t,y(t),I^{\beta_1}h_1\big(t,y(t)\big),I^{\beta_2}h_2\big(t,y(t)\big),\ldots,I^{\beta_n}h_n\big(t,y(t)\big)\big)\big| \\
&\le \phi(t)\Bigg[\left|x(t)-y(t)\right| \\
&\quad +\sum_{i=1}^{n}\left|I^{\beta_i}h_i\big(t,x(t)\big)-I^{\beta_i}h_i\big(t,y(t)\big)\right|\Bigg] \\
&\le \phi(t)\Bigg[\left|x(t)-y(t)\right| \\
&\quad +\sum_{i=1}^{n}\frac{1}{\Gamma(\beta_i)}\int_1^t\left(\ln\frac{t}{s}\right)^{\beta_i-1}\frac{|h_i(s,x(s))-h_i(s,y(s))|}{s}\,ds\Bigg] \\
&\le \phi(t)\Bigg[\left|x(t)-y(t)\right| \\
&\quad +\sum_{i=1}^{n}\frac{1}{\Gamma(\beta_i)}\int_1^t\left(\ln\frac{t}{s}\right)^{\beta_i-1}\frac{p_i(s)|x(s)-y(s)|}{s}\,ds\Bigg] \\
&\le \|\phi\|\left(\|x-y\|+\|x-y\|\sum_{i=1}^{n}\frac{\|p_i\|}{\Gamma(\beta_i+1)}\right) \\
&= \underbrace{\|\phi\|\left(1+\sum_{i=1}^{n}\frac{\|p_i\|}{\Gamma(\beta_i+1)}\right)}_{q_2}\|x-y\| = q_2\|x-y\|.
\end{aligned}$$

Since

$$M=\left\|B\big(\overline{B_r(0)}\big)\right\|=\sup_{x\in\overline{B_r(0)}}\left\{\sup Bx(t),t\in J\right\}\le\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)}+2M_0+M_1,$$

we have

$$\begin{aligned}
Mq_1+q_2 &\le \left(\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)}+2M_0+M_1\right)\|\psi\|\left(1+\sum_{i=1}^{m}\frac{1}{\Gamma(\gamma_i+1)}\right) \\
&\quad +\|\phi\|\left(1+\sum_{i=1}^{n}\frac{\|p_i\|}{\Gamma(\beta_i+1)}\right)<\frac{1}{2}.
\end{aligned}$$

Hence, all the conditions of Theorem 9 hold. Let $u\in\mathcal{X}$, $\|u\|=r$, $\mu>1$, and $\mu u\in AuBu+Cu$. Then we have

$$\begin{aligned}
\left|u(t)\right| &\le \left|g\big(t,u(t),I^{\gamma_1}u(t),I^{\gamma_2}u(t),\ldots,I^{\gamma_m}u(t)\big)\right| \\
&\quad\times\left|\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}\,ds+a_0\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(a_1-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\ln\frac{1}{s}\right)^{\alpha-1}\frac{y(s)}{s}\,ds-a_0\right)\ln t\Bigg| \\
&+\left|f\big(t,u(t),I^{\beta_1}h_1\big(t,u(t)\big),I^{\beta_2}h_2\big(t,u(t)\big),\dots,I^{\beta_n}h_n\big(t,u(t)\big)\big)\right| \\
&\le\Big[\big|g\big(t,u(t),I^{\gamma_1}u(t),I^{\gamma_2}u(t),\dots,I^{\gamma_m}u(t)\big)-g(t,\underbrace{0,0,\dots,0}_{m+1})\big| \\
&+\big|g(t,\underbrace{0,0,\dots,0}_{m+1})\big|\Big]\left(\frac{2\|p\|\varphi(\|u\|)}{\Gamma(\alpha+1)}+2M_0+M_1\right) \\
&+\big|f\big(t,u(t),I^{\beta_1}h_1\big(t,u(t)\big),I^{\beta_2}h_2\big(t,u(t)\big),\dots,I^{\beta_n}h_n\big(t,u(t)\big)\big) \\
&-f(t,\underbrace{0,0,\dots,0}_{n+1})\big|+\big|f(t,\underbrace{0,0,\dots,0}_{n+1})\big| \\
&\le\left[\psi(t)\left(\big|u(t)\big|+\sum_{i=1}^m\big|I^{\gamma_i}u(t)\big|\right)+G_0\right]\left[\frac{2\|p\|\varphi(\|u\|)}{\Gamma(\alpha+1)}+2M_0+M_1\right] \\
&+\phi(t)\left(\big|u(t)\big|+\sum_{i=1}^n\big|I^{\beta_i}h_i\big(t,u(t)\big)\big|\right)+F_0 \\
&\le\left[\|\psi\|\|u\|\left(1+\sum_{i=1}^m\frac{1}{\Gamma(\gamma_i+1)}\right)+G_0\right]\left[\frac{2\|p\|\varphi(\|u\|)}{\Gamma(\alpha+1)}+2M_0+M_1\right] \\
&+\|\phi\|\left(\|u\|+\sum_{i=1}^n\frac{\|p_i\|\|u\|+H_{i,0}}{\Gamma(\beta_i+1)}\right)+F_0
\end{aligned}
$$

for some $y\in S_{K,u}$. Therefore

$$
\begin{aligned}
\|u\|\le{}&\left[\|\psi\|\|u\|\left(1+\sum_{i=1}^m\frac{1}{\Gamma(\gamma_i+1)}\right)+G_0\right]\left[\frac{2\|p\|\varphi(\|u\|)}{\Gamma(\alpha+1)}+2M_0+M_1\right] \\
&+\|\phi\|\left(\|u\|+\sum_{i=1}^n\frac{\|p_i\|\|u\|+H_{i,0}}{\Gamma(\beta_i+1)}\right)+F_0.
\end{aligned}
$$

As a result, we have

$$
\|u\|\le\frac{G_0[\frac{2\|p\|\varphi(\|u\|)}{\Gamma(\alpha+1)}+2M_0+M_1]+\|\phi\|\sum_{i=1}^n\frac{H_{i,0}}{\Gamma(\beta_i+1)}+F_0}{1-\|\psi\|(1+\sum_{i=1}^m\frac{1}{\Gamma(\gamma_i+1)})(\frac{2\|p\|\varphi(\|u\|)}{\Gamma(\alpha+1)}+2M_0+M_1)-\|\phi\|(1+\sum_{i=1}^n\frac{\|p_i\|}{\Gamma(\beta_i+1)})}.
$$

Since $\|u\|=r$, we have

$$
r\le\frac{G_0[\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)}+2M_0+M_1]+\|\phi\|\sum_{i=1}^n\frac{H_{i,0}}{\Gamma(\beta_i+1)}+F_0}{1-\|\psi\|(1+\sum_{i=1}^m\frac{1}{\Gamma(\gamma_i+1)})(\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)}+2M_0+M_1)-\|\phi\|(1+\sum_{i=1}^n\frac{\|p_i\|}{\Gamma(\beta_i+1)})}<r.
$$

This is an apparent contradiction, and conclusion (ii) of Theorem 9 does not hold. Hence, there exists $x\in\overline{B_r(0)}$ such that $x\in AxBx+Cx$, and it is a solution of hybrid fractional differential inclusion (1). □

In the following, we review the existence of solution for the hybrid fractional differential equation

$$\begin{cases} {}_H^C D^\alpha [\frac{x(t)-f(t,x(t),I^{\beta_1}h_1(t,x(t)),I^{\beta_2}h_2(t,x(t)),\dots,I^{\beta_n}h_n(t,x(t)))}{g(t,x(t),I^{\gamma_1}x(t),I^{\gamma_2}x(t),\dots,I^{\gamma_m}x(t))}] = k(t,x(t)), \\ x(1) = \mu(x), \qquad x(e) = \eta(x), \end{cases} \tag{5}$$

where ${}_H^C D^\alpha$ and I^α denote the Caputo–Hadamard fractional derivative and Hadamard integral of order α, respectively, $t \in J = [1,e]$, $n, m \in \mathbb{N}$, $1 < \alpha \le 2$, $\beta_i > 0$ ($i = 1,2,\dots,n$), $\gamma_i > 0$ ($i = 1,2,\dots,m$), the functions $f : J \times \mathbb{R}^{n+1} \to \mathbb{R}$, $g : J \times \mathbb{R}^{m+1} \to \mathbb{R} - \{0\}$, $h_i : J \times \mathbb{R} \to \mathbb{R}(i = 1,2,\dots,n)$, $k : J \times \mathbb{R} \to \mathbb{R}$, and $\mu, \eta : C(J,\mathbb{R}) \to \mathbb{R}$ satisfy certain conditions. It is clear that equation (5) is a special case of inclusion (1), where $K(t,x(t)) = \{k(t,x(t))\}$. With argument similar to the proof of Theorem 11, and applying Theorem 8, we can prove the following theorem.

Theorem 12 *Suppose that*

(H_1') *The function $k : J \times \mathbb{R} \to \mathbb{R}$ is a continuous map such that $|k(t,x)| \le p(t)\varphi(|x|)$ for all $(t,x) \in J \times \mathbb{R}$, where $p \in C(J,\mathbb{R}^+)$ and $\varphi : [0,\infty) \to (0,\infty)$ is a continuous and increasing function*;

(H_2') *If $\frac{(x_1,x_2,\dots,x_{m+1})}{g(t,x_1,x_2,\dots,x_{m+1})} = \frac{(y_1,y_2,\dots,y_{m+1})}{g(t,y_1,y_2,\dots,y_{m+1})}$, then*

$$(x_1,x_2,\dots,x_{m+1}) = (y_1,y_2,\dots,y_{m+1})$$

for all $t \in J$ and $(x_1,x_2,\dots,x_{m+1}) \in \mathbb{R}^{m+1}$.

Also, assume that conditions (H_2), (H_3), (H_4), and (H_5) in Theorem 11 hold. If

$$\left(\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)} + 2M_0 + M_1\right)\|\psi\|\left(1 + \sum_{i=1}^m \frac{1}{\Gamma(\gamma_i+1)}\right) + \|\phi\|\left(1 + \sum_{i=1}^n \frac{\|p_i\|}{\Gamma(\beta_i+1)}\right) < \frac{1}{2},$$

where r is the constant that was introduced in Theorem 11, then the fractional differential equation (5) has at least one solution on J.

Example 1 Consider the following hybrid Caputo–Hadamard fractional differential inclusion:

$$\begin{cases} {}_H^C D^{\frac{3}{2}} [\frac{x(t)-3-\frac{t^2}{100(1+t^2)}(\sin x(t)+\cos(I^{\frac{1}{3}}(\sin t \tan^{-1} x(t)))+\frac{|I^{\frac{1}{2}}(\frac{t^2|x(t)|}{5(1+|x(t)|)})|}{1+|I^{\frac{1}{2}}(\frac{t^2|x(t)|}{5(1+|x(t)|)})|})}{1+\frac{e^{-\pi t}}{1000(1+e^{-\pi t})}(\frac{|x(t)|+|I^{\frac{2}{3}}x(t)|+|I^{\frac{3}{4}}x(t)|}{1+|x(t)|+|I^{\frac{2}{3}}x(t)|+|I^{\frac{3}{4}}x(t)|})}] \in [0, e^t \sin t + 2], \\ x(1) = \sin(x(\frac{3}{2})), \qquad x(e) = \cos(x(\frac{5}{2})). \end{cases} \tag{6}$$

Here $\alpha = \frac{3}{2}$, $m = n = 2$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{1}{2}$, $\gamma_1 = \frac{2}{3}$, $\gamma_2 = \frac{3}{4}$,

$$g(t,x_1,x_2,x_3) = 1 + \frac{(|x_1| + |x_2| + |x_3|)e^{-\pi t}}{1000(1 + (|x_1| + |x_2| + |x_3|)(1 + e^{-\pi t}))},$$

$$f(t,x_1,x_2,x_3) = 3 + \frac{t^2}{100(1+t^2)}\left(\sin x_1 + \cos x_2 + \frac{|x_3|}{1+|x_3|}\right),$$

$K(t,x) = [0, e^t \sin x + 2]$, $h_1(t,x) = \sin t \tan^{-1} x$, $h_2(t,x) = \frac{t^2|x|}{5(1+|x|)}$, $\mu(y) = \sin(y(\frac{3}{2}))$, and $\eta(y) = \cos(y(\frac{5}{2}))$ for all x, x_1, x_2, and $x_3 \in \mathbb{R}$, $t \in J$ and $y \in C(J,\mathbb{R})$. Obviously,

$$\bigl|g(t,x_1,x_2,x_3) - g\bigl(t,x_1',x_2',x_3'\bigr)\bigr| \le \frac{e^{-\pi t}}{1000(1+e^{-\pi t})}\sum_{i=1}^{3}\bigl|x_i - x_i'\bigr|,$$

$$\bigl|f(t,x_1,x_2,x_3) - f\bigl(t,x_1',x_2',x_3'\bigr)\bigr| \le \frac{t^2}{100(1+t^2)}\sum_{i=1}^{3}\bigl|x_i - x_i'\bigr|,$$

$|h_1(t,x_1) - h_1(t,x_1')| \le \sin t|x_1 - x_1'|$, $|h_2(t,x_1) - h_2(t,x_1')| \le \frac{t^2}{5}|x_1 - x_1'|$,

$$\left|\frac{\eta(y) - f(t,y(t),I^{\beta_1}h_1(t,y(t)),I^{\beta_2}h_2(t,y(t)))|_{t=e}}{g(t,x(t),I^{\gamma_1}x(t),I^{\gamma_2}x(t)|_{t=e}}\right| < 4.03,$$

$$\left|\frac{\mu(y) - f(1,\mu(y),0,0)}{g(1,\mu(y),0,0)}\right| < 4.03,$$

$\|K(t,x)\| \le e^t + 2$, $F_0 \le 3.01$, $G_0 = 1$, and $H_{1,0} = H_{2,0} = 0$ for all x_1, x_2, x_3, x_1', x_2', x_3', and $x \in \mathbb{R}$, $t \in J$, and $y \in C(J,\mathbb{R})$. Now, by setting $p(t) = e^t + 2$, $\varphi(x) = 1$, $\psi(t) = \frac{e^{-\pi t}}{1000(1+e^{-\pi t})}$, $\phi(t) = \frac{t^2}{100(1+t^2)}$, $p_1(t) = \sin t$, $p_2(t) = \frac{t^2}{5}$, $M_0 = 4.03$, and $M_1 = 4.03$ for all $t \in J$ and $x \in \mathbb{R}^+$, we have $\|p\| = e^e + 2$, $\|\phi\| \le \frac{1}{100}$, $\|\psi\| \le \frac{1}{1000}$, $\|p_1\| = 1$, and $\|p_2\| = \frac{e^2}{5}$. Hence

$$\begin{aligned}&\left(\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)} + 2M_0 + M_1\right)\|\psi\|\left(1 + \sum_{i=1}^{m}\frac{1}{\Gamma(\gamma_i+1)}\right)\\&\quad + \|\phi\|\left(1 + \sum_{i=1}^{n}\frac{\|p_i\|}{\Gamma(\beta_i+1)}\right) \le 0.14606 < \frac{1}{2}.\end{aligned}$$

Therefore, all the conditions in Theorem 11 are satisfied, and problem (6) has at least one solution on $\overline{B_r(0)}$, where

$$r > \frac{G_0[\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)} + 2M_0 + M_1] + \|\phi\|\sum_{i=1}^{n}\frac{H_{i,0}}{\Gamma(\beta_i+1)} + F_0}{1 - \|\psi\|(1 + \sum_{i=1}^{m}\frac{1}{\Gamma(\gamma_i+1)})(\frac{2\|p\|\varphi(r)}{\Gamma(\alpha+1)} + 2M_0 + M_1) - \|\phi\|(1 + \sum_{i=1}^{n}\frac{\|p_i\|}{\Gamma(\beta_i+1)})} \ge 47.68.$$

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