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Note on weakly fractional differential equations

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Abstract

In this paper, we compare solutions of q-order fractional differential equations of Caputo type for q near 1 with solutions of the corresponding 1-order ordinary differential equations. By establishing the explicit lower and upper bounds of Mittag-Leffler functions, we obtain the effective convergence results. It is shown that the limit cases $q \to 1_+$ and $q \to 1_-$ are different. A simple illustrative example is also presented.

Keywords: Weakly fractional differential equation; Caputo fractional derivative; Comparison; Mittag-Leffler function

1 Introduction

Fractional differential equations (FDEs) are a rapidly developing area of mathematics with many stimulating applications [1–4]. Recently, plenty interesting existence and controllability results on the theory of solutions of FDEs or fractional inclusions have been given in [5–22]. Mathematical modeling approaches using fractional derivatives are presented in [17–22] with numerical simulations on various challenging topics.

On the one hand, several properties of ordinary or partial differential equations (DEs) appear in FDEs as well, like asymptotic properties of solutions or equilibria. On the other hand, unlike to DEs, FDEs have no nonconstant periodic solutions and they do not create dynamical systems, which is one of the most obvious characteristics in studying FDEs. So there is a natural question to study the relationship between solutions of FDEs and DEs when the order q of FDEs is near to a natural number $n \in \mathbb{N}$. Here, we call such FDEs weakly fractional, which can be used to seek numerically the solutions of DEs.

In this paper, we investigate for simplicity the case when q is near to $n = 1$, but our method can be directly extended to any n. We study two cases: $q \to 1_-$ in Sect. 2 and $q \to 1_+$ in Sect. 3. We derive error estimates in both cases. A simple numerical illustrative example is given to demonstrate theoretical results. Our next step will be to extend this paper for weakly fractional semilinear evolution equations in Banach spaces.

2 The case $q \to 1_-$

Consider a fractional differential equation

$$\begin{aligned} &{}^{C}D_0^q x(t) = f\bigl(t, x(t)\bigr), \quad t \in \mathbb{R}_+ = [0, \infty), \\ &x(0) = x_0, \end{aligned} \tag{1}$$

where D_0^q is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit at zero,

$$D_0^q x(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} \big(x(s) - x(0)\big)\, ds,$$

and $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ along with an ordinary differential equation

$$\begin{aligned} &y'(t) = f\big(t, y(t)\big), \quad t \in \mathbb{R}_+, \\ &y(0) = y_0, \end{aligned} \tag{2}$$

where $x_0, y_0 \in \mathbb{R}^n$. We suppose

(H) There are nonnegative constants M and L such that $\|f(t,x)\| \le M$ and $\|f(t,x) - f(t,y)\| \le L\|x-y\|$ for any $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^n$, where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

It is well known [4] that problem (1) is equivalent to the following integral equation:

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\big(s, x(s)\big)\, ds.$$

Then we derive

$$\begin{aligned} &\big\|x(t) - y(t)\big\| \\ &\quad \le \|x_0 - y_0\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big\|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big\|\, ds \\ &\qquad + \int_0^t \left|1 - \frac{1}{\Gamma(q)}(t-s)^{q-1}\right| \big\|f\big(s,y(s)\big)\big\|\, ds \\ &\quad \le \|x_0 - y_0\| + \frac{L}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big\|x(s) - y(s)\big\|\, ds + M \int_0^t \left|1 - \frac{1}{\Gamma(q)}(t-s)^{q-1}\right| ds \\ &\quad = \|x_0 - y_0\| + \frac{L}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big\|x(s) - y(s)\big\|\, ds + M \int_0^t \left|1 - \frac{1}{\Gamma(q)} s^{q-1}\right| ds. \end{aligned}$$

Thus by the Henry–Gronwall inequality (see [23, Corollary 2]), we get

$$\big\|x(t) - y(t)\big\| \le \left(\|x_0 - y_0\| + M \int_0^t \left|1 - \frac{1}{\Gamma(q)} s^{q-1}\right| ds\right) E_q\big(Lt^q\big)$$

for any $t \in \mathbb{R}_+$, where E_q is the Mittag-Leffler function [24]. We continue with the case $x_0 = y_0$. Then we get

$$\big\|x(t) - y(t)\big\| \le M\theta_q(t), \qquad \theta_q(t) := \int_0^t \left|1 - \frac{1}{\Gamma(q)} s^{q-1}\right| ds E_q\big(Lt^q\big) \tag{3}$$

for any $t \in \mathbb{R}_+$. The equation

$$1 - \frac{1}{\Gamma(q)} s^{q-1} = 0$$

has the only solution $s_0 > 0$ given by

$$s_0 = s_0(q) = \Gamma(q)^{\frac{1}{q-1}}. \tag{4}$$

Note that the function $s_0(q)$ is increasing on $(0,1)$ with $\lim_{q\to 0_+} s_0(q) = 0$ and

$$\lim_{q\to 1_-} s_0(q) = e^{\lim_{q\to 1_-} \frac{\ln[\Gamma[q]]}{q-1}} = e^{\lim_{q\to 1_-} \frac{\Gamma'[q]}{\Gamma[q]}} = e^{-\gamma} \doteq 0.561459$$

for the Euler constant γ. Next, clearly, we have

$$1 - \frac{1}{\Gamma(q)} s^{q-1} \begin{cases} < 0 & \text{for } 0 < s < s_0, \\ = 0 & \text{for } s = s_0, \\ > 0 & \text{for } s > s_0. \end{cases}$$

Consequently, we obtain

$$\int_0^t \left| 1 - \frac{1}{\Gamma(q)} s^{q-1} \right| ds = \begin{cases} \frac{t^q}{\Gamma(q+1)} - t & \text{for } 0 < t < s_0, \\ \lambda(q) := \frac{\Gamma(q)^{\frac{q}{q-1}}}{\Gamma(q+1)} - \Gamma(q)^{\frac{1}{q-1}} & \text{for } t = s_0, \\ \frac{-t^q + t\Gamma(q+1) + 2\Gamma(q)^{\frac{q}{q-1}}}{\Gamma(q+1)} - 2\Gamma(q)^{\frac{1}{q-1}} & \text{for } t > s_0. \end{cases} \tag{5}$$

We can check numerically that $\lambda''(q) > 0$ for $q \in (0,1)$, then that $\lambda'(q)$ is increasing from $-\infty$ to $-e^{-\gamma} \doteq -0.561459$, and then that $\lambda(q)$ is decreasing from 1 to 0. So we consider $q \in (1/2,1)$ and then $-0.751988 \le \lambda'(q) \le -0.561459$. This implies that

$$0 < \lambda(q) \le 0.8(1-q) \tag{6}$$

for $q \in (1/2,1)$. Next, by [25, Lemma 2], we have the following.

Lemma 2.1 *For all $t \in \mathbb{R}_+$, $q \in (0,1)$, and $\kappa > 0$, it holds*

$$1 \le E_q(\kappa t^q) \le \frac{e^{\kappa^{\frac{1}{q}} t}}{q}.$$

Furthermore, (5) implies

$$\int_0^t \left| 1 - \frac{1}{\Gamma(q)} s^{q-1} \right| ds \le \left| t - \frac{t^q}{\Gamma(q+1)} \right| + 2\lambda(q)$$

for $t \in \mathbb{R}_+$. So if $q \in (1/2,1)$, then by Lemma 2.1 we get

$$\theta_q(t) \le \frac{e^{L^{\frac{1}{q}} t}}{q} \left(\left| t - \frac{t^q}{\Gamma(q+1)} \right| + 2\lambda(q) \right) \le 2e^{t\tilde{L}} \left(\left| t - \frac{t^q}{\Gamma(q+1)} \right| + 2\lambda(q) \right) \tag{7}$$

for $\tilde{L} = \max\{L, L^2\}$.

Now we are ready to deal with (3). First, (3) immediately implies the following expected result.

Theorem 2.2 *Under assumption* (H), *the solution $x(t)$ of* (1) *uniformly converges on any finite interval $[0,T]$, $T > 0$, of $\mathbb{R}_+$ to the solution $y(t)$ of* (2) *if $q \to 1_-$ and $x_0 = y_0$.*

Proof The proof follows directly from (3), (5) and by

$$\lim_{q\to 1_-}\left|t-\frac{t^q}{\Gamma(q+1)}\right|=0$$

uniformly for $t\in[0,T]$ and any fixed $T>0$. □

Next, we take any $\epsilon>0$ and consider an equation

$$\theta_q(t)=\epsilon. \tag{8}$$

Clearly, $\theta_q(t)$ is increasing on $\mathbb{R}_+$ from 0 to ∞. Thus (8) has the only solution $\bar{t}(\epsilon,q)\in\mathbb{R}_+$. By the above observations we can easily see that $\lim_{\epsilon\to 0_+}\bar{t}(\epsilon,q)=0$ and $\lim_{q\to 1_-}\bar{t}(\epsilon,q)=\infty$.

Furthermore, the function $t\mapsto t-\frac{t^q}{\Gamma(q+1)}$ is nonpositive on $[0,r_0]$ and nonnegative on $[r_0,\infty)$ for

$$r_0=r_0(q)=\Gamma(q+1)^{\frac{1}{q-1}}. \tag{9}$$

Note that the function $r_0(q)$ is increasing on $(0,1)$ from $\lim_{q\to 0_+}r_0(q)=1$ to $\lim_{q\to 1_-}r_0(q)=e^{1-\gamma}\doteq 1.526205$.

Next, we study the function $\phi_t(q):=\frac{t^q}{\Gamma(q+1)}$ on $(0,1)$ for $t>0$. We have

$$\phi_t'(q)=\frac{t^q\ln t}{\Gamma(q+1)}-\frac{t^q\Gamma'(q+1)}{\Gamma^2(q+1)}. \tag{10}$$

For $t\in(0,1]$ and $q\in(1/2,1)$, we get

$$\left|\phi_t'(q)\right|\le-\frac{\sqrt{t}\ln t}{\Gamma(q+1)}+\left|\frac{\Gamma'(q+1)}{\Gamma^2(q+1)}\right|\le 1.253,$$

while for $1\le t\le T$ and $q\in(1/2,1)$, we get

$$\left|\phi_t'(q)\right|\le 1.12838T^q\ln T+0.422784T^q$$

for $T>1$. Consequently, we have

$$\left|\phi_t'(q)\right|\le 1.253+1.12838T^q\ln T+0.422784T^q$$

for $t\in(0,T]$, $T>1$, and $q\in(1/2,1)$. This implies

$$\left|t-\frac{t^q}{\Gamma(q+1)}\right|=\left|\phi_t(1)-\phi_t(q)\right|\le\left(1.253+1.12838T^q\ln T+0.422784T^q\right)(1-q) \tag{11}$$

for $t\in[0,T]$, $T>1$, and $q\in(1/2,1)$. Using (6), (7), and (11), we arrive at

$$\theta_q(t)\le 2e^{T\tilde{L}}\left(3+2T^q\ln T+T^q\right)(1-q)$$

for $t\in[0,T]$, $T>1$, and $q\in(1/2,1)$. Now, we consider instead of (8) the following one:

$$\eta_{L,q}(T):=2e^{T\tilde{L}}\left(3+2T^q\ln T+T^q\right)=\frac{1}{\sqrt{1-q}}. \tag{12}$$

The function $\eta_{L,q}(T)$ is increasing from $8e^{\tilde{L}}$ to ∞ on $[1,\infty)$. So, for any

$$q > 1 - \frac{1}{64e^{2\tilde{L}}}, \quad q \in (1/2,1), \tag{13}$$

(12) has a unique solution $T_L(q) > 1$. Note

$$\lim_{q\to 1_-} T_L(q) = \infty.$$

Summarizing, we have the following result.

Theorem 2.3 *Under assumption* (H) *and for any q fulfilling* (13), *the solutions $x(t)$ and $y(t)$ of* (1) *and* (2) *with $x_0 = y_0$, respectively, satisfy*

$$\|x(t) - y(t)\| \le M\sqrt{1-q} \tag{14}$$

for any $t \in [0, T_L(q)]$, where $T_L(q) > 1$ is the unique solution of (12).

3 The case $q \to 1_+$

Consider a fractional differential equation

$$\begin{aligned} &D_0^q x(t) = f\big(t,x(t)\big), \quad t \in \mathbb{R}_+, \\ &x(0) = x_0, \\ &x'(0) = x_1, \end{aligned} \tag{15}$$

where $q \in (1,2)$ and $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ along with an ordinary differential equation

$$\begin{aligned} &y'(t) = f\big(t,y(t)\big) + y_1, \quad t \in \mathbb{R}_+, \\ &y(0) = y_0, \end{aligned} \tag{16}$$

where $x_0, x_1, y_0, y_1 \in \mathbb{R}^n$. Again, we suppose assumption (H). It is known [2, Theorem 3.24] that initial value problem (15) is equivalent to the integral equation

$$x(t) = x_0 + x_1 t + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\big(s,x(s)\big)\,ds.$$

Analogously to the previous section, we derive

$$\begin{aligned} \|x(t) - y(t)\| \le{}& \|x_0 - y_0\| + \|x_1 - y_1\| t \\ &+ \frac{L}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s) - y(s)\|\,ds + M \int_0^t \left|1 - \frac{1}{\Gamma(q)} s^{q-1}\right| ds, \end{aligned}$$

and the Henry–Gronwall inequality yields

$$\|x(t) - y(t)\| \le \left(\|x_0 - y_0\| + \|x_1 - y_1\| t + M \int_0^t \left|1 - \frac{1}{\Gamma(q)} s^{q-1}\right| ds \right) E_q\big(Lt^q\big).$$

Hence, for $x_0 = y_0$, $x_1 = y_1$, estimation (3) follows for any $t \in \mathbb{R}_+$. Function $s_0(q)$ of (4) is increasing on $(1,2)$ from

$$\lim_{q\to 1_+} s_0(q) = e^{\lim_{q\to 1_+} \frac{\ln[\Gamma[q]]}{q-1}} = e^{\lim_{q\to 1_+} \frac{\Gamma'[q]}{\Gamma[q]}} = e^{-\gamma} \doteq 0.561459$$

to 1. So this time,

$$1 - \frac{1}{\Gamma(q)} s^{q-1} \begin{cases} > 0 & \text{for } 0 < s < s_0, \\ = 0 & \text{for } s = s_0, \\ < 0 & \text{for } s > s_0. \end{cases}$$

Consequently, we have (compare with (5))

$$\int_0^t \left| 1 - \frac{1}{\Gamma(q)} s^{q-1} \right| ds = \begin{cases} t - \frac{t^q}{\Gamma(q+1)} & \text{for } 0 < t < s_0, \\ -\lambda(q) & \text{for } t = s_0, \\ \frac{t^q - t\Gamma(q+1) - 2\Gamma(q)^{\frac{q}{q-1}}}{\Gamma(q+1)} + 2\Gamma(q)^{\frac{1}{q-1}} & \text{for } t > s_0, \end{cases} \tag{17}$$

where $\lambda(q)$ is given by (5). One can check numerically that $-\lambda''(q) < 0$ for $q \in (1,2)$. So, $-\lambda'(q)$ is decreasing from $e^{-\gamma} \doteq 0.561459$ to $\frac{3}{4} - \frac{\gamma}{2} \doteq 0.461392$, and $-\lambda(q)$ is increasing from 0 to 1/2. Hence for $q \in (1,2)$, we can estimate

$$0 \le -\lambda(q) \le 0.6(q-1). \tag{18}$$

Next, we need the following analog to Lemma 2.1.

Lemma 3.1 *For all* $t \in \mathbb{R}_+$, $q \in (1,4/3)$, *and* $\kappa > 0$, *it holds*

$$1 \le E_q\left(\kappa t^q\right) \le \frac{e^{\kappa^{\frac{1}{q}} t}}{q} + \frac{4\sqrt{3}\sin\frac{\pi q}{2}}{9q}.$$

Proof Using Dzherbashyan's recursion formula [26],

$$E_{\alpha,\beta}(z) = \frac{1}{m} \sum_{h=0}^{m-1} E_{\frac{\alpha}{m},\beta}\left(z^{\frac{1}{m}} e^{\frac{2\pi\iota h}{m}}\right)$$

for $\alpha, \beta > 0$, $z \in \mathbb{R}$, $m \in \mathbb{N}$, where $\iota = \sqrt{-1}$, we can write

$$E_q(z) = \frac{E_{\frac{q}{2}}(\sqrt{z}) + E_{\frac{q}{2}}(-\sqrt{z})}{2} \tag{19}$$

for any $z > 0$. Next, from [27, Theorem 2.1] we know

$$E_\alpha(z) = \frac{-z\sin\pi\alpha}{\pi\alpha} \int_0^\infty \frac{e^{-r^{\frac{1}{\alpha}}}\, dr}{r^2 - 2rz\cos\pi\alpha + z^2}$$

for any $\alpha > 0$, $z < 0$. So, using $\cos\frac{\pi q}{2} \geq -1/2$ for $q \in (1, 4/3)$, we get

$$\begin{aligned} E_{\frac{q}{2}}(-\sqrt{z}) &= \frac{2\sqrt{z}\sin\frac{\pi q}{2}}{\pi q}\int_0^\infty \frac{e^{-r^{\frac{2}{q}}}\,dr}{r^2 + 2r\sqrt{z}\cos\frac{\pi q}{2} + z} \\ &\leq \frac{2\sqrt{z}\sin\frac{\pi q}{2}}{\pi q}\int_0^\infty \frac{dr}{r^2 - r\sqrt{z} + z} = \frac{2\sqrt{z}\sin\frac{\pi q}{2}}{\pi q}\frac{4\sqrt{3}\pi}{9\sqrt{z}} = \frac{8\sqrt{3}\sin\frac{\pi q}{2}}{9q}. \end{aligned}$$

Finally, applying this estimation and Lemma 2.1 to (19) results in

$$E_q(\kappa t^q) \leq \frac{1}{2}\left(\frac{2e^{\kappa^{\frac{1}{q}}t}}{q} + \frac{8\sqrt{3}\sin\frac{\pi q}{2}}{9q}\right) = \frac{e^{\kappa^{\frac{1}{q}}t}}{q} + \frac{4\sqrt{3}\sin\frac{\pi q}{2}}{9q}. \qquad \square$$

Since by (17),

$$\int_0^t \left|1 - \frac{1}{\Gamma(q)}s^{q-1}\right| ds \leq \left|t - \frac{t^q}{\Gamma(q+1)}\right| - 2\lambda(q)$$

for all $t \in \mathbb{R}_+$, Lemma 3.1 implies

$$\begin{aligned} \theta_q(t) &\leq \left(\frac{e^{L^{\frac{1}{q}}t}}{q} + \frac{4\sqrt{3}\sin\frac{\pi q}{2}}{9q}\right)\left(\left|t - \frac{t^q}{\Gamma(q+1)}\right| - 2\lambda(q)\right) \\ &\leq \left(e^{t\overline{L}} + \frac{4\sqrt{3}}{9}\right)\left(\left|t - \frac{t^q}{\Gamma(q+1)}\right| - 2\lambda(q)\right) \end{aligned} \tag{20}$$

for $q \in (1, 4/3)$, where $\overline{L} = \max\{L, L^{3/4}\}$. So we obtain a result on the uniform convergence.

Theorem 3.2 *Under assumption* (H), *the solution* $x(t)$ *of* (15) *uniformly converges on any finite interval* $[0, T]$, $T > 0$, *of* $\mathbb{R}_+$ *to the solution* $y(t)$ *of* (16) *if* $q \to 1_+$ *and* $x_0 = y_0$, $x_1 = y_1$.

Proof The statement can be proved analogously to Theorem 2.2. $\square$

Next, we consider equation (8) for an arbitrary $\epsilon > 0$ and $q \in (1, 4/3)$. Clearly, $\theta_q(t)$ is increasing on $\mathbb{R}_+$ from 0 to ∞, implying that (8) has the only solution $\overline{t}(\epsilon, q) \in \mathbb{R}_+$ for which $\lim_{\epsilon\to 0_+}\overline{t}(\epsilon, q) = 0$ and $\lim_{q\to 1_+}\overline{t}(\epsilon, q) = \infty$ hold. Moreover, the function $t \mapsto t - \frac{t^q}{\Gamma(q+1)}$ is nonnegative on $[0, r_0]$ and nonpositive on $[r_0, \infty)$ for r_0 given by (9). Note that $r_0(q)$ is increasing on $(1, \infty)$ from $\lim_{q\to 1_+} r_0(q) = e^{1-\gamma} \doteq 1.526205$ to ∞.

Next, we consider the function $\phi_t(q) := \frac{t^q}{\Gamma(q+1)}$ on $(1, 4/3)$ for $t > 0$. From (10), we obtain

$$\left|\phi_t'(q)\right| \leq -\frac{t\ln t}{\Gamma(q+1)} + \frac{\Gamma'(q+1)}{\Gamma^2(q+1)} \leq -t\ln t + \frac{\Gamma'(\frac{7}{3})}{\Gamma^2(\frac{7}{3})} \leq 1.038041$$

for $t \in (0, 1]$, and

$$\left|\phi_t'(q)\right| \leq T^q \ln T + 0.51902T^q$$

for $t \in (1, T]$, $T > 1$. As a consequence, we have

$$\left|\phi_t'(q)\right| \leq 1.038041 + T^q \ln T + 0.51902T^q$$

for all $t \in (0,T]$, $T>1$, $q \in (1,4/3)$. This implies

$$\left| t - \frac{t^q}{\Gamma(q+1)} \right| = \left|\phi_t(1) - \phi_t(q)\right| \le \left(1.038041 + T^q \ln T + 0.51902 T^q\right)(q-1) \tag{21}$$

for $t \in (0,T]$, $T>1$, $q \in (1,4/3)$. Using (18), (20), and (21), we arrive at

$$\theta_q(t) \le \left(e^{T\overline{L}} + \frac{4\sqrt{3}}{9}\right)\left(3 + T^q \ln T + T^q\right)(q-1)$$

for $t \in (0,T]$, $T>1$, $q \in (1,4/3)$. Now, we consider the equation

$$\mu_{L,q}(T) := \left(e^{T\overline{L}} + \frac{4\sqrt{3}}{9}\right)\left(3 + T^q \ln T + T^q\right) = \frac{1}{\sqrt{q-1}}. \tag{22}$$

The function $\mu_{L,q}(T)$ is increasing from $4(e^{\overline{L}} + 4\sqrt{3}/9)$ to ∞ on $[1,\infty)$. So, for any

$$q < 1 + \frac{1}{16(e^{\overline{L}} + \frac{4\sqrt{3}}{9})^2}, \quad q \in (1,4/3), \tag{23}$$

(22) has a unique solution $T_L(q) > 1$. Note that $\lim_{q\to 1_+} T_L(q) = \infty$. Summarizing, we have the following result.

Theorem 3.3 *Under assumption* (H) *and for any q fulfilling* (23), *the solutions $x(t)$ and $y(t)$ of* (15) *and* (16) *with $x_0 = y_0$, $x_1 = y_1$, respectively, satisfy*

$$\left\|x(t) - y(t)\right\| \le M\sqrt{q-1} \tag{24}$$

for any $t \in [0, T_L(q)]$, where $T_L(q) > 1$ is the unique solution of (22).

Next, we present a simple example illustrating the convergence results when the order q is close to 1.

Example 3.4 Let us consider the following initial-value problems:

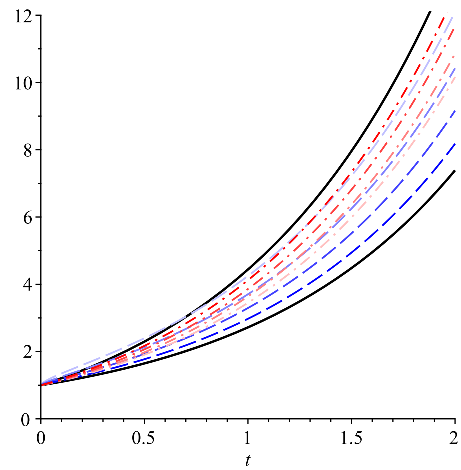
$$\begin{aligned} &D_0^q x(t) = px(t), \quad t \in \mathbb{R}_+, \\ &x(0) = x_0, \end{aligned} \tag{25}$$

$$\begin{aligned} &y'(t) = py(t), \quad t \in \mathbb{R}_+, \\ &y(0) = y_0, \end{aligned} \tag{26}$$

$$\begin{aligned} &D_0^q u(t) = pu(t), \quad t \in \mathbb{R}_+, \\ &u(0) = u_0, \\ &u'(0) = u_1, \end{aligned} \tag{27}$$

$$\begin{aligned} &v'(t) = pv(t) + v_1, \quad t \in \mathbb{R}_+, \\ &v(0) = v_0, \end{aligned} \tag{28}$$

Figure 1 Convergence of solutions of Caputo fractional DEs (25) (dashed blue), (27) (dot-dashed red) to solutions of ODEs (26), (28), respectively. The closer $q \in \{0.6, 0.7, 0.8, 0.9, 1.1, 1.2, 1.3, 1.4\}$ is to 1, the more saturated the colors are



where $q \in (0,1)$ in (25) and $q \in (1,2)$ in (27). The ODEs have the solutions $y(t) = y_0 e^{pt}$, $v(t) = e^{pt}(v_0 + v_1/p) - v_1/p$. From [2, Theorem 4.3], the other solutions are $x(t) = x_0 E_q(pt^q)$ and $u(t) = u_0 E_q(pt^q) + u_1 t E_{q,2}(pt^q)$. To see the convergence, we set all the initial conditions and the parameter equal to 1, i.e., $x_0 = y_0 = u_0 = v_0 = u_1 = v_1 = p = 1$. Figure 1 depicts the convergences $x \to y$ and $u \to v$ as $q \to 1_-$ and $q \to 1_+$, respectively.

The physical significance of Fig. 1 relies on demonstration of transition of q through 1. Since (25) is a one-dimensional system depending just on x_0, its limit (26) is also one-dimensional. But passing to (27), we get a two-dimensional system depending on u_0 and u_1. Then its limit (28) as $q \to 1_+$ is also two-dimensional. This makes the difference. Note that (28) is equivalent to a second order ODE

$$
\begin{aligned}
&v''(t) = pv'(t), \quad t \in \mathbb{R}_+,\\
&v(0) = v_0,\\
&v'(0) = \tilde{v}_1 := pv_0 + v_1.
\end{aligned}
$$

The above arguments are more visible for $p < 0$. Then by [27, Formula (7)] we see that solutions of (25), (26), and (27) asymptotically tend to zero, while the one of (28) tends to $-\frac{v_1}{p}$. So all these equations are dissipative. But the limit of (27) as $q \to 2_-$ is

$$
\begin{aligned}
&z''(t) = pz(t), \quad t \in \mathbb{R}_+,\\
&z(0) = z_0,\\
&z'(0) = z_1,
\end{aligned}
\tag{29}
$$

which has all solutions oscillating for $p < 0$. Consequently, the dissipation of (25)–(28) is changing to oscillation on finite intervals as $q \to 2_-$. This is presented in Figs. 2 and 3.

These figures also support the fact that comparison estimates can be done in general only on finite intervals.

4 Conclusion

Solutions of q-order fractional differential equations of Caputo type for q near 1 are compared to solutions of the corresponding 1-order ordinary differential equations, by estab-

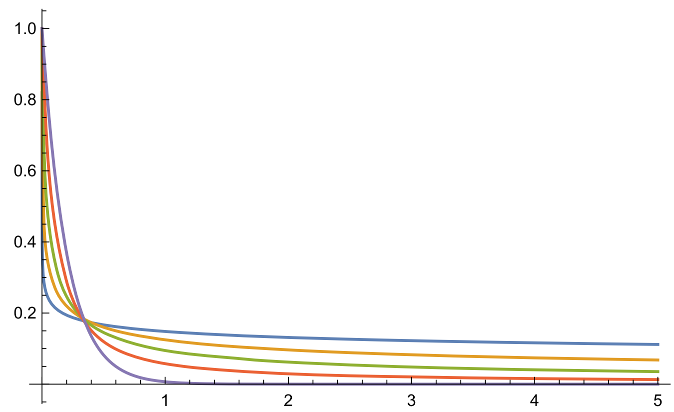


Figure 2 The solutions of Caputo fractional DEs (25) for $q \in \{0.2, 0.4, 0.6, 0.8\}$ and (26) for $p = -5$ and $x_0 = y_0 = 1$

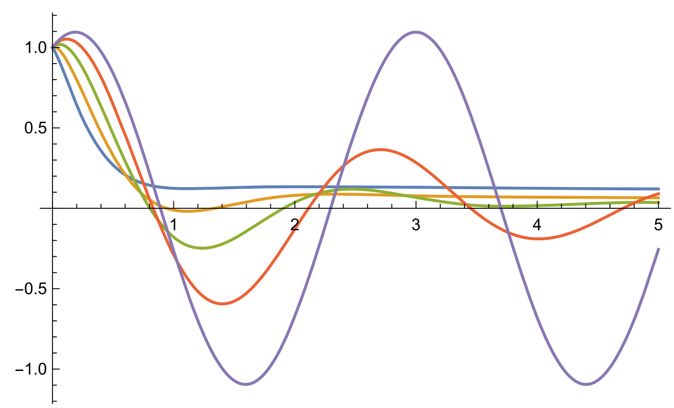


Figure 3 The solutions of Caputo fractional DEs (27) for $q \in \{1.2, 1.4, 1.6, 1.8\}$ and (29) for $p = -5$ and $u_0 = z_0 = u_1 = z_1 = 1$

lishing the effective convergence results. As a result we get that the limit cases $q \to 1_+$ and $q \to 1_-$ are different. Theoretical results are demonstrated on a simple illustrative example. Our method can be directly extended to any order q near a natural number.

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Authors' contributions
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