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On the boundedness of poles of generalized Padé approximants

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Abstract

Given a function F holomorphic on a neighborhood of some compact subset of the complex plane, we prove that if the zeros of the denominators of generalized Padé approximants (orthogonal Padé approximants and Padé–Faber approximants) for some row sequence remain uniformly bounded, then either F is a polynomial or F has a singularity in the complex plane. This result extends the known one for classical Padé approximants. Its proof relies, on the one hand, on difference equations where their coefficients relate to the coefficients of denominators of these generalized Padé approximants and, on the other hand, on a curious property of complex numbers.

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1 Introduction

Currently, Padé approximation theory emphasizes inverse-type problems where we want to describe the analytic properties of the approximated function from the knowledge of the asymptotic behavior of the poles of the approximating functions. Moreover, the theory of higher order recurrence relations (difference equations) plays very important roles in solving inverse-type problems (see, e.g., [3, 6–8, 11]). The object of the present paper is to investigate the relation between the boundedness of poles of row sequences of orthogonal Padé approximants and Padé–Faber approximants and the analyticity of the approximated function. The results are of inverse type.

In order to state a known result related to our study, we need to remind the reader of the definition of classical Padé approximants. In what follows, $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{P}_n$ is the set of all polynomials of degree at most n.

Definition 1.1 Let $F(z) = \sum_{k=0}^{\infty} f_k z^k$ be a formal power series. Fix $(n, m) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then, there exist $P \in \mathbb{P}_n$ and $Q \in \mathbb{P}_m$, $Q \not\equiv 0$, such that

$$(QF - P)(z) = \mathcal{O}\left(z^{n+m+1}\right), \quad \text{as } z \to 0.$$

The rational function $R_{n,m} := P/Q$ is called the(n, m) *classical Padé approximant of* F.

It is well-known that, for any $(n,m) \in \mathbb{N}_0 \times \mathbb{N}_0$, $R_{n,m}$ always exists and is unique. For a given pair $(n,m) \in \mathbb{N}_0 \times \mathbb{N}_0$, we write

$$R_{n,m} = \frac{P_{n,m}}{Q_{n,m}}, \tag{1}$$

where $Q_{n,m}$ is normalized so that $Q_{n,m}(0) = 1$ and it does not share zeros with $P_{n,m}$.

Many papers dealing with inverse-type problems have been limited their studies to the case when $\deg(Q_{n,m}) = m$ for all n sufficiently large. In order to find such $Q_{n,m}(z) = 1 + \sum_{j=1}^{m} q_{n,j} z^j$, one has to solve for all $k = n+1, \dots, n+m$,

$$f_k + q_{n,1} f_{k-1} + \cdots + q_{n,m} f_{k-m} = 0.$$

Indeed, the above recurrence relation has very strong connection to inverse-type problems in [8, Propositions 1–3 and Theorems 4–6] and [11, Sect. 1.1]. In particular, a generalization of the Poincaré theorem for recurrence relations developed in [8] provides a bridge connecting an inverse result for classical Padé approximation in [17] and analogous ones for several generalized Padé approximations in [3, 7, 8].

Given a formal power series $F(z) = \sum_{k=0}^{\infty} f_k z^k$, we denote by $R_0(F)$ the radius of the largest disk centered at 0 to which F can be extended holomorphically. That is, $R_0(F)$ is the radius of convergence of F. In this paper, we are interested in proving analogs of the following theorem (see [11, Theorem 1.1] or [9, Corollary 2.4]) for orthogonal Padé approximants and Padé–Faber approximants.

Theorem A *Let $m \in \mathbb{N}$ be fixed and let $\mathcal{P}_n$ be the set of all zeros of $Q_{n,m}$. Suppose that the cardinality of $\mathcal{P}_n$ is at least* 1 *for all n sufficiently large,*

$$\sup_{N \geq m} \inf_{n \geq N} \bigl\{ |\zeta| : \zeta \in \mathcal{P}_n \bigr\} > 0,$$

and

$$\inf_{N \geq m} \sup_{n \geq N} \bigl\{ |\zeta| : \zeta \in \mathcal{P}_n \bigr\} < \infty.$$

Then, F is a polynomial or $0 < R_0(F) < \infty$.

In other words, if F is not a polynomial and the poles of $R_{n,m}$ stay far from the origin and bounded for all n sufficiently large, then $0 < R_0(F) < \infty$. Up to my knowledge, this result is the first one of this sort. Moreover, the sequence $(R_{n,m})_{n \geq n_0}$, where m remains fixed, is called the *mth row sequence*.

Now, let us define four generalized Padé approximations. Let E be a bounded continuum with connected complement containing infinitely many points. From now on, the set E will satisfy the above condition. Let μ be a finite positive Borel measure with infinite support $\operatorname{supp}(\mu)$ contained in E. We write $\mu \in \mathcal{M}(E)$ and the corresponding inner product is defined by

$$\langle g, h \rangle_\mu := \int g(z) \overline{h(z)} \, d\mu(z), \quad g, h \in L_2(\mu).$$

Using this inner product, one can generate a unique sequence of orthonormal polynomials

$$(p_n)_{n\geq 0} := (\kappa_n z^n + \cdots)_{n\geq 0}$$

with positive leading coefficients $\kappa_n > 0$. By $\mathcal{H}(E)$, we denote the space of all functions holomorphic in some neighborhood of E. The first two definitions are generalized Padé approximants constructed from the sequence of orthogonal polynomials $(p_n)_{n\geq 0}$.

Definition 1.2 Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{M}(E)$. For any integers $n \geq 0$ and $m \geq 1$, there exists $Q^{\mu}_{n,m} \in \mathbb{P}_m$ such that $Q^{\mu}_{n,m} \not\equiv 0$ and $\langle Q^{\mu}_{n,m} F, p_{n+k}\rangle_{\mu} = 0$ for all $k = 1, \ldots, m$. The associated rational function

$$R^{\mu}_{n,m} := \frac{\sum_{j=0}^{n} \langle Q^{\mu}_{n,m} F, p_j\rangle_{\mu} p_j}{Q^{\mu}_{n,m}}$$

is called an (n,m) *standard orthogonal Padé approximant of F with respect to μ.*

Definition 1.3 Let $F \in \mathcal{H}(E)$ and $\mu \in \mathcal{M}(E)$. For any integers $n \geq 0$ and $m \geq 1$, there exists $\tilde{Q}^{\mu}_{n,m} \in \mathbb{P}_m$ such that $\tilde{Q}^{\mu}_{n,m} \not\equiv 0$ and $\langle z^k \tilde{Q}^{\mu}_{n,m} F, p_{n+1}\rangle_{\mu} = 0$ for all $k = 0, \ldots, m-1$. The associated rational function

$$\tilde{R}^{\mu}_{n,m} := \frac{\sum_{j=0}^{n} \langle \tilde{Q}^{\mu}_{n,m} F, p_j\rangle_{\mu} p_j}{\tilde{Q}^{\mu}_{n,m}}$$

is called an (n,m) *modified orthogonal Padé approximant of F with respect to μ.*

Let Φ the unique Riemann mapping function from $\overline{\mathbb{C}} \setminus E$ to the exterior of the closed unit disk verifying $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. For each $\rho > 1$, the *level curve of index ρ* and the *canonical domain of index ρ* are defined by

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \quad \text{and} \quad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

respectively. Given $F \in \mathcal{H}(E)$, we denote by $\rho_0(F)$ the largest index ρ such that F extends as a holomorphic function to D_ρ.

The *Faber polynomial* of E of degree n is

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t-z}\, dt, \quad z \in D_\rho, n = 0, 1, 2, \ldots.$$

One can check that

$$\Phi_n(z) = (z/\operatorname{cap}(E))^n + \text{lower degree terms},$$

where $\operatorname{cap}(E)$ is the logarithmic capacity of the set E. The nth *Faber coefficient* of $F \in \mathcal{H}(E)$ with respect to Φ_n is defined by the formula

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)}\, dt,$$

where $\rho \in (1, \rho_0(F))$.

The next two definitions are the ones of generalized Padé approximants constructed from the sequence of Faber polynomials $(\Phi_n)_{n\geq 0}$.

Definition 1.4 Let $F \in \mathcal{H}(E)$. For any integers $n \geq 0$ and $m \geq 1$, there exists $Q^E_{n,m} \in \mathbb{P}_m$ such that $Q^E_{n,m} \not\equiv 0$ and $[Q^E_{n,m}F]_{n+k} = 0$ for all $k = 1, \dots, m$. The associated rational function

$$R^E_{n,m} := \frac{\sum_{j=0}^{n} [Q^E_{n,m}F]_j \Phi_j}{Q^E_{n,m}}$$

is called an (n,m) *standard Padé–Faber approximant of* F *with respect to* E.

Definition 1.5 Let $F \in \mathcal{H}(E)$. For any integers $n \geq 0$ and $m \geq 1$, there exists $\tilde{Q}^E_{n,m} \in \mathbb{P}_m$ such that $\tilde{Q}^E_{n,m} \not\equiv 0$ and $[z^k \tilde{Q}^E_{n,m}F]_{n+1} = 0$ for all $k = 0, \dots, m-1$. The associated rational function

$$\tilde{R}^E_{n,m} := \frac{\sum_{j=0}^{n} [\tilde{Q}^E_{n,m}F]_j \Phi_j}{\tilde{Q}^E_{n,m}}$$

is called an (n,m) *modified Padé–Faber approximant of* F *with respect to* E.

In order to find $Q^\mu_{n,m}$, $\tilde{Q}^\mu_{n,m}$, $Q^E_{n,m}$, or $\tilde{Q}^E_{n,m}$ in Definitions 1.2–1.5, one has to solve a system of m homogeneous linear equations on $m+1$ unknowns. Therefore, for any integers $n \geq 0$ and $m \geq 1$, polynomials $Q^\mu_{n,m}$, $\tilde{Q}^\mu_{n,m}$, $Q^E_{n,m}$, and $\tilde{Q}^E_{n,m}$ always exist but they may not be unique. Since $Q^\mu_{n,m}$, $\tilde{Q}^\mu_{n,m}$, $Q^E_{n,m}$, and $\tilde{Q}^E_{n,m}$ are not the zero function, we normalize them to be "monic" polynomials. Unlike the classical Padé approximants, for any integers $n \geq 0$ and $m \geq 1$, $R^\mu_{n,m}$, $\tilde{R}^\mu_{n,m}$, $R^E_{n,m}$, and $\tilde{R}^E_{n,m}$ may not be unique. The rational functions $R^\mu_{n,m}$ and $R^E_{n,m}$ are natural extensions of $R_{n,m}$ and were introduced by Maehly [12] in 1960. Suetin [15, 16] was the first to give necessary and sufficient conditions for the convergence with geometric rate of the denominators of standard orthogonal Padé and standard Padé–Faber approximants on row sequences. The rational functions $\tilde{R}^\mu_{n,m}$ and $\tilde{R}^E_{n,m}$ were recently introduced (in the vector forms) in order to solve some inverse-type problems about detecting the poles of a vector of functions nearest the set E (see [5, 6] for more details). Note that in general, the approximations in Definitions 1.2 and 1.4 are not the same as the ones in Definitions 1.3 and 1.5, respectively.

An outline of this paper is as follows. In Sect. 2, we state analogs of Theorem A which are in our main result. We relegate all lemmas in Sect. 3. The proof of the main result is in Sect. 4.

2 Main result

Before stating the main result, we need to define two subclasses of $\mathcal{M}(E)$. The measure $\mu \in \mathbf{Reg}_1(E)$ if and only if

$$\lim_{n\to\infty} \left|p_n(z)\right|^{1/n} = \left|\Phi(z)\right|, \tag{2}$$

uniformly on each compact subset of $\mathbb{C} \setminus E$. This is the minimum requirement to have the limit formula for $\rho_0(F)$ and the convergence of the orthogonal polynomial expansion in Lemma 3.1 in Sect. 3. Moreover, the class $\mathbf{Reg}_1(E)$ is exactly the regular class in [13, Definition 3.1.2] when E is convex. The measure $\mu \in \mathbf{Reg}_1^*(E)$ when $\mu \in \mathbf{Reg}_1(E)$ and

$$\frac{\kappa_{n-1}}{\kappa_n} \geq c, \quad n \geq n_0, \tag{3}$$

for some $c > 0$ and $n_0 \in \mathbb{N}$. There are many examples for which $\mu \in \mathbf{Reg}_1^*(E)$. For example, if the sequence of orthonormal polynomials satisfies ratio asymptotics (see [1, 2]) or strong asymptotics (see, e.g., [18]), then the corresponding measure is in $\mathbf{Reg}_1^*(E)$.

The main result of this paper is the following.

Theorem 2.1 *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}_1^*(E)$. Fix $m \in \mathbb{N}$ and denote by $\mathcal{P}_n^\mu$, $\mathcal{P}_n^E$, $\tilde{\mathcal{P}}_n^\mu$, and $\tilde{\mathcal{P}}_n^E$ the sets of all zeros of $Q_{n,m}^\mu$, $Q_{n,m}^E$, $\tilde{Q}_{n,m}^\mu$, and $\tilde{Q}_{n,m}^E$, respectively. Assume that one of the following conditions holds.*

(a) *The cardinality of $\mathcal{P}_n^\mu$ is at least 1 for all n sufficiently large and*

$$\inf_{N \geq m} \sup_{n \geq N} \left\{ |\zeta| : \zeta \in \mathcal{P}_n^\mu \right\} < \infty.$$

(b) *The cardinality of $\mathcal{P}_n^E$ is at least 1 for all n sufficiently large and*

$$\inf_{N \geq m} \sup_{n \geq N} \left\{ |\zeta| : \zeta \in \mathcal{P}_n^E \right\} < \infty.$$

(c) *The cardinality of $\tilde{\mathcal{P}}_n^\mu$ is at least 1 for all n sufficiently large and*

$$\inf_{N \geq m} \sup_{n \geq N} \left\{ |\zeta| : \zeta \in \tilde{\mathcal{P}}_n^\mu \right\} < \infty.$$

(d) *The cardinality of $\tilde{\mathcal{P}}_n^E$ is at least 1 for all n sufficiently large and*

$$\inf_{N \geq m} \sup_{n \geq N} \left\{ |\zeta| : \zeta \in \tilde{\mathcal{P}}_n^E \right\} < \infty.$$

Then either F is a polynomial or $\rho_0(F) < \infty$.

Note that we are not interested in proving $\rho_0(F) > 1$ because this is a direct consequence of $F \in \mathcal{H}(E)$.

3 Auxiliary lemmas

Recall that the *nth Fourier coefficient* of $F \in \mathcal{H}(E)$ corresponding to p_n is defined as follows:

$$\langle F \rangle_n := \langle F, p_n \rangle_\mu = \int F(z) \overline{p_n(z)}\, d\mu(z).$$

The first lemma (see, e.g., [5, Lemma 2.1] and [14] for its proof) concerns the convergence of orthogonal and Faber polynomial expansions.

Lemma 3.1 *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}_1(E)$. Then*

$$\rho_0(F) = \left(\limsup_{n \to \infty} \left| \langle F \rangle_n \right|^{1/n} \right)^{-1} = \left(\limsup_{n \to \infty} \left| [F]_n \right|^{1/n} \right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty}\langle F\rangle_n p_n$ *and* $\sum_{n=0}^{\infty}[F]_n\Phi_n$ *converge uniformly to* F *on each compact subset of* $D_{\rho_0(F)}$.

An estimate of $\|\Phi_n\|_{\Gamma_\rho}$ is given in the succeeding lemma (see [10, p. 583] or [14, p. 43]).

Lemma 3.2 *Fix* $\rho>1$. *Then there exists* $c>0$ *such that, for all* $n\in\mathbb{N}_0$,

$$\|\Phi_n\|_{\Gamma_\rho}\le c\rho^n.$$

A curious relation of complex numbers (see [4, Lemma 3] for its proof) which serves as the cornerstone for the proof of our main result is the following.

Lemma 3.3 *If a sequence of complex numbers* $(A_N)_{N\in\mathbb{N}}$ *has the following properties*:
(i) $\lim_{N\to\infty}|A_N|^{1/N}=0$;
(ii) *there exist* $N_0\in\mathbb{N}$ *and* $C>0$ *such that* $|A_N|\le C\sum_{k=N+1}^{\infty}|A_k|$, *for all* $N\ge N_0$,
then there exists $N_1\in\mathbb{N}$ *such that* $A_N=0$ *for all* $N\ge N_1$.

4 Proof of main result

Proof of Theorem 2.1 By Lemma 3.1, since $F\in\mathcal{H}(E)$ and $\mu\in\mathbf{Reg}_1^*(E)$,

$$F(z):=\sum_{\nu=0}^{\infty}\langle F\rangle_\nu p_\nu(z)=\sum_{\nu=0}^{\infty}[F]_\nu\Phi_\nu(z)$$

for all $z\in D_{\rho_0(F)}$.

Assume that (a) holds. We will show that if F is an entire function, then F is a polynomial. Let

$$Q_{n,m}^{\mu}(z):=\prod_{k=1}^{m_n}(z-\zeta_{n,k})=\sum_{j=0}^{m_n}q_{n,j}z^j.$$

Note that $m_n\ge1$ for all n sufficiently large and $q_{n,m_n}=1$.

From the definition of $Q_{n,m}^{\mu}$, we have, for all $k=1,\dots,m$,

$$\begin{aligned}0=\langle Q_{n,m}^{\mu}F\rangle_{n+k}&=\sum_{j=0}^{m_n}\sum_{\nu=0}^{\infty}\langle F\rangle_\nu q_{n,j}\langle z^jp_\nu\rangle_{n+k}=\sum_{j=0}^{m_n}\sum_{\nu=n+k-j}^{\infty}\langle F\rangle_\nu q_{n,j}\langle z^jp_\nu\rangle_{n+k}\\&=\sum_{\nu=n+k-m_n}^{\infty}\langle F\rangle_\nu\langle z^{m_n}p_\nu\rangle_{n+k}+\sum_{j=0}^{m_n-1}\sum_{\nu=n+k-j}^{\infty}\langle F\rangle_\nu q_{n,j}\langle z^jp_\nu\rangle_{n+k}\\&=\frac{\kappa_{n+k-m_n}}{\kappa_{n+k}}\langle F\rangle_{n+k-m_n}+\sum_{\nu=n+k-m_n+1}^{\infty}\langle F\rangle_\nu\langle z^{m_n}p_\nu\rangle_{n+k}+\sum_{j=0}^{m_n-1}\sum_{\nu=n+k-j}^{\infty}q_{n,j}\langle F\rangle_\nu\langle z^jp_\nu\rangle_{n+k}.\end{aligned}\tag{4}$$

Using the Vieta formulas, since

$$\inf_{N\ge m}\sup_{n\ge N}\{|\zeta|:\zeta\in\mathcal{P}_n^{\mu}\}<\infty,$$

there exists $c_1 > 0$ such that

$$\sup\{|q_{n,j}| : 0 \le j \le m_n, n \ge n_0\} \le c_1. \tag{5}$$

From the Cauchy–Schwarz inequality and the orthonormality of p_ν, for all $n, \nu, k \in \mathbb{N}_0$ and $j = 0, \dots, m$, there exists $c_2 > 0$ such that

$$\left|\langle z^j p_\nu\rangle_{n+k}\right| = \left|\langle z^j p_\nu, p_{n+k}\rangle_\mu\right| \le c_2. \tag{6}$$

Because $\mu \in \mathbf{Reg}_1^*(E)$, there exists $c_3 > 0$ such that

$$\frac{\kappa_{n+k-m_n}}{\kappa_{n+k}} \ge c_3, \quad n \ge n_0, \tag{7}$$

where c_3 does not depend on k and m_n. Combining (5), (6), and (7), it is easy to check that (4) implies that, for all $k = 1, \dots, m$, and for all $n \ge n_0$,

$$\left|\langle F\rangle_{n+k-m_n}\right| \le c_4 \sum_{\nu=n+k-m_n+1}^{\infty} \left|\langle F\rangle_\nu\right|,$$

where c_4 is a positive constant that does not depend on n, k and m_n. For each $n \ge n_0$, we choose $k = m_n$ in the previous inequality and we obtain, for all $n \ge n_0$,

$$\left|\langle F\rangle_n\right| \le c_4 \sum_{\nu=n+1}^{\infty} \left|\langle F\rangle_\nu\right|.$$

Using Lemma 3.3 when $A_n = \langle F\rangle_n$, because the above inequality holds and

$$\lim_{n\to\infty} |A_n|^{1/n} = 0,$$

$\langle F\rangle_n = 0$ for all sufficiently large n and F is a polynomial as desired.

Now, we assume that (b) holds. We will follow the same plan by proving that if F is an entire function, then F is a polynomial. Let

$$Q_{n,m}^E(z) := \prod_{k=1}^{m_n} (z - \zeta_{n,k}) = \sum_{j=0}^{m_n} q_{n,j} z^j,$$

where $q_{n,m_n} = 1$. Arguments analogous to those used to derive (4) show that, for $k = 1, \dots, m$,

$$\begin{aligned} 0 = {} & \left(\operatorname{cap}(E)\right)^{m_n} [F]_{n+k-m_n} \\ & + \sum_{\nu=n+k-m_n+1}^{\infty} [F]_\nu \left[z^{m_n} \Phi_\nu\right]_{n+k} + \sum_{j=0}^{m_n-1} \sum_{\nu=n+k-j}^{\infty} q_{n,j} [F]_\nu \left[z^j \Phi_\nu\right]_{n+k}. \end{aligned} \tag{8}$$

Moreover, there exists $c_5 > 0$ such that

$$\sup\{|q_{n,j}| : 0 \le j \le m_n, n \ge n_0\} \le c_5. \tag{9}$$

Take $\rho > 1$. Using Lemma 3.2, for $j = 0, 1, \dots, m$, $k = 1, \dots, m$, and $n, \nu \in \mathbb{N}_0$, we obtain

$$\left|\left[z^j \Phi_\nu\right]_{n+k}\right| = \left|\frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{z^j \Phi_\nu(z)\Phi'(z)}{\Phi^{n+k+1}(z)}\,dz\right| \le c_6 \frac{\rho^\nu}{\rho^n}. \tag{10}$$

Combining (8), (9), and (10), it is easy to check that, for all $k = 1, \dots, m$, and for all $n \ge n_0$,

$$\left|[F]_{n+k-m_n}\right|\rho^n \le c_7 \sum_{\nu=n+k-m_n+1}^{\infty} \left|[F]_\nu\right|\rho^\nu,$$

where c_7 is a positive constant that does not depend on n, k and m_n. For each $n \ge n_0$, we choose $k = m_n$ in the previous inequality and we obtain, for all $n \ge n_0$,

$$\left|[F]_n\right|\rho^n \le c_7 \sum_{\nu=n+1}^{\infty} \left|[F]_\nu\right|\rho^\nu. \tag{11}$$

Using Lemma 3.3 by setting $A_n = [F]_n\rho^n$, it follows that $[F]_n = 0$. Consequently, F is a polynomial.

Next, suppose that (c) holds. Our plan is to prove that if F is an entire function, then F is a polynomial. Let

$$\tilde{Q}^{\mu}_{n,m}(z) := \prod_{k=1}^{m_n}(z - \zeta_{n,k}) = \sum_{j=0}^{m_n} q_{n,j} z^j,$$

where $q_{n,m_n} = 1$. From the definition of $\tilde{Q}^{\mu}_{n,m}$, we have, for all $k = 0, \dots, m-1$,

$$\begin{aligned}
0 &= \left\langle z^k \tilde{Q}^{\mu}_{n,m} F\right\rangle_{n+1} = \sum_{j=0}^{m_n}\sum_{\nu=0}^{\infty} \langle F\rangle_\nu q_{n,j}\left\langle z^{k+j} p_\nu\right\rangle_{n+1} = \sum_{j=0}^{m_n}\sum_{\nu=n+1-k-j}^{\infty} \langle F\rangle_\nu q_{n,j}\left\langle z^{k+j} p_\nu\right\rangle_{n+1} \\
&= \sum_{\nu=n+1-k-m_n}^{\infty} \langle F\rangle_\nu \left\langle z^{k+m_n} p_\nu\right\rangle_{n+1} + \sum_{j=0}^{m_n-1}\sum_{\nu=n+1-k-j}^{\infty} \langle F\rangle_\nu q_{n,j}\left\langle z^{k+j} p_\nu\right\rangle_{n+1} \\
&= \frac{\kappa_{n+1-k-m_n}}{\kappa_{n+1}} \langle F\rangle_{n+1-k-m_n} + \sum_{\nu=n-k-m_n+2}^{\infty} \langle F\rangle_\nu \left\langle z^{k+m_n} p_\nu\right\rangle_{n+1} \\
&\quad + \sum_{j=0}^{m_n-1}\sum_{\nu=n+1-k-j}^{\infty} \langle F\rangle_\nu q_{n,j}\left\langle z^{k+j} p_\nu\right\rangle_{n+1}.
\end{aligned} \tag{12}$$

Applying exactly the same arguments as in (5), (6), and (7), there exists $c_8 > 0$ such that

$$\sup\left\{|q_{n,j}| : 0 \le j \le m_n, n \ge n_0\right\} \le c_8, \tag{13}$$

there exists $c_9 > 0$ such that, for all $k = 0, \dots, m-1$, $j = 0, \dots, m$, and $n, \nu \in \mathbb{N}_0$,

$$\left|\left\langle z^{k+j} p_\nu\right\rangle_{n+1}\right| = \left|\left\langle z^j p_\nu, p_{n+k}\right\rangle_\mu\right| \le c_9, \tag{14}$$

there exists $c_{10} > 0$ such that, for all $k = 0, \dots, m-1$, $m_n = 1, \dots, m$, and $n \ge n_0$,

$$\frac{\kappa_{n+1-k-m_n}}{\kappa_{n+1}} \ge c_{10}. \tag{15}$$

Using (12), (13), (14), and (15), we have, for all $k = 0,\dots,m-1$ and $n \geq n_0$,

$$\left|\langle F\rangle_{n-k-m_n+1}\right| \leq c_{11} \sum_{\nu=n-k-m_n+2}^{\infty} \left|\langle F\rangle_\nu\right|.$$

For each $n \geq n_0$, we choose $k = m - m_n$ in the previous inequality and we obtain, for all $n \geq n_0$,

$$\left|\langle F\rangle_{n-m+1}\right| \leq c_{11} \sum_{\nu=n-m+2}^{\infty} \left|\langle F\rangle_\nu\right|.$$

Setting $N = n - m + 1$, we have

$$\left|\langle F\rangle_N\right| \leq c_{11} \sum_{\nu=N+1}^{\infty} \left|\langle F\rangle_\nu\right|, \quad N \geq N_0.$$

Applying $A_N = \langle F\rangle_N$ in Lemma 3.3, we can conclude that F is a polynomial.

Finally, we suppose that (d) holds. Following the same plan by assuming that F is an entire function, we want to show that F is a polynomial. Let

$$\tilde{Q}^E_{n,m}(z) := \prod_{k=1}^{m_n}(z - \zeta_{n,k}) = \sum_{j=0}^{m_n} q_{n,j} z^j,$$

where $q_{n,m_n} = 1$. Arguments analogous to those used to derive (12) show that, for all $k = 0,\dots,m-1$,

$$\begin{aligned} 0 &= \left(\operatorname{cap}(E)\right)^{m_m+k}[F]_{n+1-k-m_n} + \sum_{\nu=n-k-m_n+2}^{\infty} [F]_\nu \left[z^{k+m_n}\Phi_\nu\right]_{n+1} \\ &\quad + \sum_{j=0}^{m_n-1} \sum_{\nu=n+1-k-j}^{\infty} [F]_\nu q_{n,j}\left[z^{k+j}\Phi_\nu\right]_{n+1}. \end{aligned} \tag{16}$$

Moreover, there exists $c_{12} > 0$ such that

$$\sup\left\{|q_{n,j}| : 0 \leq j \leq m_n, n \geq n_0\right\} \leq c_{12}. \tag{17}$$

Take $\rho > 1$. Using Lemma 3.2, we obtain, for all $j = 0,1,\dots,m$, $k = 0,\dots,m-1$, and $n,\nu \in \mathbb{N}_0$,

$$\left|\left[z^{k+j}\Phi_\nu\right]_{n+1}\right| = \left|\frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{z^{k+j}\Phi_\nu(z)\Phi'(z)}{\Phi^{n+2}(z)}\,dz\right| \leq c_{13}\frac{\rho^\nu}{\rho^n}. \tag{18}$$

By (16), (17), and (18), we have, for all $k = 0,\dots,m-1$ and $n \geq n_0$,

$$\left|[F]_{n+1-k-m_n}\right|\rho^{n-m+1} \leq c_{14} \sum_{\nu=n-k-m_n+2}^{\infty} \left|[F]_\nu\right|\rho^\nu.$$

For each $n \geq n_0$, we choose $k = m - m_n$ in the previous inequality and we obtain, for all $n \geq n_0$,

$$\left|[F]_{n-m+1}\right|\rho^{n-m+1} \leq c_{14} \sum_{\nu=n-m+2}^{\infty} \left|[F]_\nu\right|\rho^\nu.$$

Setting $N = n - m + 1$ and $A_N = [F]_N \rho^N$ in Lemma 3.3, we arrive at the same conclusion. $\square$

5 Conclusion

We prove that if the zeros of the denominators of four generalized Padé approximations based on orthogonal and Faber polynomials stay uniformly bounded, then either the approximated function is a polynomial or it has a singularity in the complex plane. This result extends the well-known one for classical Padé approximants.

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The author was the only one to contribute to the writing of this paper. The author conceived of the study, participated in its design and coordination, and read and approved the final manuscript.

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