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Existence and Ulam–Hyers stability for Caputo conformable differential equations with four-point integral conditions

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Abstract

In this article, we investigate the existence and uniqueness of solutions for conformable derivatives in the Caputo setting with four-point integral conditions, applying standard fixed point theorems such as Banach contraction mapping principle, Krasnoselskii's fixed point theorem, and Leray–Schauder nonlinear alternative. Further, we present Ulam–Hyers stability results by using direct analysis methods. Different types of Ulam stability, such as Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability, are studied. Examples which support our theoretical results are also presented.

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1 Introduction

Fractional calculus extends the theory of differentiation and integration of integer order to real or complex order. Recently, there has been shown a great interest in the study of differential equations and inclusions with non-integer order, since fractional order models are more accurate than integer order models. Fractional derivatives provide an excellent instrument for the description of systems with memory and hereditary properties. Many books and monographs are devoted to the development of fractional calculus, see for instance [1–8] and the references therein.

One of the most prominent research areas in the field of fractional differential equations, which has attracted great attention from the researchers, is devoted to the existence theory of solutions. For theoretical development of the topic, we refer the reader to papers [9–23] and the references cited therein. Another important and interesting area of research, which has got great attention from the researchers recently, is devoted to the stability analysis of differential equations for classical and fractional order. The notion of Ulam stability, which can be considered as a special type of data dependence, was initiated by Ulam [24, 25]. Hyers, Aoki, Rassias, and Obloza contributed in the development of this field (see [26–30] and the references therein). Meanwhile, there have been few works

considering the Ulam stability of a variety of classes of fractional differential equations [31–44].

In this paper, we study the existence, uniqueness, and Ulam–Hyers stability of solutions for conformable derivatives in the Caputo setting with four-point integral conditions:

$$\begin{cases} ({}_a^C \mathfrak{D}^{\alpha, \rho} x)(t) = f(t, x(t)), & 1 < \alpha \leq 2, a < t < T, \\ x(a) = \mu_1 x(\xi) + \mu_2, & x(T) = \lambda ({}_a \mathfrak{J}^{\beta, \rho} x)(\sigma), \end{cases} \quad (1.1)$$

where ${}_a^C \mathfrak{D}^{\alpha, \rho}$ denotes the left conformable derivative in the Caputo setting of order α with $\rho \in (0, 1]$, $f : [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $a \geq 0$, the points $\xi, \sigma \in (a, T)$, given constants $\mu_1, \mu_2, \lambda \in \mathbb{R}$, and ${}_a \mathfrak{J}^{\beta, \rho}$ is the left conformable integral operator of order $\beta > 0$. Indeed, we study existence and uniqueness results by applying standard fixed point theorem such as the Banach contraction mapping principle, Krasnoselskii's fixed point theorem, and Leray–Schauder nonlinear alternative. In addition, we study different types of Ulam stability: Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability for problem (1.1).

The paper is organized as follows. A brief review of the fractional calculus theory is given in Sect. 2. In Sect. 3, we prove the existence and uniqueness of solutions for problem (1.1). In Sect. 4, we discuss the Ulam–Hyers stability results. Finally, examples are given in Sect. 5 to illustrate the usefulness of our main results.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later.

Definition 2.1 The left conformable derivative starting at a point a of the function $f : [a, \infty) \rightarrow \mathbb{R}$ of order $0 < \rho \leq 1$ is defined by

$$({}_a D^\rho f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon(t-a)^{1-\rho}) - f(t)}{\epsilon}.$$

If $({}_a D^\rho f)(t)$ exists on (a, b) , then $({}_a D^\rho f)(a) = \lim_{t \rightarrow a^+} ({}_a D^\rho f)(t)$. If f is differentiable, then

$$({}_a D^\rho f)(t) = (t-a)^{1-\rho} f'(t). \quad (2.1)$$

The corresponding left conformable integral is defined as

$${}_a I^\rho f(x) = \int_a^x f(t) \frac{dt}{(t-a)^{1-\rho}}, \quad 0 < \rho \leq 1.$$

For the extension to the higher order $\rho > 1$, see [45].

Definition 2.2 ([45]) The left Riemann–Liouville conformable integral of a function $f : [a, \infty) \rightarrow \mathbb{R}$ of order α with $0 < \rho \leq 1$ is defined by

$${}_a \mathfrak{J}^{\alpha, \rho} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{(x-a)^\rho - (t-a)^\rho}{\rho} \right)^{\alpha-1} \frac{f(t) dt}{(t-a)^{1-\rho}}, \quad (2.2)$$

where $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$.

Definition 2.3 ([45]) The left Riemann–Liouville conformable derivative of a function $f : [a, \infty) \rightarrow \mathbb{R}$ of order $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$ with $0 < \rho \leq 1$ is defined by

$$\begin{aligned} {}_a\mathfrak{D}^{\alpha,\rho}f(x) &= {}^m_aD^\rho({}_a\mathfrak{J}^{m-\alpha,\rho}f)(x) \\ &= \frac{{}^m_aD^\rho}{\Gamma(m-\alpha)} \int_a^x \left(\frac{(x-a)^\rho - (t-a)^\rho}{\rho} \right)^{m-\alpha-1} \frac{f(t)dt}{(t-a)^{1-\rho}}, \end{aligned} \quad (2.3)$$

where $m = \lceil \Re(\alpha) \rceil = \min\{m \in \mathbb{Z} | m \geq \Re(\alpha)\}$, ${}^m_aD^\rho = \underbrace{{}_aD^\rho {}_aD^\rho \cdots {}_aD^\rho}_{m\text{-times}}$, and ${}_aD^\rho$ is the left conformable differential operator presented in Definition 2.1.

Definition 2.4 ([45]) The left Caputo conformable derivative of a function $f : [a, \infty) \rightarrow \mathbb{R}$ of order $\alpha \in \mathbb{C}$, $\Re(\alpha) \geq 0$ with $0 < \rho \leq 1$ is defined by

$$\begin{aligned} {}^C_a\mathfrak{D}^{\alpha,\rho}f(x) &= {}_a\mathfrak{J}^{m-\alpha,\rho}({}^m_aD^\rho f)(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_a^x \left(\frac{(x-a)^\rho - (t-a)^\rho}{\rho} \right)^{m-\alpha-1} \frac{{}^m_aD^\rho f(t)dt}{(t-a)^{1-\rho}}, \end{aligned} \quad (2.4)$$

provided the right-hand side exists.

Lemma 2.1 ([45]) Let $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $\Re(\nu) > 0$. Then the following formulas hold:

$$({}_a\mathfrak{J}^{\alpha,\rho}({}_a\mathfrak{J}^{\beta,\rho}f))(x) = ({}_a\mathfrak{J}^{\alpha+\beta,\rho}f)(x) \quad (2.5)$$

and

$${}_a\mathfrak{J}^{\beta,\rho}(t-a)^{\rho(v-1)}(x) = \frac{1}{\rho^\beta} \frac{\Gamma(\nu)}{\Gamma(\beta+\nu)} (x-a)^{\rho(v-1)+\rho\beta}. \quad (2.6)$$

Lemma 2.2 ([45]) Let $f \in C_{\alpha,a}^n[a, b]$, $\alpha \in \mathbb{C}$. Then

$${}_a\mathfrak{J}^{\alpha,\rho}({}^C_a\mathfrak{D}^{\alpha,\rho}f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{{}_aD^\rho f(a)(x-a)^{\rho k}}{\rho^k k!}, \quad (2.7)$$

where n is the smallest integer greater than or equal to α .

Lemma 2.3 For $\alpha > 0$, the general solution of the fractional differential equation ${}^C_a\mathfrak{D}^{\alpha,\rho}u(t) = 0$ is given by

$$u(t) = c_0 + c_1(t-a)^\rho + c_2(t-a)^{2\rho} + \cdots + c_{n-1}(t-a)^{\rho(n-1)}, \quad (2.8)$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \dots, n-1$, n is the smallest integer greater than or equal to α .

In view of Lemma 2.3, it follows that

$${}_a\mathfrak{J}^{\alpha,\rho}({}^C_a\mathfrak{D}^{\alpha,\rho}u(t)) = u(t) + c_0 + c_1(t-a)^\rho + \cdots + c_{n-1}(t-a)^{\rho(n-1)} \quad (2.9)$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \dots, n-1$.

For convenience we set constants

$$\eta_1 = \frac{\lambda(\sigma - a)^{\rho\beta}}{\rho^\beta \Gamma(\beta + 1)} - 1, \quad \eta_2 = \frac{\lambda(\sigma - a)^{\rho(\beta+1)}}{\rho^\beta \Gamma(\beta + 2)} - (T - a)^\rho. \quad (2.10)$$

Lemma 2.4 Let $1 < \alpha \leq 2$, $0 < \rho \leq 1$, $\beta > 0$, $\xi, \sigma \in (a, T)$, $\lambda, \mu_1, \mu_2 \in \mathbb{R}$, and $y \in C([a, T], \mathbb{R})$ and a constant

$$\mathcal{J} = \eta_2(1 - \mu_1) + \eta_1\mu_1(\xi - a)^\rho \neq 0.$$

Then the problem

$$({}_a^C \mathfrak{D}^{\alpha, \rho} x)(t) = y(t), \quad a < t < T, \quad (2.11)$$

$$x(a) = \mu_1 x(\xi) + \mu_2, \quad x(T) = \lambda({}_a \mathfrak{J}^{\beta, \rho} x)(\sigma), \quad (2.12)$$

has a unique solution given by

$$x(t) = {}_a \mathfrak{J}^{\alpha, \rho} y(t) + \lambda_1(t) {}_a \mathfrak{J}^{\alpha, \rho} y(\xi) + \lambda_2(t) {}_a \mathfrak{J}^{\alpha, \rho} y(T) - \lambda_3(t) {}_a \mathfrak{J}^{\alpha+\beta, \rho} y(\sigma) + \lambda_4(t), \quad (2.13)$$

where

$$\begin{aligned} \lambda_1(t) &= \frac{\mu_1}{\mathcal{J}} \{ \eta_2 - (t - a)^\rho \eta_1 \}, & \lambda_2(t) &= \frac{1}{\mathcal{J}} \{ (t - a)^\rho (1 - \mu_1) + \mu_1(\xi - a)^\rho \}, \\ \lambda_3(t) &= \frac{\lambda}{\mathcal{J}} \{ \mu_1(\xi - a)^\rho + (1 - \mu_1)(t - a)^\rho \}, & \lambda_4(t) &= \frac{1}{\mathcal{J}} \{ \eta_2 \mu_2 - (t - a)^\rho \eta_1 \mu_2 \}. \end{aligned}$$

Proof Using Lemma 2.3, (2.11) can be expressed as an equivalent integral equation

$$x(t) = {}_a \mathfrak{J}^{\alpha, \rho} y(t) + c_0 + c_1(t - a)^\rho \quad (2.14)$$

for arbitrary constants $c_0, c_1 \in \mathbb{R}$.

Taking the left-fractional conformable integral operator of order $\beta > 0$ for (2.14), we have

$${}_a \mathfrak{J}^{\beta, \rho} x(t) = {}_a \mathfrak{J}^{\alpha+\beta, \rho} y(t) + c_0 \frac{(t - a)^{\rho\beta}}{\rho^\beta \Gamma(\beta + 1)} + c_1 \frac{(t - a)^{\rho(\beta+1)}}{\rho^\beta \Gamma(\beta + 2)}. \quad (2.15)$$

From the first condition of (2.12), it follows that

$$(1 - \mu_1)c_0 - \mu_1(\xi - a)^\rho c_1 = \mu_1 {}_a \mathfrak{J}^{\alpha, \rho} y(\xi) + \mu_2. \quad (2.16)$$

The second condition of (2.12) and (2.15) implies

$$\eta_1 c_0 + \eta_2 c_1 = {}_a \mathfrak{J}^{\alpha, \rho} y(T) - \lambda {}_a \mathfrak{J}^{\alpha+\beta, \rho} y(\sigma). \quad (2.17)$$

From (2.16) and (2.17), we obtain two constants as follows:

$$c_0 = \frac{1}{\mathcal{J}} \{ \eta_2 \mu_1 {}_a \mathfrak{J}^{\alpha, \rho} y(\xi) + \eta_2 \mu_2 + \mu_1(\xi - a)^\rho [{}_a \mathfrak{J}^{\alpha, \rho} y(T) - \lambda {}_a \mathfrak{J}^{\alpha+\beta, \rho} y(\sigma)] \}$$

and

$$c_1 = \frac{1}{\mathcal{J}} \left\{ (1 - \mu_1) [{}_a\mathcal{J}^{\alpha,\rho} y(T) - \lambda {}_a\mathcal{J}^{\alpha+\beta,\rho} y(\sigma)] - \eta_1 [\mu_1 {}_a\mathcal{J}^{\alpha,\rho} y(\xi) + \mu_2] \right\}.$$

Substituting constants c_0 and c_1 into (2.14), we obtain (2.13). The converse follows by direct computation. The proof is completed. $\square$

3 Existence and uniqueness results

Let $\mathcal{C} = C([a, T], \mathbb{R})$ denote the Banach space of all continuous functions from $[a, T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t \in [a, T]} |x(t)|$. Throughout this paper, for convenience, the expression ${}_a\mathcal{J}^{b,\rho} f(s, x(s))(c)$ means

$${}_a\mathcal{J}^{b,\rho} f(s, x(s))(c) = \frac{1}{\Gamma(b)} \int_a^c \left(\frac{(c-a)^\rho - (s-a)^\rho}{\rho} \right)^{b-1} \frac{f(s, x(s)) ds}{(s-a)^{1-\rho}}, \quad t \in [a, T],$$

where $b \in \{\alpha, \alpha + \beta\}$ and $c \in \{t, T, \xi, \sigma\}$.

In view of Lemma 2.4, we define an operator $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}$ by

$$\begin{aligned} (\mathcal{F}x)(t) = & {}_a\mathcal{J}^{\alpha,\rho} f(s, x(s))(t) + \lambda_1(t) {}_a\mathcal{J}^{\alpha,\rho} f(s, x(s))(\xi) \\ & + \lambda_2(t) {}_a\mathcal{J}^{\alpha,\rho} f(s, x(s))(T) - \lambda_3(t) {}_a\mathcal{J}^{\alpha+\beta,\rho} f(s, x(s))(\sigma) + \lambda_4(t). \end{aligned} \quad (3.1)$$

It should be noticed that problem (1.1) has solutions if and only if the operator $\mathcal{F}$ has fixed points. In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (1.1) by using a variety of fixed point theorems. In addition, we set

$$|\lambda_1(t)| \leq M_1 := \frac{|\mu_1|}{|\mathcal{J}|} \{ |\eta_2| + (T-a)^\rho |\eta_1| \}, \quad (3.2)$$

$$|\lambda_2(t)| \leq M_2 := \frac{1}{|\mathcal{J}|} \{ (T-a)^\rho (1 + |\mu_1|) + |\mu_1| (\xi-a)^\rho \}, \quad (3.3)$$

$$|\lambda_3(t)| \leq M_3 := \frac{|\lambda|}{|\mathcal{J}|} \{ |\mu_1| (\xi-a)^\rho + (T-a)^\rho (1 + |\mu_1|) \}, \quad (3.4)$$

$$|\lambda_4(t)| \leq M_4 := \frac{1}{|\mathcal{J}|} \{ |\mu_2| (|\eta_2| + (T-a)^\rho |\eta_1|) \}, \quad (3.5)$$

and

$$\Phi = \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_1 \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_2 \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_3 \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)}. \quad (3.6)$$

3.1 Existence and uniqueness result via Banach's fixed point theorem

The first existence and uniqueness result is based on the Banach contraction mapping principle (Banach's fixed point theorem).

Theorem 3.1 Assume that $f : [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that

(H₁) there exists a constant $L > 0$ such that $|f(t, x) - f(t, y)| \leq L|x - y|$ for each $t \in [a, T]$ and $x, y \in \mathbb{R}$.

If

$$L\Phi < 1, \quad (3.7)$$

where Φ is defined by (3.6), then the boundary value problem (1.1) has a unique solution on $[a, T]$.

Proof We transform problem (1.1) into a fixed point problem, $x = \mathcal{F}x$, where the operator $\mathcal{F}$ is defined as in (3.1). Observe that the fixed points of the operator $\mathcal{F}$ are solutions of problem (1.1). Applying the Banach contraction mapping principle, we shall show that $\mathcal{F}$ has a unique fixed point. Now, we let $\sup_{t \in [a, T]} |f(t, 0)| = M < \infty$ and choose a positive constant r satisfying

$$r \geq \frac{\Phi M + M_4}{1 - L\Phi}.$$

Next, we show that $\mathcal{F}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For any $x \in B_r$, we have

$$\begin{aligned} & |(\mathcal{F}x)(t)| \\ &= \left| {}_a\mathcal{J}^{\alpha, \rho} f(s, x(s))(t) + \lambda_1(t) {}_a\mathcal{J}^{\alpha, \rho} f(s, x(s))(\xi) + \lambda_2(t) {}_a\mathcal{J}^{\alpha, \rho} f(s, x(s))(T) \right. \\ &\quad \left. - \lambda_3(t) {}_a\mathcal{J}^{\alpha+\beta, \rho} f(s, x(s))(\sigma) + \lambda_4(t) \right| \\ &\leq {}_a\mathcal{J}^{\alpha, \rho} |f(s, x(s))|(t) + |\lambda_1(t)| {}_a\mathcal{J}^{\alpha, \rho} |f(s, x(s))|(\xi) + |\lambda_2(t)| {}_a\mathcal{J}^{\alpha, \rho} |f(s, x(s))|(T) \\ &\quad + |\lambda_3(t)| {}_a\mathcal{J}^{\alpha+\beta, \rho} |f(s, x(s))|(\sigma) + |\lambda_4(t)| \\ &\leq {}_a\mathcal{J}^{\alpha, \rho} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|)(T) \\ &\quad + M_1 {}_a\mathcal{J}^{\alpha, \rho} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|)(\xi) \\ &\quad + M_2 {}_a\mathcal{J}^{\alpha, \rho} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|)(T) \\ &\quad + M_3 {}_a\mathcal{J}^{\alpha+\beta, \rho} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|)(\sigma) + M_4 \\ &\leq (Lr + M) {}_a\mathcal{J}^{\alpha, \rho} (1)(T) + (Lr + M)M_1 {}_a\mathcal{J}^{\alpha, \rho} (1)(\xi) + (Lr + M)M_2 {}_a\mathcal{J}^{\alpha, \rho} (1)(T) \\ &\quad + (Lr + M)M_3 {}_a\mathcal{J}^{\alpha+\beta, \rho} (1)(\sigma) + M_4 \\ &= (Lr + M) \left[\frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_1 \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_2 \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \right. \\ &\quad \left. + M_3 \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right] + M_4 \\ &= (Lr + M)\Phi + M_4 \leq r, \end{aligned}$$

which implies that $\|\mathcal{F}x\| \leq r$ and therefore $\mathcal{F}B_r \subset B_r$.

Next, we let $x, y \in \mathcal{C}$. Then, for $t \in [a, T]$, we have

$$\begin{aligned} & |(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| \leq {}_a\mathcal{J}^{\alpha, \rho} |f(s, x(s)) - f(s, y(s))|(t) \\ &\quad + |\lambda_1(t)| {}_a\mathcal{J}^{\alpha, \rho} |f(s, x(s)) - f(s, y(s))|(\xi) \\ &\quad + |\lambda_2(t)| {}_a\mathcal{J}^{\alpha, \rho} |f(s, x(s)) - f(s, y(s))|(T) \end{aligned}$$

$$\begin{aligned}
& + |\lambda_3(t)| {}_a\mathcal{I}^{\alpha+\beta,\rho} |f(s, x(s)) - f(s, y(s))|(\sigma) \\
& \leq L \|x - y\| {}_a\mathcal{I}^{\alpha,\rho}(1)(T) + L \|x - y\| M_1 {}_a\mathcal{I}^{\alpha,\rho}(1)(\xi) \\
& \quad + L \|x - y\| M_2 {}_a\mathcal{I}^{\alpha,\rho}(1)(T) + L \|x - y\| M_3 {}_a\mathcal{I}^{\alpha+\beta,\rho}(1)(\sigma) \\
& = L \left(\frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_1 \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_2 \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \right. \\
& \quad \left. + M_3 \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right) \|x - y\| \\
& = L\Phi \|x - y\|,
\end{aligned}$$

which implies that $\|\mathcal{F}x - \mathcal{F}y\| \leq L\Phi \|x - y\|$. As $L\Phi < 1$, $\mathcal{F}$ is a contraction operator. Therefore, we deduce, by the Banach contraction mapping principle, that $\mathcal{F}$ has a fixed point which is the unique solution of problem (1.1) on $[a, T]$. The proof is completed. $\square$

3.2 Existence result via Krasnoselskii's fixed point theorem

The next existence theorem is based on Krasnoselskii's fixed point theorem.

Lemma 3.1 (Krasnoselskii's fixed point theorem [46]) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X . Let A, B be the operators such that (a) $Ax + By \in M$ whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.2 *Let $f : [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying (H_1) . In addition we assume that*

$$(H_2) \quad |f(t, x)| \leq \delta(t), \quad \forall (t, x) \in [a, T] \times \mathbb{R}, \text{ and } \delta \in C([a, T], \mathbb{R}^+).$$

Then the boundary value problem (1.1) has at least one solution on $[a, T]$ provided

$$M_1 \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_2 \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_3 \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} < 1. \quad (3.8)$$

Proof Setting $\sup_{t \in [a, T]} |\delta(t)| = \|\delta\|$ and choosing

$$\bar{r} \geq \|\delta\| \Phi + M_4, \quad (3.9)$$

where Φ is defined by (3.6), we consider $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \leq \bar{r}\}$. Let us define the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $B_{\bar{r}}$ by

$$\begin{aligned}
\mathcal{F}_1 x(t) &= {}_a\mathcal{I}^{\alpha,\rho} f(s, x(s))(t), \quad t \in [a, T], \\
\mathcal{F}_2 x(t) &= \lambda_1(t) {}_a\mathcal{I}^{\alpha,\rho} f(s, x(s))(\xi) + \lambda_2(t) {}_a\mathcal{I}^{\alpha,\rho} f(s, x(s))(T) \\
&\quad - \lambda_3(t) {}_a\mathcal{I}^{\alpha+\beta,\rho} f(s, x(s))(\sigma) + \lambda_4(t), \quad t \in [a, T].
\end{aligned}$$

For any $x, y \in B_{\bar{r}}$, we have

$$\begin{aligned}
\|\mathcal{F}_1 x + \mathcal{F}_2 y\| &\leq \sup_{t \in [a, T]} \left\{ {}_a\mathcal{I}^{\alpha,\rho} |f(s, x(s))|(t) + |\lambda_1(t)| {}_a\mathcal{I}^{\alpha,\rho} |f(s, y(s))|(\xi) \right. \\
&\quad \left. + |\lambda_2(t)| {}_a\mathcal{I}^{\alpha,\rho} |f(s, y(s))|(T) + |\lambda_3(t)| {}_a\mathcal{I}^{\alpha+\beta,\rho} |f(s, y(s))|(\sigma) + |\lambda_4(t)| \right\}
\end{aligned}$$

$$\begin{aligned} &\leq \|\delta\| \left[M_1 \frac{(\xi - a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha + 1)} + M_2 \frac{(T - a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha + 1)} + M_3 \frac{(\sigma - a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha + \beta + 1)} \right] + M_4 \\ &= \|\delta\| \Phi + M_4 \leq \bar{r}. \end{aligned}$$

This shows that $\mathcal{F}_1 x + \mathcal{F}_2 y \in B_{\bar{r}}$ which satisfies condition (a) of Lemma 3.1. It is easy to see, using (3.8), that $\mathcal{F}_2$ is a contraction mapping and also condition (c) of Lemma 3.1 holds.

To show that condition (b) of Lemma 3.1 is fulfilled, we apply the continuity of a function f , which leads to operator $\mathcal{F}_1$ being continuous. Also, the set $\mathcal{F}_1 B_{\bar{r}}$ is uniformly bounded as

$$\|\mathcal{F}_1 x\| \leq \frac{(T - a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha + 1)} \|\delta\|.$$

Next, we prove the compactness of the operator $\mathcal{F}_1$ by setting $\sup_{(t,x) \in [a,T] \times B_{\bar{r}}} |f(t,x)| = \bar{f} < \infty$. Then, for $a \leq t_1 \leq t_2 \leq T$, we have

$$\begin{aligned} &|\mathcal{F}_1 x(t_2) - \mathcal{F}_1 x(t_1)| \\ &= |{}_a \mathcal{J}^{\alpha,\rho} f(s, x(s))(t_2) - {}_a \mathcal{J}^{\alpha,\rho} f(s, x(s))(t_1)| \\ &\leq \left| \frac{1}{\Gamma(\alpha)} \left[\int_a^{t_1} \left(\frac{(t_2 - a)^\rho - (s - a)^\rho}{\rho} \right)^{\alpha-1} - \int_a^{t_1} \left(\frac{(t_1 - a)^\rho - (s - a)^\rho}{\rho} \right)^{\alpha-1} \right] \right. \\ &\quad \times \left. \frac{f(s, x(s)) ds}{(s - a)^{1-\rho}} + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\frac{(t_2 - a)^\rho - (s - a)^\rho}{\rho} \right)^{\alpha-1} \frac{f(s, x(s)) ds}{(s - a)^{1-\rho}} \right| \\ &\leq \frac{\bar{f}}{\rho^\alpha \Gamma(\alpha + 1)} [2|(t_2 - a)^{\rho\alpha} - (t_1 - a)^{\rho\alpha}| + |(t_2 - a)^{\rho\alpha} - (t_1 - a)^{\rho\alpha}|], \end{aligned}$$

which is independent of x and tends to zero as $t_2 \rightarrow t_1$. Thus, the set $\mathcal{F}_1 B_{\bar{r}}$ is equicontinuous. So the set $\mathcal{F}_1 B_{\bar{r}}$ is relatively compact. Hence, by the Arzelà–Ascoli theorem, the operator $\mathcal{F}_1$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Lemma 3.1 are satisfied. So, the conclusion of Lemma 3.1 implies that the boundary value problem (1.1) has at least one solution on $[a, T]$. The proof is completed. $\square$

3.3 Existence result via Leray–Schauder’s nonlinear alternative

By using Leray–Schauder’s nonlinear alternative, we give in this subsection our last existence theorem.

Lemma 3.2 (Nonlinear alternative for single-valued maps [47]) *Let E be a Banach space, C be a closed, convex subset of E , X be an open subset of C , and $0 \in X$. Suppose that $F : \bar{X} \rightarrow C$ is a continuous, compact (that is, $F(\bar{X})$ is a relatively compact subset of C) map. Then either*

- (i) F has a fixed point in $\bar{X}$, or
- (ii) there is $x \in \partial X$ (the boundary of X in C) and $\lambda \in (0, 1)$ with $x = \lambda F(x)$.

Theorem 3.3 *Assume that:*

- (H₃) *there exist a continuous nondecreasing function $\psi : [0, \infty) \rightarrow (0, \infty)$ and a function $p \in C([a, T], \mathbb{R}^+)$ such that*

$$|f(t, x)| \leq p(t) \psi(\|x\|) \quad \text{for each } (t, x) \in [a, T] \times \mathbb{R};$$

(H₄) there exists a constant $N > 0$ such that

$$\frac{N}{\|p\|\psi(N)\Phi + M_4} > 1,$$

where Φ is defined by (3.6).

Then the boundary value problem (1.1) has at least one solution on $[a, T]$.

Proof Let the operator $\mathcal{F}$ be defined by (3.1). Firstly, we shall show that $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $\mathcal{C}$. For a number $R > 0$, let $B_R = \{x \in \mathcal{C} : \|x\| \leq R\}$ be a bounded ball in $\mathcal{C}$. Then, for $t \in [a, T]$, we have

$$\begin{aligned} & |(\mathcal{F}x)(t)| \\ & \leq {}_a\mathcal{J}^{\alpha,\rho} |f(s, x(s))|(t) + |\lambda_1(t)| {}_a\mathcal{J}^{\alpha,\rho} |f(s, x(s))|(\xi) + |\lambda_2(t)| {}_a\mathcal{J}^{\alpha,\rho} |f(s, x(s))|(T) \\ & \quad + |\lambda_3(t)| {}_a\mathcal{J}^{\alpha+\beta,\rho} |f(s, x(s))|(\sigma) + |\lambda_4(t)| \\ & \leq \|p\|\psi(\|x\|) \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + \|p\|\psi(\|x\|) M_1 \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \\ & \quad + \|p\|\psi(\|x\|) M_2 \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + \|p\|\psi(\|x\|) M_3 \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} + M_4, \end{aligned}$$

which leads to

$$\begin{aligned} \|\mathcal{F}x\| & \leq \|p\|\psi(R) \left\{ \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_1 \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_2 \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \right. \\ & \quad \left. + M_3 \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right\} + M_4 \\ & := K. \end{aligned}$$

Secondly, we show that $\mathcal{F}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $v_1, v_2 \in [a, T]$ with $v_1 < v_2$ and $x \in B_R$. Then we have

$$\begin{aligned} & |(\mathcal{F}x)(v_2) - (\mathcal{F}x)(v_1)| \\ & = |{}_a\mathcal{J}^{\alpha,\rho} f(s, x(s))(v_2) + \lambda_1(v_2) {}_a\mathcal{J}^{\alpha,\rho} f(s, x(s))(\xi) + \lambda_2(v_2) {}_a\mathcal{J}^{\alpha,\rho} f(s, x(s))(T) \\ & \quad - \lambda_3(v_2) {}_a\mathcal{J}^{\alpha+\beta,\rho} f(s, x(s))(\sigma) + \lambda_4(v_2) - {}_a\mathcal{J}^{\alpha,\rho} f(s, x(s))(v_1) \\ & \quad - \lambda_1(v_1) {}_a\mathcal{J}^{\alpha,\rho} f(s, x(s))(\xi) - \lambda_2(v_1) {}_a\mathcal{J}^{\alpha,\rho} f(s, x(s))(T) \\ & \quad + \lambda_3(v_1) {}_a\mathcal{J}^{\alpha+\beta,\rho} f(s, x(s))(\sigma) - \lambda_4(v_1)| \\ & \leq \frac{\|p\|\psi(R)}{\Gamma(\alpha)} \left| \int_a^{v_1} \left[\left(\frac{(v_2-a)^\rho - (s-a)^\rho}{\rho} \right)^{\alpha-1} - \left(\frac{(v_1-a)^\rho - (s-a)^\rho}{\rho} \right)^{\alpha-1} \right] \frac{ds}{(s-a)^{1-\rho}} \right. \\ & \quad \left. + \int_{v_1}^{v_2} \left(\frac{(v_2-a)^\rho - (s-a)^\rho}{\rho} \right)^{\alpha-1} \frac{ds}{(s-a)^{1-\rho}} \right| \\ & \quad + \|p\|\psi(R) \left\{ \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} |\lambda_1(v_2) - \lambda_1(v_1)| \right. \\ & \quad \left. + \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} |\lambda_2(v_2) - \lambda_2(v_1)| + \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} |\lambda_3(v_2) - \lambda_3(v_1)| \right\} \end{aligned}$$

$$\begin{aligned}
& + |\lambda_4(v_2) - \lambda_4(v_1)| \\
& \leq \frac{\|p\|\psi(R)}{\rho^\alpha \Gamma(\alpha+1)} [2|(t_2-a)^{\rho\alpha} - (t_1-a)^{\rho\alpha}| + |(t_2-a)^{\rho\alpha} - (t_1-a)^{\rho\alpha}|] \\
& + \|p\|\psi(R) \left\{ \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} |\lambda_1(v_2) - \lambda_1(v_1)| \right. \\
& + \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} |\lambda_2(v_2) - \lambda_2(v_1)| + \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} |\lambda_3(v_2) - \lambda_3(v_1)| \Big\} \\
& + |\lambda_4(v_2) - \lambda_4(v_1)|.
\end{aligned}$$

Obviously, the above inequality tends to zero independently of $x \in B_R$ as $v_2 \rightarrow v_1$. Therefore it follows from the Arzelà–Ascoli theorem that $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}$ is completely continuous.

Finally, we show that there exists an open set $X \subseteq \mathcal{C}$ with $x \neq \theta \mathcal{F}(x)$ for $\theta \in (0, 1)$ and $x \in \partial X$.

Let $x \in \mathcal{C}$ be a solution of $x = \theta \mathcal{F}x$ for $\theta \in [0, 1]$. Then, for $t \in [a, T]$, we have

$$\begin{aligned}
|x(t)| & = |v(\mathcal{F}x)(t)| \\
& \leq \|p\|\psi(\|x\|) \left\{ \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_1 \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_2 \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \right. \\
& \quad \left. + M_3 \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right\} + M_4 \\
& = \|p\|\psi(\|x\|)\Phi + M_4,
\end{aligned}$$

which, on taking the norm for $t \in [a, T]$, implies that

$$\|x\| \leq \|p\|\psi(\|x\|)\Phi + M_4.$$

Consequently, we have

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\Phi + M_4} \leq 1.$$

In view of (H_4) , there exists N such that $\|x\| \neq N$. Let us set

$$X = \{x \in \mathcal{C} : \|x\| < N\} \quad \text{and} \quad Y = X \cap B_R.$$

Note that the operator $\mathcal{F} : \overline{Y} \rightarrow \mathcal{C}$ is continuous and completely continuous. From the choice of Y , there is no $x \in \partial Y$ such that $x = \theta \mathcal{F}x$ for some $\theta \in (0, 1)$. Consequently, by the nonlinear alternative of Leray–Schauder type (Lemma 3.2), we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{Y}$ which is a solution of the boundary value problem (1.1). This completes the proof. $\square$

4 Ulam–Hyers stability analysis

In this section, we study Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias stability of problem (1.1).

Definition 4.1 Problem (1.1) is Ulam–Hyers stable if there exists a real constant $\kappa > 0$ such that, for $\varepsilon > 0$ and for every solution $y \in \mathcal{C}$ of the inequality

$$\left| {}^C_a \mathfrak{D}^{\alpha, \rho} y(t) - f(t, y(t)) \right| \leq \varepsilon, \quad t \in [a, T], \quad (4.1)$$

there exists a solution $x \in \mathcal{C}$ of problem (1.1) with

$$|y(t) - x(t)| \leq \kappa \varepsilon, \quad t \in [a, T].$$

Definition 4.2 Problem (1.1) is generalized Ulam–Hyers stable if there is $\Psi_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\Psi_f(0) = 0$ such that for every solution $y \in \mathcal{C}$ of inequality (4.1) there exists a solution $x \in \mathcal{C}$ of problem (1.1) which satisfies the following inequality:

$$|y(t) - x(t)| \leq \Psi_f(\varepsilon), \quad t \in [a, T].$$

Definition 4.3 Problem (1.1) is Ulam–Hyers–Rassias stable with respect to $\varphi : [a, T] \rightarrow \mathbb{R}^+$ if there exists a real constant $\kappa_\varphi > 0$ such that, for $\varepsilon > 0$ and for every solution $y \in \mathcal{C}$ of the inequality

$$\left| {}^C_a \mathfrak{D}^{\alpha, \rho} y(t) - f(t, y(t)) \right| \leq \varepsilon \varphi(t), \quad t \in [a, T], \quad (4.2)$$

there exists a solution $x \in \mathcal{C}$ of problem (1.1) with

$$|y(t) - x(t)| \leq \kappa_\varphi \varepsilon \varphi(t), \quad t \in [a, T].$$

Definition 4.4 Problem (1.1) is generalized Ulam–Hyers–Rassias stable with respect to $\varphi : [a, T] \rightarrow \mathbb{R}^+$ if there exists a real constant $\kappa_\varphi > 0$ such that, for $\varepsilon > 0$ and for every solution $y \in \mathcal{C}$ of the inequality

$$\left| {}^C_a \mathfrak{D}^{\alpha, \rho} y(t) - f(t, y(t)) \right| \leq \varphi(t), \quad t \in [a, T], \quad (4.3)$$

there exists a solution $x \in \mathcal{C}$ of problem (1.1) with

$$|y(t) - x(t)| \leq \kappa_\varphi \varphi(t), \quad t \in [a, T].$$

Remark 4.1 A function $y \in \mathcal{C}$ is a solution of inequality (4.1) if and only if there exists a function $g \in \mathcal{C}$ (which depends on y) such that

- (i) $|g(t)| < \varepsilon$, $t \in [a, T]$,
- (ii) ${}^C_a \mathfrak{D}^{\alpha, \rho} y(t) = f(t, y(t)) + g(t)$, $t \in [a, T]$.

By Remark 4.1, the solution of the equation

$${}^C_a \mathfrak{D}^{\alpha, \rho} y(t) = f(t, y(t)) + g(t), \quad t \in [a, T],$$

can be formulated by

$$\begin{aligned} y(t) = & {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(t) + \lambda_1(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(\xi) + \lambda_2(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(T) \\ & - \lambda_3(t) {}_a\mathfrak{J}^{\alpha+\beta,\rho}f(s,y(s))(\sigma) + \lambda_4(t) {}_a\mathfrak{J}^{\alpha,\rho}g(t) + \lambda_1(t) {}_a\mathfrak{J}^{\alpha,\rho}g(\xi) \\ & + \lambda_2(t) {}_a\mathfrak{J}^{\alpha,\rho}g(T) - \lambda_3(t) {}_a\mathfrak{J}^{\alpha+\beta,\rho}g(\sigma). \end{aligned}$$

Then we have the following estimation.

Remark 4.2 Let $y \in \mathcal{C}$ be a solution of inequality (4.1). Then y is a solution of the following integral inequality:

$$\begin{aligned} & |y(t) - {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(t) - \lambda_1(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(\xi) - \lambda_2(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(T) \\ & \quad + \lambda_3(t) {}_a\mathfrak{J}^{\alpha+\beta,\rho}f(s,y(s))(\sigma) - \lambda_4(t)| \\ & = |{}_a\mathfrak{J}^{\alpha,\rho}g(t) + \lambda_1(t) {}_a\mathfrak{J}^{\alpha,\rho}g(\xi) + \lambda_2(t) {}_a\mathfrak{J}^{\alpha,\rho}g(T) - \lambda_3(t) {}_a\mathfrak{J}^{\alpha+\beta,\rho}g(\sigma)| \\ & \leq \varepsilon \left[\frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_1 \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_2 \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_3 \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right] \\ & = \varepsilon \Phi. \end{aligned}$$

Now we are ready to state our Ulam–Hyers stability result.

Theorem 4.1 Assume that $f : [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and (H_1) holds with $L < \Phi^{-1}$. Then problem (1.1) is Ulam–Hyers stable on $[a, T]$ and consequently generalized Ulam–Hyers stable.

Proof Let $y \in \mathcal{C}$ be the solution of inequality (4.1) and let $x \in \mathcal{C}$ be the unique solution of

$$\begin{cases} ({}_a^C\mathfrak{D}^{\alpha,\rho}x)(t) = f(t, x(t)), & 1 < \alpha \leq 2, a < t < T, \\ x(a) = \mu_1 x(\xi) + \mu_2, & x(T) = \lambda({}_a\mathfrak{J}^{\beta,\rho}x)(\sigma), \quad \xi, \sigma \in (a, T). \end{cases}$$

Then consider

$$\begin{aligned} & |y(t) - x(t)| \\ & = |y(t) - {}_a\mathfrak{J}^{\alpha,\rho}f(s,x(s))(t) - \lambda_1(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,x(s))(\xi) - \lambda_2(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,x(s))(T) \\ & \quad + \lambda_3(t) {}_a\mathfrak{J}^{\alpha+\beta,\rho}f(s,x(s))(\sigma) - \lambda_4(t)| \\ & = |y(t) - {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(t) - \lambda_1(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(\xi) - \lambda_2(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(T) \\ & \quad + \lambda_3(t) {}_a\mathfrak{J}^{\alpha+\beta,\rho}f(s,y(s))(\sigma) - \lambda_4(t) + {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(t) + \lambda_1(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(\xi) \\ & \quad + \lambda_2(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,y(s))(T) - \lambda_3(t) {}_a\mathfrak{J}^{\alpha+\beta,\rho}f(s,y(s))(\sigma) + \lambda_4(t) - {}_a\mathfrak{J}^{\alpha,\rho}f(s,x(s))(t) \\ & \quad - \lambda_1(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,x(s))(\xi) - \lambda_2(t) {}_a\mathfrak{J}^{\alpha,\rho}f(s,x(s))(T) + \lambda_3(t) {}_a\mathfrak{J}^{\alpha+\beta,\rho}f(s,x(s))(\sigma) \\ & \quad - \lambda_4(t)| \\ & \leq \varepsilon \Phi + L\Phi |y(t) - x(t)|, \end{aligned}$$

which yields that

$$|y(t) - x(t)| \leq \frac{\varepsilon \Phi}{1 - L\Phi}. \quad (4.4)$$

Taking for simplicity

$$\kappa = \frac{\Phi}{1 - L\Phi}$$

such that $L\Phi < 1$, then (4.4) becomes

$$|y(t) - x(t)| \leq \kappa \varepsilon, \quad t \in [a, T].$$

Thus problem (1.1) is Ulam–Hyers stable. Further, using $\Psi_f(\varepsilon) = \kappa \varepsilon$, $\Psi_f(0) = 0$ implies that solution of (1.1) is generalized Ulam–Hyers stable. This completes the proof. $\square$

Remark 4.3 A function $y \in \mathcal{C}$ is a solution of inequality (4.2) if and only if there exists a function $h \in \mathcal{C}$ (which depends on y) such that

- (i) $|h(t)| < \varepsilon \varphi(t)$, $t \in [a, T]$,
- (ii) ${}_a^C \mathcal{D}^{\alpha, \rho} y(t) = f(t, y(t)) + h(t)$, $t \in [a, T]$.

By Remark 4.3, the solution of the equation

$${}_a^C \mathcal{D}^{\alpha, \rho} y(t) = f(t, y(t)) + h(t), \quad t \in [a, T],$$

can be formulated by

$$\begin{aligned} y(t) = & {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(t) + \lambda_1(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(\xi) + \lambda_2(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(T) \\ & - \lambda_3(t) {}_a \mathcal{I}^{\alpha + \beta, \rho} f(s, y(s))(\sigma) + \lambda_4(t) {}_a \mathcal{I}^{\alpha, \rho} h(t) + \lambda_1(t) {}_a \mathcal{I}^{\alpha, \rho} h(\xi) \\ & + \lambda_2(t) {}_a \mathcal{I}^{\alpha, \rho} h(T) - \lambda_3(t) {}_a \mathcal{I}^{\alpha + \beta, \rho} h(\sigma). \end{aligned}$$

Then we have the following estimation.

Remark 4.4 Let $y \in \mathcal{C}$ be a solution of inequality (4.2). Then y is a solution of the following integral inequality:

$$\begin{aligned} & |y(t) - {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(t) - \lambda_1(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(\xi) - \lambda_2(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(T) \\ & \quad + \lambda_3(t) {}_a \mathcal{I}^{\alpha + \beta, \rho} f(s, y(s))(\sigma) - \lambda_4(t)| \\ & = |{}_a \mathcal{I}^{\alpha, \rho} h(t) + \lambda_1(t) {}_a \mathcal{I}^{\alpha, \rho} h(\xi) + \lambda_2(t) {}_a \mathcal{I}^{\alpha, \rho} h(T) - \lambda_3(t) {}_a \mathcal{I}^{\alpha + \beta, \rho} h(\sigma)| \\ & \leq \varepsilon \varphi(t) \left[\frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_1 \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_2 \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_3 \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right] \\ & \leq \varepsilon \varphi(t) \Phi. \end{aligned}$$

Now we are ready to state our Ulam–Hyers–Rassias stability result.

Theorem 4.2 Assume that $f : [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying (H_3) . If (H_4) holds and there exists a function $h(t)$ satisfying Remark 4.3 with $2N \leq h(t)$, $t \in [a, T]$, then problem (1.1) is Ulam–Hyers–Rassias stable and consequently generalized Ulam–Hyers–Rassias stable.

Proof Let $y \in \mathcal{C}$ be the solution of inequality (4.2) and $x \in \mathcal{C}$ be a solution of

$$\begin{cases} ({}^C_a \mathcal{D}^{\alpha, \rho} x)(t) = f(t, x(t)), & 1 < \alpha \leq 2, a < t < T, \\ x(a) = \mu_1 x(\xi) + \mu_2, & x(T) = \lambda({}_a \mathcal{I}^{\beta, \rho} x)(\sigma), \quad \xi, \sigma \in (a, T). \end{cases}$$

Then we consider

$$\begin{aligned} & |y(t) - x(t)| \\ &= |y(t) - {}_a \mathcal{I}^{\alpha, \rho} f(s, x(s))(t) - \lambda_1(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, x(s))(\xi) - \lambda_2(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, x(s))(T) \\ &\quad + \lambda_3(t) {}_a \mathcal{I}^{\alpha+\beta, \rho} f(s, x(s))(\sigma) - \lambda_4(t)| \\ &\leq |y(t) - {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(t) - \lambda_1(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(\xi) - \lambda_2(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(T) \\ &\quad + \lambda_3(t) {}_a \mathcal{I}^{\alpha+\beta, \rho} f(s, y(s))(\sigma) - \lambda_4(t)| \\ &\quad + |{}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(t) + \lambda_1(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(\xi) + \lambda_2(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, y(s))(T) \\ &\quad - \lambda_3(t) {}_a \mathcal{I}^{\alpha+\beta, \rho} f(s, y(s))(\sigma) + \lambda_4(t)| \\ &\quad + |{}_a \mathcal{I}^{\alpha, \rho} f(s, x(s))(t) + \lambda_1(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, x(s))(\xi) + \lambda_2(t) {}_a \mathcal{I}^{\alpha, \rho} f(s, x(s))(T) \\ &\quad - \lambda_3(t) {}_a \mathcal{I}^{\alpha+\beta, \rho} f(s, x(s))(\sigma) + \lambda_4(t)| \\ &\leq \varepsilon \varphi(t) \Phi + \|p\| \psi(\|x\|) \Phi + \|p\| \psi(\|y\|) \Phi + 2M_4 \\ &\leq \varepsilon \varphi(t) \Phi + 2N \leq \varepsilon \varphi(t) \Phi + \varepsilon \varphi(t), \end{aligned}$$

which yields that

$$|y(t) - x(t)| \leq \varepsilon(1 + \Phi) \varphi(t). \quad (4.5)$$

Taking for simplicity $\kappa_\varphi = (1 + \Phi)$, then (4.5) becomes

$$|y(t) - x(t)| \leq \kappa_\varphi \varepsilon \varphi(t), \quad t \in [a, T].$$

Thus problem (1.1) is Ulam–Hyers–Rassias stable. Further, in the same fashion, it can be shown that problem (1.1) is generalized Ulam–Hyers–Rassias stable. $\square$

5 Examples

In this section, we present examples to illustrate our results.

Example 5.1 Consider the following four-point integral boundary value problem:

$$\begin{cases} ({}_0^C \mathcal{D}^{\frac{3}{2}, 1} x)(t) = f(t, x(t)), & 0 < t < 1, \\ x(0) = \frac{1}{2} x(\frac{1}{2}) + \frac{3}{4}, & x(1) = 109({}_0 \mathcal{I}^{\frac{1}{3}, 1} x)(\frac{1}{3}). \end{cases} \quad (5.1)$$

Here $\alpha = 3/2$, $\rho = 1$, $a = 0$, $T = 1$, $\mu_1 = 1/2$, $\mu_2 = 3/4$, $\xi = 1/2$, $\lambda = 109$, $\sigma = 1/3$, $\beta = 1/3$. From information, we can find that $\mathcal{J} = 30.98772937 \neq 0$ and

$$\begin{aligned}\Phi &= \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_1 \frac{(\xi-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_2 \frac{(T-a)^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + M_3 \frac{(\sigma-a)^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \\ &= 1.71645006.\end{aligned}$$

(i) If

$$f(t, x) = \frac{2}{57} \sin((t+1)^2) \left(\frac{|x|}{1+|x|} + \frac{1}{2+|x|} \right) |x|,$$

then $|f(t, x) - f(t, y)| \leq 2/57|x - y|$, and consequently (H_1) is satisfied with $L = 2/57$. Thus $L\Phi = 0.06022632 < 1$. Hence, by Theorem 3.1, problem (5.1) has a unique solution on $[0, 1]$.

Further, we can find that $\kappa = \Phi/(1 - L\Phi) = 1.82645044 > 0$. Hence, by Theorem 4.1, problem (5.1) is Ulam–Hyers stable and also generalized Ulam–Hyers stable.

(ii) If

$$f(t, x) = \frac{1}{19}(t-1)^2 \left(\frac{x^2}{1+|x|} + \frac{|x|+2}{3+|x|} \right),$$

it follows that

$$|f(t, x)| \leq \frac{1}{19}(t-1)^2(|x|+1).$$

Choosing $p(t) = (1/19)(t-1)^2$ and $\psi(|x|) = |x| + 1$, we can show that

$$\frac{N}{\|p\| \psi(N)\Phi + M_4} > 1$$

implies that $N > 2.860892580$. Hence, by Theorem 3.3, problem (5.1) has at least one solution on $[0, 1]$.

Further, by choosing $h(t) = 7e^{(t+1)^2}$ and $N = 3$, then we have $2N \leq h(t)$ for all $t \in [0, 1]$. Now we set $\varphi(t) = e^{(t+1)^2}$ and we find that $\kappa_\varphi = (1 + \Phi) = 2.71645006 > 0$. Hence, by Theorem 4.2, problem (5.1) is Ulam–Hyers–Rassias stable and also generalized Ulam–Hyers–Rassias stable on $[0, 1]$.

6 Conclusion

Existence and uniqueness of solutions for conformable derivatives in the Caputo setting with four-point integral conditions are investigated. The existence results are proved via Krasnoselskii's fixed point theorem and Leray–Schauder nonlinear alternative, while the uniqueness result is obtained by applying the Banach contraction mapping principle. Further, different types of Ulam stability, such as Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability, are presented. Examples illustrating the obtained results are also included.

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Availability of data and materials

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Competing interests

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Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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