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Existence of positive solution and Hyers–Ulam stability for a nonlinear singular-delay-fractional differential equation

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Abstract

In this article, we consider a study of a general class of nonlinear singular fractional DEs with p -Laplacian for the existence and uniqueness (EU) of a positive solution and the Hyers–Ulam (HU) stability. To proceed, we use classical fixed point theorem and properties of a p -Laplacian operator. The fractional DE is converted into an integral alternative form with the help of the Green's function. The Green's function is analyzed as regards its nature and then, with the help of a fixed point approach, the existence of a positive solution and uniqueness are studied. After the EU of a positive solution, the HU-stability and an application are considered. The suggested singular fractional DE with ϕ_p is more general than the one considered in (Khan et al. in *Eur. Phys. J. Plus* 133:26, 2018)

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1 Introduction

Fractional order models have attracted the attentions of researchers of various discipline, in the last two decades. In fact, fractional order models can be met with in numerous fields such as processing, control theory, signals, biology, fluid dynamics, modern physics, set theory, hydrodynamics, viscoelastic theory, computer networking and others. For detailed literature connected to these topics, we refer the reader to some famous books [2–8]. Various mathematical procedures have been considered by scientists through different research oriented aspects of fraction DEs.

Recently, some authors explored fractional DEs with singularity with the help of different mathematical techniques. For example, Bai and Qiu [9] established EU of solutions for a nonlinear singular boundary value problem (BVP) of fractional DEs with the help of the Leray–Schauder and Krasnosel'skii's fixed point theorems. They also provided some applications to illuminate the results. Agarwal et al. [10, 11] investigated EU for a singular fractional BVP involving the Riemann–Liouville fractional derivative. Bai and Fang [12] explored a singular coupled system of nonlinear fractional DEs for the EU of solution with the use of Leray–Schauder and Krasnosel'skii's fixed point technique. Vong [13] studied fractional DEs with singularity and non-local boundary conditions by using a Schauder

fixed point approach and using upper-lower solution techniques. Zhang et al. [14] investigated positive solutions of a multipoint BVP with singularity and applied the results to a specific example to illuminate the results. Khan et al. [15] studied a nonlinear singular fractional DE with singularity and p -Laplacian for the EU of solutions and stability analysis. They also presented an important application of their work.

Hybrid fractional DEs were studied by several authors for the EU of solutions with the help of different mathematical procedures. For instance, Kumam et al. [16] highlighted the study of EU of solutions for a coupled system of high order hybrid functional DEs using a classical fixed point approach by Krasnosel’skii. Chasreechai et al. [17] explored a hybrid fractional sum-difference initial and BVP in a Banach algebra and presented an instructive example.

Some authors worked on the EU of solutions for fractional DEs with p -Laplacian operator. For instance; Li [18] studied a fractional DE for the EU of positive solutions having integral boundary conditions and the nonlinear p -Laplacian operator. Wang [19] examined the EU of positive solutions for a class of mixed fractional BVP and p -Laplacian. Khan et al. [1] studied a nonlinear fractional DE with p -Laplacian operator for the EU of solutions and a vital application was presented. On the other hand, existence and uniqueness of delay fractional differential equations were first investigated in [20, 21] one decade ago. For more existence and uniqueness results in fractional calculus one may refer to [22–25].

From the literature we note that no such investigation to p -Laplacian operator in the frame of delay fractional DEs with singularity has been considered. Therefore, inspired by the aforementioned work we investigate the following proposal for the existence of a positive solution (EPS) and stability analysis:

$$\begin{cases} \mathcal{D}^\beta [\Phi_p[\mathcal{D}^{\vartheta_0}[x(t) - y_2^*(t, x(t))]]] = -\mathcal{A}^*(t)y_1^*(t, x(t - \tau)), \\ (\Phi_p[\mathcal{D}^{\vartheta_0}x(t) - y_2^*(t, x(t))])|_{t=0} = 0 = (\Phi_p[\mathcal{D}^{\vartheta_0}x(t) - y_2^*(t, x(t))])'|_{t=0}, \\ x(0) = 0 = x'(1), \quad \mathcal{I}^{2-\vartheta_0}[x(t) - y_2^*(t, x(t))]|_{t=0} = 0, \end{cases} \tag{1.1}$$

where $\beta \in (1, 2]$, $\vartheta_0 \in (2, 3]$ and y_1^* , \mathcal{W} are continuous but singular at some points. The fractional derivatives \mathcal{D}^β and $\mathcal{D}^{\vartheta_0}$ are taken in the Caputo sense and in the Riemann–Liouville sense, respectively, $\Phi_p(r) = |r|^{p-2}r$ is a nonlinear Φ_p -operator satisfying $1/p + 1/q = 1$ and $\Phi_p^{-1} = \Phi_q$. By the positive solution $x(t)$ of the suggested fractional DE (1.1) we mean $x(t) > 0$ for $t \in (0, 1]$, satisfying (1.1). Our projected p -Laplacian delay fractional DEs with singularity operator Φ_p is more general than the problems considered in [26–28].

In this paper, we are involved in the study of EU of solutions and stability for the fractional DEs with operator p -Laplacian relating time and space singularity and $\tau > 0$ delay. To the best of our knowledge, in the literature there is no article investigating the mentioned problem. In order to study this problem, we will change it to fractional integral form with the help of the Green function $\mathcal{G}^{\vartheta_0}(t, s)$. Throughout the investigation, the monotonicity properties of the Green function will be considered in the given interval $(1, 2]$. Further, using fixed point theorems the EU of solutions will be demonstrated and the stability will be checked. As an application, we give and analyse an example. The reader may further consider multiplicity results using different definitions of the fractional derivatives. We take a help of in our work from [29–41].

Definition 1.1 The Riemann–Liouville fractional integral of a function ψ of order $y_1 > 0$, $\psi : (0, +\infty) \rightarrow \mathbb{R}$, is given by

$$\mathcal{I}^{y_1} \psi(t) = \frac{1}{\Gamma(y_1)} \int_0^t (t-s)^{y_1-1} \psi(s) ds,$$

where for $\text{Re}(y_1) > 0$ we have

$$\Gamma(y_1) = \int_0^{+\infty} e^{-s} s^{y_1-1} ds.$$

Definition 1.2 The fractional order derivative in the Caputo sense for a continuous function $\psi(t) : (0, +\infty) \rightarrow \mathbb{R}$, is given as

$$\mathcal{D}^{\vartheta_0} \psi(t) = \frac{1}{\Gamma(k-y_1)} \int_0^t (t-s)^{k-y_1-1} \psi^{(k)}(s) ds,$$

for $k = [y_1] + 1$, where $[y_1]$ is integer part of y_1 , such that the integral is well defined on $(0, +\infty)$ interval.

Lemma 1.1 ([29]) For a fractional order $y_1 \in (n-1, n]$, $\psi \in C^{n-1}$, the following is satisfied:

$$\mathcal{I}^{y_1} \mathcal{D}_1^y \psi(t) = \psi(t) + m_0 + m_1 t + m_2 t^2 + \dots + m_{n-1} t^{n-1},$$

for the $m_k \in \mathbb{R}$ for $k = 0, 1, 2, \dots, n-1$.

We utilize the well-known Guo–Krasnosel’skii theorem for the existence of positive solution.

Theorem 1.2 ([42, 43]) Consider a Banach space \mathcal{Y} and let $P \in \mathcal{Y}$ be a cone. Suppose that $\mathcal{W}_1, \mathcal{W}_2$ are two bounded subsets of \mathcal{Y} such that $0 \in \mathcal{W}_1, \overline{\mathcal{W}_1} \subset \mathcal{W}_2$, and the operator $\mathcal{F}^* : P \cap (\overline{\mathcal{W}_2} \setminus \mathcal{W}_1) \rightarrow P$ be continuous such that

$$(\mathcal{N}_1) \quad \|\mathcal{F}^* x\| \leq \|x\| \text{ if } x \in P \cap \partial \mathcal{W}_1 \text{ and } \|\mathcal{F}^* x\| \geq \|x\| \text{ if } x \in P \cap \partial \mathcal{W}_2, \text{ or}$$

$$(\mathcal{N}_2) \quad \|\mathcal{F}^* x\| \geq \|x\| \text{ if } x \in P \cap \partial \mathcal{W}_1 \text{ and } \|\mathcal{F}^* x\| \leq \|x\| \text{ if } x \in P \cap \partial \mathcal{W}_2.$$

Then \mathcal{F}^* has a fixed point in $P \cap (\overline{\mathcal{W}_2} \setminus \mathcal{W}_1)$.

Lemma 1.3 ([15]) Let Φ_p be the nonlinear Φ_p -operator. Then

(1) for $1 < p \leq 2$, $\gamma_1^* \gamma_2 > 0$ and $|\gamma_1^*|, |\gamma_2^*| \geq \rho > 0$, then

$$|\Phi_p(\gamma_1^*) - \Phi_p(\gamma_2^*)| \leq (p-1) \rho^{p-2} |\gamma_1^* - \gamma_2^*|.$$

(2) If $p > 2$, and $|\gamma_1^*|, |\gamma_2^*| \leq \rho^*$, then

$$|\Phi_p(\gamma_1^*) - \Phi_p(\gamma_2^*)| \leq (p-1) \rho^{*p-2} |\gamma_1^* - \gamma_2^*|.$$

2 Green function and its properties

Theorem 2.1 For $\beta \in (1, 2]$, $\vartheta \in (2, 3]$ a function $y_1^* \in C[0, 1]$ satisfies (1.1) if and only if

$$x(t) = y_2^*(t, x(t)) + \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s-\zeta)^{\beta-1} f(\zeta) y_1^*(\zeta, x(\zeta)) \right) d\zeta ds, \quad (2.1)$$

where $\mathcal{G}^{\vartheta_0}(t, s)$ is a Green's function given by

$$\mathcal{G}^{\vartheta_0}(t, s) = \begin{cases} \frac{-(t-s)^{\vartheta_0-1}}{\Gamma(\vartheta_0)} + \frac{t^{\vartheta_0-1}(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} & s \leq t, \\ \frac{t^{\vartheta_0-1}(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)}, & s \geq t. \end{cases} \tag{2.2}$$

Proof If we apply the fractional integral operator \mathcal{I}^β on (1.1) and make use of Lemma 1.1, then problem (1.1) becomes

$$\Phi_p[\mathcal{D}^\beta x(t) - y_2^*(t, x(t))] = -\mathcal{I}^\beta [f(t)y_1^*(t, x(t-\tau))] + c_1 + c_2t. \tag{2.3}$$

The conditions $(\Phi_p(\mathcal{D}^{\vartheta_0}[x(t) - y_2^*(t, x(t))])|_{t=0} = 0 = (\Phi_p(\mathcal{D}^{\vartheta_0}[x(t) - y_2^*(t, x(t))])'|_{t=0}$, imply that $c_1 = c_2 = 0$. Also, we get

$$\Phi_p(\mathcal{D}^{\vartheta_0}[x(t) - y_2^*(t, x(t))]) = -\mathcal{I}^\alpha [f(t)y_1^*(t, x(t-\tau))]. \tag{2.4}$$

From (2.4), we further have

$$\mathcal{D}^{\vartheta_0}[x(t) - y_2^*(t, x(t))] = -\Phi_q(\mathcal{I}^\beta [f(t)y_1^*(t, x(t-\tau))] dt). \tag{2.5}$$

Applying integral operator of fractional order $\mathcal{I}^{\vartheta_0}$ on (2.5) and using Lemma 1.1 again, we have

$$\begin{aligned} x(t) - y_2^*(t, x(t)) &= -\mathcal{I}^{\vartheta_0}(\Phi_q(\mathcal{I}^\beta [f(t)y_1^*(t, x(t-\tau))])) \\ &\quad + c_1^*t^{\vartheta_0-1} + c_2^*t^{\vartheta_0-2} + c_3^*t^{\vartheta_0-3}, \end{aligned} \tag{2.6}$$

where $x(0) = 0$ implies that $c_3^* = 0$, $\mathcal{I}^{2-\vartheta_0}[x(t) - y_2^*(t, x(t))]|_{t=0} = 0$ implies that $c_2^* = 0$ Now for c_1^* , using condition $x'(1) = 0$, we have

$$c_1^* = \frac{1}{\vartheta_0 - 1} \mathcal{I}^{\vartheta_0-1}(\Phi_q(\mathcal{I}^\beta [f(t)y_1^*(t, x(t-\tau))]))|_{t=1}. \tag{2.7}$$

Putting the values of c_1^* and c_2^* in (2.6), we get

$$\begin{aligned} x(t) &= y_2^*(t, x(t)) - \mathcal{I}^{\vartheta_0}(\Phi_q(\mathcal{I}^\beta [f(t)y_1^*(t, x(t-\tau))])) \\ &\quad + \frac{t^{\vartheta_0-1}}{\vartheta_0 - 1} \mathcal{I}^{\vartheta_0-1}(\Phi_q(\mathcal{I}^\beta [f(t)y_1^*(t, x(t-\tau))]))|_{t=1} \\ &= y_2^*(t, x(t)) + \left[-\int_0^t \frac{(t-s)^{\vartheta_0-1}}{\Gamma(\vartheta_0)} + t^{\vartheta_0-1} \int_0^1 \frac{(1-s)^{\vartheta_0-1}}{\Gamma(\vartheta_0)} \right] \\ &\quad \Phi_q\left(\frac{1}{\Gamma(\vartheta_0)} \int_0^s (s-\zeta)^{\beta-1} [f(\zeta)y_1^*(\zeta, x(\zeta-\tau))] d\zeta\right) ds \\ &= y_2^*(t, x(t)) \\ &\quad + \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s-\zeta)^{\beta-1} [f(t)y_1^*(\zeta, x(\zeta-\tau))] d\zeta\right) ds, \end{aligned} \tag{2.8}$$

where $\mathcal{G}^{\vartheta_0}(t, s)$ is well defined by (2.2). □

Lemma 2.2 *The function $\mathcal{G}^{\vartheta_0}(t, s)$ defined by Eq. (2.2), satisfies:*

- (\mathcal{N}_1) $0 < \mathcal{G}^{\vartheta_0}(t, s)$ for all $s, t \in (0, 1)$;
- (\mathcal{N}_2) the function $\mathcal{G}^{\vartheta_0}(t, s)$ is increasing and $\mathcal{G}^{\vartheta_0}(1, s) = \max_{t \in [0,1]} \mathcal{G}^{\vartheta_0}(t, s)$ and
- (\mathcal{N}_3) $\mathcal{G}^{\vartheta_0}(t, s) \geq t^{\vartheta_0-1} \max_{t \in [0,1]} \mathcal{G}^{\vartheta_0}(t, s)$ for $s, t \in (0, 1)$.

Proof To evaluate the proof of (\mathcal{N}_1) , we assume two cases.

Case 1: For $s \leq t$, consider

$$\begin{aligned} \mathcal{G}^{\vartheta_0}(t, s) &= -\frac{(t-s)^{\vartheta_0-1}}{\Gamma(\vartheta_0)} + t^{\vartheta_0-1} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} \\ &= -t^{\vartheta_0-1} \frac{(1-\frac{s}{t})^{\vartheta_0-1}}{\Gamma(\vartheta_0)} + t^{\vartheta_0-1} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} \\ &\geq -t^{\vartheta_0-1} \frac{(1-s)^{\vartheta_0-1}}{\Gamma(\vartheta_0)} + t^{\vartheta_0-1} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} \geq 0. \end{aligned} \tag{2.9}$$

Case 2: For $t \leq s$, we have

$$\mathcal{G}^{\vartheta_0}(t, s) = t^{\vartheta_0-1} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} > 0. \tag{2.10}$$

With (2.9) and (2.10), it is proved that $\mathcal{G}^{\vartheta_0}(t, s) > 0$ for all $0 < s, t < 1$.

For (\mathcal{N}_2) , we consider that:

Case 1 when $s \leq t$, we proceed to find

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{G}^{\vartheta_0}(t, s) &= -(\vartheta_0 - 1) \frac{(t-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} + (\vartheta_0 - 1)t^{\vartheta_0-2} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} \\ &= -t^{\vartheta_0-2} \frac{(1-\frac{s}{t})^{\vartheta_0-2}}{\Gamma(\vartheta_0 - 1)} + t^{\vartheta_0-2} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0 - 1)} \\ &= -t^{\vartheta_0-2} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0 - 1)} + t^{\vartheta_0-2} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0 - 1)} > 0. \end{aligned} \tag{2.11}$$

Case 2: For $t \leq s$, we evaluate

$$\frac{\partial}{\partial t} \mathcal{G}^{\vartheta_0}(t, s) = (\vartheta_0 - 1)t^{\vartheta_0-2} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} > 0. \tag{2.12}$$

With (2.11), (2.12), we have $\frac{\partial}{\partial t} \mathcal{G}^{\vartheta_0}(t, s) > 0$ for $s, t \in (0, 1)$. This implies that the Green’s function $\mathcal{G}^{\vartheta_0}(t, s)$ is increasing with respect to t . Thus, for $t \geq s$, we arrive at

$$\max_{t \in [0,1]} \mathcal{G}^{\vartheta_0}(t, s) = \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} = \mathcal{G}^{\vartheta_0}(1, s). \tag{2.13}$$

Similarly, for $s \geq t$, we have

$$\max_{t \in [1,2]} \mathcal{G}^{\vartheta_0}(t, s) = \frac{(1-s)^{\vartheta_0-1}}{\Gamma(\vartheta_0 - 1)} + \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0 - 1)} = \mathcal{G}^{\vartheta_0}(1, s). \tag{2.14}$$

For (\mathcal{N}_3) , we presume two cases.

Case 1: For $t \geq s$, then

$$\begin{aligned} \mathcal{G}^{\vartheta_0}(t,s) &= -\frac{(t-s)^{\vartheta_0-1}}{\Gamma(\vartheta_0)} + t^{\vartheta_0-1} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} \\ &\geq -\frac{t^{\vartheta_0-1}(1-\frac{s}{t})^{\vartheta_0-1}}{\Gamma(\vartheta_0)} + \frac{t^{\vartheta_0-1}(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} \\ &\geq -\frac{t^{\vartheta_0-1}(1-s)^{\vartheta_0-1}}{\Gamma(\vartheta_0)} + \frac{t^{\vartheta_0-1}(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} \geq 0. \end{aligned} \tag{2.15}$$

Case 2: For $s \geq t$, then

$$\mathcal{G}^{\vartheta_0}(t,s) = t^{\vartheta_0-1} \frac{(1-s)^{\vartheta_0-2}}{\Gamma(\vartheta_0)} = t^{\vartheta_0-1} \max_{t \in [0,1]} \mathcal{G}^{\vartheta_0}(t,s) = t^{\vartheta_0-1} \mathcal{G}^{\vartheta_0}(1,s). \tag{2.16}$$

By (2.15) and (2.16), the proof of (\mathcal{N}_3) is accomplished. □

3 Existence results

Consider a Banach space $\mathcal{Y} = \mathcal{C}[0, 1]$ with a norm $\|x\| = \max_{t \in [0,1]} \{|x(t)| : x \in \mathcal{Y}\}$ and let P be a cone containing non-negative functions in the space \mathcal{Y} , where $P = \{x \in \mathcal{Y} : x(t) \geq t^\epsilon \|x\|, t \in [0, 1]\}$. Let $\mathcal{W}(r) = \{x \in P : \|x\| < r\}$, $\partial \mathcal{W}(r) = \{x \in P : \|x\| = r\}$.

With the help of Theorem 2.1, an alternate form of (1.1) is

$$\begin{aligned} x(t) &= y_2^*(t, x(t)) \\ &\quad + \int_0^1 \mathcal{G}^{\vartheta_0}(t,s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s-\zeta)^{\beta-1} [\mathcal{A}^*(t) y_1^*(t, x(t-\tau))] d\zeta ds \right). \end{aligned} \tag{3.1}$$

Define $\mathcal{F}^* : P \setminus \{0\} \rightarrow \mathcal{Y}$ by

$$\begin{aligned} \mathcal{F}^* x(t) &= y_2^*(t, x(t)) \\ &\quad + \int_0^1 \mathcal{G}^{\vartheta_0}(t,s) \Phi_q \left(\int_0^1 (s-\zeta)^\beta (s,\zeta) [\mathcal{A}^*(t) y_1^*(t, x(t-\tau))] dt \right) ds. \end{aligned} \tag{3.2}$$

With the help of Theorem 2.1, the solution of (1.1), i.e., $x(t)$ is equivalent to a fixed point of \mathcal{F} , that is,

$$x(t) = \mathcal{F}^* x(t). \tag{3.3}$$

Presume the following conditions:

- (\mathcal{P}_1) $y^* : ((0, 1) \times (0, +\infty)) \rightarrow [0, +\infty)$ is continuous.
- (\mathcal{P}_2) $A : (0, 1) \rightarrow [0, +\infty)$ is discontinuous on $(0, 1)$ and non vanishing and $\|A\| = \max_{t \in [0,1]} |A^*(t)| < +\infty$.
- (\mathcal{P}_3) For $a_1, a_2, \chi_1^*, \chi_2^*$ positive constants and $k_1 \in [0, 1]$, the functions y_1^*, y_2^* satisfy

$$\begin{aligned} |y_1^*(t, x(t))| &\leq \Phi_p(a_1 |x(t)|^{k_1} + \chi_1^*); \\ |y_2^*(t, x(t))| &\leq \phi_p(a_2 |x(t)|^{k_2} + \chi_2^*). \end{aligned}$$

(\mathcal{P}_4) For a constant value $\lambda_1, \lambda_2 > 0$ and $u, v \in \mathcal{Y}$, the functions y_1^*, y_2^* satisfy

$$\begin{aligned} |y_1^*(t, x(t)) - y_1^*(t, v(t))| &\leq \lambda_1 |x(t) - v(t)|, \\ |y_2^*(t, x(t - \tau)) - y_2^*(t, v(t - \tau))| &\leq \lambda_2 |x(t) - v(t)|. \end{aligned}$$

Theorem 3.1 *Assume that conditions (\mathcal{P}_1)–(\mathcal{P}_3) are satisfied. Then \mathcal{F}^* is completely continuous operator.*

Proof For any $x \in \overline{\mathcal{W}(r_2)} \setminus \mathcal{W}(r_1)$, we have from Lemma 2.2 and (3.2) that

$$\begin{aligned} \mathcal{F}^*x(t) &= y_2^*(t, x(t)) + \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \left(\frac{1}{\Gamma(\vartheta_0)} \int_0^s (s - \zeta)^{\vartheta_0 - 1} [\mathcal{A}^*(t) y_1^*(t, x(t - \tau))] dt \right) ds \\ &\leq \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q \left(\frac{1}{\Gamma(\vartheta_0)} \int_0^s (s - \zeta)^{\vartheta_0 - 1} [A(t) y_1^*(t, x(t - \tau))] dt \right) ds, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \mathcal{F}^*x(t) &= y_2^*(t, x(t)) + \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\vartheta_0 - 1} [\mathcal{A}^*(t) y_1^*(t, x(t - \tau))] dt \right) ds \\ &\geq y_2^*(t, x(t)) \\ &\quad + t^{\vartheta_0 - 1} \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta - 1} [\mathcal{A}^*(t) y_1^*(t, x(t - \tau))] dt \right) ds. \end{aligned} \tag{3.5}$$

With help of (3.4) and (3.5), we proceed to find

$$\mathcal{F}^*x(t) \geq t^{\vartheta_0 - 1} \|\mathcal{F}^*x(t)\|, \quad t \in [0, 1]. \tag{3.6}$$

This implies $\mathcal{F}^* : \overline{\mathcal{W}(r_2)} \setminus \mathcal{W}(r_1) \rightarrow P$. Now, in order to show that \mathcal{F}^* is continuous, we prove $\|\mathcal{F}^*(x_n) - \mathcal{F}^*(x)\| \rightarrow 0$ as $n \rightarrow \infty$, given by

$$\begin{aligned} &\|\mathcal{F}^*x_n(t) - \mathcal{F}^*x(t)\| \\ &= \left| y_2^*(t, x_n(t)) - y_2^*(t, x(t)) \right. \\ &\quad + \left(\int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \right) \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta - 1} [\mathcal{A}^*(t) y_1^*(t, x_n(t - \tau))] dt \right) ds \\ &\quad \left. - \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta - 1} [\mathcal{A}^*(t) y_1^*(t, x(t - \tau))] dt \right) ds \right| \\ &\leq \int_0^1 |\mathcal{G}^{\vartheta_0}(t, s)| \left| \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta - 1} [\mathcal{A}^*(t) y_1^*(t, x_n(t - \tau))] dt \right) \right. \\ &\quad \left. - \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta - 1} [\mathcal{A}^*(t) y_1^*(t, x(t - \tau))] dt \right) \right| ds. \end{aligned} \tag{3.7}$$

With the help of (3.7), and continuity of y_1^*, y_2^* we have $|\mathcal{F}^*x_n(t) - \mathcal{F}^*x(t)| \rightarrow 0$ as $n \rightarrow +\infty$, which shows that \mathcal{F}^* is continuous. Now, for the uniformly boundedness of \mathcal{F}^* , by (3.2)

and (P_2) , we get

$$\begin{aligned}
 |\mathcal{F}^*x(t)| &= \left| y_2^*(t, x(t)) \right. \\
 &\quad \left. + \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [\mathcal{A}^*(t)y_1^*(t, x(t - \tau))] dt \right) ds \right| \\
 &= |y_2^*(t, x(t))| \\
 &\quad + \int_0^1 |\mathcal{G}^{\vartheta_0}(t, s)| \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [|\mathcal{A}^*(t)y_1^*(t, x(t - \tau))|] dt \right) ds \\
 &\leq |y_2^*(t, x(t))| \\
 &\quad + \int_0^1 |\mathcal{G}^{\vartheta_0}(1, s)| \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} \|A\| \Phi_p(a_1 \|x\|^{k_1} + \chi^* y^*) dt \right) ds \\
 &\leq \left(\frac{2}{\Gamma(\vartheta_0 + 1)} + \frac{1}{\Gamma(\vartheta_0)} \right) \left[\frac{1}{\Gamma(\beta + 1)} \right]^{q-1} \|\mathcal{A}\|^{q-1} (a_1 \|x\|^{k_1} + \chi_1^*) \\
 &\quad + a_2 \|x\|_2^k + \chi^* \\
 &= \Delta_1 \|\mathcal{A}\|^{q-1} (a_1 \|x\| + \chi^*) + a_2 \|x\|_2^k + \chi_2^*, \tag{3.8}
 \end{aligned}$$

where $\Delta_1 = \left(\frac{1}{\Gamma(\vartheta_0 + 1)} + \frac{1}{(\vartheta_0 - 1)\Gamma(\vartheta_0)} \right) \left[\frac{1}{\Gamma(\vartheta_0 + 1)} \right]^{q-1}$. By (3.8), the operator $\mathcal{F}^* : \overline{\mathcal{W}(r_2)} \setminus \mathcal{W}(r_1)$ is uniformly bounded. Now for the equicontinuity of the operator \mathcal{F}^* , by (P_3) , Theorem 2.1 and (3.2), for any $t_1, t_2 \in [0, 1]$, we have

$$\begin{aligned}
 &|\mathcal{F}^*x(t_1) - \mathcal{F}^*x(t_2)| \\
 &\leq |y_2^*(t_1, x(t_1)) - y_2^*(t_2, x(t_2))| \\
 &\quad + \left| \int_0^1 \mathcal{G}^{\vartheta_0}(t_1, s) \Phi_q \left(\frac{1}{\Gamma(\vartheta_0)} \int_0^s (s - \zeta)^{\vartheta_0-1} [\mathcal{A}^*(t)y_1^*(t, x(t - \tau))] dt \right) ds \right. \\
 &\quad \left. - \int_0^1 \mathcal{G}^{\vartheta_0}(t_2, s) \Phi_q \left(\frac{1}{\Gamma(\vartheta_0)} \int_0^s (s - \zeta)^{\vartheta_0-1} [\mathcal{A}^*(t)y_1^*(t, x(t - \tau))] dt \right) ds \right| \\
 &\leq \int_0^1 |\mathcal{G}^{\vartheta_0}(t_1, s) - \mathcal{G}^{\vartheta_0}(t_2, s)| \\
 &\quad \times \Phi_q \left(\frac{1}{\Gamma(\vartheta_0)} \int_0^s (s - \zeta)^{\vartheta_0-1} \|A\| \Phi_p(a_1 \|x\|^{k_1} + \chi_1^*) d\zeta \right) ds \\
 &\leq \left(\frac{|t_1^{\vartheta_0} - t_2^{\vartheta_0}|}{\Gamma(\vartheta_0 + 1)} + \frac{|t_1^{\vartheta_0-1} - t_2^{\vartheta_0-1}|}{\Gamma(\vartheta_0)} \right) \left[\frac{1}{\Gamma(\beta + 1)} \right]^{q-1} \|\mathcal{A}\|^{q-1} (a_1 \|x\|^{k_1} + \chi_1^*). \tag{3.9}
 \end{aligned}$$

As $t_1 \rightarrow t_2$, we find that (3.9) goes to zero. Hence $\mathcal{F}^*(\overline{\mathcal{W}(r_2)} \setminus \mathcal{W}(r_1))$ is an equicontinuous operator and by the Arzela–Ascoli theorem $\mathcal{F}^*(\overline{\mathcal{W}(r_2)} \setminus \mathcal{W}(r_1))$ is compact. This proves that \mathcal{F}^* is compact in $\overline{\mathcal{W}(r_2)} \setminus \mathcal{W}(r_1)$. Consequently; $\mathcal{F}^* : \overline{\mathcal{W}(r_2)} \setminus \mathcal{W}(r_1) \rightarrow P$ is completely continuous. \square

Here, we define height for $y^*(t, x(t))$ for $r > 0$, and

$$\begin{cases} \Phi_{\max}(t, r) = \max_{t \in (0,1)} \{y_1^*(t, x(t - \tau)) : t^{\vartheta_0-1}r \leq x \leq r\}, \\ \Phi_{\min}(t, r) = \min_{t \in (0,1)} \{y_1^*(t, x(t - \tau)) : t^{\vartheta_0-1}r \leq x \leq r\}. \end{cases} \tag{3.10}$$

Theorem 3.2 *Assume that (\mathcal{P}_1) – (\mathcal{P}_3) hold true and there exist $a, b \in \mathbb{R}^+$ such that (\mathcal{W}_1) $a \leq \|y_2^*(t, x(t))\| + \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q\left(\frac{1}{\Gamma(\vartheta_0)} \int_0^s (s - \zeta)^{\vartheta_0-1} \Phi_{\min}(\zeta, a) dt\right) ds < +\infty$ and $\|y_2^*(t, x(t))\| + \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [f(\zeta) \Phi_{\max}(\zeta, b)] d\zeta\right) ds \leq b$, or (\mathcal{W}_2) $\|y_2^*(t, x(t))\| + \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q\left(\frac{1}{\Gamma(\vartheta_0)} \int_0^s (s - \zeta)^{\vartheta_0-1} [\Omega(\zeta) \Phi_{\max}(\zeta, a)] dt\right) ds < a$ and $b \leq \|y_2^*(t, x(t))\| + \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q\left(\frac{1}{\Gamma(\vartheta_0)} \int_0^s (s - \zeta)^{\vartheta_0-1} [\Omega(\zeta) \Phi_{\min}(\zeta, b)] dt\right) ds < +\infty$, is satisfied. Then the fractional DE with operator Φ_p (1.1) has a positive solution $x \in P$ and $a \leq \|x\| \leq b$.*

Proof With no loss of generalization, consider the case (\mathcal{W}_1) . If $x \in \partial\mathcal{W}(a)$, then we have $\|x\| = a$ and $t^{\vartheta_0-1}a \leq x(t) \leq a, t \in [0, 1]$. With the help of (3.10), for $t \in (0, 1)$, we have $\Phi_{\min}(t, u) \leq y_1^*(t, x(t - \tau))$, which implies

$$\begin{aligned} \|\mathcal{F}^*x(t)\| &= \max_{t \in [0,1]} \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [\mathcal{A}^*(t)y^*(t, x(t - \tau))] dt\right) ds \\ &\geq t^{\vartheta_0-1} \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [f(\zeta)y^*(t, x(t - \tau))] dt\right) ds \\ &\geq \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [\mathcal{A}^*(t)\Phi_{\min}(t, a)] dt\right) ds \geq a \\ &= \|x\|. \end{aligned} \tag{3.11}$$

If $x(t) \in \partial\mathcal{W}(b)$, then $\|x\| = b$ and $t^{\vartheta_0-1}b \leq u \leq b$, for $0 \leq t \leq 1$. By (3.10), we get $\Phi_{\max}(t, u) \geq y^*(t, x(t - \tau))$, and we proceed to find

$$\begin{aligned} \|\mathcal{F}^*x(t)\| &= \max_{t \in [0,1]} [y_2^*(t, x(t)) \\ &\quad + \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [\mathcal{A}^*(t)y^*(t, x(t - \tau))] dt\right) ds \\ &\leq t^{\vartheta_0-1} \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [\mathcal{A}^*(t)y^*(t, x(t - \tau))] dt\right) ds \\ &\leq \|y_2^*(t, x(t))\| \\ &\quad + \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [f(\zeta)\Phi_{\max}(\zeta, b)] d\zeta\right) ds \\ &\leq b = \|x\|. \end{aligned} \tag{3.12}$$

By Lemma 1.2, \mathcal{F}^* has a fixed point say $x \in \overline{\mathcal{W}(b)} \setminus \mathcal{W}(a)$. Thus, we have $a \leq \|x^*\| \leq b$, which by Lemma 2.2 and Theorem 2.1 implies $x^*(t) \geq t^{\vartheta_0-1}\|x^*\| \geq at^{\vartheta_0-1} > 0$, for $t \in (0, 1)$. Thus x^* is an increasing positive solution

$$\begin{aligned} \frac{\partial}{\partial t}x^*(t) &= \frac{\partial}{\partial t}\mathcal{F}^*x(t) \\ &= \frac{\partial}{\partial t}y_2^*(t, x(t)) \\ &\quad + \int_0^1 \frac{\partial}{\partial t}\mathcal{G}^{\vartheta_0}(t, s) \Phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [\mathcal{A}^*(t)y^*(t, x^*(t - \tau))] dt\right) ds \\ &> 0. \end{aligned} \tag{3.13}$$

□

4 Stability analysis

Here, a stability analysis is presented for the fractional DE with nonlinear Φ_p -operator of the problem (1.1). The following definition is given on the basis of [1, 28] and related literature.

Definition 4.1 We say that (3.1) is Hyers–Ulam stable if for every $\lambda > 0$, there exists a constant value $D^* > 0$, such that the following holds true:

if

$$\left| x(t) - y_2^*(t, x(t)) - \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [\mathcal{A}^*(t) y_1^*(\zeta, x(\zeta - \tau))] d\zeta \right) ds \right| \leq \lambda, \tag{4.1}$$

there exists $u(t)$ satisfying that

$$u(t) = \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [\mathcal{A}^*(\zeta) y_1^*(\zeta, u(\zeta - \tau))] d\zeta \right) ds, \tag{4.2}$$

such that

$$|x(t) - u(t)| \leq \mathcal{L}^* \lambda^*. \tag{4.3}$$

Theorem 4.1 *The singular fractional DE with delay and Φ_p -operator, the problem (1.1) is Hyers–Ulam stable provided that (P_1) , (P_2) and (P_4) are satisfied.*

Proof By Theorem 3.2 and Definition 4.1, we assume that $x(t)$ is a solution of the fractional DE with delay (3.1) and $y(t)$ is an approximate solution and satisfying (4.2). Then we have

$$\begin{aligned} & |x(t) - u(t)| \\ &= |y_2^*(t, x(t)) - y_2^*(t, u(t))| \\ &+ \left| \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [\mathcal{A}^*(t) y_1^*(t, x(t - \tau))] dt \right) ds \right. \\ &\quad \left. - \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [\mathcal{A}^*(t) y_1^*(t, u(\zeta - \tau))] dt \right) ds \right| \\ &\leq (p - 1) \rho^{p-2} \|\mathcal{A}^*\|^{q-1} \left(\int_0^1 |\mathcal{G}^{\vartheta_0}(t, s)| \left| \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [y_1^*(t, x(t - \tau))] dt \right) \right. \right. \\ &\quad \left. \left. - \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s - \zeta)^{\beta-1} [y_1^*(t, u(t - \tau))] dt \right) \right| ds \right) \\ &\leq (p - 1) \rho^{p-2} \lambda_{y_1^*} \left(\frac{1}{\Gamma(\vartheta_0 + 1)} + \frac{1}{(\vartheta_0 - 1)\Gamma(\vartheta_0)} \right) \left[\frac{1}{\Gamma(\beta + 1)} \right]^{q-1} \\ &\quad \times \|\mathcal{A}\|^{q-1} \|x - u\|, \end{aligned} \tag{4.4}$$

where $\mathcal{L}^* = (p - 1) \rho^{p-2} \lambda_{y_1^*} \left(\frac{1}{\Gamma(\vartheta_0 + 1)} + \frac{1}{(\vartheta_0 - 1)\Gamma(\vartheta_0)} \right) \left[\frac{1}{\Gamma(\beta + 1)} \right]^{q-1} \|\mathcal{A}\|^{q-1}$. Hence (4.4) is Hyers–Ulam stable. Consequently, the singular fractional DE with delay and operator Φ_p (1.1) is Hyers–Ulam stable. \square

5 Illustrative example

In this section, an application of the results which have proved in Sects. 3 and 4, is provided.

Example 5.1 For $\psi_1(t, x(t - \tau)) = x^5(t) + \frac{-\tau+1}{\sqrt{x(t)}}$, $t \in [0, 1]$, $p = 3$, $q = 1.5$, $\beta = \vartheta_0 = 1.5$, $\tau = 0.3$, $\mathcal{A}(t) = \frac{1}{\sqrt{1-t}}$, $y_1^*(t, x(t)) = x^5(t) + \frac{1}{5\sqrt{x(t)}}$, $y_2^*(t, x(t)) = 0.002$, clearly $\mathcal{A} \in C((0, 1), [0, +\infty))$, $y_1^* \in C((0, 1) \times (0, +\infty), [0, +\infty))$. Presume a singular fractional DE with Φ_p -operator:

$$\begin{cases} \mathcal{D}^\beta [\Phi_p[\mathcal{D}^{\vartheta_0}(x(t) - y_2^*(t, x(t)))] + [\frac{1-\tau}{\sqrt{x(t)}} + x^5(t)] \frac{1}{\sqrt{1-t}}] = 0, \\ (\Phi_p[\mathcal{D}^\beta(x(t) - y_2^*(t, x(t)))]|_{t=0} = 0 = (\phi_p(\mathcal{D}^{\vartheta_0}(x(t) - y_2^*(t, x(t))))'|_{t=0}, \\ x(1) = 0 = x'(0). \end{cases} \tag{5.1}$$

We consider

$$\begin{aligned} \Phi_{\max}(t, r) &= \max \left\{ x^5 + \frac{1-\tau}{x^{\frac{3}{35}}} : t^{\frac{5}{3}}r \leq x \leq r \right\} \leq r^5 + \frac{1}{5} \frac{0.7}{t^{\frac{1}{7}}r^{\frac{3}{35}}}, \\ \Phi_{\min}(t, r) &= \min \left\{ x^5 + \frac{1-\tau}{x^{\frac{3}{35}}} : t^{\frac{5}{3}}r \leq x \leq r \right\} \geq t^5r^3 + \frac{1}{5r^{\frac{3}{35}}}, \end{aligned}$$

as height functions and

$$y_2^*(t, x(t)) = \begin{cases} 0 & \text{if } t = 0; \\ 0.002 & \text{if } t \in (0, 1); \\ 0 & \text{if } t = 1. \end{cases} \tag{5.2}$$

Then, for $t \in (0, 1)$, we have

$$\begin{aligned} &0.002 + \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s-\zeta)^{\beta-1} [\mathcal{A}(\zeta) \Phi_{\max}(\zeta, b)] d\zeta \right) ds \\ &= 0.002 + \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s-\zeta)^{\beta-1} [\mathcal{A}(\zeta) \Phi_{\max}(\zeta, 1)] d\zeta \right) ds \\ &\leq 0.002 + \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \left(\frac{(s-\zeta)^\beta}{\Gamma(\beta+1)} \frac{1}{\sqrt{1-\zeta}} \left(1 + \frac{1}{5\zeta^{\frac{1}{7}}} \right) \right)^{1.5} d\zeta ds \\ &= 0.535426 < 1, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &0.002 + \int_0^1 \mathcal{G}^{\vartheta_0}(t, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s-\zeta)^{\beta-1} [\mathcal{A}(\zeta) \psi_{\min}(\zeta, a)] d\zeta \right) ds \\ &= 0.002 + \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s-\zeta)^{\beta-1} \left[\mathcal{A}(\zeta) \psi_{\min} \left(\zeta, \frac{1}{1000} \right) \right] d\zeta \right) ds \\ &\geq 0.002 \\ &\quad + \int_0^1 \mathcal{G}^{\vartheta_0}(1, s) \Phi_q \left(\frac{1}{\Gamma(\beta)} \int_0^s (s-\zeta)^{\beta-1} \left[\frac{1}{\sqrt{1-\zeta}} \left(\zeta^5 \frac{1}{1000^3} + \frac{1000^{\frac{1}{3}}}{35} \right) \right] d\zeta \right) ds \\ &= 0.262296 > \frac{1}{1000}. \end{aligned} \tag{5.4}$$

With the help of Theorem 3.2, (5.1) has a solution u^* and satisfies $\frac{1}{1000} \leq \|x^*\| \leq 1$.

6 Conclusion

In this paper we have considered a general class of hybrid fractional DEs (1.1) involving time and space singularities and the nonlinear operator ϕ_p for the EU of solutions and HU-stability. For these objectives, we utilized the classical results and obtained an alternate integral formulation of the singular fractional DE (1.1). The integral equation is based on the Green's function. The Green's function was examined and proved a positive increasing function. Then with the help of fixed point approach, the existence and uniqueness of solutions were evaluated and then HU-stability was also explored. A comprehensive example was studied to deduce the applicability of the results. We suggest the reader to re-consider the problem for multiplicity and exponential stability. Also, it would be of interest to consider the existing results of this article for fractional differential operators of nonsingular kernels. The recent articles [44, 45], with the formulation of Green's type functions, are recommended in this direction.

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