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Variable coefficient KdV equation with time-dependent variable coefficient topographic forcing term and atmospheric blocking

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Abstract

This paper mainly focuses on solitary waves excited by topography with time-dependent variable coefficient. By making use of multiple scale expansion and multiple level approximation method, a variable coefficient KdV equation with variable coefficient topographic forcing term is derived from barotropic and potential vorticity equation on a beta-plane including topography effect. In the derivation, removing *y*-average trick, a higher order term of stream function including five arbitrary functions and forced topography is introduced. Taking the strict solution of the standard constant coefficient KdV equation as the initial value, the approximate analytical solution of the derived equation is obtained by means of homotopy analysis method. Based on the new equation and its analytical solution, some complicated and changeable atmospheric blocking phenomena might be explained when some functions are selected appropriately.

PACS Codes: 02.30.Jr; 05.45.Yv

Keywords: Atmospheric blocking; Time-dependent variable coefficient topographic forcing term; Higher order term including forcing topography; Approximate analytical solution

1 Introduction

Atmospheric blocking is an important large-scale weather phenomenon. When atmospheric blocking occurs at mid-high latitudes, it often causes extraordinary flood, extreme drought or extreme cold, and other abnormal phenomena. Usually an atmospheric blocking anticyclone has three types of patterns: monopole-type blocking, dipole-type blocking, and multi-pole type blocking. In the past decades, a large number of constant-coefficient nonlinear equations have been derived, for example, the famous Korteweg de Vries (KdV), modified KdV (mKdV), nonlinear Schrödinger (NLS) equations, and so on. However, analytical solutions of these deduced constant coefficient equations are impossible to describe the formation, development, and extinction of the complicated atmospheric blocking life cycle. The nonlinear variable coefficient equations can be used to explain atmospheric blocking phenomena. Ref. [1] used the derived variable coefficient KdV system to explain well the life cycle of a blocking system. In fact, blocking phenomena

are related to beta effect, energy dispersion, topography, and other forcing factors. Yang and Chen [2–4] derived Boussinesq-BO equation, generalized Boussinesq equation, and generalized inhomogeneous KdV-Burgers equation considering dissipation effect and topography. The topography plays very important roles in atmospheric blocking evolution. A series of research works about topography forcing effect have been carried out [5–8], most of which still involved constant coefficient nonlinear equations.

In this paper, combining variable coefficient thought with the effect of topographic forcing, starting from barotropic and potential vorticity equation on a beta-plane including topographic effect, a variable coefficient KdV equation with time-dependent variable coefficient forced term is derived.

At present, there are many analytical methods [9–15] for solving nonlinear partial differential equations, for example, q-homotopy analysis transform method [16–18], homotopy analysis method [19, 20], homotopy perturbation method [21], and so on. However, solving analytical solutions of variable coefficient equation with forced term is a long-standing difficulty. Consider the KdV equation with forced term in Ref. [22]

$$U_T + a(T)UU_X + b(T)U^2U_X + c(T)U_{XXX} + d(T)U_X + \gamma(T)U = \Gamma(T), \tag{1}$$

where the analytical solutions were obtained, and the interaction of multiple wave solutions was presented. In Ref. [23], the new exact solitary wave solution and periodic wave solution of Eq. (2) KdV-Burgers equation with forcing term were obtained.

$$u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx} = f(t). \tag{2}$$

Obviously, forcing terms of Eqs. (1)–(2) are all functions of time. But research into exact analytical solutions of KdV type equation with forced term about variable x or X is almost blank. The homotopy analysis method (HAM) [24] is very powerful, we select it to solve the approximate analytic solution of the derived equation in this paper.

The main purpose of this paper is to derive a new variable coefficient KdV equation with forced term and investigate the evolution process of atmospheric blocking. The structure of this paper is as follows. A new model is derived from a non-dimensional barotropic and potential vorticity equation on a beta-plane including topography effect in Sect. 2. In Sect. 3, an approximate analytic solution of the derived equation is obtained by HAM, and the evolution process of stream function field is investigated in different parameters. In Sect. 4, discussion is presented. Finally, our conclusions are given.

2 Derivation of variable-coefficient KdV equation with time-dependent variable coefficient topographic forcing term

In barotropic atmosphere, the known barotropic and potential vorticity equation including topography effect on a beta-plane can be written in the form [25]

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + J\left(\psi, \nabla^2\psi + h\right) + \beta\frac{\partial\psi}{\partial x} = 0, \tag{3}$$

where $\psi = \psi(x, y, t)$ is a non-dimensional stream function, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the horizontal Laplace operator, $J(a, b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ is the Jacobian of a and b, $\beta = (L^2/U)\beta_0$ and $\beta_0 =$

$(2\omega_0/a_0)\cos\varphi_0$, ω_0 is the angular frequency of the earth's rotation, a_0 is the radius of the earth, φ_0 is the latitude, $F = (L/R_0)^2$, U and L are the characteristic horizontal velocity and the characteristic horizontal length, respectively, R_0 is the Rossby deformation radius, $h = h(x,y)$ is non-dimensional topography height distribution.

In order to consider weakly nonlinear perturbations on a zonal flow, we introduce a small parameter $\varepsilon \ll 1$, then assume

$$\psi(x,y,t) = \Psi_0(y,t) + \varepsilon\psi'(x,y,t), \tag{4}$$

where the basic flow $\Psi_0(y,t)$ is a function about y, t, $\psi'(x,y,t)$ is a perturbation stream function.

Substituting Eq. (4) into Eq. (3) and omitting $'$ of $\psi'(x,y,t)$, we obtain the following equation for perturbation stream function:

$$\frac{\partial}{\partial t}(\Psi_{0yy} - F\Psi_0) + \varepsilon\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + \varepsilon\frac{\partial\psi}{\partial x}\Psi_{0yyy} + \varepsilon^2\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}$$

$$- \varepsilon\Psi_{0y}\frac{\partial\nabla^2\psi}{\partial x} - \varepsilon^2\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}$$

$$+ \varepsilon\frac{\partial\psi}{\partial x}\frac{\partial h}{\partial y} - \Psi_{0y}\frac{\partial h}{\partial x} - \varepsilon\frac{\partial\psi}{\partial y}\frac{\partial h}{\partial x} + \beta\varepsilon\frac{\partial\psi}{\partial x} = 0. \tag{5}$$

We adopt the following transform:

$$X = \varepsilon^{\frac{1}{2}}(x - c_0 t), \qquad T = \varepsilon^{\frac{3}{2}}t, \qquad h(x,y) = \varepsilon^2 H(X,y). \tag{6}$$

Suppose topography can be separated as

$$H(X,y) = H_1(X) + H_2(y), \tag{7}$$

where $H_2(y)$ is a function of latitude, $H_1(X)$ is a function about X.

According to Eq. (6), we obtain

$$\frac{\partial}{\partial x} = \varepsilon^{\frac{1}{2}}\frac{\partial}{\partial X}, \qquad \frac{\partial}{\partial t} = -\varepsilon^{\frac{1}{2}}c_0\frac{\partial}{\partial X} + \varepsilon^{\frac{3}{2}}\frac{\partial}{\partial T}, \qquad \frac{\partial^2}{\partial x^2} = \varepsilon\frac{\partial}{\partial X}. \tag{8}$$

Substitution of Eqs. (6)–(8) into Eq. (5) gives

$$\varepsilon^{\frac{3}{2}}\frac{\partial}{\partial T}(\Psi_{0yy} - F\Psi_0) + \varepsilon\left(-\varepsilon^{\frac{1}{2}}c_0\frac{\partial}{\partial X} + \varepsilon^{\frac{3}{2}}\frac{\partial}{\partial T}\right)\left(\varepsilon\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial y^2} - F\psi\right)$$

$$+ \varepsilon^{\frac{3}{2}}\Psi_{0yyy}\frac{\partial\psi}{\partial X} + \varepsilon^{\frac{5}{2}}\frac{\partial\psi}{\partial X}\frac{\partial}{\partial y}\left(\varepsilon\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial y^2}\right) - \varepsilon^{\frac{5}{2}}\frac{\partial\psi}{\partial y}\frac{\partial}{\partial X}\left(\varepsilon\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial y^2}\right)$$

$$- \varepsilon^{\frac{3}{2}}\Psi_{0y}\frac{\partial}{\partial X}\left(\varepsilon\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \varepsilon^{\frac{7}{2}}\frac{\partial\psi}{\partial X}\frac{\partial H_2}{\partial y} - \varepsilon^{\frac{7}{2}}\frac{\partial\psi}{\partial y}\frac{\partial H_1}{\partial X} - \varepsilon^{\frac{5}{2}}\Psi_{0y}\frac{\partial H_1}{\partial X}$$

$$+ \varepsilon^{\frac{3}{2}}\beta\frac{\partial\psi}{\partial X} = 0. \tag{9}$$

Equation (9) can be further simplified as follows:

$$
\frac{\partial}{\partial T}(\Psi_{0yy} - F\Psi_0) - (c_0 + \Psi_{0y})\frac{\partial}{\partial X}\frac{\partial^2 \psi}{\partial y^2} + (c_0 F + \Psi_{0yyy} + \beta)\frac{\partial \psi}{\partial X}
$$

$$
+ \varepsilon\left[-(c_0 + \Psi_{0y})\frac{\partial^3 \psi}{\partial X^3} + \frac{\partial}{\partial T}\left(\frac{\partial^2 \psi}{\partial y^2} - F\psi\right) \right.
$$

$$
\left. + \frac{\partial \psi}{\partial X}\frac{\partial}{\partial y}\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial X}\frac{\partial^2 \psi}{\partial y^2} - \Psi_{0y}\frac{\partial H_1}{\partial X} \right] + O(\varepsilon^2) = 0. \tag{10}
$$

The asymptotic expansion is imported for a perturbation stream function ψ in the form

$$
\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \cdots. \tag{11}
$$

Taking the substitution of Eq. (11) into Eq. (10) and requiring all the coefficients of different powers of ε to be zero yield

$$
\varepsilon^0 : (c_0 + \Psi_{0y})\frac{\partial}{\partial X}\frac{\partial^2 \psi_0}{\partial y^2} - (c_0 F + \Psi_{0yyy} + \beta)\frac{\partial \psi_0}{\partial X} = \frac{\partial}{\partial T}(\Psi_{0yy} - F\Psi_0), \tag{12}
$$

$$
\varepsilon^1 : (c_0 + \Psi_{0y})\frac{\partial}{\partial X}\frac{\partial^2 \psi_1}{\partial y^2} - (c_0 F + \Psi_{0yyy} + \beta)\frac{\partial \psi_1}{\partial X}
$$

$$
= -(c_0 + \Psi_{0y})\frac{\partial^3 \psi_0}{\partial X^3} + \frac{\partial}{\partial T}\left(\frac{\partial^2 \psi_0}{\partial y^2} - F\psi_0\right) + \frac{\partial \psi_0}{\partial X}\frac{\partial}{\partial y}\frac{\partial^2 \psi_0}{\partial y^2}
$$

$$
- \frac{\partial \psi_0}{\partial y}\frac{\partial}{\partial X}\frac{\partial^2 \psi_0}{\partial y^2} - \Psi_{0y}\frac{\partial H_1}{\partial X}. \tag{13}
$$

We suppose that Eq. (12) has the following variable separation solution:

$$
\psi_0 = G_0(y, T)A(X, T). \tag{14}
$$

Taking the substitution of Eq. (14) into Eq. (12) yields

$$
\Psi_{0yy} = F\Psi_0 + F_1(y), \tag{15}
$$

$$
(c_0 + \Psi_{0y})\frac{\partial^2 G_0}{\partial y^2} - (c_0 F + F\Psi_{0y} + F_{1y} + \beta)G_0 = 0, \tag{16}
$$

where $F_1(y)$ is an arbitrary integration function about y, F_{1y} is the first order differential to y.

In order to derive an arbitrary variable coefficient KdV equation with forced term, we introduce the following higher order term ψ_1 with forced term:

$$
\psi_1 = B_1(y, T)A^2 + B_2(y, T)A_{XX} + B_3(y, T) + B_4(y, T)\int A\, dX + B_5(y, T)H_1(X)
$$

$$
= B_1 A^2 + B_2 A_{XX} + B_3 + B_4 \int A\, dX + B_5 H_1, \tag{17}
$$

where B_i ($i = 1, 2, 3, 4, 5$) are arbitrary functions about y and T.

By virtue of Eq. (17), we obtain

$$\frac{\partial}{\partial X}\frac{\partial^2 \psi_1}{\partial y^2} = B_{1yy}2AA_X + B_{2yy}A_{XXX} + B_{4yy}A + B_{5yy}H_{1X}, \tag{18}$$

$$\frac{\partial \psi_1}{\partial X} = B_1 2AA_X + B_2 A_{XXX} + B_4 A + B_5 H_{1X}. \tag{19}$$

Suppose

$$F_2 = -(c_0 + \Psi_{0y})\frac{\partial^3 \psi_0}{\partial X^3} + \frac{\partial}{\partial T}\left(\frac{\partial^2 \psi_0}{\partial y^2} - F\psi_0\right)$$

$$+ \frac{\partial \psi_0}{\partial X}\frac{\partial}{\partial y}\frac{\partial^2 \psi_0}{\partial y^2} - \frac{\partial \psi_0}{\partial y}\frac{\partial}{\partial X}\frac{\partial^2 \psi_0}{\partial y^2} - \Psi_{0y}\frac{\partial H_1}{\partial X}. \tag{20}$$

Substituting Eqs. (14) and (16) into Eq. (20) leads to

$$\frac{\partial \psi_0}{\partial X}\frac{\partial}{\partial y}\frac{\partial^2 \psi_0}{\partial y^2} - \frac{\partial \psi_0}{\partial y}\frac{\partial}{\partial X}\frac{\partial^2 \psi_0}{\partial y^2} = AA_X G_0^2 \left(\frac{c_0 F + F\Psi_{0y} + F_{1y} + \beta}{c_0 + \Psi_{0y}}\right)_y, \tag{21}$$

$$F_2 = -(c_0 + \Psi_{0y})G_0 A_{XXX} + A_T(G_{0yy} - FG_0) + (G_{0yy} - FG_0)_T A$$

$$+ G_0^2\left(\frac{c_0 F + F\Psi_{0y} + F_{1y} + \beta}{c_0 + \Psi_{0y}}\right)_y AA_X - \Psi_{0y}H_{1X}$$

$$= -(c_0 + \Psi_{0y})G_0 A_{XXX} + \frac{F_{1y} + \beta}{c_0 + \Psi_{0y}}G_0 A_T + G_0^2\left(\frac{c_0 F + F\Psi_{0y} + F_{1y} + \beta}{c_0 + \Psi_{0y}}\right)_y AA_X$$

$$+ \left(\frac{F_{1y} + \beta}{c_0 + \Psi_{0y}}\right)_T A - \Psi_{0y}H_{1X}$$

$$= -(c_0 + \Psi_{0y})G_0 A_{XXX} + \frac{F_{1y} + \beta}{c_0 + \Psi_{0y}}G_0 A_T$$

$$+ \frac{(c_0 + \Psi_{0y})(F\Psi_{0yy} + F_{1yy}) - \Psi_{0yy}(c_0 F + F\Psi_{0y} + F_{1y} + \beta)}{(c_0 + \Psi_{0y})^2}G_0^2 AA_X$$

$$- \frac{\Psi_{0yT}(F_{1y} + \beta)G_0}{(c_0 + \Psi_{0y})^2}A - \Psi_{0y}H_{1X}. \tag{22}$$

Substituting Eqs. (18)–(19) and (22) into Eq. (13), we have

$$A_T + \left[\frac{(c_0 + \Psi_{0y})(F\Psi_{0yy} + F_{1yy}) - \Psi_{0yy}(c_0 F + F\Psi_{0y} + F_{1y} + \beta)}{(c_0 + \Psi_{0y})^2}G_0^2\right.$$

$$\left. - 2(c_0 + \Psi_{0y})B_{1yy} + 2(c_0 F + F\Psi_{0y} + F_{1y} + \beta)B_1\right]\frac{c_0 + \Psi_{0y}}{(F_{1y} + \beta)G_0}AA_X$$

$$- \left[(c_0 + \Psi_{0y})G_0 + (c_0 + \Psi_{0y})B_{2yy} - (c_0 F + F\Psi_{0y} + F_{1y} + \beta)B_2\right]\frac{c_0 + \Psi_{0y}}{(F_{1y} + \beta)G_0}A_{XXX}$$

$$- \left[\frac{\Psi_{0yT}(F_{1y} + \beta)}{(c_0 + \Psi_{0y})^2} + (c_0 + \Psi_{0y})B_{4yy} - (c_0 F + F\Psi_{0y} + F_{1y} + \beta)B_4\right]\frac{c_0 + \Psi_{0y}}{(F_{1y} + \beta)G_0}A$$

$$- \left[\Psi_{0y} + (c_0 + \Psi_{0y})B_{5yy} - (c_0 F\right.$$

$$\left. + F\Psi_{0y} + F_{1y} + \beta)B_5\right]\frac{c_0 + \Psi_{0y}}{(F_{1y} + \beta)G_0}H_{1X} = 0. \tag{23}$$

Therefore, when B_i $(i = 1, 2, 3, 4, 5)$ satisfy the following equations:

$$(c_0 + \Psi_{0y})^2(F\Psi_{0yy} + F_{1yy}) - \Psi_{0yy}(c_0 + \Psi_{0y})(c_0 F + F\Psi_{0y} + F_{1y} + \beta)G_0^2$$
$$- 2(c_0 + \Psi_{0y})^3 B_{1yy} + 2(c_0 + \Psi_{0y})^2(c_0 F$$
$$+ F\Psi_{0y} + F_{1y} + \beta)B_1 - a_1 G_0(c_0 + \Psi_{0y})(F_{1y} + \beta) = 0, \tag{24}$$

$$(c_0 + \Psi_{0y})^2 G_0 + (c_0 + \Psi_{0y})^2 B_{2yy} - (c_0 + \Psi_{0y})(c_0 F + F\Psi_{0y} + F_{1y} + \beta)B_2$$
$$+ a_2 G_0(F_{1y} + \beta) = 0, \tag{25}$$

$$\Psi_{0yT}(F_{1y} + \beta) + (c_0 + \Psi_{0y})^3 B_{4yy} - (c_0 + \Psi_{0y})^2(c_0 F + F\Psi_{0y} + F_{1y} + \beta)B_4$$
$$+ a_3 G_0(F_{1y} + \beta) = 0, \tag{26}$$

$$\Psi_{0y}(c_0 + \Psi_{0y}) + (c_0 + \Psi_{0y})^2 B_{5yy} - (c_0 + \Psi_{0y})(c_0 F + F\Psi_{0y} + F_{1y} + \beta)B_5$$
$$- a_4 G_0(F_{1y} + \beta) = 0, \tag{27}$$

where B_3 is an arbitrary function, we can obtain a new variable coefficient KdV equation with forcing term

$$\frac{\partial A}{\partial t} + a_1 A \frac{\partial A}{\partial X} + a_2 \frac{\partial^3 A}{\partial X^3} + a_3 A = a_4 \frac{\partial H_1}{\partial X}. \tag{28}$$

In Eqs. (24)–(28), $a_i = a_i(T)$ $(i = 1, 2, 3, 4)$ are arbitrary functions about T. Obviously, forcing term of Eq. (28) contains time-dependent variable coefficient $a_4(T)$ and topographic forcing term $\frac{\partial H_1}{\partial X}$. Consequently, Eq. (28) is called variable coefficient KdV equation with time-dependent variable coefficient topographic forcing term.

3 Approximate analytic solution of derived equation and complicated atmospheric blocking phenomena

In this section, we first solve approximate analytical solution of Eq. (28) by HAM in Ref. [24]. Equation (28) can be rewritten as

$$LA(X, T) + PA(X, T) + BA(X, T) = 0, \tag{29}$$

where $L = \frac{\partial}{\partial T}$ is a linear operator, $LA(X, T) = \frac{\partial A}{\partial T}$, P is an auxiliary linear operator, $PA(X, T) = a_2 \frac{\partial^3 A}{\partial X^3} + a_3 A$, B is a nonlinear operator, $BA(X, T) = \frac{a_1}{2} \frac{\partial A^2}{\partial X} - a_4 \frac{\partial H}{\partial X}$.

Taking variable coefficients as

$$
\begin{aligned}
a_1 &= c_1 \sin(T), & a_2 &= c_2, \\
a_3 &= c_3 \cos(T), & a_4 &= c_4,
\end{aligned} \tag{30}
$$

where c_1, c_2, c_3, c_4 are arbitrary constants, we take forced topography as

$$H(X) = n_0 \cos(mX), \tag{31}$$

where n_0, m are arbitrary constants.

We make use of the strict solution of the standard constant coefficient KdV equation in the solitary theory as initial value, namely

$$A_0(X, T) = 2k^2 \operatorname{sech}^2(kX), \tag{32}$$

where $k \neq 0$ is a constant.

When $m = 1$, we obtain

$$
\begin{aligned}
A_1(X, T) &= \hbar \int_0^T \left[PA_0(X, T) + BA_0(X, T) \right] d\tau \\
&= \hbar \big[\tanh(kX) \operatorname{sech}^4(kX) \left(48k^5 c_2 T - 8k^5 c_1 + 8k^5 c_1 \cos(T) \right) \\
&\quad + 2k^2 c_3 \operatorname{sech}^2(kX) \sin(T) + c_4 m n_0 \sin(Xm) T \\
&\quad - 16k^5 c_2 \tanh(kX) \operatorname{sech}^2(kX) T \big],
\end{aligned}
\tag{33}
$$

where $\hbar$ is an auxiliary parameter.

When $m > 1$, we obtain the following equations:

$$PA_{m-1}(X, T) = a_2 \frac{\partial^3 A_{m-1}(X, T)}{\partial X^3} + a_3 A_{m-1}(X, T), \tag{34}$$

$$BA_{m-1}(X, T) = \frac{a_1}{2} \frac{\partial}{\partial X} \left(\sum_{j=0}^{m-1} A_j(X, T) A_{m-1-j}(X, T) \right) - a_4 \frac{\partial H}{\partial X}, \tag{35}$$

$$A_m(X, T) = (1 + \hbar) A_{m-1}(X, T) + \hbar \int_0^t \left[PA_{m-1}(X, T) + BA_{m-1}(X, T) \right] d\tau. \tag{36}$$

Substituting Eqs. (33)–(35) into Eq. (36), we obtain

$$
\begin{aligned}
A_2(X, T) &= k^2 \hbar^2 \operatorname{sech}^2(kX) \big[c_3^2 - c_3^2 \cos^2(T) + 2c_3 \sin(T) + 4c_1 c_4 m n_0 \sin(Xm) \sin(T) \\
&\quad - 4c_1 c_4 m n_0 \sin(Xm) \cos(T) T + 64k^6 T^2 c_2^2 + 2c_3 \sin(T) \big] \\
&\quad + 1680 k^8 \hbar^2 c_2 \operatorname{sech}^8(kX) \left(c_1 T - 3T^2 c_2 - c_1 \sin(T) \right) \\
&\quad + 160 k^8 \hbar^2 c_2 \operatorname{sech}^6(kX) \left[13T^2 c_2 - 42c_1 T + 42c_1 \sin(T) \right] \\
&\quad + 4k^4 \hbar^2 \operatorname{sech}^4(kX) \big[c_1 c_3 T - 504 k^4 c_2^2 T^2 \\
&\quad + 103 k^4 c_1 c_2 T - 103 k^4 c_1 c_2 \sin(T) - 0.5 c_1 c_3 \\
&\quad \cdot \sin(2T) \big] + 2k^5 \hbar \tanh(kX) \operatorname{sech}^4(kX) \big[4c_1 \hbar \cos(T) - 4c_1 \\
&\quad + 24 c_2 T + 24 \hbar c_2 T + 4c_1 \cos(T) + 2\hbar c_1 c_3 T - 4\hbar c_1 c_3 \\
&\quad \cdot \sin(T) - 32 k^2 \hbar c_1 c_2 \sin(T) + c_1 c_3 \hbar \sin(2T) + 32 k^2 c_1 c_2 \hbar \cos(T) T \\
&\quad + 24 c_2 c_3 \hbar \sin(T) T \big] + 16 k^7 \hbar^2 c_1 \tanh(kX) \\
&\quad \cdot \operatorname{sech}^6(kX) \big[2c_1 \cos(T) - c_1 - c_1 \cos^2(T) + 12 c_2 \sin(T) - 12 c_2 \cos(T) T \big] \\
&\quad - 16 k^5 \hbar c_2 \tanh(kX) \operatorname{sech}^2(kX) \big[2T \\
&\quad - c_3 \hbar \sin(T) \big] + T c_4 \hbar^2 m n_0 \sin(mX) - c_3 c_4 \hbar^2 m n_0 \sin(mX)
\end{aligned}
$$

$$+ 2c_4\hbar mn_0\sin(mX)T - 0.5T^2c_2c_4\hbar^2m^4n_0\cos(mX)$$

$$+ c_3c_4\hbar^2mn_0\sin(mX)\cos(T) + c_3c_4\hbar^2mn_0\sin(mX)\sin(T)T. \tag{37}$$

Then the second-order HAM approximate analytical solution of Eq. (28) is as follows:

$$A(X,T) = A_0(X,T) + A_1(X,T) + A_2(X,T). \tag{38}$$

When the basic flow $\Psi_0(y,t)$ is taken to be

$$\Psi_0 = b_0 y + b_1, \tag{39}$$

where b_0, b_1 are arbitrary constants, and using Eqs. (15)–(16), we obtain

$$G_0(y,T) = R(T)\sin\left(\sqrt{-\frac{c_0 F + \beta}{b_0 + c_0}}\, y\right), \tag{40}$$

where $R(T)$ is an arbitrary function about T.

Substituting Eqs. (11) and (38)–(40) into Eq. (4) and returning the variables x and t, we obtain the first-order approximate solution of the original equation Eq. (3) in the form

$$
\begin{aligned}
\psi(x,y,t) &\approx b_0 y + b_1 + \varepsilon R\big(\varepsilon^{\frac{3}{2}}t\big)\sin\left(\sqrt{-\frac{c_0 F + \beta}{b_0 + c_0}}\, y\right)\Big\{2k^2\,\mathrm{sech}^2\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\\
&\quad + \hbar\Big[\tanh\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\\
&\quad \cdot \mathrm{sech}^4\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\big(48k^5 c_2\varepsilon^{\frac{3}{2}}t - 8k^5 c_1 + 8k^5 c_1\cos\big(\varepsilon^{\frac{3}{2}}t\big)\big)\\
&\quad + 2k^2 c_3\,\mathrm{sech}^2\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\sin\big(\varepsilon^{\frac{3}{2}}t\big) + c_4 mn_0\cdot\sin\big(m\sqrt{\varepsilon}(x - c_0 t)\big)\varepsilon^{\frac{3}{2}}t\\
&\quad - 16k^5 c_2\tanh\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\,\mathrm{sech}^2\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\varepsilon^{\frac{3}{2}}t\big]k^2\hbar^2\\
&\quad \cdot \mathrm{sech}^2\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\big[c_3^2 - c_3^2\cos^2\big(\varepsilon^{\frac{3}{2}}t\big)\\
&\quad + 2c_3\sin\big(\varepsilon^{\frac{3}{2}}t\big) + 4c_1 c_4 mn_0\sin\big(m\sqrt{\varepsilon}(x - c_0 t)\big)\sin\big(\varepsilon^{\frac{3}{2}}t\big)\\
&\quad - 4c_1 c_4 mn_0\sin\big(m\sqrt{\varepsilon}(x - c_0 t)\big)\cos\big(\varepsilon^{\frac{3}{2}}t\big)\\
&\quad \cdot \varepsilon^{\frac{3}{2}}t + 64k^6\varepsilon^3 t^2 c_2^2 + 2c_3\sin\big(\varepsilon^{\frac{3}{2}}t\big)\big]\\
&\quad + 1680k^8\hbar^2 c_2\,\mathrm{sech}^8\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\big(c_1\varepsilon^{\frac{3}{2}}t - 3T^2 c_2 - c_1\sin\big(\varepsilon^{\frac{3}{2}}t\big)\big)\\
&\quad + 160k^8\hbar^2 c_2\,\mathrm{sech}^6\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\big[13\varepsilon^3 t^2 c_2 - 42c_1\varepsilon^{\frac{3}{2}}t + 42c_1\sin\big(\varepsilon^{\frac{3}{2}}t\big)\big]\\
&\quad + 4k^4\hbar^2\,\mathrm{sech}^4\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\big[c_1 c_3\varepsilon^{\frac{3}{2}}t - 504k^4 c_2^2\cdot\varepsilon^3 t^2 + 103k^4 c_1 c_2\varepsilon^{\frac{3}{2}}t\\
&\quad - 103k^4 c_1 c_2\sin\big(\varepsilon^{\frac{3}{2}}t\big) - 0.5c_1 c_3\sin\big(2\varepsilon^{\frac{3}{2}}t\big)\big]\\
&\quad + 2k^5\hbar\tanh\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\,\mathrm{sech}^4\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\\
&\quad \cdot \big[4c_1\hbar\cos\big(\varepsilon^{\frac{3}{2}}t\big) - 4c_1 + 24\hbar c_2\varepsilon^{\frac{3}{2}}t + 4c_1\cos\big(\varepsilon^{\frac{3}{2}}t\big) + 2\hbar c_1 c_3\varepsilon^{\frac{3}{2}}t\\
&\quad - 4\hbar c_1 c_3\sin\big(\varepsilon^{\frac{3}{2}}t\big) - 32k^2\hbar c_1 c_2\sin\big(\varepsilon^{\frac{3}{2}}t\big) + c_1 c_3\hbar\\
&\quad \cdot \sin\big(2\varepsilon^{\frac{3}{2}}t\big) + 32k^2 c_1 c_2\hbar\cos\big(\varepsilon^{\frac{3}{2}}t\big)\varepsilon^{\frac{3}{2}}t + 24c_2 c_3\hbar\sin\big(\varepsilon^{\frac{3}{2}}t\big)\varepsilon^{\frac{3}{2}}t\big]\\
&\quad + 16k^7\hbar^2 c_1\tanh\big(k\sqrt{\varepsilon}(x - c_0 t)\big)\,\mathrm{sech}^6\big(k\sqrt{\varepsilon}(x
\end{aligned}
$$

$$
\begin{aligned}
&- c_0 t)) \big[2c_1 \cos\big(\varepsilon^{\frac{3}{2}} t\big) - c_1 - c_1 \cos^2\big(\varepsilon^{\frac{3}{2}} t\big) + 12 c_2 \sin\big(\varepsilon^{\frac{3}{2}} t\big) \\
&- 12 c_2 \cos\big(\varepsilon^{\frac{3}{2}} t\big) \varepsilon^{\frac{3}{2}} t \big] - 16 k^5 \hbar c_2 \tanh(kX) \operatorname{sech}^2\big(k\sqrt{\varepsilon}(x \\
&- c_0 t)) \big[2\varepsilon^{\frac{3}{2}} t - c_3 \hbar \sin\big(\varepsilon^{\frac{3}{2}} t\big) \big] + T c_4 \hbar^2 m n_0 \sin\big(m\sqrt{\varepsilon}(x - c_0 t)\big) \\
&- c_3 c_4 \hbar^2 m n_0 \sin\big(m\sqrt{\varepsilon}(x - c_0 t)\big) + 2 c_4 \hbar m n_0 \sin\big(m\sqrt{\varepsilon} \\
&\cdot (x - c_0 t)\big) \varepsilon^{\frac{3}{2}} t - 0.5 T^2 c_2 c_4 \hbar^2 m^4 n_0 \cos\big(m\sqrt{\varepsilon}(x - c_0 t)\big) \\
&+ c_3 c_4 \hbar^2 m n_0 \sin\big(m\sqrt{\varepsilon}(x - c_0 t)\big) \cos\big(\varepsilon^{\frac{3}{2}} t\big) \\
&+ c_3 c_4 \hbar^2 m n_0 \sin\big(m\sqrt{\varepsilon}(x - c_0 t)\big) \sin\big(\varepsilon^{\frac{3}{2}} t\big) \varepsilon^{\frac{3}{2}} t \big\},
\end{aligned}
\tag{41}
$$

where $R(\varepsilon^{\frac{3}{2}} t)$ is an arbitrary function about t.

When taking $R(\varepsilon^{\frac{3}{2}} t)$ in the form

$$
R\big(\varepsilon^{\frac{3}{2}} t\big) = k_1 \operatorname{sech}\big[20\varepsilon^{\frac{3}{2}}(t - 9)\big],
\tag{42}
$$

other parameters are as follows:

$$
\begin{aligned}
&F = 1.5, \qquad c_0 = -2.4, \qquad c_1 = 2.7, \qquad c_2 = 5.3, \qquad c_3 = 1.3, \qquad c_4 = 0.6, \\
&\beta = 16.3, \qquad b_0 = -18, \\
&b_1 = 5, \qquad n_0 = 5.2, \qquad m = 1.4, \qquad \varepsilon = 0.1, \qquad k = 6, \qquad k_1 = 10.
\end{aligned}
\tag{43}
$$

The evolution of stream function field is shown in Fig. 1.

When

$$
\begin{aligned}
&a_1 = c_1 \sin\big(\varepsilon^{3/2} t\big), \qquad a_2 = c_2, \qquad a_3 = c_3 \cos\big(\varepsilon^{3/2} t\big), \\
&a_4 = c_4 e^{\varepsilon^{3/2} t}, \qquad R = k_1 \operatorname{sech}\big[30\varepsilon^{\frac{3}{2}}(t - 4)\big],
\end{aligned}
\tag{44}
$$

where c_1, c_2, c_3, c_4 are arbitrary constants, and $c_4 = 9.6$, $\beta = 16.3$, $b_0 = -8$, $b_1 = 5$, $n_0 = 5.2$, $m = 2.8$, $\varepsilon = 0.05$, $k_1 = 20$, other parameters are the same as those for Fig. 1, Ψ_0, G_0 are shown in Eqs. (39)–(40), solving process is similar to above, no longer repeated, the evolution of stream function field is shown in Fig. 2.

In order to enhance the practical significance of our work, we take a real observational blocking case that happened during 14 July 2003 to 27 July 2003 as a simple example. Making use of the National Centers for Environmental Prediction–National Center for Atmospheric Research (NCEP-NCAR) reanalysis data, a dipole blocking event is depicted in Fig. 3.

4 Discussions

Figure 1 clearly demonstrates the formation, maintenance, and decay of a complicated dipole-type blocking, which gradually moves westward in the whole process. At the beginning (day 0), zonal flow flows from west to east (Fig. 1(a)). From the first day to the sixth day, stream function field presents north-south direction meridional flow in the form of multi-pole gradually moving westward (Fig. 1(b)–Fig. 1(e)), and then the multi-pole disappears (Fig. 1(f)). On the seventh day, monopoles appear (Fig. 1(g)); on the eighth day, dipoles appear (Fig. 1(h)); on the tenth day, they are at the strongest status (Fig. 1(j)) and
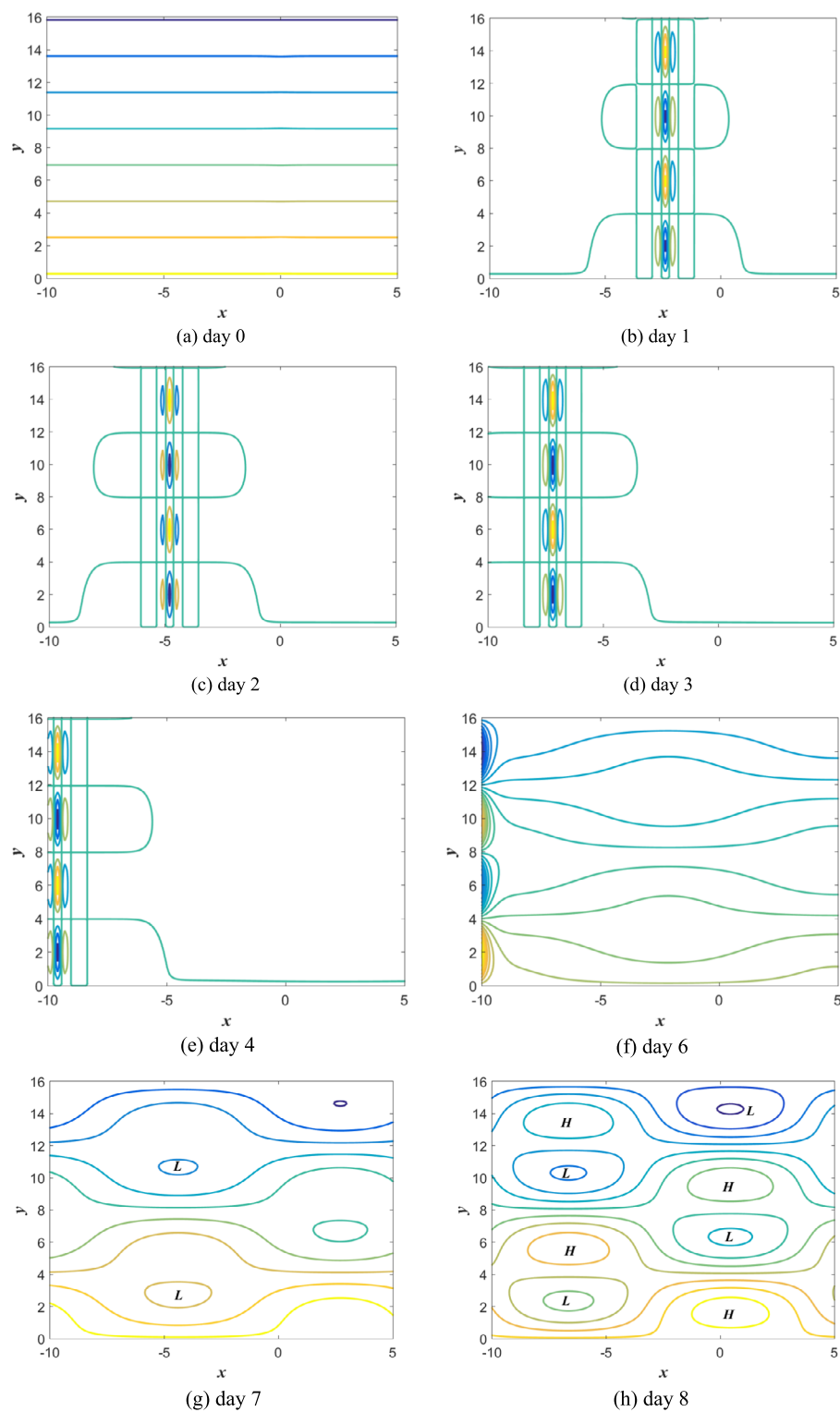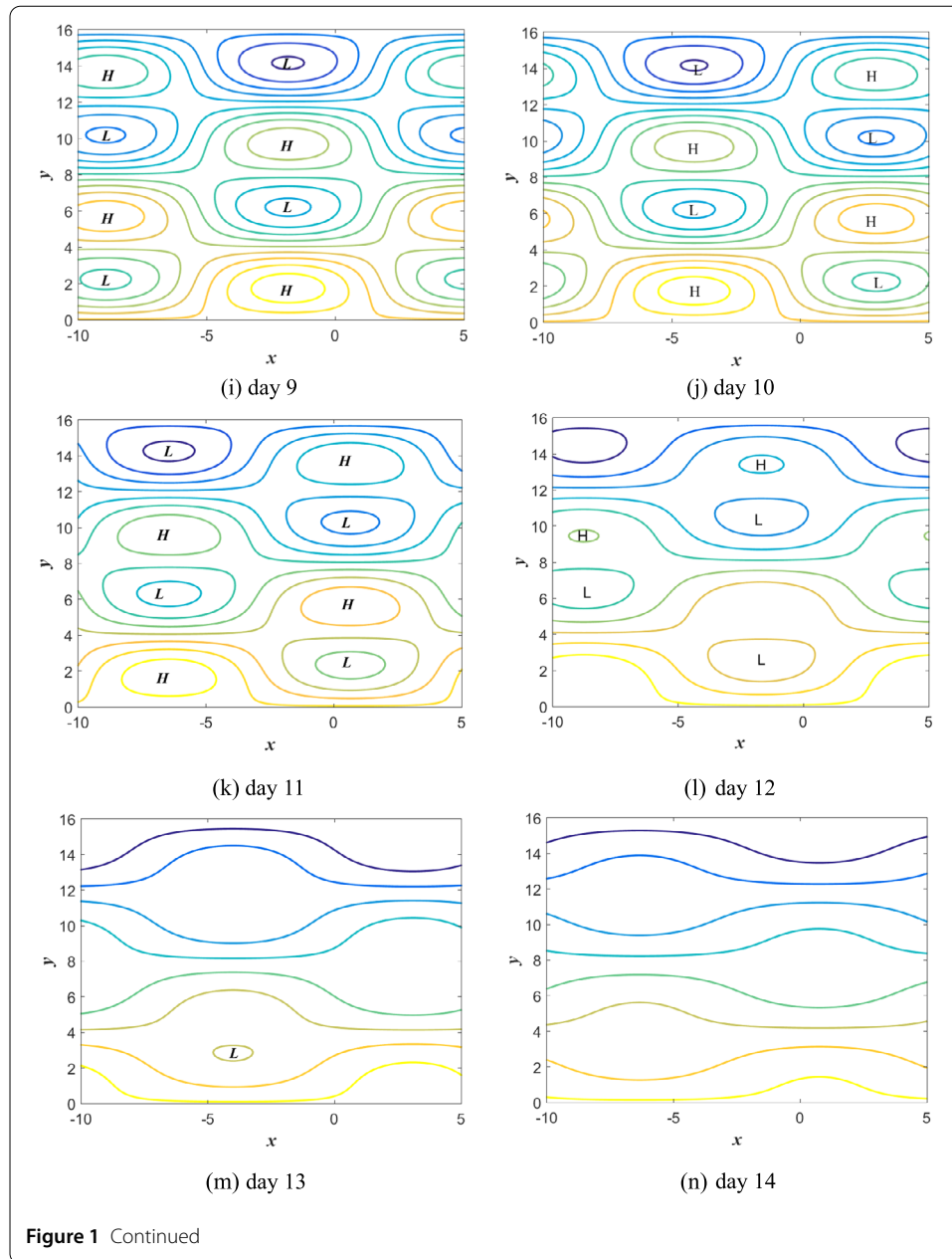
Figure 1 When $a_1 = c_1 \sin(\varepsilon^{3/2} t)$, $a_2 = c_2$, $a_3 = c_3 \cos(\varepsilon^{3/2} t)$, $a_4 = c_4$, the evolution of stream function field, the stream function analytic expression is Eq. (43), where $R = k_1 \mathrm{sech}[20\varepsilon^{\frac{3}{2}}(t-9)]$, $F = 1.5$, $c_0 = -2.4$, $c_1 = 2.7$, $c_2 = 5.3$, $c_3 = 1.3$, $c_4 = 0.6$, $\beta = 16.3$, $b_0 = -18$, $b_1 = 5$, $n_0 = 5.2$, $m = 1.4$, $\varepsilon = 0.1$, $k = 6$, $k_1 = 10$

Figure 1 Continued

then become weaker, finally disappearing after the fourth day (Fig. 1(h)). We still call it dipole blocking, and its life-time is fourteen days.

Figure 2 reveals a complicated dipole blocking process. At the initial time, a weak dipole is presented (Fig. 2(a)). From the second to sixth day, stream function field presents multipole status (Fig. 2(b)–(f)); on the seventh day, multi-dipoles appear (Fig. 2(g)), on the fourteenth day, multi-dipoles disappear (Fig. 2(n)), then a dipole appears (Fig. 2(o)). After the twenty-fourth day, the dipole disappears (Fig. 2(x)). Obviously, stream function field possesses a process from dipole to multi-pole then to multi-dipole, again to dipole, finally disappearing, but we still call it dipole blocking, and its life-time is twenty-four days.

From Fig. 3, it is easily found that geopotential height experiences the developing process of dipole (14 July 2003), multi-pole (15 July 2003), and the onset, maintenance, and collapse
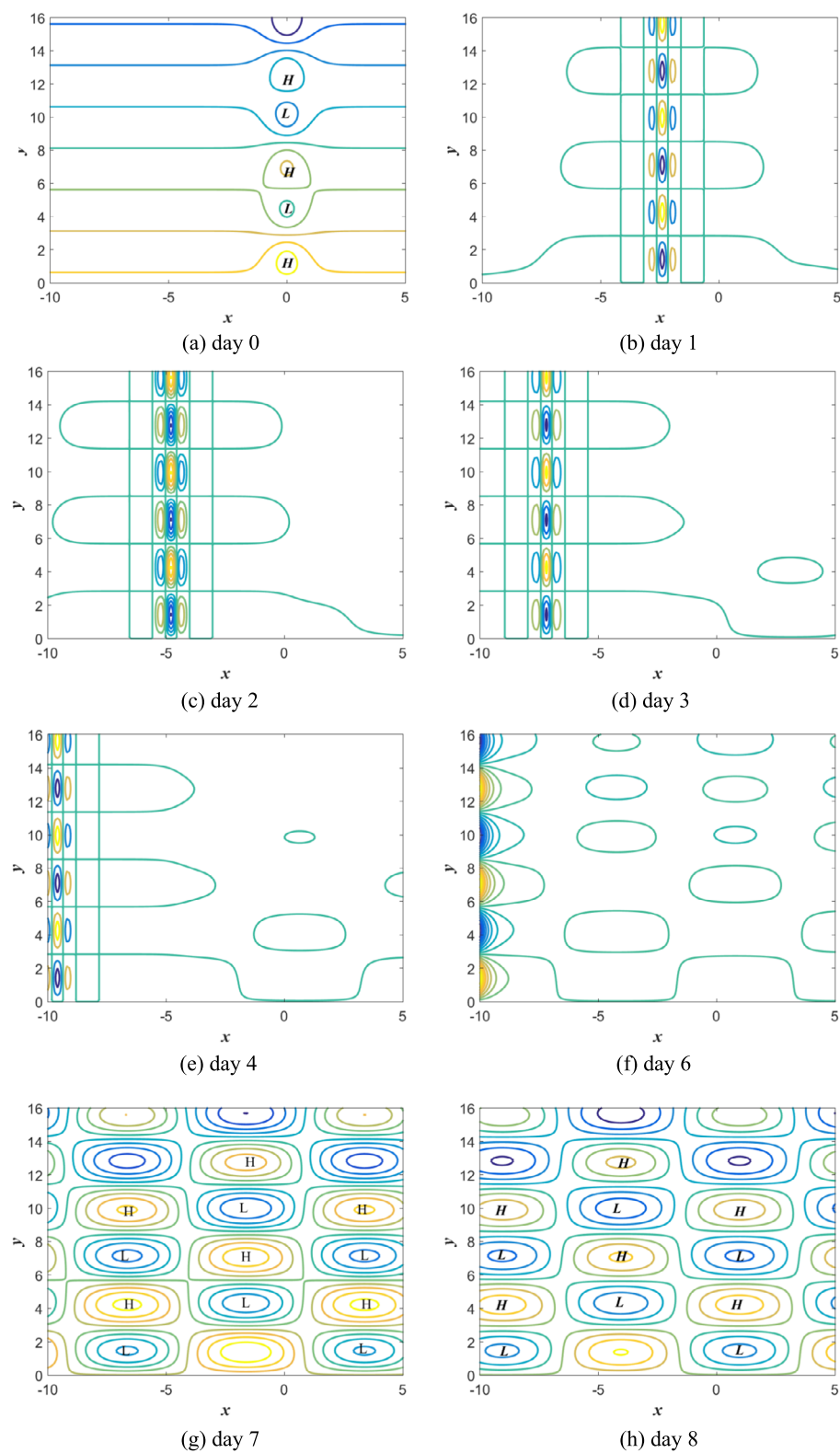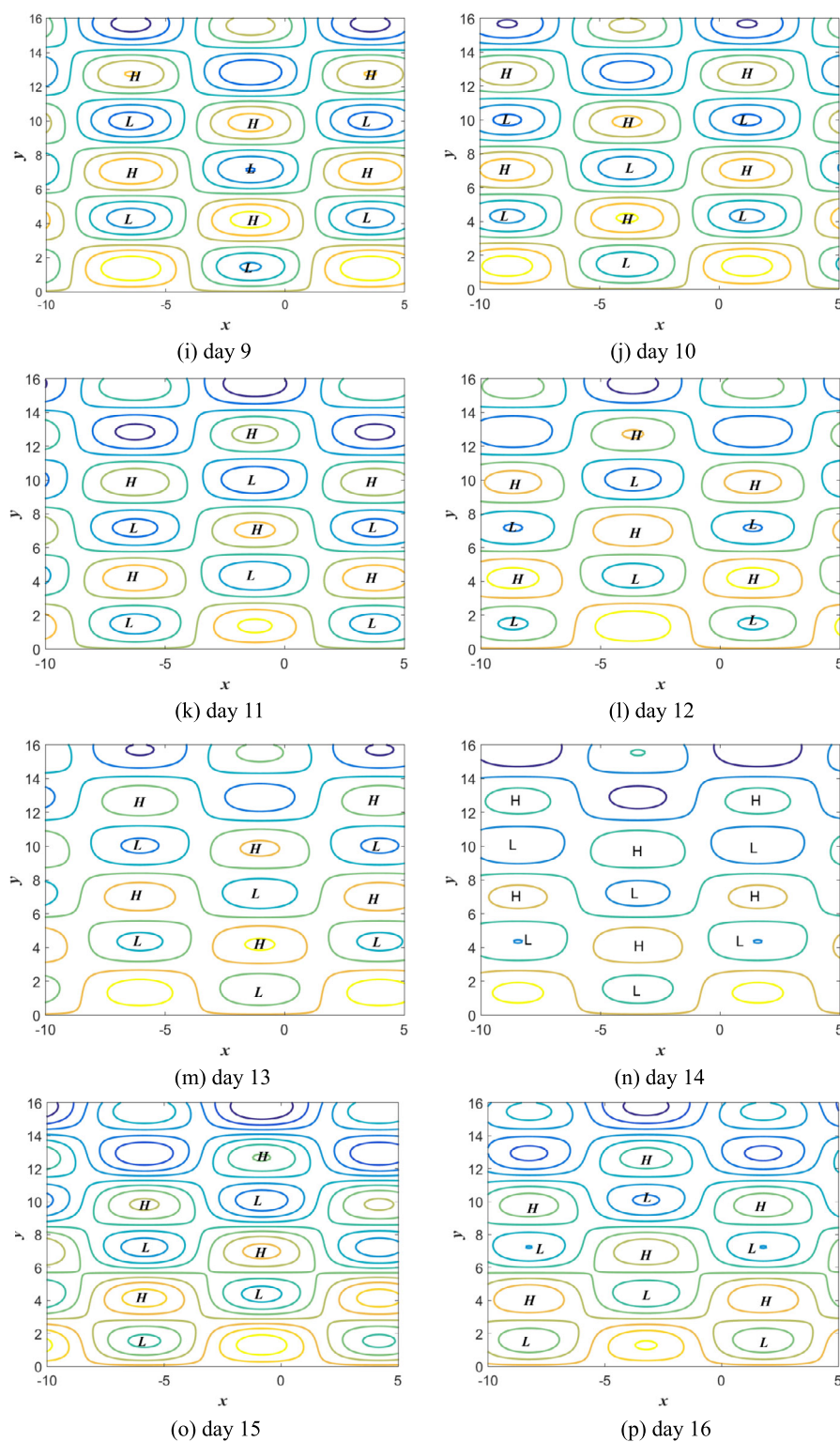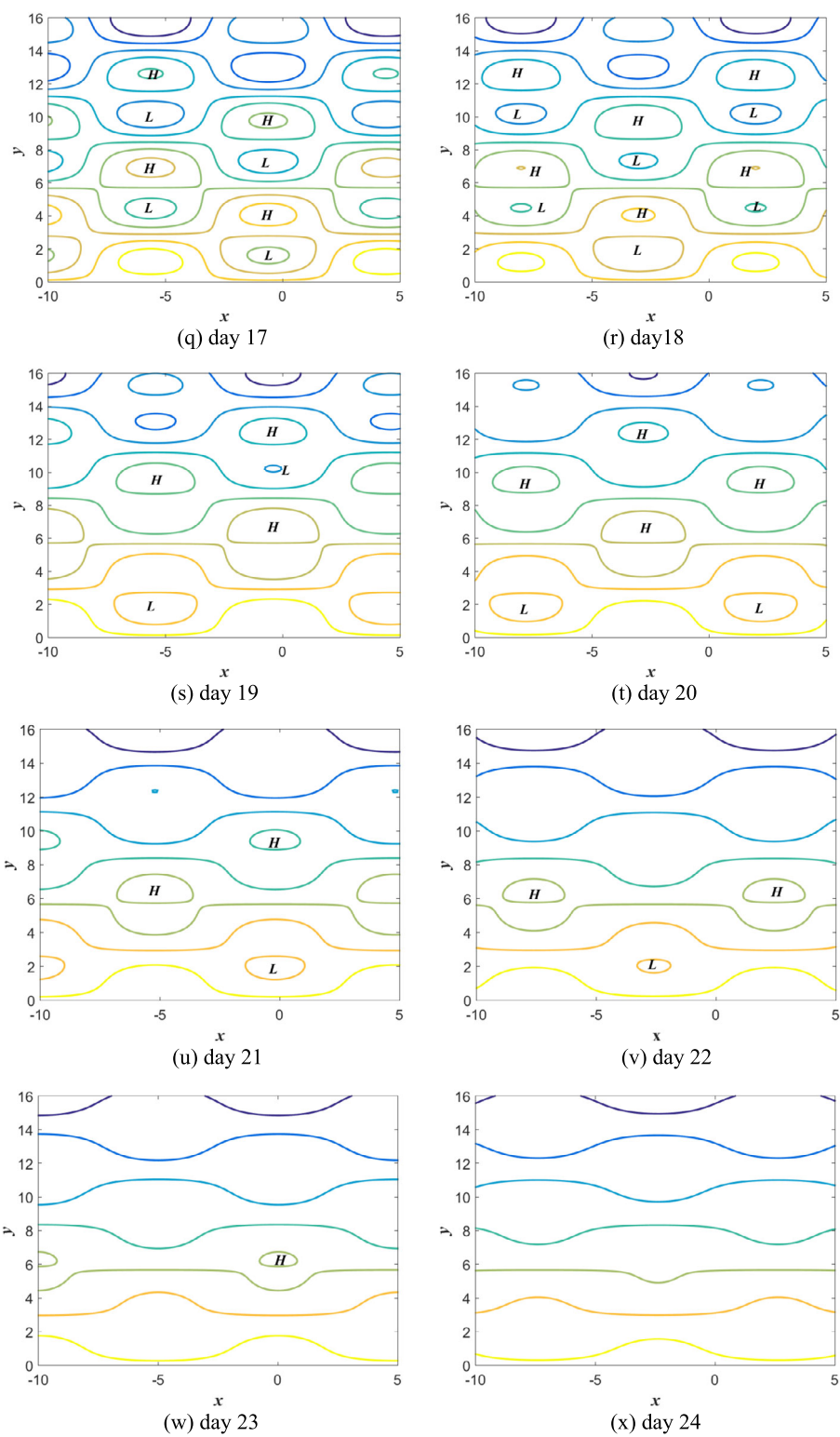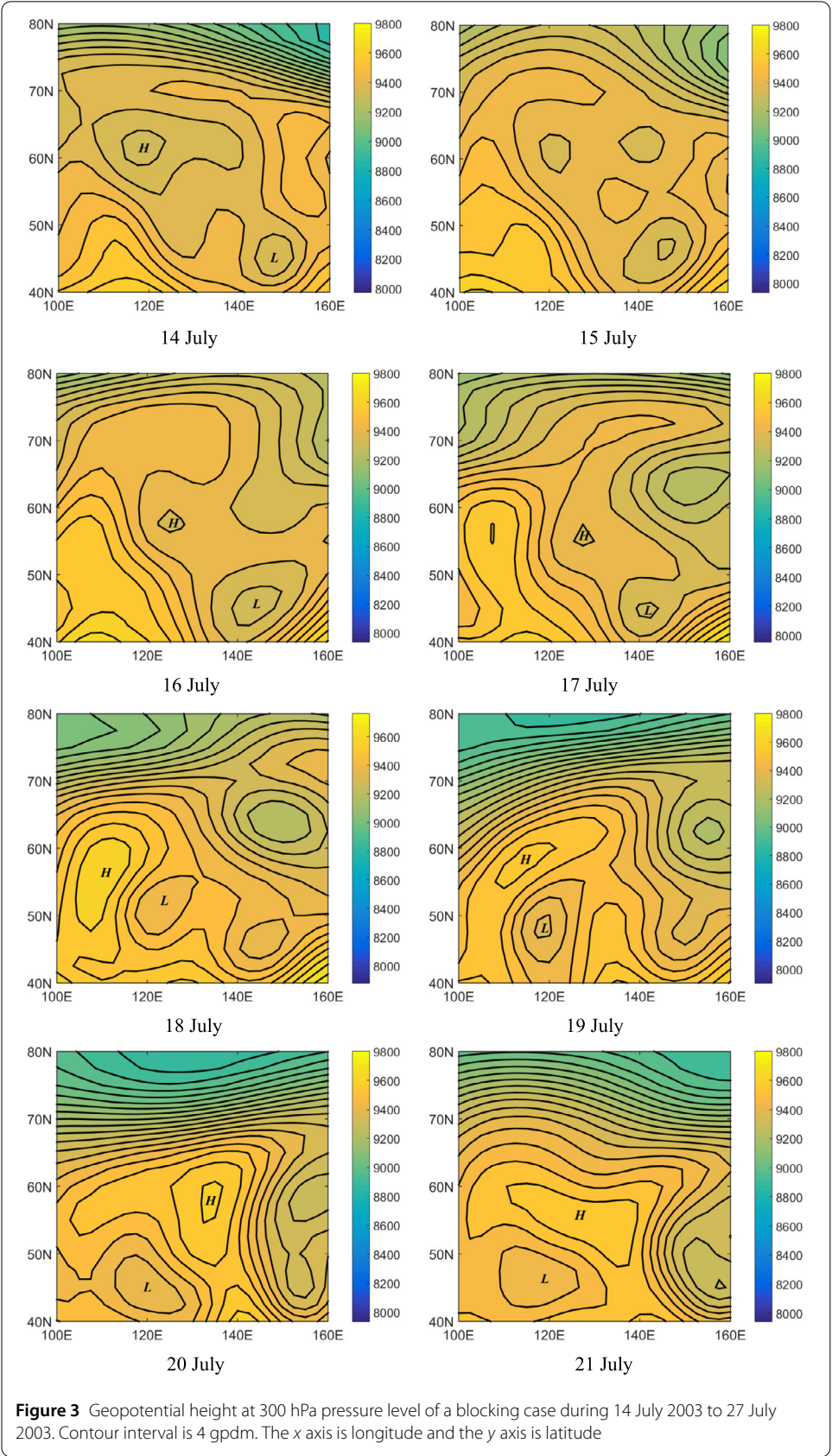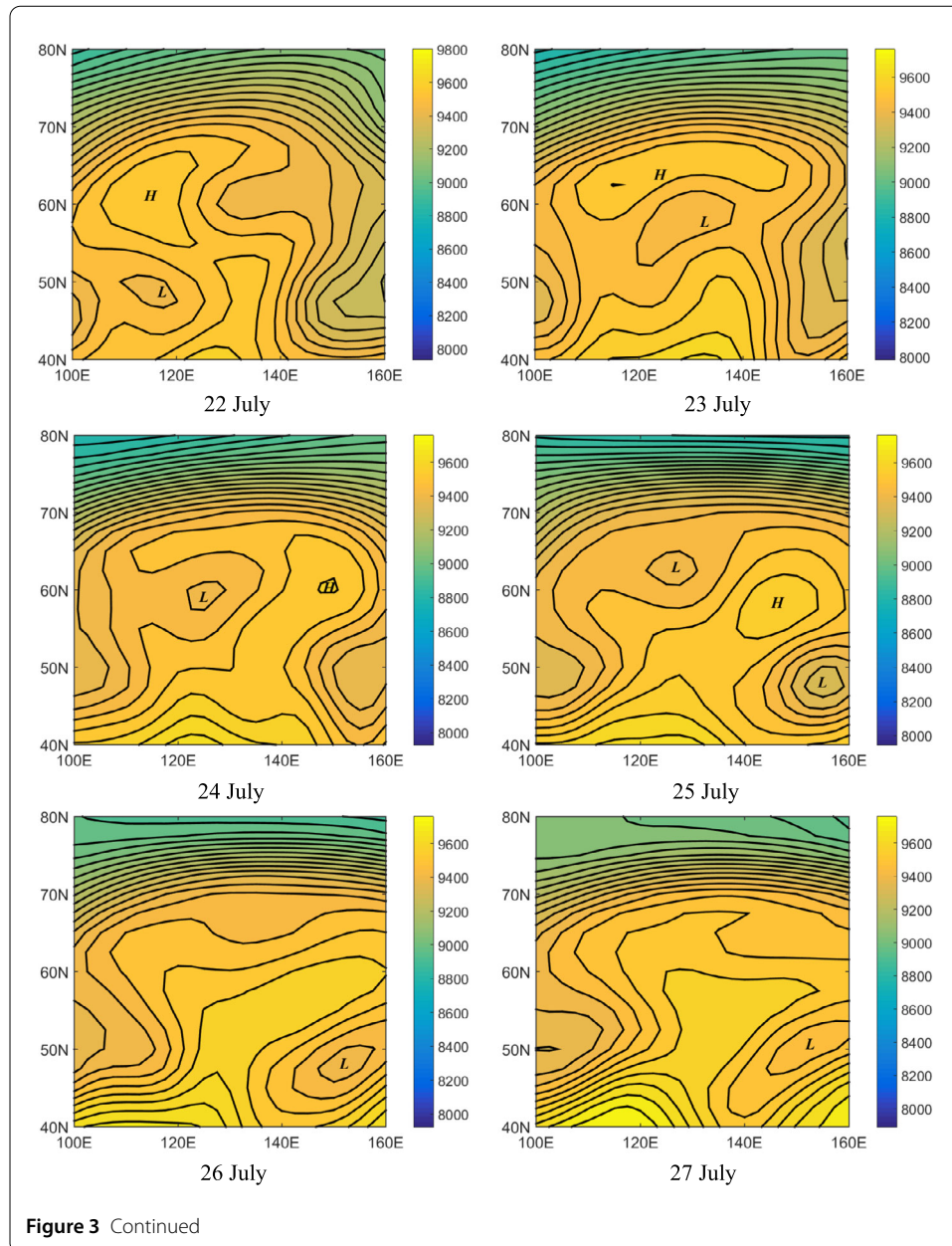
Figure 2 When $a_1 = c_1 \sin(\varepsilon^{3/2} t)$, $a_2 = c_2$, $a_3 = c_3 \cos(\varepsilon^{3/2} t)$, $a_4 = c_4 e^{\varepsilon^{3/2} t}$, the evolution of stream function field, $R = k_1 \text{sech}[20\varepsilon^{3/2}(t - 9)]$, $F = 1.5$, $c_0 = -2.4$, $c_1 = 2.7$, $c_2 = 5.3$, $c_3 = 1.3$, $c_4 = 9.6$, $\boldsymbol{\beta} = 16.3$, $b_0 = -8$, $b_1 = 5$, $n_0 = 5.2$, $m = 2.8$, $\varepsilon = 0.05$, $k = 6$, $k_1 = 20$

(i) day 9

(j) day 10

(k) day 11

(l) day 12

(m) day 13

(n) day 14

(o) day 15

(p) day 16

Figure 2 Continued

(q) day 17

(r) day18

(s) day 19

(t) day 20

(u) day 21

(v) day 22

(w) day 23

(x) day 24

Figure 2 Continued

Figure 3 Geopotential height at 300 hPa pressure level of a blocking case during 14 July 2003 to 27 July 2003. Contour interval is 4 gpdm. The *x* axis is longitude and the *y* axis is latitude

Figure 3 Continued

of dipole (16–27 July 2003). We still call it dipole blocking. Besides, from 16 July 2003 to 27 July 2003, the dipole gradually moves westward.

Figure 1 and Fig. 2 reveal the complexity of dipole blocking evolution. However, Fig. 2 basically corresponds well to the real observational blocking case (Fig. 3), although they have different life cycles, they have the same developing process, namely from dipole to multi-pole, then to dipole. Consequently, the introduction of a time-dependent variable coefficient topographic forcing term is more suitable to explain the complicated blocking phenomenon. Besides, we suspect that the real blocking process is very complicated and changeable, may be accompanied by monopole, dipole, and multi-pole three forms.

5 Conclusions

In this paper, combining topographic forcing effect with time-varying property and introducing a higher order term of the stream function with five arbitrary functions and forced topography, a variable coefficient KdV equation with time-dependent variable coefficient topographic forcing term is obtained. When selecting some proper parameters, stream function field presents a complicated and changeable dipole-type blocking. Consequently, time-dependent variable coefficient and forced topography have great influence on the evolution of stream function field. With appropriate selection of arbitrary functions and constants, many kinds of real atmospheric phenomena may be successfully explained.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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