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Some new solutions of the Caudrey–Dodd–Gibbon (CDG) equation using the conformable derivative

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Abstract

New exact solutions of the space–time conformable Caudrey–Dodd–Gibbon (CDG) equation have been derived by implementing the conformable derivative. The generalized Riccati equation mapping method is applied to figure out twenty-seven forms of exact solutions, which are soliton, rational, and periodic ones. Also, for some suitable values of parameters, the exact solutions are found, namely dark, bell type, periodic, soliton, singular soliton, and several others, by using the conformable derivative. These types of solutions have not been proclaimed so far. 2D and 3D graphical patterns of some solutions are also given for clarification of physical features. The conformable derivative is one of the excellent choices to solve the nonlinear conformable problems arising in theory of solitons and many other areas. The results are new and very interesting for the large community of researchers working in the field of mathematics and mathematical physics.

Keywords: Generalized Riccati equation mapping method; Caudrey–Dodd–Gibbon (CDG) equation; Soliton solution; Periodic solution; Rational solution; Conformable derivative

1 Introduction

Fractional calculus [1, 2] containing differential equations of fractional order in numerous physical phenomena has put a revolutionary impact since these particular equations reflect generalization of evolution equations of integer order. Derivatives and integrals of real fractional order or complex order can be applied in many fields. These fields are geochemistry, plasma physics, mechanics, control theory, optical fibers, solid state physics, system identification, chemical kinematics, biogenetics, etc. These phenomena involve diffusion, dispersion, dissipation convection, and reaction.

Finding the extraction of exact traveling wave solutions of nonlinear differential equations has become very important for discussing wave phenomena of nonlinear nature. Many analytical, approximate and numerical techniques have been used for finding exact solutions of nonlinear differential equations including homotopy analysis transform method [3], homotopy analysis samadu transform method [4], fractional homotopy analysis transform method [5], the differential transform method [6], F-expansion method [7], first integral method [8], fractional sub-equation method [9], sine-cosine method [10],

variational iteration method [11, 12], exponential rational function method [13, 14], exp-function method [15–18], $(\frac{G'}{G})$ -expansion method [19–21], $\exp(-\phi(\xi))$ -expansion method [22], $\tan(\frac{\phi(\xi)}{2})$ expansion method [23, 24], modified trial equation method [25, 26], etc. Some recent advancements include fractional nonlinear dynamics [27], new fractional order view of HIV and CD4+ T-cells [28], life cycle involving poor nutrition [29], and control theory [30]. Further, a new analysis regarding Mittag-Leffler type kernel [31], study of computer viruses [32] are the recent advancements in fractional derivatives and their applications.

To seek out exact solutions of some nonlinear evolution equations (NEEs), a Riccati equation mapping method is introduced by considering the nonlinear ODE: $\varphi(\xi) = r + p\varphi(\xi) + q\varphi^2(\xi)$. This approach was formulated from the $(\frac{G'}{G})$ method by utilizing Cole–Hopf transformation and yielded many solutions as compared to the $(\frac{G'}{G})$ -expansion method. The generalized Riccati equation mapping is simple, straightforward, convenient for homogeneous balancing order and in working out a system of algebraic equations and can produce a variety of exact solutions of differential equations. This work discovers twenty-seven types of different solutions comprising hyperbolic, periodic, and rational solutions.

In [33], Khalil introduced the notion of conformable derivative, suitably compatible with integer order derivative and satisfying some conventional properties such as the chain rule. Atangana discussed some features of this particular derivative in [34]. The author presented proofs of linked theorems and gave new definitions. This operator is also investigated in [35, 36], leading to fruitful discussion. By using sub-equation method in conjunction with conformable operator, Rezazadeh in [37] found solutions of traveling wave form for conformable generalized Kuramoto–Sivashinsky equation. The resulting solutions were presented in hyperbolic and trigonometric forms. In [38], the authors extracted exact solutions of conformable regularized long-wave equations, while some other analytical techniques are given in [39, 40]. A new form of conformable derivative is proposed by Atangana [41] referred to as beta fractional operator, overcoming the limitation of conformable derivative.

In this work, we focus on the relatively modern fractional differentiation known as Atangana–Baleanu fractional differentiation [42]. Abdon Atangana, an African researcher and mathematician, presented his new idea of fractional operators. This particular work attracted many investigators to apply it to a number of problems of practical applications. Atangana et al. gave new properties of the conformable fractional operator or derivative. Abdon Atangana and Dumitru Baleanu studied the Mittag-Leffler function which is a more generalized function. Consequently, they introduced a fractional derivative, based on non-local kernel, describing complex nonlinear applied problems of complex nature. The power and exponential decay law was also considered. This new version derivative is found to be more appealing and advantageous than the already established ones.

The solutions of space–time conformable Caudrey–Dodd–Gibbon (CDG) equation are sought by different methods, e.g., Wazwaz [43] used the tanh method to obtain solitary wave solutions. Zhou et al. [44] applied the exp-function method to derive generalized solitary solution. In 2011 [45], the authors obtained traveling wave solutions by the $(\frac{G'}{G})$ -expansion method. By studying the generalized Riccati equation mapping method [46], the conformable derivative of Atangana’s version finds the hyperbolic, rational, and trigonometric solutions of (CDG) equation, not brought to light before in the literature so far.

Next section therefore gives some new solutions of traveling wave form of the space–time conformable Caudrey–Dodd–Gibbon (CDG) equation.

Analysis of the generalized Riccati equation mapping is presented in Sect. 2. We have implemented the same mapping for attaining new exact soliton solutions to the time and space conformable Caudrey–Dodd–Gibbon (CDG) equation in Sect. 3. Physical interpretation and comparison will be presented in Sects. 4 and 5, whereas the last section includes our conclusion.

2 The improved generalized Riccati equation mapping method

The focal steps of this technique [47–49] are as follows.

Step 1. We consider the conformable differential equation including independent variables x_1, x_2, \dots, x_m, t and some unknown function u :

$$p(u, {}_0^A D_t^\alpha u, {}_0^A D_{x_1}^\alpha u, {}_0^A D_{x_2}^\alpha u, {}_0^A D_{x_3}^\alpha u, \dots, {}_0^A D_t^{2\alpha} u, {}_0^A D_{x_1}^{2\alpha} u, {}_0^A D_{x_2}^{2\alpha} u, \dots) = 0, \tag{1}$$

where ${}_0^A D_t^\alpha$ and ${}_0^A D_x^\alpha$ are known as Atangana’s conformable differential operators.

Step 2. For the said operator, we use the variable transformation:

$$\begin{aligned} u(x_1, x_2, \dots, x_m, t) &= U(\xi), \\ \xi &= \frac{\chi 1}{\alpha} \left(x_1 + \frac{1}{\Gamma(\alpha)}\right)^\alpha + \frac{\chi 2}{\alpha} \left(x_2 + \frac{1}{\Gamma(\alpha)}\right)^\alpha + \dots + \frac{\chi m}{\alpha} \left(x_m + \frac{1}{\Gamma(\alpha)}\right)^\alpha \\ &\quad + \frac{\lambda}{\alpha} \left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha, \end{aligned} \tag{2}$$

where the constants λ and χ_i are to be determined. The conformable differential equation (1) leads to the nonlinear conformable ODE for $u(x, t) = u(\xi)$:

$$R = R(u(\xi), u'(\xi), u''(\xi), \dots) = 0, \tag{3}$$

where $u'(\xi) = \frac{du(\xi)}{d\xi}$ indicates a derivative in terms of ξ .

Step 3. We consider an assumption that the solution of Eq. (3) can be given by a polynomial in $\phi(\xi)$ as follows:

$$u(\xi) = \sum_{i=-n}^n a_i \phi(\xi)^i, \tag{4}$$

where a_i ($i = 0, \pm 1, \pm 2, \dots, \pm n$) are constants, which are to be determined provided $a_i \neq 0$. The function $\phi = \phi(\xi)$ satisfies the Riccati differential equation [49]

$$\phi'(\xi) = r + p\phi(\xi) + \phi(\xi)^2. \tag{5}$$

Step 4. Positive integer N is found by using homogeneous balance between the nonlinear term and the highest order derivatives in Eq. (3).

Step 5. Plugging Eq. (4) along with Eq. (5) into Eq. (3), followed by collecting all the same order terms ϕ^i together, we get the polynomial equation in ϕ^i and ϕ^{-i} , where ($i = 0, 1, 2, \dots$). Equalizing coefficients of the resulting polynomial to zero, we get an over-determined system of algebraic equations for a_i , where $i = 0, \pm 1, \pm 2, \dots, \pm n$.

Step 6. By using Maple, we work out the system described in Step 5 for obtaining a_i , where $i = 0, \pm 1, \pm 2, \dots, \pm n$. We plug the obtained values in Eq. (4) along with solutions of Eq. (5) using the transformation in Eq. (2) to obtain several exact solutions of Eq. (1). For general solutions of Eq. (1), the generalized Riccati equation mapping technique gives twenty-seven solutions of the Riccati equation in Eq. (5) presenting four different families [49] as follows.

Family 1: When $p^2 - 4qr > 0$ and $pq \neq 0$ or $qr \neq 0$, the hyperbolic function solutions of Eq. (5) are as follows:

$$\begin{aligned} \phi_1(\xi) &= -\frac{1}{2q} \left[p + \sqrt{p^2 - 4qr} \tanh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) \right], \\ \phi_2(\xi) &= -\frac{1}{2q} \left[p + \sqrt{p^2 - 4qr} \coth\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) \right], \\ \phi_3(\xi) &= -\frac{1}{2q} \left[p + \sqrt{p^2 - 4qr} (\tanh(\sqrt{p^2 - 4qr} \xi) \pm i \operatorname{sech}(\sqrt{p^2 - 4qr} \xi)) \right], \\ \phi_4(\xi) &= -\frac{1}{2q} \left[p + \sqrt{p^2 - 4qr} (\coth(\sqrt{p^2 - 4qr} \xi) \pm \operatorname{csch}(\sqrt{p^2 - 4qr} \xi)) \right], \\ \phi_5(\xi) &= -\frac{1}{4q} \left[2p + \sqrt{p^2 - 4qr} \left(\tanh\left(\frac{\sqrt{p^2 - 4qr}}{4} \xi\right) \pm \coth\left(\frac{\sqrt{p^2 - 4qr}}{4} \xi\right) \right) \right], \\ \phi_6(\xi) &= \frac{1}{2q} \left[-p + \frac{\sqrt{(A^2 + B^2)(p^2 - 4qr)} - A\sqrt{p^2 - 4qr} \cosh(\sqrt{p^2 - 4qr} \xi)}{A \sinh(\sqrt{p^2 - 4qr} \xi) + B} \right], \\ \phi_7(\xi) &= \frac{1}{2q} \left[-p - \frac{\sqrt{(B^2 - A^2)(p^2 - 4qr)} + A\sqrt{p^2 - 4qr} \sinh(\sqrt{p^2 - 4qr} \xi)}{A \cosh(\sqrt{p^2 - 4qr} \xi) + B} \right], \end{aligned}$$

where two non-zero real constants A and B satisfy $B^2 - A^2 > 0$,

$$\begin{aligned} \phi_8(\xi) &= \frac{2r \cosh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right)}{\sqrt{p^2 - 4qr} \sinh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) - p \cosh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right)}, \\ \phi_9(\xi) &= \frac{-2r \sinh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right)}{p \sinh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) - \sqrt{p^2 - 4qr} \cosh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right)}, \\ \phi_{10}(\xi) &= \frac{2r \cosh(\sqrt{p^2 - 4qr} \xi)}{\sqrt{p^2 - 4qr} \sinh(\sqrt{p^2 - 4qr} \xi) - p \cosh(\sqrt{p^2 - 4qr} \xi) \pm i\sqrt{p^2 - 4qr}}, \\ \phi_{11}(\xi) &= \frac{2r \sinh(\sqrt{p^2 - 4qr} \xi)}{-p \sinh(\sqrt{p^2 - 4qr} \xi) + \sqrt{p^2 - 4qr} \cosh(\sqrt{p^2 - 4qr} \xi) \pm \sqrt{p^2 - 4qr}}, \\ \phi_{12}(\xi) &= \frac{4r \sinh\left(\frac{\sqrt{p^2 - 4qr}}{4} \xi\right) \cosh\left(\frac{\sqrt{p^2 - 4qr}}{4} \xi\right)}{\left(-2p \sinh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) \cosh\left(\frac{\sqrt{p^2 - 4qr}}{4} \xi\right) \right.} \\ &\quad \left. + 2\sqrt{p^2 - 4qr} \cosh^2\left(\frac{\sqrt{p^2 - 4qr}}{4} \xi\right) - \sqrt{p^2 - 4qr} \right)}. \end{aligned}$$

Family 2: When $p^2 - 4qr < 0$ and $pq \neq 0$ or $qr \neq 0$, the trigonometric solutions of Eq. (5) are as follows:

$$\phi_{13}(\xi) = \frac{1}{2q} \left[-p + \sqrt{4qr - p^2} \tan\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right) \right],$$

$$\begin{aligned} \phi_{14}(\xi) &= -\frac{1}{2q} \left[p + \sqrt{4qr - p^2} \cot\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right) \right], \\ \phi_{15}(\xi) &= \frac{1}{2q} \left[-p + \sqrt{4qr - p^2} (\tan(\sqrt{4qr - p^2} \xi) \pm \sec(\sqrt{4qr - p^2} \xi)) \right], \\ \phi_{16}(\xi) &= -\frac{1}{2q} \left[p + \sqrt{4qr - p^2} (\cot(\sqrt{4qr - p^2} \xi) \pm \csc(\sqrt{4qr - p^2} \xi)) \right], \\ \phi_{17}(\xi) &= \frac{1}{4q} \left[-2p + \sqrt{4qr - p^2} \left(\tan\left(\frac{\sqrt{4qr - p^2}}{4} \xi\right) - \cot\left(\frac{\sqrt{4qr - p^2}}{4} \xi\right) \right) \right], \\ \phi_{18}(\xi) &= \frac{1}{2q} \left[-p + \frac{\pm \sqrt{(A^2 - B^2)(4qr - p^2)} - A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2} \xi)}{A \sin(\sqrt{4qr - p^2} \xi) + B} \right], \\ \phi_{19}(\xi) &= \frac{1}{2q} \left[-p - \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} + A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2} \xi)}{A \sin(\sqrt{4qr - p^2} \xi) + B} \right], \end{aligned}$$

where two non-zero real constants A and B satisfy $A^2 - B^2 > 0$,

$$\begin{aligned} \phi_{20}(\xi) &= -\frac{2r \cos\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)}{\sqrt{4qr - p^2} \sin\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right) + p \cos\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)}, \\ \phi_{21}(\xi) &= \frac{2r \sin\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)}{-p \sin\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right) + \sqrt{4qr - p^2} \cos\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)}, \\ \phi_{22}(\xi) &= -\frac{2r \cos(\sqrt{4qr - p^2} \xi)}{\sqrt{4qr - p^2} \sin(\sqrt{4qr - p^2} \xi) + p \cos(\sqrt{4qr - p^2} \xi) \pm \sqrt{4qr - p^2}}, \\ \phi_{23}(\xi) &= \frac{2r \sin(\sqrt{4qr - p^2} \xi)}{-p \sin(\sqrt{4qr - p^2} \xi) + \sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2} \xi) \pm \sqrt{4qr - p^2}}, \\ \phi_{24}(\xi) &= \frac{4r \sin\left(\frac{\sqrt{4qr - p^2}}{4} \xi\right) \cos\left(\frac{\sqrt{4qr - p^2}}{4} \xi\right)}{\left(\begin{aligned} &-2p \sin\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right) \cos\left(\frac{\sqrt{4qr - p^2}}{4} \xi\right) \\ &+ 2\sqrt{4qr - p^2} \cos^2\left(\frac{\sqrt{4qr - p^2}}{4} \xi\right) - \sqrt{4qr - p^2} \end{aligned} \right)}. \end{aligned}$$

Family 3: When $r = 0$ and $pq \neq 0$, the solutions of Eq. (5) are as follows:

$$\begin{aligned} \phi_{25}(\xi) &= -\frac{pd}{q(d + \cosh(p\xi) - \sinh(p\xi))}, \\ \phi_{26}(\xi) &= \frac{-p(\cosh(p\xi) + \sinh(p\xi))}{q(d + \cosh(p\xi) - \sinh(p\xi))}, \end{aligned}$$

where d in the above solution is an arbitrary constant.

Family 4: When $r = p = 0$ and $q \neq 0$, the rational solution of Eq. (5) is

$$\phi_{27}(\xi) = -\frac{1}{q\xi + c},$$

where c in the above solution is an arbitrary constant.

3 Space–time conformable Caudrey–Dodd–Gibbon equation

Consider the space–time conformable Caudrey–Dodd–Gibbon (CDG) equation [44] via Atangana’s conformable derivative in the form

$${}^A_0D_t^\alpha u + {}^A_0D_x^{5\alpha} u + 30u {}^A_0D_x^{3\alpha} u + 30 {}^A_0D_x^\alpha u {}^A_0D_x^{2\alpha} u + 180u^2 {}^A_0D_x^\alpha u = 0. \tag{6}$$

We first transfer Eq. (6) from FPDE to the following nonlinear conformable ordinary differential equation by using the variable transformation and integrating once

$$u(x, t) = u(\xi),$$

$$\xi = \frac{\chi}{\alpha} \left(x + \frac{1}{\Gamma(\alpha)} \right)^\alpha + \frac{\lambda}{\alpha} \left(t + \frac{1}{\Gamma(\alpha)} \right)^\alpha, \tag{7}$$

in the following form:

$$60\chi u^3(\xi) + \lambda u(\xi) + 30\chi^3 u(\xi)u''(\xi) + u''''(\xi)\chi^5 = 0, \tag{8}$$

where $u' = \frac{du}{d\xi}$.

Here, positive integer number M is found by using the homogeneous balance between the highest order derivative u'''' and a nonlinear term of the highest order $u(\xi)u''(\xi)$ present in Eq. (8), we get $M = 2$.

Suppose that the solution of Eq. (8) has the following form:

$$u(\xi) = a_{-2}\phi(\xi)^{-2} + a_{-1}\phi(\xi)^{-1} + a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2. \tag{9}$$

Substituting Eq. (9) along with Eq. (5) into Eq. (8), we obtain a polynomial in $\phi^i(\xi)$, where $(i = -6, -5, -4, \dots, 6)$. Setting all terms of the like power to zero, we get a system of algebraic equations. Solving the set of over-determined algebraic equations by Maple, we come up with different solutions. Implementing these solutions then into Eq. (6), we justify the solutions to be exact.

Case 1:

$$a_0 = -\chi^2qr, \quad a_1 = -\chi^2pq, \quad a_{-1} = 0, \quad a_{-2} = 0,$$

$$a_2 = -\chi^2q^2, \quad \lambda = -\chi^5(p^2 - 4qr)^2. \tag{10}$$

Inserting these values into Eq. (9) by using Families 1–4 of Eq. (5) and Eq. (7), we obtain the hyperbolic, periodic, and rational solutions as follows.

Family 1: When $p^2 - 4qr > 0$ and $pq \neq 0$ or $qr \neq 0$, the hyperbolic function solutions of Eq. (6) are as follows:

$$u_{1,1}(\xi) = -\chi^2qr - \chi^2pq \left(-\frac{1}{2q} \left(p + \sqrt{p^2 - 4qr} \tanh \left(\frac{\sqrt{p^2 - 4qr}}{2} \xi \right) \right) \right)$$

$$- \chi^2q^2 \left(-\frac{1}{2q} \left(p + \sqrt{p^2 - 4qr} \tanh \left(\frac{\sqrt{p^2 - 4qr}}{2} \xi \right) \right) \right)^2, \tag{11}$$

$$\begin{aligned}
 u_{1,2}(\xi) = & -\chi^2qr - \chi^2pq \left(-\frac{1}{2q} \left(p + \sqrt{p^2 - 4qr} \coth \left(\frac{\sqrt{p^2 - 4qr}}{2} \xi \right) \right) \right) \\
 & - \chi^2q^2 \left(-\frac{1}{2q} \left(p + \sqrt{p^2 - 4qr} \coth \left(\frac{\sqrt{p^2 - 4qr}}{2} \xi \right) \right) \right)^2, \tag{12}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,3}(\xi) = & -\chi^2qr \\
 & - \chi^2pq \left(-\frac{1}{2q} \left[\frac{p + \sqrt{p^2 - 4qr}}{\tanh(\sqrt{p^2 - 4qr}\xi) \pm i \operatorname{sech}(\sqrt{p^2 - 4qr}\xi)} \right] \right) \\
 & - \chi^2q^2 \left(-\frac{1}{2q} \left[\frac{p + \sqrt{p^2 - 4qr}}{\tanh(\sqrt{p^2 - 4qr}\xi) \pm i \operatorname{sech}(\sqrt{p^2 - 4qr}\xi)} \right] \right)^2, \tag{13}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,4}(\xi) = & -\chi^2qr \\
 & - \chi^2pq \left(-\frac{1}{2q} \left[\frac{p + \sqrt{p^2 - 4qr}}{\coth(\sqrt{p^2 - 4qr}\xi) \pm \operatorname{csch}(\sqrt{p^2 - 4qr}\xi)} \right] \right) \\
 & - \chi^2q^2 \left(-\frac{1}{2q} \left[\frac{p + \sqrt{p^2 - 4qr}}{\coth(\sqrt{p^2 - 4qr}\xi) \pm \operatorname{csch}(\sqrt{p^2 - 4qr}\xi)} \right] \right)^2, \tag{14}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,5}(\xi) = & -\chi^2qr \\
 & - \chi^2pq \left(-\frac{1}{4q} \left[2p + \sqrt{p^2 - 4qr} \left(\tanh \left(\frac{\sqrt{p^2 - 4qr}}{4} \xi \right) \pm \coth \left(\frac{\sqrt{p^2 - 4qr}}{4} \xi \right) \right) \right] \right) \\
 & - \chi^2q^2 \left(-\frac{1}{4q} \left[2p + \sqrt{p^2 - 4qr} \left(\tanh \left(\frac{\sqrt{p^2 - 4qr}}{4} \xi \right) \pm \coth \left(\frac{\sqrt{p^2 - 4qr}}{4} \xi \right) \right) \right] \right)^2, \tag{15}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,6}(\xi) = & -\chi^2qr \\
 & - \chi^2pq \left(\frac{1}{2q} \left[-p + \frac{\sqrt{(A^2 + B^2)(p^2 - 4qr)} - A\sqrt{p^2 - 4qr} \cosh(\sqrt{p^2 - 4qr}\xi)}{A \sinh(\sqrt{p^2 - 4qr}\xi) + B} \right] \right) \\
 & - \chi^2q^2 \left(\frac{1}{2q} \left[-p + \frac{\sqrt{(A^2 + B^2)(p^2 - 4qr)} - A\sqrt{p^2 - 4qr} \cosh(\sqrt{p^2 - 4qr}\xi)}{A \sinh(\sqrt{p^2 - 4qr}\xi) + B} \right] \right)^2, \tag{16}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,7}(\xi) = & -\chi^2qr \\
 & - \chi^2pq \left(\frac{1}{2q} \left[-p - \frac{\sqrt{(B^2 - A^2)(p^2 - 4qr)} + A\sqrt{p^2 - 4qr} \sinh(\sqrt{p^2 - 4qr}\xi)}{A \cosh(\sqrt{p^2 - 4qr}\xi) + B} \right] \right) \\
 & - \chi^2q^2 \left(\frac{1}{2q} \left[-p - \frac{\sqrt{(B^2 - A^2)(p^2 - 4qr)} + A\sqrt{p^2 - 4qr} \sinh(\sqrt{p^2 - 4qr}\xi)}{A \cosh(\sqrt{p^2 - 4qr}\xi) + B} \right] \right)^2, \tag{17}
 \end{aligned}$$

where A and B are two non-zero real constants satisfying $B^2 - A^2 > 0$

$$u_{1,8}(\xi) = -\chi^2qr$$

$$\begin{aligned}
 & -\chi^2 pq \left(\frac{2r \cosh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right)}{\sqrt{p^2-4qr} \sinh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right) - p \cosh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right)} \right) \\
 & -\chi^2 q^2 \left(\frac{2r \cosh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right)}{\sqrt{p^2-4qr} \sinh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right) - p \cosh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right)} \right)^2, \tag{18}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,9}(\xi) &= -\chi^2 qr \\
 & -\chi^2 pq \left(\frac{-2r \sinh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right)}{p \sinh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right) - \sqrt{p^2-4qr} \cosh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right)} \right) \\
 & -\chi^2 q^2 \left(\frac{-2r \sinh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right)}{p \sinh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right) - \sqrt{p^2-4qr} \cosh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right)} \right)^2, \tag{19}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,10}(\xi) &= -\chi^2 qr \\
 & -\chi^2 pq \left(\frac{2r \cosh(\sqrt{p^2-4qr}\xi)}{\sqrt{p^2-4qr} \sinh(\sqrt{p^2-4qr}\xi) - p \cosh(\sqrt{p^2-4qr}\xi) \pm i\sqrt{p^2-4qr}} \right) \\
 & -\chi^2 q^2 \left(\frac{2r \cosh(\sqrt{p^2-4qr}\xi)}{\sqrt{p^2-4qr} \sinh(\sqrt{p^2-4qr}\xi) - p \cosh(\sqrt{p^2-4qr}\xi) \pm i\sqrt{p^2-4qr}} \right)^2, \tag{20}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,11}(\xi) &= -\chi^2 qr \\
 & -\chi^2 pq \left(\frac{2r \sinh(\sqrt{p^2-4qr}\xi)}{-p \sinh(\sqrt{p^2-4qr}\xi) + \sqrt{p^2-4qr} \cosh(\sqrt{p^2-4qr}\xi) \pm \sqrt{p^2-4qr}} \right) \\
 & -\chi^2 pq \left(\frac{2r \sinh(\sqrt{p^2-4qr}\xi)}{-p \sinh(\sqrt{p^2-4qr}\xi) + \sqrt{p^2-4qr} \cosh(\sqrt{p^2-4qr}\xi) \pm \sqrt{p^2-4qr}} \right)^2, \tag{21}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,12}(\xi) &= -\chi^2 qr - \chi^2 pq \\
 & \times \left(\frac{4r \sinh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right) \cosh\left(\frac{\sqrt{p^2-4qr}}{4} \xi\right)}{-2p \sinh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right) \cosh\left(\frac{\sqrt{p^2-4qr}}{4} \xi\right)} \right. \\
 & \left. + 2\sqrt{p^2-4qr} \cosh^2\left(\frac{\sqrt{p^2-4qr}}{4} \xi\right) - \sqrt{p^2-4qr} \right) \\
 & -\chi^2 q^2 \left(\frac{4r \sinh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right) \cosh\left(\frac{\sqrt{p^2-4qr}}{4} \xi\right)}{-2p \sinh\left(\frac{\sqrt{p^2-4qr}}{2} \xi\right) \cosh\left(\frac{\sqrt{p^2-4qr}}{4} \xi\right)} \right. \\
 & \left. + 2\sqrt{p^2-4qr} \cosh^2\left(\frac{\sqrt{p^2-4qr}}{4} \xi\right) - \sqrt{p^2-4qr} \right)^2. \tag{22}
 \end{aligned}$$

Family 2: When $p^2 - 4qr < 0$ and $pq \neq 0$ or $qr \neq 0$, the trigonometric solutions of Eq. (6) are as follows:

$$\begin{aligned}
 u_{1,13}(\xi) &= -\chi^2 qr - \chi^2 pq \left(\frac{1}{2q} \left(-p + \sqrt{4qr - p^2} \tan \left(\frac{\sqrt{4qr - p^2}}{2} \xi \right) \right) \right) \\
 &\quad - \chi^2 q^2 \left(\frac{1}{2q} \left(-p + \sqrt{4qr - p^2} \tan \left(\frac{\sqrt{4qr - p^2}}{2} \xi \right) \right) \right)^2, \tag{23}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,14}(\xi) &= -\chi^2 qr - \chi^2 pq \left(-\frac{1}{2q} \left(p + \sqrt{4qr - p^2} \cot \left(\frac{\sqrt{4qr - p^2}}{2} \xi \right) \right) \right) \\
 &\quad - \chi^2 q^2 \left(-\frac{1}{2q} \left(p + \sqrt{4qr - p^2} \cot \left(\frac{\sqrt{4qr - p^2}}{2} \xi \right) \right) \right)^2, \tag{24}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,15}(\xi) &= -\chi^2 qr \\
 &\quad - \chi^2 pq \left(\frac{1}{2q} \left[-p + \sqrt{4qr - p^2} (\tan(\sqrt{4qr - p^2} \xi) \pm \sec(\sqrt{4qr - p^2} \xi)) \right] \right) \\
 &\quad - \chi^2 q^2 \left(\frac{1}{2q} \left[-p + \sqrt{4qr - p^2} (\tan(\sqrt{4qr - p^2} \xi) \pm \sec(\sqrt{4qr - p^2} \xi)) \right] \right)^2, \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,16}(\xi) &= -\chi^2 qr \\
 &\quad - \chi^2 pq \left(-\frac{1}{2q} \left[p + \sqrt{4qr - p^2} (\cot(\sqrt{4qr - p^2} \xi) \pm \csc(\sqrt{4qr - p^2} \xi)) \right] \right) \\
 &\quad - \chi^2 q^2 \left(-\frac{1}{2q} \left[p + \sqrt{4qr - p^2} (\cot(\sqrt{4qr - p^2} \xi) \pm \csc(\sqrt{4qr - p^2} \xi)) \right] \right)^2, \tag{26}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,17}(\xi) &= -\chi^2 qr \\
 &\quad - \chi^2 pq \left(\frac{1}{4q} \left[-2p + \sqrt{4qr - p^2} \left(\tan \left(\frac{\sqrt{4qr - p^2}}{4} \xi \right) - \cot \left(\frac{\sqrt{4qr - p^2}}{4} \xi \right) \right) \right] \right) \\
 &\quad - \chi^2 q^2 \left(\frac{1}{4q} \left[-2p + \sqrt{4qr - p^2} \left(\tan \left(\frac{\sqrt{4qr - p^2}}{4} \xi \right) - \cot \left(\frac{\sqrt{4qr - p^2}}{4} \xi \right) \right) \right] \right)^2, \tag{27}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,18}(\xi) &= -\chi^2 qr \\
 &\quad - \chi^2 pq \left(\frac{1}{2q} \left[-p + \frac{\pm \sqrt{(A^2 - B^2)(4qr - p^2)} - A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2} \xi)}{A \sin(\sqrt{4qr - p^2} \xi) + B} \right] \right) \\
 &\quad - \chi^2 q^2 \left(\frac{1}{2q} \left[-p + \frac{\pm \sqrt{(A^2 - B^2)(4qr - p^2)} - A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2} \xi)}{A \sin(\sqrt{4qr - p^2} \xi) + B} \right] \right)^2, \tag{28}
 \end{aligned}$$

$$\begin{aligned}
 u_{1,19}(\xi) &= -\chi^2 qr
 \end{aligned}$$

$$\begin{aligned}
 & -\chi^2 pq \left(\frac{1}{2q} \left[-p - \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} + A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2}\xi)}{A \sin(\sqrt{4qr - p^2}\xi) + B} \right] \right) \\
 & -\chi^2 q^2 \left(\frac{1}{2q} \left[-p - \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} + A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2}\xi)}{A \sin(\sqrt{4qr - p^2}\xi) + B} \right] \right)^2,
 \end{aligned} \tag{29}$$

where A and B are two non-zero real constants satisfying $A^2 - B^2 > 0$,

$$\begin{aligned}
 u_{1,20}(\xi) &= -\chi^2 qr \\
 & -\chi^2 pq \left(-\frac{2r \cos(\frac{\sqrt{4qr-p^2}}{2}\xi)}{\sqrt{4qr-p^2} \sin(\frac{\sqrt{4qr-p^2}}{2}\xi) + p \cos(\frac{\sqrt{4qr-p^2}}{2}\xi)} \right) \\
 & -\chi^2 q^2 \left(-\frac{2r \cos(\frac{\sqrt{4qr-p^2}}{2}\xi)}{\sqrt{4qr-p^2} \sin(\frac{\sqrt{4qr-p^2}}{2}\xi) + p \cos(\frac{\sqrt{4qr-p^2}}{2}\xi)} \right)^2,
 \end{aligned} \tag{30}$$

$$\begin{aligned}
 u_{1,21}(\xi) &= -\chi^2 qr \\
 & -\chi^2 pq \left(\frac{2r \sin(\frac{\sqrt{4qr-p^2}}{2}\xi)}{-p \sin(\frac{\sqrt{4qr-p^2}}{2}\xi) + \sqrt{4qr-p^2} \cos(\frac{\sqrt{4qr-p^2}}{2}\xi)} \right) \\
 & -\chi^2 q^2 \left(\frac{2r \sin(\frac{\sqrt{4qr-p^2}}{2}\xi)}{-p \sin(\frac{\sqrt{4qr-p^2}}{2}\xi) + \sqrt{4qr-p^2} \cos(\frac{\sqrt{4qr-p^2}}{2}\xi)} \right)^2,
 \end{aligned} \tag{31}$$

$$\begin{aligned}
 u_{1,22}(\xi) &= -\chi^2 qr \\
 & -\chi^2 pq \left(-\frac{2r \cos(\sqrt{4qr-p^2}\xi)}{\sqrt{4qr-p^2} \sin(\sqrt{4qr-p^2}\xi) + p \cos(\sqrt{4qr-p^2}\xi) \pm \sqrt{4qr-p^2}} \right) \\
 & -\chi^2 q^2 \left(-\frac{2r \cos(\sqrt{4qr-p^2}\xi)}{\sqrt{4qr-p^2} \sin(\sqrt{4qr-p^2}\xi) + p \cos(\sqrt{4qr-p^2}\xi) \pm \sqrt{4qr-p^2}} \right)^2,
 \end{aligned} \tag{32}$$

$$\begin{aligned}
 u_{1,23}(\xi) &= -\chi^2 qr \\
 & -\chi^2 pq \left(\frac{2r \sin(\sqrt{4qr-p^2}\xi)}{-p \sin(\sqrt{4qr-p^2}\xi) + \sqrt{4qr-p^2} \cos(\sqrt{4qr-p^2}\xi) \pm \sqrt{4qr-p^2}} \right) \\
 & -\chi^2 q^2 \left(\frac{2r \sin(\sqrt{4qr-p^2}\xi)}{-p \sin(\sqrt{4qr-p^2}\xi) + \sqrt{4qr-p^2} \cos(\sqrt{4qr-p^2}\xi) \pm \sqrt{4qr-p^2}} \right)^2,
 \end{aligned} \tag{33}$$

$$\begin{aligned}
 u_{1,24}(\xi) &= -\chi^2 qr \\
 & -\chi^2 pq \left(\frac{4r \sin(\frac{\sqrt{4qr-p^2}}{4}\xi) \cos(\frac{\sqrt{4qr-p^2}}{4}\xi)}{-2p \sin(\frac{\sqrt{4qr-p^2}}{2}\xi) \cos(\frac{\sqrt{4qr-p^2}}{4}\xi)} \right. \\
 & \left. + 2\sqrt{4qr-p^2} \cos^2(\frac{\sqrt{4qr-p^2}}{4}\xi) - \sqrt{4qr-p^2} \right)
 \end{aligned}$$

$$-\chi^2 q^2 \left(\frac{4r \sin\left(\frac{\sqrt{4qr-p^2}}{4} \xi\right) \cos\left(\frac{\sqrt{4qr-p^2}}{4} \xi\right)}{-2p \sin\left(\frac{\sqrt{4qr-p^2}}{2} \xi\right) \cos\left(\frac{\sqrt{4qr-p^2}}{4} \xi\right) + 2\sqrt{4qr-p^2} \cos^2\left(\frac{\sqrt{4qr-p^2}}{4} \xi\right) - \sqrt{4qr-p^2}} \right)^2. \tag{34}$$

Family 3: When $r = 0$ and $pq \neq 0$, the solutions of Eq. (6) are as follows:

$$u_{1,25}(\xi) = -\chi^2 qr - \chi^2 pq \left(-\frac{pd}{q(d + \cosh(p\xi) - \sinh(p\xi))} \right) - \chi^2 q^2 \left(-\frac{pd}{q(d + \cosh(p\xi) - \sinh(p\xi))} \right)^2, \tag{35}$$

$$u_{1,26}(\xi) = -\chi^2 qr - \chi^2 pq \left(\frac{-p(\cosh(p\xi) + \sinh(p\xi))}{q(d + \cosh(p\xi) - \sinh(p\xi))} \right) - \chi^2 q^2 \left(\frac{-p(\cosh(p\xi) + \sinh(p\xi))}{q(d + \cosh(p\xi) - \sinh(p\xi))} \right)^2. \tag{36}$$

Family 4: When $r = p = 0$ and $q \neq 0$, the rational solution of Eq. (6) is

$$u_{1,27}(\xi) = -\chi^2 qr - \chi^2 pq \left(-\frac{1}{q\xi + c} \right) - \chi^2 q^2 \left(-\frac{1}{q\xi + c} \right)^2, \tag{37}$$

where $\xi = \frac{x}{\alpha} \left(x + \frac{1}{\Gamma(\alpha)}\right)^\alpha - \frac{\chi^5(p^2-4qr)^2}{\alpha} \left(t + \frac{1}{\Gamma(\alpha)}\right)^\alpha$.

The graphical representation of solutions of Case 1 are given in Figs. 1–8.

Case 2:

$$a_0 = -\frac{1}{120} \chi^2 ((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr), \quad a_1 = 0, \tag{38}$$

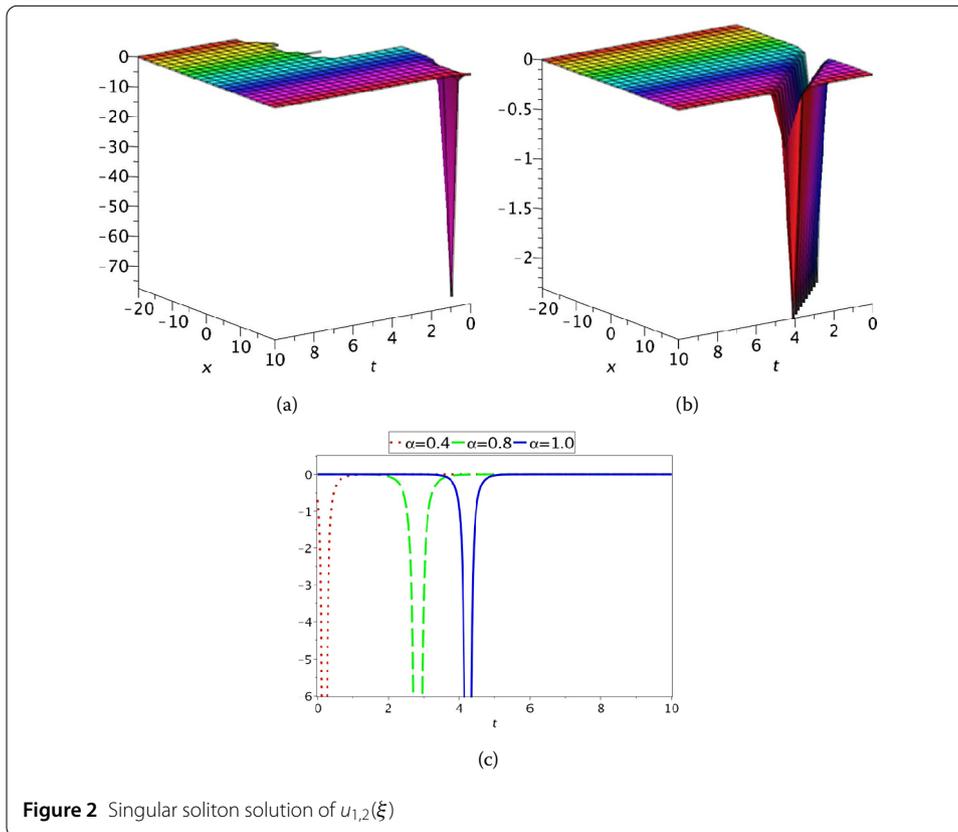
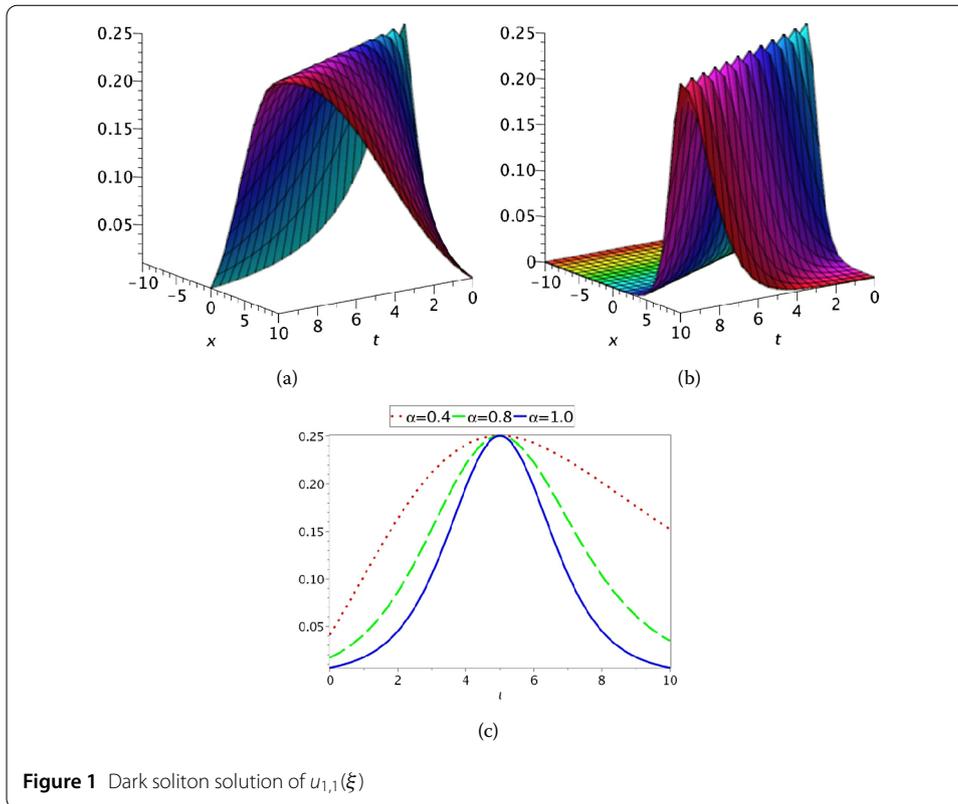
$$a_{-1} = -\chi^2 pr, \quad a_{-2} = -\chi^2 r^2, \quad a_2 = 0, \quad \lambda = -\frac{1}{8} \chi^5 (p^2 - 4qr)^2 (\sqrt{105} + 11).$$

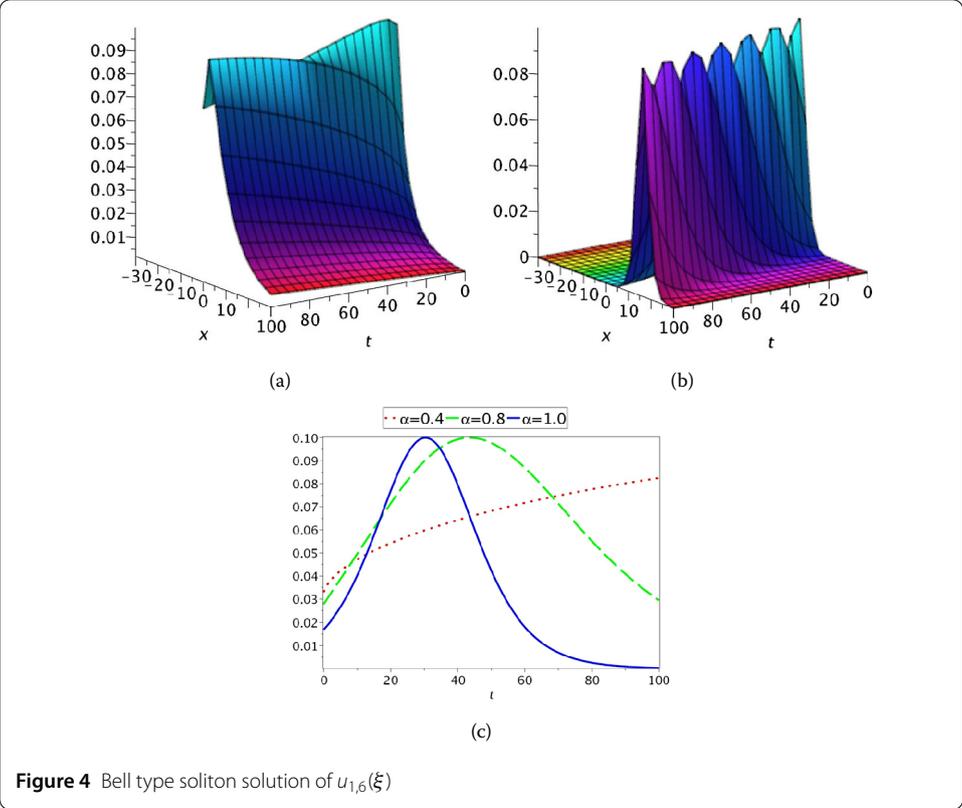
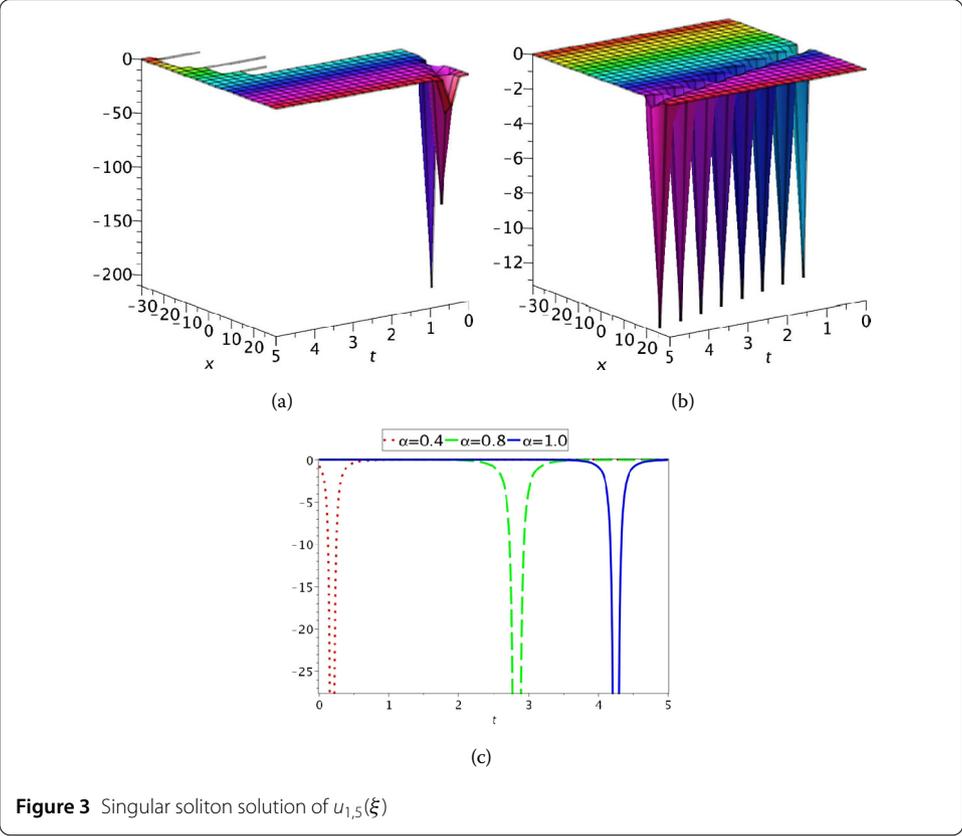
Inserting these values into Eq. (9) along with Families 1–4 of Eq. (5) and Eq. (7), we obtain the hyperbolic, periodic, and rational solutions as follows.

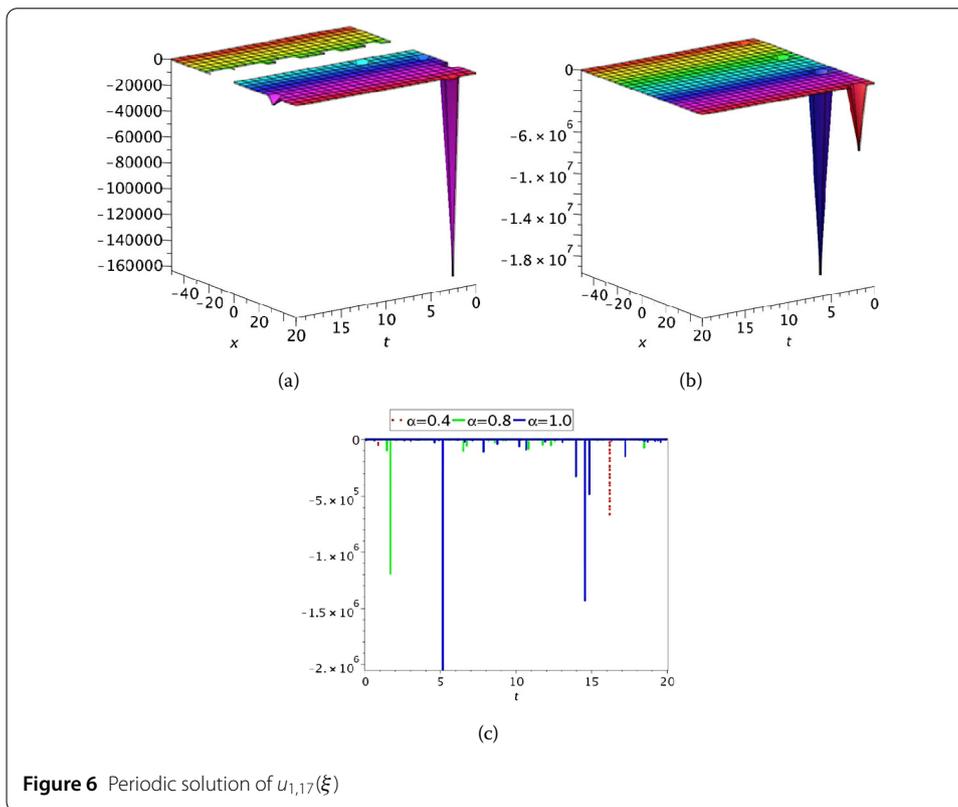
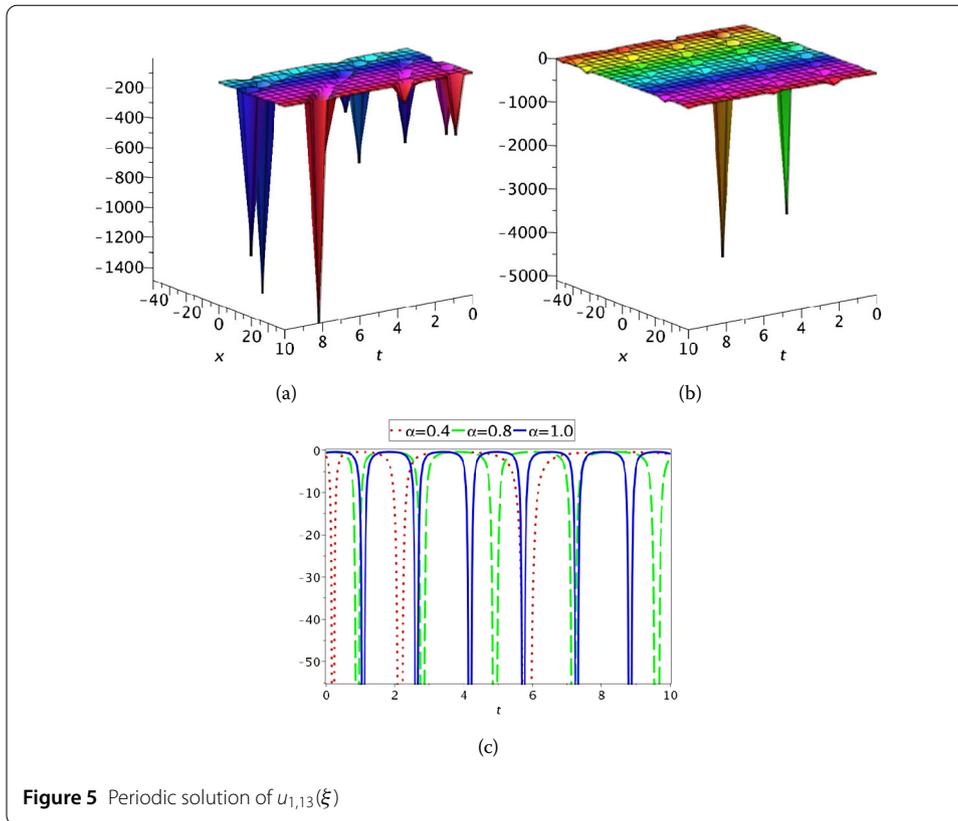
Family 1: When $p^2 - 4qr > 0$ and $pq \neq 0$ or $qr \neq 0$, the hyperbolic function solutions of Eq. (6) are as follows:

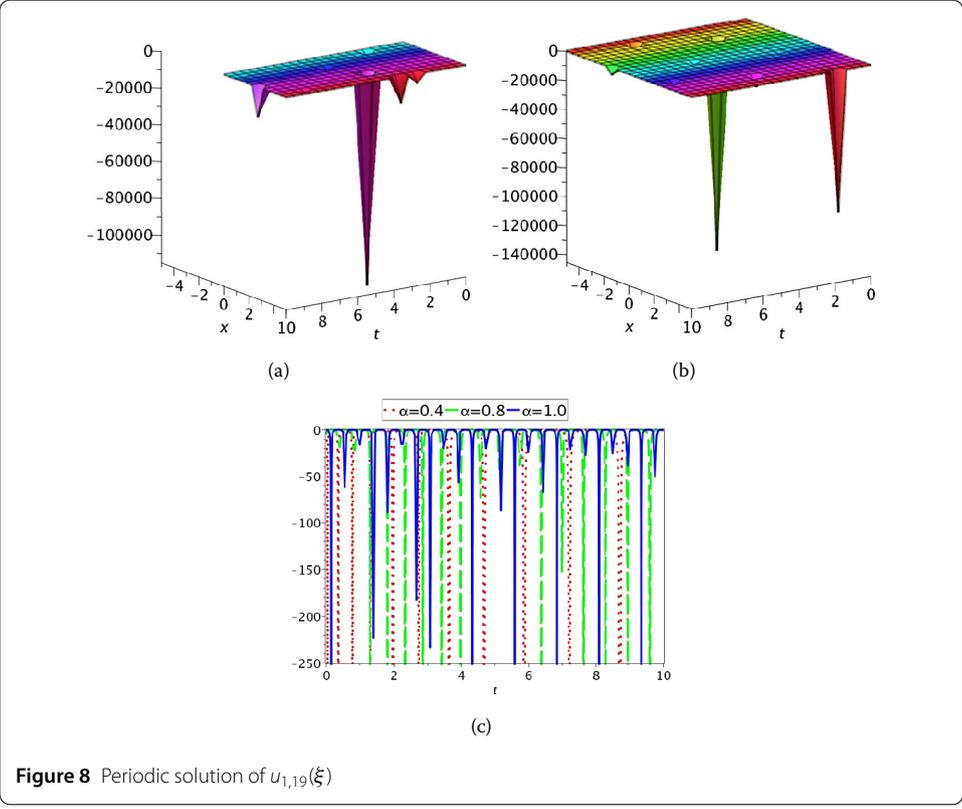
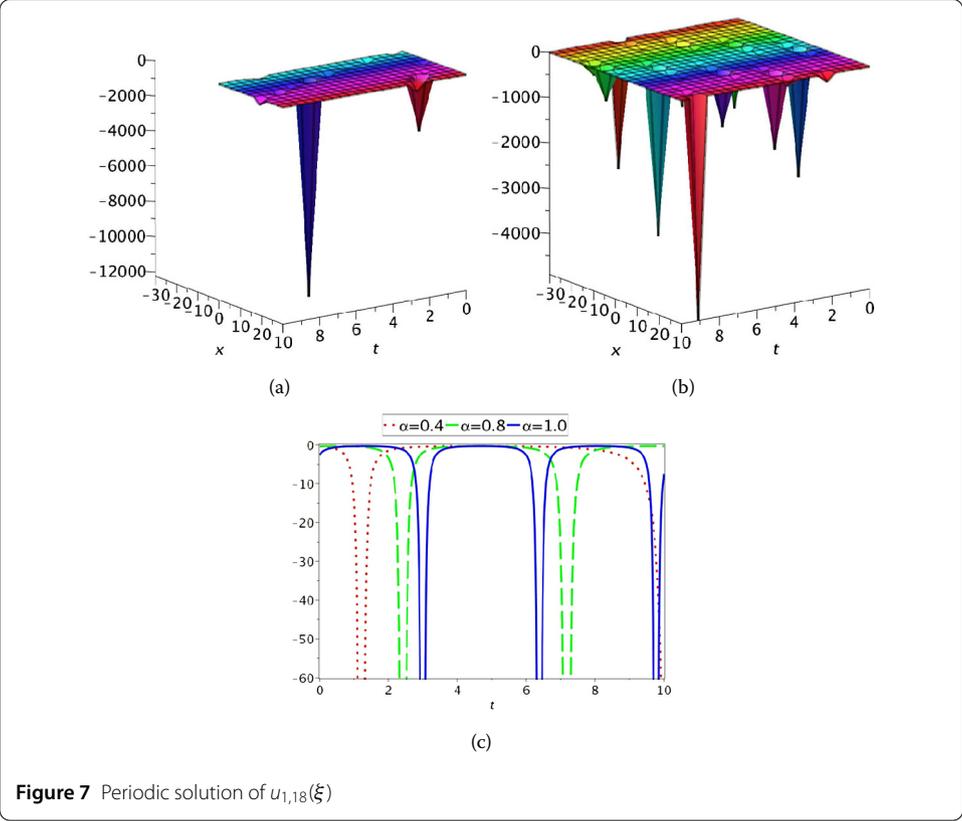
$$u_{2,1}(\xi) = -\frac{1}{120} \chi^2 ((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) - \chi^2 pr \left(-\frac{1}{2q} \left(p + \sqrt{p^2 - 4qr} \tanh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) \right) \right)^{-1} - \chi^2 r^2 \left(-\frac{1}{2q} \left(p + \sqrt{p^2 - 4qr} \tanh\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) \right) \right)^{-2}, \tag{39}$$

$$u_{2,2}(\xi) = -\frac{1}{120} \chi^2 ((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) - \chi^2 pr \left(-\frac{1}{2q} \left(p + \sqrt{p^2 - 4qr} \coth\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) \right) \right)^{-1} - \chi^2 q^2 \left(-\frac{1}{2q} \left(p + \sqrt{p^2 - 4qr} \coth\left(\frac{\sqrt{p^2 - 4qr}}{2} \xi\right) \right) \right)^{-2}, \tag{40}$$









$$\begin{aligned}
 u_{2,3}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(-\frac{1}{2q} [p + \sqrt{p^2 - 4qr}(\tanh(\sqrt{p^2 - 4qr}\xi) \pm i \operatorname{sech}(\sqrt{p^2 - 4qr}\xi))] \right)^{-1} \\
 & - \chi^2 r^2 \left(-\frac{1}{2q} [p + \sqrt{p^2 - 4qr}(\tanh(\sqrt{p^2 - 4qr}\xi) \pm i \operatorname{sech}(\sqrt{p^2 - 4qr}\xi))] \right)^{-2},
 \end{aligned} \tag{41}$$

$$\begin{aligned}
 u_{2,4}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(-\frac{1}{2q} [p + \sqrt{p^2 - 4qr}(\coth(\sqrt{p^2 - 4qr}\xi) \pm \operatorname{csch}(\sqrt{p^2 - 4qr}\xi))] \right)^{-1} \\
 & - \chi^2 r^2 \left(-\frac{1}{2q} [p + \sqrt{p^2 - 4qr}(\coth(\sqrt{p^2 - 4qr}\xi) \pm \operatorname{csch}(\sqrt{p^2 - 4qr}\xi))] \right)^{-2},
 \end{aligned} \tag{42}$$

$$\begin{aligned}
 u_{2,5}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(-\frac{1}{4q} \left[\frac{2p + \sqrt{p^2 - 4qr}}{\tanh(\frac{\sqrt{p^2 - 4qr}}{4}\xi) \pm \coth(\frac{\sqrt{p^2 - 4qr}}{4}\xi)} \right] \right)^{-1} \\
 & - \chi^2 r^2 \left(-\frac{1}{4q} \left[\frac{2p + \sqrt{p^2 - 4qr}}{\tanh(\frac{\sqrt{p^2 - 4qr}}{4}\xi) \pm \coth(\frac{\sqrt{p^2 - 4qr}}{4}\xi)} \right] \right)^{-2},
 \end{aligned} \tag{43}$$

$$\begin{aligned}
 u_{2,6}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(\frac{1}{2q} \left[-p + \frac{\sqrt{(A^2 + B^2)(p^2 - 4qr)} - A\sqrt{p^2 - 4qr} \cosh(\sqrt{p^2 - 4qr}\xi)}{A \sinh(\sqrt{p^2 - 4qr}\xi) + B} \right] \right)^{-1} \\
 & - \chi^2 r^2 \left(\frac{1}{2q} \left[-p + \frac{\sqrt{(A^2 + B^2)(p^2 - 4qr)} - A\sqrt{p^2 - 4qr} \cosh(\sqrt{p^2 - 4qr}\xi)}{A \sinh(\sqrt{p^2 - 4qr}\xi) + B} \right] \right)^{-2},
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 u_{2,7}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(\frac{1}{2q} \left[-p - \frac{\sqrt{(B^2 - A^2)(p^2 - 4qr)} + A\sqrt{p^2 - 4qr} \sinh(\sqrt{p^2 - 4qr}\xi)}{A \cosh(\sqrt{p^2 - 4qr}\xi) + B} \right] \right)^{-1} \\
 & - \chi^2 r^2 \left(\frac{1}{2q} \left[-p - \frac{\sqrt{(B^2 - A^2)(p^2 - 4qr)} + A\sqrt{p^2 - 4qr} \sinh(\sqrt{p^2 - 4qr}\xi)}{A \cosh(\sqrt{p^2 - 4qr}\xi) + B} \right] \right)^{-2},
 \end{aligned} \tag{45}$$

where A and B are two non-zero real constants satisfying $B^2 - A^2 > 0$,

$$\begin{aligned}
 u_{2,8}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(\frac{2r \cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{\sqrt{p^2 - 4qr} \sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) - p \cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)} \right)^{-1} \\
 & - \chi^2 r^2 \left(\frac{2r \cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{\sqrt{p^2 - 4qr} \sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) - p \cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)} \right)^{-2}, \tag{46}
 \end{aligned}$$

$$\begin{aligned}
 u_{2,9}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(\frac{-2r \sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{p \sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) - \sqrt{p^2 - 4qr} \cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)} \right)^{-1} \\
 & - \chi^2 r^2 \left(\frac{-2r \sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)}{p \sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) - \sqrt{p^2 - 4qr} \cosh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right)} \right)^{-2}, \tag{47}
 \end{aligned}$$

$$\begin{aligned}
 u_{2,10}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(\frac{2r \cosh(\sqrt{p^2 - 4qr}\xi)}{\sinh(\sqrt{p^2 - 4qr}\xi) - p \cosh(\sqrt{p^2 - 4qr}\xi) \pm i\sqrt{p^2 - 4qr}} \right)^{-1} \\
 & - \chi^2 r^2 \left(\frac{2r \cosh(\sqrt{p^2 - 4qr}\xi)}{\sinh(\sqrt{p^2 - 4qr}\xi) - p \cosh(\sqrt{p^2 - 4qr}\xi) \pm i\sqrt{p^2 - 4qr}} \right)^{-2}, \tag{48}
 \end{aligned}$$

$$\begin{aligned}
 u_{2,11}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(\frac{2r \sinh(\sqrt{p^2 - 4qr}\xi)}{-p \sinh(\sqrt{p^2 - 4qr}\xi) + \sqrt{p^2 - 4qr} \cosh(\sqrt{p^2 - 4qr}\xi) \pm \sqrt{p^2 - 4qr}} \right)^{-1} \\
 & - \chi^2 r^2 \left(\frac{2r \sinh(\sqrt{p^2 - 4qr}\xi)}{-p \sinh(\sqrt{p^2 - 4qr}\xi) + \sqrt{p^2 - 4qr} \cosh(\sqrt{p^2 - 4qr}\xi) \pm \sqrt{p^2 - 4qr}} \right)^{-2}, \tag{49}
 \end{aligned}$$

$$u_{2,12}(\xi) = -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr)$$

$$\begin{aligned}
 & -\chi^2 pr \left(\frac{4r \sinh\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right) \cosh\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right)}{-2p \sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) \cosh\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right)} + 2\sqrt{p^2-4qr} \cosh^2\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right) - \sqrt{p^2-4qr} \right)^{-1} \\
 & -\chi^2 r^2 \left(\frac{4r \sinh\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right) \cosh\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right)}{-2p \sinh\left(\frac{\sqrt{p^2-4qr}}{2}\xi\right) \cosh\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right)} + 2\sqrt{p^2-4qr} \cosh^2\left(\frac{\sqrt{p^2-4qr}}{4}\xi\right) - \sqrt{p^2-4qr} \right)^{-2}. \tag{50}
 \end{aligned}$$

Family 2: When $p^2 - 4qr < 0$ and $pq \neq 0$ or $qr \neq 0$, the trigonometric solutions of Eq. (6) are as follows:

$$\begin{aligned}
 u_{2,13}(\xi) &= -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(\frac{1}{2q} \left(-p + \sqrt{4qr - p^2} \tan\left(\frac{\sqrt{4qr - p^2}}{2}\xi\right) \right) \right)^{-1} \\
 & -\chi^2 r^2 \left(\frac{1}{2q} \left(-p + \sqrt{4qr - p^2} \tan\left(\frac{\sqrt{4qr - p^2}}{2}\xi\right) \right) \right)^{-2}, \tag{51}
 \end{aligned}$$

$$\begin{aligned}
 u_{2,14}(\xi) &= -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(-\frac{1}{2q} \left(p + \sqrt{4qr - p^2} \cot\left(\frac{\sqrt{4qr - p^2}}{2}\xi\right) \right) \right)^{-1} \\
 & -\chi^2 r^2 \left(-\frac{1}{2q} \left(p + \sqrt{4qr - p^2} \cot\left(\frac{\sqrt{4qr - p^2}}{2}\xi\right) \right) \right)^{-2}, \tag{52}
 \end{aligned}$$

$$\begin{aligned}
 u_{2,15}(\xi) &= -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(\frac{1}{2q} [-p + \sqrt{4qr - p^2}(\tan(\sqrt{4qr - p^2}\xi) \pm \sec(\sqrt{4qr - p^2}\xi))] \right)^{-1} \\
 & -\chi^2 r^2 \left(\frac{1}{2q} [-p + \sqrt{4qr - p^2}(\tan(\sqrt{4qr - p^2}\xi) \pm \sec(\sqrt{4qr - p^2}\xi))] \right)^{-2}, \tag{53}
 \end{aligned}$$

$$\begin{aligned}
 u_{2,16}(\xi) &= -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(-\frac{1}{2q} [p + \sqrt{4qr - p^2}(\cot(\sqrt{4qr - p^2}\xi) \pm \csc(\sqrt{4qr - p^2}\xi))] \right)^{-1} \\
 & -\chi^2 r^2 \left(-\frac{1}{2q} [p + \sqrt{4qr - p^2}(\cot(\sqrt{4qr - p^2}\xi) \pm \csc(\sqrt{4qr - p^2}\xi))] \right)^{-2}, \tag{54}
 \end{aligned}$$

$$\begin{aligned}
 & u_{2,17}(\xi) \\
 & = -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr)
 \end{aligned}$$

$$\begin{aligned}
 & -\chi^2 pr \left(\frac{1}{4q} \left[-2p + \sqrt{4qr - p^2} \left(\tan\left(\frac{\sqrt{4qr - p^2}}{4} \xi\right) - \cot\left(\frac{\sqrt{4qr - p^2}}{4} \xi\right) \right) \right] \right)^{-1} \\
 & -\chi^2 r^2 \left(\frac{1}{4q} \left[-2p + \sqrt{4qr - p^2} \left(\tan\left(\frac{\sqrt{4qr - p^2}}{4} \xi\right) - \cot\left(\frac{\sqrt{4qr - p^2}}{4} \xi\right) \right) \right] \right)^{-2},
 \end{aligned} \tag{55}$$

$$\begin{aligned}
 u_{2,18}(\xi) &= -\frac{1}{120} \chi^2 ((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(\frac{1}{2q} \left[-p + \frac{\pm\sqrt{(A^2 - B^2)(4qr - p^2)} - A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2}\xi)}{A \sin(\sqrt{4qr - p^2}\xi) + B} \right] \right)^{-1} \\
 & -\chi^2 r^2 \left(\frac{1}{2q} \left[-p + \frac{\pm\sqrt{(A^2 - B^2)(4qr - p^2)} - A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2}\xi)}{A \sin(\sqrt{4qr - p^2}\xi) + B} \right] \right)^{-2},
 \end{aligned} \tag{56}$$

$$\begin{aligned}
 u_{2,19}(\xi) &= -\frac{1}{120} \chi^2 ((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(\frac{1}{2q} \left[-p - \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} + A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2}\xi)}{A \sin(\sqrt{4qr - p^2}\xi) + B} \right] \right)^{-1} \\
 & -\chi^2 r^2 \left(\frac{1}{2q} \left[-p - \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} + A\sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2}\xi)}{A \sin(\sqrt{4qr - p^2}\xi) + B} \right] \right)^{-2},
 \end{aligned} \tag{57}$$

where A and B are two non-zero real constants satisfying $A^2 - B^2 > 0$,

$$\begin{aligned}
 u_{2,20}(\xi) &= -\frac{1}{120} \chi^2 ((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(-\frac{2r \cos\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)}{\sqrt{4qr - p^2} \sin\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right) + p \cos\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)} \right)^{-1} \\
 & -\chi^2 r^2 \left(-\frac{2r \cos\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)}{\sqrt{4qr - p^2} \sin\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right) + p \cos\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)} \right)^{-2},
 \end{aligned} \tag{58}$$

$$\begin{aligned}
 u_{2,21}(\xi) &= -\frac{1}{120} \chi^2 ((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(\frac{2r \sin\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)}{-p \sin\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right) + \sqrt{4qr - p^2} \cos\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)} \right)^{-1} \\
 & -\chi^2 r^2 \left(\frac{2r \sin\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)}{-p \sin\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right) + \sqrt{4qr - p^2} \cos\left(\frac{\sqrt{4qr - p^2}}{2} \xi\right)} \right)^{-2},
 \end{aligned} \tag{59}$$

$$\begin{aligned}
 u_{2,22}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(\frac{2r \cos(\sqrt{4qr - p^2}\xi)}{\sin(\sqrt{4qr - p^2}\xi) + p \cos(\sqrt{4qr - p^2}\xi) \pm \sqrt{4qr - p^2}} \right)^{-1} \\
 & -\chi^2 r^2 \left(\frac{2r \cos(\sqrt{4qr - p^2}\xi)}{\sin(\sqrt{4qr - p^2}\xi) + p \cos(\sqrt{4qr - p^2}\xi) \pm \sqrt{4qr - p^2}} \right)^{-2}, \tag{60}
 \end{aligned}$$

$$\begin{aligned}
 u_{2,23}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(\frac{2r \sin(\sqrt{4qr - p^2}\xi)}{-p \sin(\sqrt{4qr - p^2}\xi) + \sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2}\xi) \pm \sqrt{4qr - p^2}} \right)^{-1} \\
 & -\chi^2 r^2 \left(\frac{2r \sin(\sqrt{4qr - p^2}\xi)}{-p \sin(\sqrt{4qr - p^2}\xi) + \sqrt{4qr - p^2} \cos(\sqrt{4qr - p^2}\xi) \pm \sqrt{4qr - p^2}} \right)^{-2}, \tag{61}
 \end{aligned}$$

$$\begin{aligned}
 u_{2,24}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(\frac{4r \sin(\frac{\sqrt{4qr - p^2}}{4}\xi) \cos(\frac{\sqrt{4qr - p^2}}{4}\xi)}{-2p \sin(\frac{\sqrt{4qr - p^2}}{2}\xi) \cos(\frac{\sqrt{4qr - p^2}}{4}\xi) + 2\sqrt{4qr - p^2} \cos^2(\frac{\sqrt{4qr - p^2}}{4}\xi) - \sqrt{4qr - p^2}} \right)^{-1} \\
 & -\chi^2 r^2 \left(\frac{4r \sin(\frac{\sqrt{4qr - p^2}}{4}\xi) \cos(\frac{\sqrt{4qr - p^2}}{4}\xi)}{-2p \sin(\frac{\sqrt{4qr - p^2}}{2}\xi) \cos(\frac{\sqrt{4qr - p^2}}{4}\xi) + 2\sqrt{4qr - p^2} \cos^2(\frac{\sqrt{4qr - p^2}}{4}\xi) - \sqrt{4qr - p^2}} \right)^{-2}. \tag{62}
 \end{aligned}$$

Family 3: When $r = 0$ and $pq \neq 0$, the solutions of Eq. (6) are as follows:

$$\begin{aligned}
 u_{2,25}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & -\chi^2 pr \left(-\frac{pd}{q(d + \cosh(p\xi) - \sinh(p\xi))} \right)^{-1} \\
 & -\chi^2 r^2 \left(-\frac{pd}{q(d + \cosh(p\xi) - \sinh(p\xi))} \right)^{-2}, \tag{63}
 \end{aligned}$$

$$\begin{aligned}
 u_{2,26}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(\frac{-p(\cosh(p\xi) + \sinh(p\xi))}{q(d + \cosh(p\xi) - \sinh(p\xi))} \right)^{-1} \\
 & - \chi^2 r^2 \left(\frac{-p(\cosh(p\xi) + \sinh(p\xi))}{q(d + \cosh(p\xi) - \sinh(p\xi))} \right)^{-2}.
 \end{aligned} \tag{64}$$

Family 4: When $r = p = 0$ and $q \neq 0$, the rational solution of Eq. (6) is

$$\begin{aligned}
 u_{2,27}(\xi) = & -\frac{1}{120}\chi^2((p^2 - 4qr)\sqrt{105} + 15p^2 + 60qr) \\
 & - \chi^2 pr \left(-\frac{1}{q\xi + c} \right)^{-1} - \chi^2 r^2 \left(-\frac{1}{q\xi + c} \right)^{-2},
 \end{aligned} \tag{65}$$

where $\xi = \frac{x}{\alpha} \left(x + \frac{1}{\Gamma(\alpha)} \right)^\alpha - \frac{1}{8} \frac{\chi^5 (p^2 - 4qr)^2 (\sqrt{105} + 11)}{\alpha} \left(t + \frac{1}{\Gamma(\alpha)} \right)^\alpha$.

The graphical representation of solutions of Case 2 are given in Figs. 9–14.

4 Discussion and physical interpretation

This section gives visualization of our obtained solutions of the space–time conformable Cauchy–Dodd–Gibbon (CDG) equation. Using the generalized Riccati equation mapping (GREM) method, we came up with the solitary wave solutions of the (CDG) equation. These are generalized and closed form solutions of traveling wave type. The solutions include singular soliton solution, soliton solution, dark soliton, periodic wave solution, and bell shape soliton. For 2D graphical representation, we have opted for $\alpha = 0.4, 0.8, 1$, while

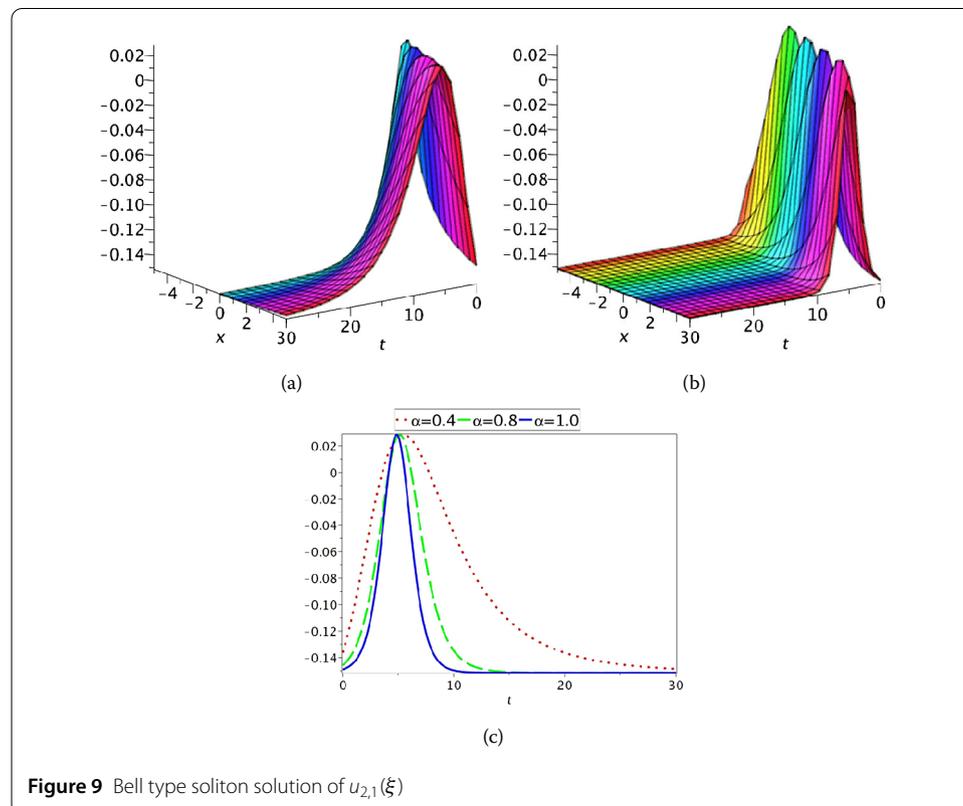
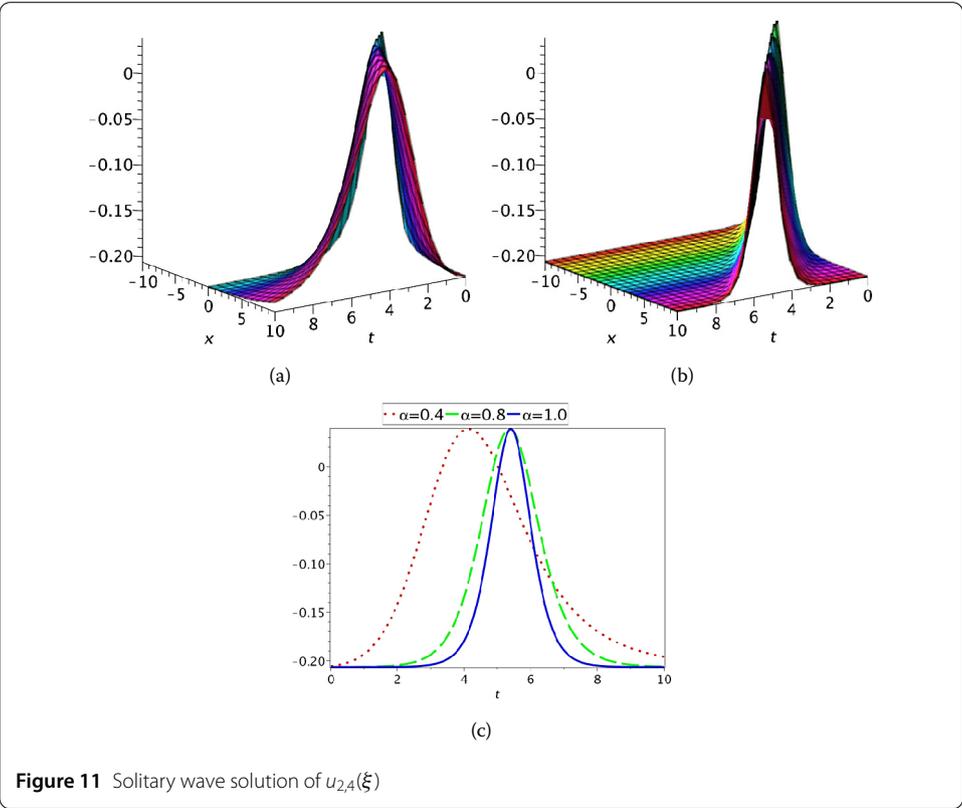
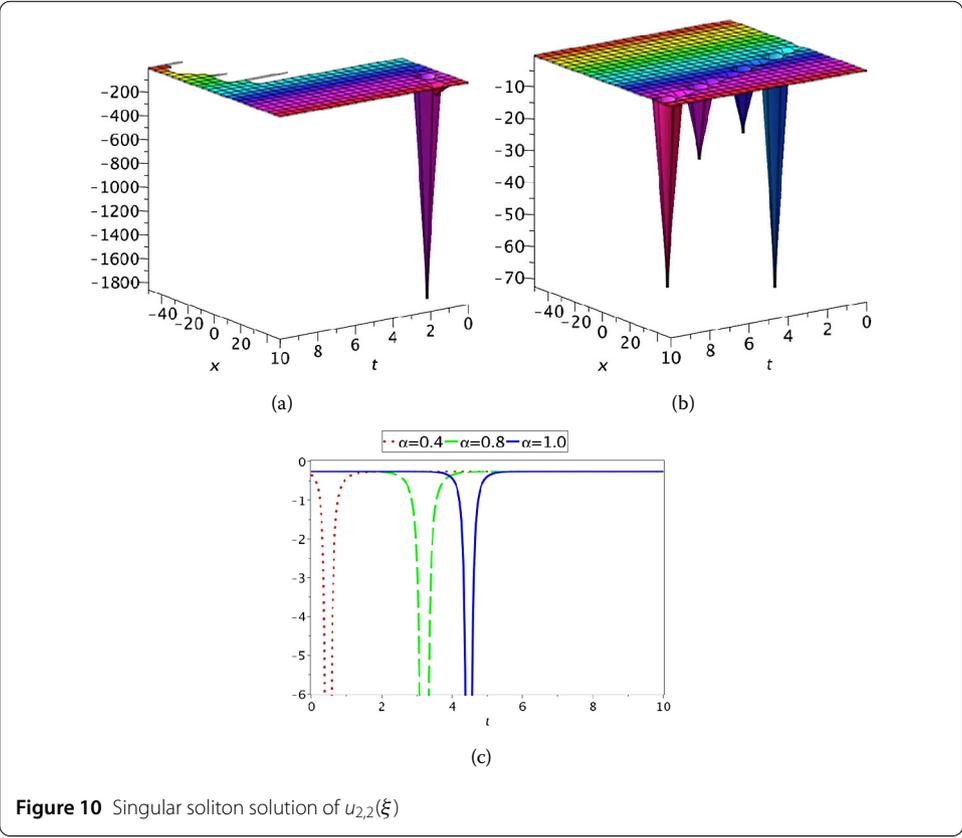


Figure 9 Bell type soliton solution of $u_{2,1}(\xi)$



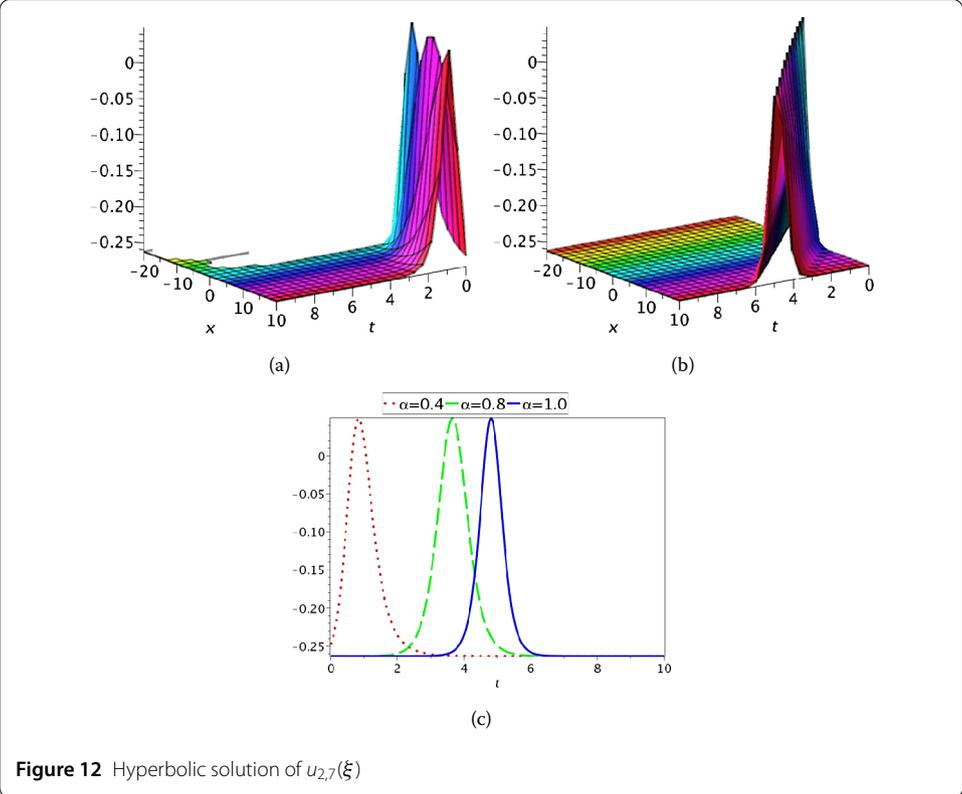


Figure 12 Hyperbolic solution of $u_{2,7}(\xi)$

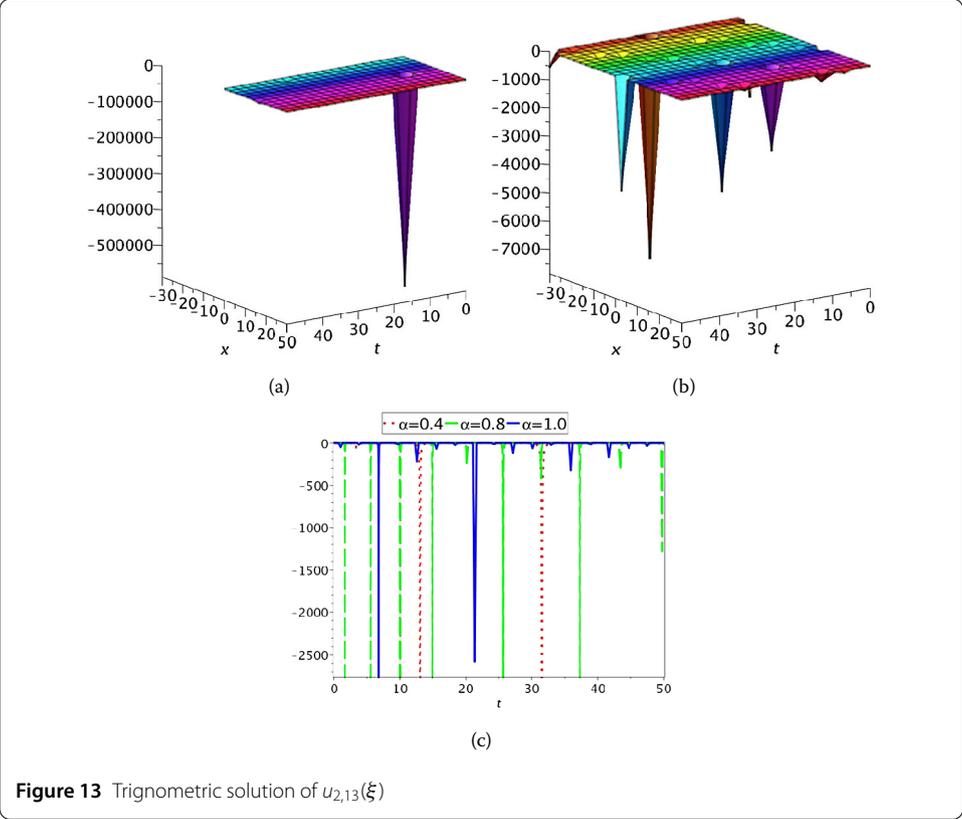
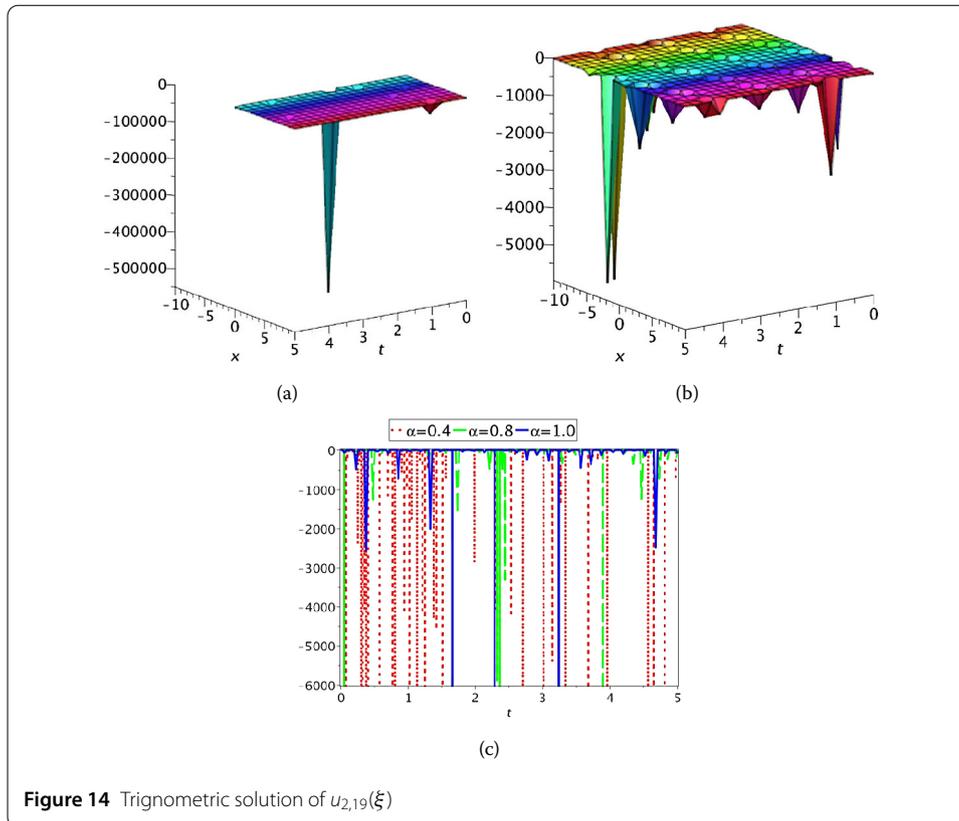


Figure 13 Trigonometric solution of $u_{2,13}(\xi)$



for 3D graphical representation, we have opted for $\alpha = 0.4, 1$, depicted in Figs. 1–14, respectively.

- I. Fig. 1 gives a dark soliton solution of $u_{1,1}(\xi)$ by considering $p = 5, q = 2, r = 3, \chi = 1$ within the interval $-10 \leq x \leq 10, 0 \leq t < 10$, and Fig. 2 gives a singular soliton solution of $u_{1,2}(\xi)$ with $p = 2, q = 0.5, r = 1, \chi = 1$ within the interval $-20 \leq x \leq 20, 0 \leq t < 10$. Figure 3 represents the singular soliton solution of $u_{1,5}(\xi)$ by considering $p = 4, q = 0.5, r = 4, \chi = 0.5$ within the interval $-30 \leq x \leq 30, 0 \leq t < 5$. The solutions $u_{1,6}(\xi)$ that describe the bell type soliton solution with $p = 4, q = 3, r = -2, \chi = 0.1, A = 3, B = 2$ for $-30 \leq x \leq 30, 0 \leq t < 100$ are depicted in Fig. 4. The solutions $u_{1,13}(\xi)$ and $u_{1,17}(\xi)$ describe the periodic solution with $p = 3, q = 4, r = 1, \chi = 0.5$ within the interval $-40 \leq x \leq 40, 0 \leq t < 10$ and $p = 5, q = 3, r = 4, \chi = 0.7$ within the interval $-50 \leq x \leq 50, 0 \leq t < 20$, respectively. Figure 7 and Fig. 8 show the profile of Eq. (28) and Eq. (29) which is periodic, by considering $p = 4, q = 3, r = 2, \chi = 0.4, A = 5, B = 3$ within the interval $-30 \leq x \leq 30, 0 \leq t < 10$ and $p = 2, q = 1, r = 3, \chi = 0.5, B = 1, A = 2$ for $-5 \leq x \leq 5, 0 \leq t < 10$, respectively.
- II. Fig. 9 shows the graphical representation of a bell type soliton solution of $u_{2,1}(\xi)$ with $p = 4, q = 2, r = 1, \chi = 0.3$ for $-5 \leq x \leq 5, 0 \leq t < 30$. The solution $u_{2,2}(\xi)$ depicts the properties of a singular soliton solution of Eq. (40) with $p = 3, q = 1, r = 1, \chi = 0.5$ which are graphed in Fig. 10. Figure 11 represents the solitary wave solution of $u_{2,4}(\xi)$ by considering $p = 5, q = 0.25, r = 0.5, \chi = 0.2$ within the interval $-10 \leq x \leq 10, 0 \leq t < 10$. The hyperbolic solution of Eq. (45) with $p = 3, q = 0.5, r = 2, \chi = 0.5, A = 2, B = 5$ for $-20 \leq x \leq 20, 0 \leq t < 10$ is depicted in Fig. 12. The trigonometric solution of $u_{2,13}(\xi)$ and $u_{2,19}(\xi)$ with $p = 1, q = 3, r = 2, \chi = 0.2$ for

$$\begin{aligned} & -30 \leq x \leq 30, 0 \leq t < 50 \text{ and } p = 3, q = 4, r = 2, \chi = 0.5, A = 5, B = 2 \text{ for} \\ & -10 \leq x \leq 10, 0 \leq t < 5, \text{ respectively.} \end{aligned}$$

5 Comparison of the obtained solutions

In this section, we discuss and make a comparison of the exact solutions. We will find the relationship between the exact solutions obtained by the generalized Riccati equation mapping method with Atangana–Baleanu fractional derivative applied on (CDG) equation and the exact solutions obtained by other methods. To the best of our knowledge, these exact solutions are new and not available in the literature. The tanh-coth [50] method was applied to investigate the fifth order Caudrey–Dodd–Gibbon equation and find out only four solutions. Neamaty et al. [51] executed the $(\frac{G'}{G})$ -expansion method and obtained only five travelling wave solutions. Yaslan et al. [52] investigated the same equation through the $(\frac{G'}{G^2})$ -expansion method and obtained only six solutions. The obtained solutions are expressed by hyperbolic, trigonometric, and rational solutions. However, we have attained twenty-seven travelling wave solutions by applying the generalized Riccati equation mapping method with Atangana–Baleanu fractional derivative. The obtained solutions might be useful to analyze the physical significance. Furthermore, for the definite values of the parameters, diverse known solitons, such as soliton, singular soliton, periodic, bell type, and dark solutions, are originated. Therefore, we might conclude that our obtained results are more general than those attained by other methods.

6 Conclusion

Generalized Riccati equation mapping approach is used to obtain periodic, hyperbolic, and rational solutions of space–time conformable Caudrey–Dodd–Gibbon (CDG) equation. These solutions include solitary wave solutions, which are dark, singular, and bell type soliton solutions, for suitable values of parameters. Our work (not reported yet in the literature) successfully finds solutions by using the conformable derivative. All the solutions are verified with the help of Maple 16. 2D and 3D graphical representations of some solutions are illustrated for the fractional orders $\alpha = 0.4, 0.8, 1$ and $\alpha = 0.4, 1$, respectively. This method can be very promising for nonlinear equations and their systems in the study of mathematical physics.

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Authors' contributions

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