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Boundary controllability of nonlocal Hilfer fractional stochastic differential systems with fractional Brownian motion and Poisson jumps

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Abstract

By using stochastic analysis, fractional analysis, compact semigroups and the Schauder fixed-point theorem, we discuss the approximate boundary controllability of a nonlocal Hilfer fractional stochastic differential system with fractional Brownian motion and a Poisson jump. In addition, we establish the sufficient conditions for exact null controllability for the same problem. Finally, an example is given to illustrate the results obtained.

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1 Introduction

Fractional calculus has been applied to the description of problems that arise in a variety of fields, including finance, physics, geomagnetics, thermodynamics, and optimal control. Fractional Brownian motion (fBm) is for a family of Gaussian processes that is indexed by the Hurst parameter $H \in (0,1)$ (see [1]). When $H = 1/2$, the fBm is a standard Brownian motion. When $H \neq 1/2$, it behaves in a way completely different from the standard Brownian motion, in particular it is neither a semi-martingale nor a Markov process. Especially, when $H > 1/2$, fBm has a long range dependence. This property makes this process useful as driving noise in models appearing in finance markets, physics, telecommunication networks, hydrology and medicine etc. (see [2–4]). Stochastic differential equations driven by fractional Brownian motion have been considered extensively by research community in various aspects due to the salient features for real world problems (see [5–12]). In addition, controllability problems for different kinds of dynamical systems have been studied by several authors (see [13–21]), and the references therein. Few authors studied the controllability for linear and nonlinear systems when the control is on the boundary (see [22–30]). Also, few authors studied the stochastic fractional differential equations with Poisson jumps. Muthukumar and Thiagu [31] studied the existence of solutions and the approximate controllability of fractional nonlocal neutral impulsive stochastic differential equations of order $1 < q < 2$ with infinite delay and Poisson jumps. Rihan et al. [32] studied the fractional stochastic differential equations with Hilfer fractional derivative, with Poisson jumps and optimal control. Chadha and Bora [33] obtained the sufficient conditions for the approximate controllability of impulsive neutral stochastic differential equations driven by Poisson jumps. Ahmed and Wang [34] established the sufficient conditions for exact null controllability of Sobolev type Hilfer fractional stochastic differential equations with fractional Brownian motion and Poisson jumps. However, the approximate boundary controllability and the null boundary controllability results for Hilfer fractional stochastic differential system with fractional Browonian motion and Poisson jumps have not yet been considered in the literature. Motivated by these facts, we in this paper investigate the sufficient conditions for approximate boundary controllability and null boundary controllability of nonlocal Hilfer fractional stochastic differential systems with fractional Brownian motion and Poisson jump in the following form:

$$\begin{cases} D_{0+}^{\nu,\mu}x(t) = \sigma x(t) + f_1(t,x(t)) + f_2(t,x(t))\frac{d\omega(t)}{dt} + G(t,x(t))\frac{dB^H(t)}{dt} \\ \qquad\qquad + \int_V h(t,x(t),\nu)\tilde{N}(dt,d\nu), \quad t \in J = (0,b], \\ \tau x(t) = B_1 u(t), \quad t \in [0,b], \\ I_{0+}^{(1-\nu)(1-\mu)}x(0) + g(x) = x_0, \end{cases} \tag{1.1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative, $0 \le \nu \le 1$, $\frac{1}{2} < \mu < 1$, the state $x(\cdot)$ takes values in the separable Hilbert space X with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and the control function $u(\cdot)$ is given in $L_2(J,U)$, the Hilbert space of admissible control functions with U a Hilbert space. Let σ be a closed, densely defined linear operator with domain $D(\sigma) \subset C(J, L_2(\Omega, X))$ and range $R(\sigma) \subset X$ and let $\tau : D(\tau) \subset C(J, L_2(\Omega, X)) \to R(\tau) \subset X$ is a linear operator. Here, $B_1 : U \to X$ is a linear continuous operator. Let $A : X \to X$ be the linear operator defined by $D(A) = \{x \in D(\sigma); \tau x = 0\}$, $Ax = \sigma x$, for $x \in D(A)$.

Suppose $\{\omega(t)\}_{t\ge 0}$ is a Wiener process defined on $(\Omega, F, \{F_t\}_{t\ge0}, P)$ with values in the Hilbert space K and $\{B^H(t)\}_{t\ge0}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$ defined on $(\Omega, F, \{F_t\}_{t\ge0}, P)$ with values in Hilbert space Y. The nonlinear operators $f_1 : J \times X \to X$, $f_2 : J \times X \to L(K, X)$, $h : J \times X \times V \to X$, $G : J \times X \to L_2^0(Y, X)$ and $g : C(J, X) \to X$ are given.

2 Preliminaries

In order to study the approximate boundary controllability and the null boundary controllability of nonlocal Hilfer fractional stochastic differential equations with fractional Brownian motion and Poisson jumps, we need the following basic definitions and lemmas.

Definition 2.1 (see [35]) The fractional integral operator of order $\mu > 0$ for a function f can be defined as

$$I^{\mu}f(t) = \frac{1}{\Gamma(\mu)}\int_0^t \frac{f(s)}{(t-s)^{1-\mu}}\,ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 (see [36]) The Hilfer fractional derivative of order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ is defined as

$$D_{0+}^{\nu,\mu} f(t) = I_{0+}^{\nu(1-\mu)} \frac{d}{dt} I_{0+}^{(1-\nu)(1-\mu)} f(t).$$

Fix a time interval $[0,b]$ and let (Ω, η, P) be a complete probability space furnished with a complete family of right continuous increasing sub σ-algebras $\{\eta_t : t \in [0,b]\}$ satisfying $\eta_t \subset \eta$. Let $(V, \Phi, \rho(dv))$ be a σ-finite measurable space. Consider the stationary Poisson point process $(p_t)_{t\geq 0}$, which is defined on (Ω, η, P) with values in V and with characteristic measure ρ. We will denote by $N(t, dv)$ the counting measure of p_t such that $\tilde{N}(t, \Theta) := E(N(t, \Theta)) = t\rho(\Theta)$ for $\Theta \in \Psi$. Define $\tilde{N}(t, dv) := N(t, dv) - t\lambda(dv)$, the Poisson martingale measure generated by p_t.

Let X, K and Y be real, separable Hilbert spaces. For the sake of convenience, we shall use the same notation $\|\cdot\|$ to denote the norms in X, K, Y, $L(K,X)$ and $L(Y,X)$ where $L(K,X)$ and $L(Y,X)$ denote, respectively, the space of all bounded linear operators from K into X and Y into X. Let $Q \in L(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$ where $\lambda_n \geq 0$ $(n = 1, 2, \ldots)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2, \ldots)$ is a complete orthonormal basis in Y.

We define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where β_n^H are real, independent fBm's. The Y-valued process is Gaussian, starts from 0, and has mean zero and covariance:

$$E\langle B^H(t), x\rangle\langle B^H(s), y\rangle = R(s,t)\langle Q(x), y\rangle \quad \text{for all } x, y \in Y \text{ and } t, s \in [0,b].$$

In order to define Wiener integrals with respect to the Q-fBm, we introduce the space $L_2^0 := L_2^0(Y,X)$ of all Q-Hilbert Schmidt operators $\psi : Y \to X$. We recall that $\psi \in L(Y,X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|_{L_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2 < \infty$$

and that the space L_2^0 equipped with the inner product $\langle \vartheta, \psi\rangle_{L_2^0} = \sum_{n=1}^{\infty} \langle \vartheta e_n, \psi e_n\rangle$ is a separable Hilbert space.

Let $\phi(s)$; $s \in [0,b]$ be a function with values in $L_2^0(Y,X)$, the Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s)\, dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda} \phi(s) e_n \, d\beta_n^H = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda} K^*(\phi e_n)(s)\, d\beta_n(s), \tag{2.1}$$

where β_n is the standard Brownian motion.

Lemma 2.1 (see [1]) *If $\psi : [0,b] \to L_2^0(Y,X)$ satisfies $\int_0^b \|\psi(s)\|^2_{L_2^0} < \infty$ then the above sum in (2.1) is well defined as an X-valued random variable and we have*

$$E\left\| \int_0^t \psi(s)\,dB^H(s) \right\|^2 \leq 2Ht^{2H-1} \int_0^t \left\| \psi(s) \right\|^2_{L_2^0} ds.$$

The collection of all strongly measurable, square-integrable, X-valued random variables, denoted by $L_2(\Omega,X)$, is a Banach space equipped with norm

$$\left\| x(\cdot) \right\|_{L_2(\Omega,X)} = \left(E\left\| x(\cdot,\omega) \right\|^2 \right)^{\frac{1}{2}},$$

where the expectation, E, is defined by $E(x) = \int_\Omega x(\omega)\,dP$.

Let $C(J,L_2(\Omega,X))$ be the Banach space of all continuous maps from J into $L_2(\Omega,X)$ satisfying the condition $\sup_{t\in J} E\|x(t)\|^2 < \infty$. Define $\bar{C} = \{x : \cdot^{(1-\nu)(1-\mu)}x(\cdot) \in C(J,L_2(\Omega,X))\}$, with norm $\|\cdot\|_{\bar{C}}$ defined by

$$\|\cdot\|_{\bar{C}} = \left(\sup_{t\in J} E\left\| t^{(1-\nu)(1-\mu)}x(t) \right\|^2 \right)^{\frac{1}{2}}.$$

Obviously, $\bar{C}$ is a Banach space.

Like [35], we denote $\bar{C}^{\nu+\mu-\nu\mu} = \{x : x \in \bar{C}, D_{0+}^{\nu+\mu-\nu\mu}x \in \bar{C}\}$, $\bar{C}^{\nu,\mu} = \{x : x \in \bar{C}, D_{0+}^{\nu,\mu}x \in \bar{C}\}$. Obviously, $\bar{C}^{\nu+\mu-\nu\mu} \subseteq \bar{C}^{\nu,\mu}$.

Also, let us introduce the set $B_r = \{\nu \in \bar{C} : \|\nu\|^2_{\bar{C}} \leq r\}$, where $r > 0$.

To establish the results, we need the following hypotheses.

(H1) $D(\sigma) \subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous relative to graph norm of $D(\sigma)$.

(H2) The operator A is the infinitesimal generator of a compact semigroup $T(t)$ on X and there exists a constant $M > 0$ such that $\|T(t)\| \leq M$.

(H3) There exists a linear operator $B : U \to X$ such that for all $u \in U$ we have $Bu \in D(\sigma)$, $\tau(Bu) = B_1u$ and $\|Bu\| \leq C\|B_1u\|$, where C is a constant.

(H4) For all $t \in (0,b]$ and $u \in U$, $T(t)Bu \in D(A)$. Moreover, there exists a positive constant $M_1 > 0$ such that $\|AT(t)\| \leq M_1$.

(H5) The function $f_1 : J \times X \to X$ satisfies the following two conditions:

(i) The function $f_1 : J \times X \to X$ is continuous;

(ii) for each positive number $r \in N$, there is a positive function $\rho_r(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \leq r} E\left\| f_1\left(t,x(t)\right) \right\|^2 \leq \rho_r(t),$$

the function $s \to \rho_r(s) \in L^1([0,t],R^+)$, and there exists a $\delta_1 > 0$ such that

$$\liminf_{r\to\infty} \frac{\int_0^t (t-s)^{\mu-1}\rho_r(s)\,ds}{r} = \delta_1 < \infty, \quad t \in J.$$

(H6) The function $f_2 : J \times X \to L(K,X)$ satisfies the following two conditions:

(i) The function $f_2 : J \times X \to L(K,X)$ is continuous;

(ii) for each positive number $r \in N$, there is a positive function $h_r(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \le r} E\big\|f_2\big(t, x(t)\big)\big\|_Q^2 \le h_r(t),$$

the function $s \to h_r(s) \in L^1([0,t], R^+)$, and there exists a $\delta_2 > 0$ such that

$$\lim_{r\to\infty} \inf \frac{\int_0^t (t-s)^{\mu-1} h_r(s)\,ds}{r} = \delta_2 < \infty, \quad t \in J.$$

(H7) The function $G : J \times X \to L_2^0(Y, X)$ satisfies the following two conditions:
(i) The function $G : J \times X \to L_2^0(Y, X)$ is continuous;
(ii) for each positive number $r \in N$, there is a positive function $k_r(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \le r} E\big\|G\big(t, x(t)\big)\big\|_{L_2^0}^2 \le k_r(t),$$

the function $s \to k_r(s) \in L^1([0,t], R^+)$, and there exists a $\delta_3 > 0$ such that

$$\lim_{r\to\infty} \inf \frac{\int_0^t (t-s)^{\mu-1} k_r(s)\,ds}{r} = \delta_3 < \infty, \quad t \in J.$$

(H8) The function $h : J \times X \times V \to X$ satisfies the following two conditions:
(i) The function $h : J \times X \times V \to X$ is continuous;
(ii) for each positive number $r \in N$, there is a positive function $\chi_r(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \le r} \int_V E\big\|h(t, x, v)\big\|^2 \lambda(dv) \le \chi_r(t),$$

the function $s \to (t-s)^{\mu-1}\chi_r(s) \in L^1([0,t], R^+)$, and there exists a $\delta_4 > 0$ such that

$$\lim_{r\to\infty} \inf \frac{\int_0^t (t-s)^{\mu-1} \chi_r(s)\,ds}{r} = \delta_4 < \infty, \quad t \in J.$$

(H9) The function $g : C(J, X) \to X$ satisfies the following two conditions:
(i) There exist positive constants M_2 and M_3 such that $\|g(x)\| \le M_2\|x\| + M_3$ for all $x \in X$;
(ii) g is completely continuous map.

Let $x(t)$ be the solution of the system (1.1). Then we can define a function $z(t) = x(t) - Bu(t)$ and from our assumption we see that $z(t) \in D(A)$. Hence (1.1) can be written in terms of A and B as

$$\begin{cases} D_{0+}^{\nu,\mu} z(t) = Az(t) + \sigma Bu(t) - BD_{0+}^{\nu,\mu} u(t) + f_1(t, x(t)) + f_2(t, x(t)) \frac{d\omega(t)}{dt} \\ \qquad\qquad + G(t, x(t)) \frac{dB^H(t)}{dt} + \int_V h(t, x(t), v)\tilde{N}(dt, dv), \quad t \in J = (0, b], \\ I_{0+}^{(1-\nu)(1-\mu)}[z(0) + Bu(0)] = I_{0+}^{(1-\nu)(1-\mu)} x(0) = x_0 - g(x). \end{cases} \tag{2.2}$$

Lemma 2.2 *From* (2.2), *problem* (1.1) *is equivalent to the integral equation*

$$\begin{aligned} x(t) &= \frac{x_0 - g(x)}{\Gamma(\nu + \mu - \nu\mu)} t^{(\nu-1)(1-\mu)} + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} Ax(s)\,ds \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} [\sigma - A] Bu(s)\,ds + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f_1\big(s, x(s)\big)\,ds \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f_2\big(s, x(s)\big)\,d\omega(s) + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} G\big(s, x(s)\big)\,dB^H(s) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} \int_V h\big(s, x(s), \nu\big) \tilde{N}(ds, d\nu). \end{aligned} \tag{2.3}$$

Proof Applying I^μ to both sides of (2.2), we obtain

$$\begin{aligned} I^\mu D_{0+}^{\nu,\mu} \big[z(t) + Bu(t)\big] &= I^\mu \bigg[Az(t) + \sigma Bu(t) + f_1\big(t, x(t)\big) + f_2\big(t, x(t)\big) \frac{d\omega(t)}{dt} \\ &\quad + G\big(t, x(t)\big) \frac{dB^H(t)}{dt} + \int_V h\big(t, x(t), \nu\big) \tilde{N}(dt, d\nu) \bigg]. \end{aligned}$$

Since $I^\mu D_{0+}^{\nu,\mu}[z(t) + Bu(t)] = I^{\mu+\nu-\mu\nu} D_{0+}^{\mu+\nu-\mu\nu}[z(t) + Bu(t)]$,

$$\begin{aligned} I^{\mu+\nu-\mu\nu} D_{0+}^{\mu+\nu-\mu\nu} \big[z(t) + Bu(t)\big] &= I^\mu \bigg[Az(t) + \sigma Bu(t) + f_1\big(t, x(t)\big) + f_2\big(t, x(t)\big) \frac{d\omega(t)}{dt} \\ &\quad + G\big(t, x(t)\big) \frac{dB^H(t)}{dt} + \int_V h\big(t, x(t), \nu\big) \tilde{N}(dt, d\nu) \bigg]. \end{aligned}$$

From properties of fractional integral, we obtain

$$\begin{aligned} z(t) + Bu(t) &= \frac{I_{0+}^{(1-\nu)(1-\mu)}[z(0) + Bu(0)]}{\Gamma(\nu + \mu - \nu\mu)} t^{(\nu-1)(1-\mu)} + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} Ax(s)\,ds \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} [\sigma - A] Bu(s)\,ds + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f_1\big(s, x(s)\big)\,ds \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f_2\big(s, x(s)\big)\,d\omega(s) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} G\big(s, x(s)\big)\,dB^H(s) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} \int_V h\big(s, x(s), \nu\big) \tilde{N}(ds, d\nu). \end{aligned}$$

Since $z(t) + Bu(t) = x(t)$,

$$\begin{aligned} x(t) &= \frac{x_0 - g(x)}{\Gamma(\nu + \mu - \nu\mu)} t^{(\nu-1)(1-\mu)} + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} Ax(s)\,ds \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} [\sigma - A] Bu(s)\,ds + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f_1\big(s, x(s)\big)\,ds \end{aligned}$$

$$+\frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} f_2\big(s,x(s)\big)\,d\omega(s)+\frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} G\big(s,x(s)\big)\,dB^H(s)$$
$$+\frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\int_V h\big(s,x(s),\nu\big)\tilde{N}(ds,d\nu).$$

For more details see [37]. □

Lemma 2.3 *If the integral equation* (2.3) *holds, then we have*

$$\begin{aligned} x(t) &= S_{\nu,\mu}(t)\big[x_0-g(x)\big]+\int_0^t \big[P_\mu(t-s)\sigma - AP_\mu(t-s)\big]Bu(s)\,ds \\ &\quad+\int_0^t P_\mu(t-s)f_1\big(s,x(s)\big)\,ds+\int_0^t P_\mu(t-s)f_2\big(s,x(s)\big)\,d\omega(s) \\ &\quad+\int_0^t P_\mu(t-s)G\big(s,x(s)\big)\,dB^H(s) \\ &\quad+\int_0^t P_\mu(t-s)\int_V h\big(s,x(s),\nu\big)\tilde{N}(ds,d\nu), \end{aligned} \tag{2.4}$$

where

$$S_{\nu,\mu}(t)=I_{0+}^{\nu(1-\mu)}P_\mu(t),\qquad P_\mu(t)=t^{\mu-1}T_\mu(t),\qquad T_\mu(t)=\int_0^\infty \mu\theta\Psi_\mu(\theta)T\big(t^\mu\theta\big)\,d\theta,$$

with

$$\Psi_\mu(\theta)=\sum_{n=1}^\infty \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\mu)},\quad 0<\mu<1,\theta\in(0,\infty),$$

is a function of Wright-type which satisfies

$$\int_0^\infty \theta^\delta\Psi_\mu(\theta)\,d\theta=\frac{\Gamma(1+\delta)}{\Gamma(1+\mu\delta)}\quad \textit{for } \theta\geq 0.$$

Proof The proof of this lemma similar to the proof of Lemma 2.12 in [38]. □

Lemma 2.4 (see [38]) *The operator $S_{\nu,\mu}$ and P_μ have the following properties*:

(i) *$\{P_\mu(t):t>0\}$ is continuous in the uniform operator topology.*

(ii) *For any fixed $t>0$, $S_{\nu,\mu}(t)$ and $P_\mu(t)$ are linear and bounded operators, and*

$$\big\|P_\mu(t)x\big\|\leq\frac{Mt^{\mu-1}}{\Gamma(\mu)}\|x\|,\qquad \big\|S_{\nu,\mu}(t)x\big\|\leq\frac{Mt^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu)+\mu)}\|x\|.$$

(iii) *$\{P_\mu(t):t>0\}$ and $\{S_{\nu,\mu}(t):t>0\}$ are strongly continuous.*

Lemma 2.5 *If the assumption* (H4) *is satisfied, then $\|AP_\mu(t)x\|\leq\frac{M_1t^{\mu-1}}{\Gamma(\mu)}\|x\|$.*

Proof We have

$$P_\mu(t)x=\mu t^{\mu-1}\int_0^\infty \mu\theta\Psi_\mu(\theta)T\big(t^\mu\theta\big)\,d\theta,\qquad \int_0^\infty \theta\Psi_\mu(\theta)\,d\theta=\frac{1}{\Gamma(1+\mu)}.$$

By using (H4),

$$\|AP_\mu(t)x\| = \left\|\mu t^{\mu-1}\int_0^\infty \mu\theta\Psi_\mu(\theta)AT(t^\mu\theta)x\,d\theta\right\| \le \frac{\mu M_1 t^{\mu-1}}{\Gamma(\mu+1)}\|x\| = \frac{M_1 t^{\mu-1}}{\Gamma(\mu)}\|x\|. \qquad \square$$

Definition 2.3 We say $x \in \bar{C}$ is a mild solution to (1.1) if it satisfies

$$\begin{aligned} x(t) &= S_{\nu,\mu}(t)\bigl[x_0 - g(x)\bigr] + \int_0^t \bigl[P_\mu(t-s)\sigma - AP_\mu(t-s)\bigr]Bu(s)\,ds \\ &\quad + \int_0^t P_\mu(t-s)f_1\bigl(s,x(s)\bigr)\,ds + \int_0^t P_\mu(t-s)f_2\bigl(s,x(s)\bigr)\,d\omega(s) \\ &\quad + \int_0^t P_\mu(t-s)G\bigl(s,x(s)\bigr)\,dB^H(s) + \int_0^t P_\mu(t-s)\int_V h\bigl(s,x(s),v\bigr)\tilde{N}(ds,dv). \end{aligned}$$

3 Approximate boundary controllability

In this section, we discuss the approximate controllability for the system (1.1), so we introduce the following linear Hilfer fractional differential system with control on the boundary:

$$\begin{cases} D_{0+}^{\nu,\mu}x(t) = \sigma x(t), & t \in J = (0,b], \\ \tau x(t) = B_1 u(t), & t \in J, \\ I_{0+}^{(1-\nu)(1-\mu)}x(0) = x_0. \end{cases} \tag{3.1}$$

It is convenient at this point to introduce the operators associated with (3.1) as

$$\begin{aligned} \Gamma_0^b &= \int_0^b (b-s)^{2(\mu-1)}\bigl[T_\mu(b-s)\sigma - AT_\mu(b-s)\bigr] \\ &\quad \times BB^*\bigl[T_\mu(b-s)\sigma - AT_\mu(b-s)\bigr]^*\,ds, \quad \frac{1}{2} < \mu < 1, \end{aligned}$$

and

$$R\bigl(b,\Gamma_0^b\bigr) = \bigl(bI + \Gamma_0^b\bigr)^{-1}, \quad b > 0,$$

where B^* and $[T_\mu(b-s)\sigma - AT_\mu(b-s)]^*$ denote the adjoint of B and $[T_\mu(b-s)\sigma - AT_\mu(b-s)]$, respectively.

Let $x(b;x_0,u)$ be the state value of (1.1) at terminal state b, corresponding to the control u and the initial value x_0. Denote by $R(b,x_0) = \{x(b;x_0,u) : u \in L_2(J,U)\}$ the reachable set of system (1.1) at terminal time b, its closure in X is denoted $\overline{R(b,x_0)}$.

Definition 3.1 (see [19]) The system (1.1) is said to be approximately controllable on the interval J if $\overline{R(b,x_0)} = L_2(\Omega,X)$.

Lemma 3.1 (see [19]) *The fractional linear control system* (3.1) *is approximately controllable on J if and only if $\lambda(\lambda I + \Gamma_0^b)^{-1} \to 0$ as $\lambda \to 0^+$.*

Lemma 3.2 *For any $\bar{x}_b \in L_2(\Omega, X)$ there exist $\bar{\psi} \in L_2(\Omega; L_2(J; L_Q))$ and $\bar{\varphi} \in L_2(\Omega; L_2(J; L_2^0))$ such that*

$$\bar{x}_b = E\bar{x}_b + \int_0^b \bar{\psi}(s)\,d\omega(s) + \int_0^b \bar{\varphi}(s)\,dB^H(s).$$

Now for any $\kappa > 0$ and $\bar{x}_b \in L_2(\Omega, X)$, we define the control function in the following form:

$$\begin{aligned} u^\kappa(t) = {} & B^*\bigl[T_\mu(b-t)\sigma - AT_\mu(b-t)\bigr]^*\bigl(\lambda I + \Gamma_0^b\bigr)^{-1}\biggl\{E\bar{x}_b - S_{\nu,\mu}(b)\bigl[x_0 - g(x)\bigr] \\ & - \int_0^b P_\mu(b-s)f_1\bigl(s,x(s)\bigr)\,ds - \int_0^b P_\mu(b-s)f_2\bigl(s,x(s)\bigr)\,d\omega(s) \\ & - \int_0^b P_\mu(b-s)G\bigl(s,x(s)\bigr)\,dB^H(s) - \int_0^b P_\mu(b-s)\int_V h\bigl(s,x(s),\nu\bigr)\tilde{N}(ds,d\nu)\biggr\}. \end{aligned}$$

Theorem 3.1 *If the assumptions* (H1)–(H9) *are satisfied, then system* (1.1) *has a mild solution on J, provided that*

$$\begin{aligned} & 36M^2\biggl[\frac{M_2^2}{\Gamma^2(\nu(1-\mu)+\mu)} + \frac{b^{\mu+2(1-\nu)(1-\mu)}[\delta_1 + \delta_2\operatorname{Tr}(Q) + 2Hb^{2H-1}\delta_3 + \delta_4]}{\mu\Gamma^2(\mu)}\biggr] \\ & \quad \times \biggl[1 + \frac{6b^{2\mu-1}\|B\|^2\|B^*\|^2(M^2\|\sigma\|^2 + M_1^2)^2}{\lambda^2(2\mu-1)\Gamma^4(\mu)}\biggr] < 1. \end{aligned} \tag{3.2}$$

Proof For any $\kappa > 0$, consider the map Φ_κ on $\bar{C}$ defined by

$$\begin{aligned} (\Phi_\kappa x)(t) = {} & S_{\nu,\mu}(t)\bigl[x_0 - g(x)\bigr] + \int_0^t \bigl[P_\mu(t-s)\sigma - AP_\mu(t-s)\bigr]Bu^\kappa(s)\,ds \\ & + \int_0^t P_\mu(t-s)f_1\bigl(s,x(s)\bigr)\,ds + \int_0^t P_\mu(t-s)f_2\bigl(s,x(s)\bigr)\,d\omega(s) \\ & + \int_0^t P_\mu(t-s)G\bigl(s,x(s)\bigr)\,dB^H(s) \\ & + \int_0^t P_\mu(t-s)\int_V h\bigl(s,x(s),\nu\bigr)\tilde{N}(ds,d\nu), \quad t \in J. \end{aligned}$$

For $t \in J$, we have

$$\begin{aligned} & E\bigl\|u^\kappa(t)\bigr\|^2 \\ & \quad \leq 6E\biggl\|B^*\bigl[T_\mu(b-t)\sigma - AT_\mu(b-t)\bigr]^*\bigl(\lambda I + \Gamma_0^b\bigr)^{-1} \\ & \qquad \times \biggl[\bar{x}_b - \int_0^b \bar{\psi}(s)\,d\omega(s) - \int_0^b \bar{\varphi}(s)\,dB^H(s)\biggr]\biggr\|^2 \\ & \qquad + 6E\bigl\|B^*\bigl[T_\mu(b-t)\sigma - AT_\mu(b-t)\bigr]^*\bigl(\lambda I + \Gamma_0^b\bigr)^{-1}S_{\nu,\mu}(b)\bigl[x_0 - g(x)\bigr]\bigr\|^2 \\ & \qquad + 6E\biggl\|B^*\bigl[T_\mu(b-t)\sigma - AT_\mu(b-t)\bigr]^*\bigl(\lambda I + \Gamma_0^b\bigr)^{-1}\int_0^b P_\mu(b-s)f_1\bigl(s,x(s)\bigr)\,ds\biggr\|^2 \\ & \qquad + 6E\biggl\|B^*\bigl[T_\mu(b-t)\sigma - AT_\mu(b-t)\bigr]^*\bigl(\lambda I + \Gamma_0^b\bigr)^{-1}\int_0^b P_\mu(b-s)f_2\bigl(s,x(s)\bigr)\,d\omega(s)\biggr\|^2 \end{aligned}$$

$$
\begin{aligned}
&+6E\left\| B^*\left[T_\mu(b-t)\sigma - AT_\mu(b-t)\right]^*\left(\lambda I + \Gamma_0^b\right)^{-1}\int_0^b P_\mu(b-s)G\big(s,x(s)\big)\,dB^H(s)\right\|^2 \\
&+36E\left\| B^*\left[T_\mu(b-t)\sigma - AT_\mu(b-t)\right]^*\left(\lambda I + \Gamma_0^b\right)^{-1}\right. \\
&\left.\times\int_0^b P_\mu(b-s)\int_V h\big(s,x(s),v\big)\tilde{N}(ds,dv)\right\|^2 \\
\le{}& \frac{6\|B\|^2(M^2\|\sigma\|^2+M_1^2)}{\lambda^2\Gamma^2(\mu)}\left[E\|\bar{x}_b\|^2 + \operatorname{Tr}(Q)\int_0^b E\big\|\bar{\psi}(s)\big\|_Q^2\,ds\right. \\
&+2Hb^{2H-1}\int_0^b E\big\|\bar{\varphi}(s)\big\|_{L_2^0}^2\,ds + \frac{M^2b^{2(\nu-1)(1-\mu)}[E\|x_0\|^2+M_2^2r+M_3^2]}{\Gamma^2(\nu(1-\mu)+\mu)} \\
&+\frac{M^2b^\mu}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}\rho_r(s)\,ds + \operatorname{Tr}(Q)\frac{M^2b^\mu}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}h_r(s)\,ds \\
&\left.+\frac{2HM^2b^{2H-1+\mu}}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}k_r(s)\,ds + \frac{M^2b^\mu}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}\chi_q(s)\,ds\right].
\end{aligned}
$$

It will be shown that the operator Φ_κ from $\bar{C}$ into itself has a fixed point.

We claim that there exists a positive number r such that $\Phi_\kappa(B_r)\subseteq B_r$. If it is not true, then, for each positive number r, there is a function $x_r(\cdot)\in B_r$, but $\Phi(x_r)\notin B_r$, that is, $\|\Phi_\kappa x_r\|_{\bar{C}}^2 > r$ for some $t = t(r)\in J$, where $t(r)$ means that t is dependent of r.

From our hypotheses together with Lemma 2.4, Lemma 2.5, the Hölder inequality and Burkholder–Gungy's inequality, we obtain

$$
\begin{aligned}
r \le{}& \|\Phi_\kappa x_r\|_{\bar{C}}^2 = \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\big\|\Phi_\kappa(x_r)(t)\big\|^2 \\
\le{}& \frac{36M^2[E\|x_0\|^2+M_2^2r+M_3^2]}{\Gamma^2(\nu(1-\mu)+\mu)} + \frac{216b^{2\mu-1}\|B\|^4(M^2\|\sigma\|^2+M_1^2)}{\lambda^2(2\mu-1)\Gamma^2(\mu)}\left[b^{2(1-\nu)(1-\mu)}\left(E\|\bar{x}_b\|^2\right.\right. \\
&\left.+\operatorname{Tr}(Q)\int_0^b E\big\|\bar{\psi}(s)\big\|_Q^2\,ds + 2Hb^{2H-1}\int_0^b E\big\|\bar{\varphi}(s)\big\|_{L_2^0}^2\,ds\right) \\
&+\frac{M^2[E\|x_0\|^2+M_2^2r+M_3^2]}{\Gamma^2(\nu(1-\mu)+\mu)} + \frac{M^2b^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}\rho_r(s)\,ds \\
&+\operatorname{Tr}(Q)\frac{M^2b^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}h_r(s)\,ds \\
&+\frac{2HM^2b^{2H-1+\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}k_r(s)\,ds \\
&\left.+\frac{M^2b^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}\chi_q(s)\,ds\right] \\
&+\frac{36M^2b^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^t(t-s)^{\mu-1}\rho_r(s)\,ds \\
&+\frac{36M^2b^{\mu+2(1-\nu)(1-\mu)}\operatorname{Tr}(Q)}{\mu\Gamma^2(\mu)}\int_0^t(t-s)^{\mu-1}h_r(s)\,ds \\
&+\frac{72HM^2b^{2H-1+\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^b(t-s)^{\mu-1}k_r(s)\,ds \\
&+\frac{36M^2b^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^t(t-s)^{\mu-1}\chi_q(s)\,ds.
\end{aligned}
\tag{3.3}
$$

Dividing both sides of (3.3) by r and taking the lower limit $r \to +\infty$, we get

$$36M^2\left[\frac{M_2^2}{\Gamma^2(\nu(1-\mu)+\mu)} + \frac{b^{\mu+2(1-\nu)(1-\mu)}[\delta_1 + \delta_2 \operatorname{Tr}(Q) + 2Hb^{2H-1}\delta_3 + \delta_4]}{\mu\Gamma^2(\mu)}\right] \\ \times \left[1 + \frac{6b^{2\mu-1}\|B\|^2\|B^*\|^2(M^2\|\sigma\|^2 + M_1^2)^2}{\lambda^2(2\mu-1)\Gamma^4(\mu)}\right] \geq 1.$$

This contradicts (3.2). Hence, for positive r, $\Phi_\kappa(B_r) \subseteq B_r$ for positive number r.

In fact, the operator Φ_κ maps B_r into a compact subset of B_r. To prove this, we first show that the set $V_r(t) = \{(\Phi_\kappa x)(t) : x \in B_r\}$ is precompact in X, for every fixed $t \in J$. This is trivial for $t = 0$, since $V_r(0) = \{x_0\}$. Let t, $0 < t \leq b$, be fixed.

For $0 < \epsilon < t$ and arbitrary $\delta > 0$, take

$$\begin{aligned}
&\bigl(\Phi_\kappa^{\epsilon,\delta} x\bigr)(t) \\
&= \frac{\mu}{\Gamma(\nu(1-\mu))}\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)T\bigl(s^\mu\theta\bigr)\bigl[x_0 - g(x)\bigr]\,d\theta\,ds \\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)\bigl[T\bigl((t-s)^\mu\theta\bigr)\sigma - AT\bigl((t-s)^\mu\theta\bigr)\bigr]Bu^\kappa(s)\,d\theta\,ds \\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta\bigr)f_1\bigl(s,x(s)\bigr)\,d\theta\,ds \\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta\bigr)f_2\bigl(s,x(s)\bigr)\,d\theta\,d\omega(s) \\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta\bigr)G\bigl(s,x(s)\bigr)\,d\theta\,dB^H(s) \\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty\int_V \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta\bigr)h\bigl(s,x(s),\nu\bigr)\tilde{N}(ds,d\nu) \\
&= \frac{\mu T(\epsilon^\mu\delta)}{\Gamma(\nu(1-\mu))}\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)T\bigl(s^\mu\theta - \epsilon^\mu\delta\bigr)\bigl[x_0 - g(x)\bigr]\,d\theta\,ds \\
&\quad + \mu T\bigl(\epsilon^\mu\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)\bigl[T\bigl((t-s)^\mu\theta - \epsilon^\mu\delta\bigr)\sigma \\
&\quad - AT\bigl((t-s)^\mu\theta - \epsilon^\mu\delta\bigr)\bigr]Bu^\kappa(s)\,d\theta\,ds \\
&\quad + \mu T\bigl(\epsilon^\mu\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta - \epsilon^\mu\delta\bigr)f_1\bigl(s,x(s)\bigr)\,d\theta\,ds \\
&\quad + \mu T\bigl(\epsilon^\mu\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta - T\bigl(\epsilon^\mu\delta\bigr)\bigr)f_2\bigl(s,x(s)\bigr)\,d\theta\,d\omega(s) \\
&\quad + \mu T\bigl(\epsilon^\mu\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta - \epsilon^\mu\delta\bigr)G\bigl(s,x(s)\bigr)\,d\theta\,dB^H(s) \\
&\quad + \mu T\bigl(\epsilon^\mu\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty\int_V \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta - \epsilon^\mu\delta\bigr) \\
&\quad \times h\bigl(s,x(s),\nu\bigr)\tilde{N}(ds,d\nu).
\end{aligned}$$

Since $u^\kappa(s)$ is bounded and $T(\epsilon^\mu\delta)$, $\epsilon^\mu\delta > 0$ is a compact operator, then the set $V_r^{\epsilon,\delta}(t) = \{(\Phi_\kappa^{\epsilon,\delta}x)(t) : x \in B_r\}$ is a precompact set in X for every ϵ, $0 < \epsilon < t$, and for all $\delta > 0$. Moreover, for every $x \in B_r$, we have

$$
\begin{aligned}
&\left\|\Phi_\kappa x - \Phi_\kappa^{\epsilon,\delta} x\right\|_{\bar{C}}^2 \\
&= \sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\|(\Phi_\kappa x)(t) - \left(\Phi_\kappa^{\epsilon,\delta} x\right)(t)\right\|^2 \\
&\leq \frac{36\mu^2 \sup_{t\in J} t^{2(1-\nu)(1-\mu)}}{\Gamma^2(\nu(1-\mu))} \\
&\quad\times E\left\|\int_0^t \int_0^\delta \theta(t-s)^{\nu(1-\mu)-1} s^{\mu-1} \Psi_\mu(\theta) T\left(s^\mu\theta\right)\left[x_0 - g(x)\right] d\theta\, ds\right\|^2 \\
&\quad + 36 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\|\mu \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\mu-1} \Psi_\mu(\theta)\left[T\left((t-s)^\mu\theta\right)\sigma\right.\right. \\
&\quad \left.\left. - AT\left((t-s)^\mu\theta\right)\right] Bu^\kappa(s)\, d\theta\, ds\right\|^2 \\
&\quad + 36 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\|\mu \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\mu-1} \Psi_\mu(\theta) T\left((t-s)^\mu\theta\right) f_1\left(s, x(s)\right) d\theta\, ds\right\|^2 \\
&\quad + 36 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} \\
&\quad\times E\left\|\mu \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\mu-1} \Psi_\mu(\theta) T\left((t-s)^\mu\theta\right) f_2\left(s, x(s)\right) d\theta\, d\omega(s)\right\| \\
&\quad + 36 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} \\
&\quad\times E\left\|\mu \int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\mu-1} \Psi_\mu(\theta) T\left((t-s)^\mu\theta\right) G\left(s, x(s)\right) d\theta\, dB^H(s)\right\|^2 \\
&\quad + 36 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} \\
&\quad\times E\left\|\mu \int_{t-\epsilon}^t \int_\delta^\infty \int_V \theta(t-s)^{\mu-1} \Psi_\mu(\theta) T\left((t-s)^\mu\theta\right) h\left(s, x(s), \nu\right) \tilde{N}(ds, d\nu)\right\|^2 \\
&\quad + \frac{36\mu^2 \sup_{t\in J} t^{2(1-\nu)(1-\mu)}}{\Gamma^2(\nu(1-\mu))} \\
&\quad\times E\left\|\int_{t-\epsilon}^t \int_\delta^\infty \theta(t-s)^{\nu(1-\mu)-1} s^{\mu-1} \Psi_\mu(\theta) T\left(s^\mu\theta\right)\left[x_0 - g(x)\right] d\theta\, ds\right\|^2 \\
&\quad + 36 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\|\mu \int_0^t \int_0^\delta \theta(t-s)^{\mu-1} \Psi_\mu(\theta)\left[T\left((t-s)^\mu\theta\right)\sigma\right.\right. \\
&\quad \left.\left. - AT\left((t-s)^\mu\theta\right)\right] Bu^\kappa(s)\, d\theta\, ds\right\|^2 \\
&\quad + 36 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\|\mu \int_0^t \int_0^\delta \theta(t-s)^{\mu-1} \Psi_\mu(\theta) T\left((t-s)^\mu\theta\right) f_1\left(s, x(s)\right) d\theta\, ds\right\|^2 \\
&\quad + 36 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\|\mu \int_0^t \int_0^\delta \theta(t-s)^{\mu-1} \Psi_\mu(\theta) T\left((t-s)^\mu\theta\right) f_2\left(s, x(s)\right) d\theta\, d\omega(s)\right\|^2 \\
&\quad + 36 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} \\
&\quad\times E\left\|\mu \int_0^t \int_0^\delta \theta(t-s)^{\mu-1} \Psi_\mu(\theta) T\left((t-s)^\mu\theta\right) G\left(s, x(s)\right) d\theta\, dB^H(s)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 36 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} \\
&\times E\left\| \mu \int_0^t \int_0^\delta \int_V \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta\big)h\big(s,x(s),\nu\big)\tilde{N}(ds,d\nu)\right\|^2.
\end{aligned}
$$

We see that, for each $x \in B_r$, $\|\Phi_\kappa x - \Phi_\kappa^{\epsilon,\delta} x\|_{\bar{C}}^2 \to 0$ as $\epsilon \to 0^+$ and $\delta \to 0^+$. Therefore, there are precompact sets arbitrarily close to the set $V_r(t)$ and so $V_r(t)$ is precompact in X.

Next we prove that the family $\{\Phi_\kappa x : x \in B_r\}$ is an equicontinuous family of functions. Let $x \in B_r$ and $t_1, t_2 \in J$ such that $0 < t_1 < t_2$, then

$$
\begin{aligned}
&E\big\|(\Phi_\kappa x)(t_2) - (\Phi_\kappa x)(t_1)\big\|^2 \\
&\quad\le 36E\big\|\big(S_{\nu,\mu}(t_2) - S_{\nu,\mu}(t_1)\big)\big[x_0 - g(x)\big]\big\|_{\bar{C}}^2 \\
&\qquad + 36E\left\|\int_{t_1}^{t_2}\big[P_\mu(t_2-s)\sigma - AP_\mu(t_2-s)\big]Bu^\kappa(s)\,ds\right\|_{\bar{C}}^2 \\
&\qquad + 36E\left\|\int_{t_1}^{t_2} P_\mu(t_2-s)f_1\big(s,x(s)\big)\,ds\right\|_{\bar{C}}^2 \\
&\qquad + 36E\left\|\int_{t_1}^{t_2} P_\mu(t_2-s)f_2\big(s,x(s)\big)\,d\omega(s)\right\|_{\bar{C}}^2 \\
&\qquad + 36E\left\|\int_{t_1}^{t_2} P_\mu(t_2-s)G\big(s,x(s)\big)\,dB^H(s)\right\|_{\bar{C}}^2 \\
&\qquad + 36E\left\|\int_{t_1}^{t_2} P_\mu(t_2-s)\int_V h\big(s,x(s),\nu\big)\tilde{N}(ds,d\nu)\right\|_{\bar{C}}^2 \\
&\qquad + 36E\bigg\|\int_0^{t_1}\big[\big(P_\mu(t_2-s)\sigma - AP_\mu(t_2-s)\big) \\
&\qquad - \big(P_\mu(t_1-s)\sigma - AP_\mu(t_1-s)\big)\big]Bu^\kappa(s)\,ds\bigg\|_{\bar{C}}^2 \\
&\qquad + 36E\left\|\int_0^{t_1}\big[P_\mu(t_2-s) - P_\mu(t_1-s)\big]f_1\big(s,x(s)\big)\,ds\right\|_{\bar{C}}^2 \\
&\qquad + 36E\left\|\int_0^{t_1}\big[P_\mu(t_2-s) - P_\mu(t_1-s)\big]f_2\big(s,x(s)\big)\,d\omega(s)\right\|_{\bar{C}}^2 \\
&\qquad + 36E\left\|\int_0^{t_1}\big[P_\mu(t_2-s) - P_\mu(t_1-s)\big]G\big(s,x(s)\big)\,dB^H(s)\right\|_{\bar{C}}^2 \\
&\qquad + 36E\left\|\int_0^{t_1}\big[P_\mu(t_2-s) - P_\mu(t_1-s)\big]\int_V h\big(s,x(s),\nu\big)\tilde{N}(ds,d\nu)\right\|_{\bar{C}}^2.
\end{aligned}
$$

From the above fact, we see that $E\|(\Phi_\kappa x)(t_2) - (\Phi_\kappa x)(t_1)\|^2$ tends to zero independently of $x \in B_r$ as $t_2 \to t_1$. Thus, $\Phi_\kappa(B_r)$ is both equicontinuous and bounded. By the Arzela–Ascoli theorem $\Phi_\kappa(B_r)$ is precompact in X. Hence Φ_κ is a completely continuous operator on X. From the Schauder fixed point theorem, Φ_κ has a fixed point in B_r. Any fixed point of Φ_κ is a mild solution of (1.1) on J. The proof is completed. □

Theorem 3.2 *Assume that* (H1)–(H9) *are satisfied. Further, if the functions* f_1, f_2, G *and* h *are uniformly bounded, then the system* (1.1) *is approximately controllable on* J.

Proof Let x_κ be a fixed point of Φ_κ. By using the stochastic Fubini theorem, it can easily be seen that

$$\begin{aligned} x^\kappa(b) &= \bar{x}_b - \lambda\big(\lambda I + \Gamma_0^b\big)^{-1}\bigg\{E\bar{x}_b - S_{\nu,\mu}(b)\big[x_0^\kappa - g\big(x^\kappa\big)\big] - \int_0^b P_\mu(b-s)f_1\big(s,x^\kappa(s)\big)\,ds \\ &\quad - \int_0^b P_\mu(b-s)f_2\big(s,x^\kappa(s)\big)\,d\omega(s) - \int_0^b P_\mu(b-s)G\big(s,x^\kappa(s)\big)\,dB^H(s) \\ &\quad - \int_0^b P_\mu(b-s)\int_V h\big(s,x(s),\nu\big)\tilde{N}(ds,d\nu) \\ &\quad + \int_0^b \bar{\psi}(s)\,d\omega(s) + \int_0^b \bar{\varphi}(s)\,dB^H(s)\bigg\}. \end{aligned} \tag{3.4}$$

It follows from the assumption on f_1, f_2, G and h that there exists $D > 0$ such that

$$\begin{aligned} &\big\|f_1\big(s,x^\kappa(s)\big)\big\|^2 \le D, \qquad \big\|f_2\big(s,x^\kappa(s)\big)\big\|_Q^2 \le D, \\ &\big\|G\big(s,x^\kappa(s)\big)\big\|_{L_2^0}^2 \le D, \qquad \big\|h\big(s,x^\kappa(s),\nu\big)\big\|^2 \le D. \end{aligned}$$

Consequently, the sequences $\{f_1(s,x^\kappa(s))\}$, $\{f_2(s,x^\kappa(s))\}$, $\{G(s,x^\kappa(s))\}$, $\{h(s,x^\kappa(s),\nu)\}$ are weakly compact in $L_2(J,X)$, $L_2(L_Q(K,X))$, $L_2(L_2^0(Y,X))$ and $L_2(J,X)$, so there are subsequences, still denoted by $\{f_1(s,x^\kappa(s))\}$, $\{f_2(s,x^\kappa(s))\}$, $\{G(s,x^\kappa(s))\}$, $\{h(s,x^\kappa(s),\nu)\}$, that are weakly convergent to $\{f_1(s)\}$, $\{f_2(s)\}$, $\{G(s)\}$, $\{h(s,\nu)\}$ in $L_2(J,X)$, $L_2(L_Q(K,X))$, $L_2(L_2^0(Y,X))$ and $L_2(J,X)$.

From Eq. (3.4), we have

$$\begin{aligned} &E\big\|x^\kappa(b) - \bar{x}_b\big\|^2 \\ &\quad \le 9E\big\|\lambda\big(\lambda I + \Gamma_0^b\big)^{-1}\big(E\bar{x}_b - S_{\nu,\mu}(b)\big[x_0^\kappa - g\big(x^\kappa\big)\big]\big)\big\|^2 \\ &\qquad + 9E\bigg\|\lambda\big(\lambda I + \Gamma_0^b\big)^{-1}\int_0^b P_\mu(b-s)\big(f_1\big(s,x^\kappa(s)\big) - f_1(s)\big)\,ds\bigg\|^2 \\ &\qquad + 9E\bigg\|\lambda\big(\lambda I + \Gamma_0^b\big)^{-1}\int_0^b P_\mu(b-s)f_1(s)\,ds\bigg\|^2 \\ &\qquad + 9E\bigg\|\lambda\big(\lambda I + \Gamma_0^b\big)^{-1}\int_0^b P_\mu(b-s)\big(f_2\big(s,x^\kappa(s)\big) - f_2(s)\big)\,d\omega(s)\bigg\|^2 \\ &\qquad + 9E\bigg\|\lambda\big(\lambda I + \Gamma_0^b\big)^{-1}\int_0^b P_\mu(b-s)f_2(s)\,d\omega(s)\bigg\|^2 \\ &\qquad + 9E\bigg\|\lambda\big(\lambda I + \Gamma_0^b\big)^{-1}\int_0^b P_\mu(b-s)\big(G\big(s,x^\kappa(s)\big) - G(s)\big)\,dB^H(s)\bigg\|^2 \\ &\qquad + 9E\bigg\|\lambda\big(\lambda I + \Gamma_0^b\big)^{-1}\int_0^b P_\mu(b-s)g(s)\,dB^H(s)\bigg\|^2 \\ &\qquad + 9E\bigg\|\lambda\big(\lambda I + \Gamma_0^b\big)^{-1}\int_0^b P_\mu(b-s)\bigg(\int_V h\big(s,x^\kappa(s),\nu\big) - \int_V h(s,\nu)\bigg)\tilde{N}(ds,d\nu)\bigg\|^2 \\ &\qquad + 9E\bigg\|\lambda\big(\lambda I + \Gamma_0^b\big)^{-1}\int_0^b P_\mu(b-s)\int_V h(s,\nu)\tilde{N}(ds,d\nu)\bigg\|^2. \end{aligned}$$

On the other hand, by Lemma 3.1, the operator $\lambda(\lambda I + \Gamma_0^b)^{-1} \to 0$ strongly as $\lambda \to 0^+$ for all $0 \le s \le b$, and $\|\lambda(\lambda I + \Gamma_0^b)^{-1}\| \le 1$ together with the Lebesgue dominated convergence theorem and the compactness of $P_\mu(t)$ implies that $E\|x_\kappa(b) - \bar{x}_b\|^2 \to 0$ as $\lambda \to 0^+$. This proves the approximate controllability of (1.1). □

4 Null boundary controllability

In this section, we investigate the sufficient conditions for exact null controllability for the system (1.1), so we consider the fractional stochastic linear system with fractional Brownian motion and control on the boundary in the form

$$\begin{cases} D_{0+}^{\nu,\mu} y(t) = \sigma\mu(t) + f_1(t) + f_2(t)\frac{d\omega(t)}{dt} + G(t)\frac{dB^H(t)}{dt}, & t \in J = (0,b], \\ \tau y(t) = B_1 u(t), & t \in J, \\ I_{0+}^{(1-\nu)(1-\mu)} y(0) = y_0, \end{cases} \tag{4.1}$$

associated with the system (1.1).

Consider

$$L_0^b u = \int_0^b \bigl[P_\mu(b-s)\sigma B - AP_\mu(b-s)B\bigr]u(s)\,ds : L_2(J,U) \to X,$$

where $L_0^b u$ has a bounded inverse operator $(L_0)^{-1}$ with values in $L_2(J,U)/\ker(L_0^b)$, and

$$\begin{aligned} &N_0^b(y, f_1, f_2, G) \\ &\quad = S_{\nu,\mu}(b)y + \int_0^b P_\mu(b-s)f_1(s)\,ds + \int_0^b P_\mu(b-s)f_2(s)\,d\omega(s) \\ &\qquad + \int_0^b P_\mu(b-s)G(s)\,dB^H(s) : X \times L_2(J,X) \to X. \end{aligned}$$

Definition 4.1 (see [16]) The system (4.1) is said to be exactly null controllable on J if $\operatorname{Im} L_0^b \supset \operatorname{Im} N_0^b$ or there exists a $\gamma > 0$ such that $\|(L_0^b)^* y\|^2 \ge \gamma \|(N_0^b)^* y\|^2$ for all $y \in X$.

Lemma 4.1 (see [39]) *Suppose that the linear system (4.1) is exactly null controllable on J. Then the linear operator $(L_0)^{-1} N_0^b : X \times L_2(J,X) \to L_2(J,U)$ is bounded and the control*

$$\begin{aligned} u(t) = -(L_0)^{-1}\biggl[&S_{\nu,\mu}(b)y_0 + \int_0^b P_\mu(b-s)f_1(s)\,ds + \int_0^b P_\mu(b-s)f_2(s)\,d\omega(s) \\ &+ \int_0^b P_\mu(b-s)G(s)\,dB^H(s)\biggr](t) \end{aligned}$$

transfers the system (4.1) from y_0 to 0, where L_0 is the restriction of L_0^b to $[\ker L_0^b]^\perp$.

To prove the null controllability for the system (1.1), we need in addition the hypothesis:

(H10) The linear system (4.1) is exactly null controllable on J.

Theorem 4.1 *If the hypotheses* (H1)–(H10) *are satisfied, then the boundary control system* (1.1) *is exactly null controllable on J provided that*

$$36M^2\left[\frac{M_2^2}{\Gamma^2(\nu(1-\mu)+\mu)}+\frac{b^{\mu+2(1-\nu)(1-\mu)}[\delta_1+\delta_2\operatorname{Tr}(Q)+2Hb^{2H-1}\delta_3+\delta_4]}{\mu\Gamma^2(\mu)}\right]$$
$$\times\left[1+\frac{b^{2\mu-1}\|(L_0)^{-1}\|^2\|B\|^2(M^2\|\sigma\|^2+M_1^2)}{(2\mu-1)\Gamma^2(\mu)}\right]<1. \tag{4.2}$$

Proof For an arbitrary $x(\cdot)$ define the operator Φ on $\bar{C}$ as follows:

$$\begin{aligned}(\Phi x)(t) &= S_{\nu,\mu}(t)\big(x_0-g(x)\big)+\int_0^t\big[P_\mu(t-s)\sigma-AP_\mu(t-s)\big]Bu(s)\,ds\\
&\quad+\int_0^t P_\mu(t-s)f_1\big(s,x(s)\big)\,ds+\int_0^t P_\mu(t-s)f_2\big(s,x(s)\big)\,d\omega(s)\\
&\quad+\int_0^t P_\mu(t-s)G\big(s,x(s)\big)\,dB^H(s)\\
&\quad+\int_0^t P_\mu(t-s)\int_V h\big(s,x(s),v\big)\tilde{N}(ds,dv),\quad t\in J,\end{aligned}$$

where

$$\begin{aligned}u(t) &= -(L_0)^{-1}\bigg\{S_{\nu,\mu}(b)\big(x_0-g(x)\big)+\int_0^b P_\mu(b-s)f_1\big(s,x(s)\big)\,ds\\
&\quad+\int_0^b P_\mu(b-s)f_2\big(s,x(s)\big)\,d\omega(s)\,ds+\int_0^b P_\mu(b-s)g\big(s,x(s)\big)\,dB^H(s)\\
&\quad+\int_0^b P_\mu(b-s)\int_V h\big(s,x(s),v\big)\tilde{N}(ds,dv)\bigg\}(t).\end{aligned}$$

It will be shown that the operator Φ from $\bar{C}$ into itself has a fixed point.

We claim that there exists a positive number r such that $\Phi(B_r)\subseteq B_r$. If it is not true, then, for each positive number r, there is a function $x_r(\cdot)\in B_r$, but $\Phi(x_r)\notin B_r$, that is, $\|\Phi x_r\|_{\bar{C}}^2>r$ for some $t=t(r)\in J$, where $t(r)$ means that t is dependent of r.

From our hypotheses together with Lemma 2.4, Lemma 2.5, the Hölder inequality and Burkholder–Gungy's inequality, we obtain

$$\begin{aligned}r &\le \|\Phi x_r\|_{\bar{C}}^2=\sup_{t\in J}t^{2(1-\nu)(1-\mu)}E\big\|\Phi(x_r)(t)\big\|^2\\
&\le \frac{36M^2[E\|x_0\|^2+M_2^2r+M_3^2]}{\Gamma^2(\nu(1-\mu)+\mu)}+\frac{36b^{2\mu-1}\|(L_0)^{-1}\|^2\|B\|^2(M^2\|\sigma\|^2+M_1^2)}{(2\mu-1)\Gamma^2(\mu)}\\
&\quad\times\bigg\{\frac{M^2[E\|x_0\|^2+M_2^2r+M_3^2]}{\Gamma^2(\nu(1-\mu)+\mu)}+\frac{M^2b^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}\rho_r(s)\,ds\\
&\quad+\operatorname{Tr}(Q)\frac{M^2b^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}h_r(s)\,ds\\
&\quad+\frac{2HM^2b^{2H-1+\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^b(b-s)^{\mu-1}k_r(s)\,ds\\
&\quad+\frac{M^2b^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)}\int_0^t(t-s)^{\mu-1}\chi_q(s)\,ds\bigg\}\end{aligned}$$

$$+ \frac{36M^2 b^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)} \int_0^t (t-s)^{\mu-1}\rho_r(s)\,ds$$
$$+ \frac{36M^2 b^{\mu+2(1-\nu)(1-\mu)}\operatorname{Tr}(Q)}{\mu\Gamma^2(\mu)} \int_0^t (t-s)^{\mu-1}h_r(s)\,ds$$
$$+ \frac{72HM^2 b^{2H-1+\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)} \int_0^b (t-s)^{\mu-1}k_r(s)\,ds$$
$$+ \frac{36M^2 b^{\mu+2(1-\nu)(1-\mu)}}{\mu\Gamma^2(\mu)} \int_0^t (t-s)^{\mu-1}\chi_q(s)\,ds. \tag{4.3}$$

Dividing both sides of (4.3) by r and taking the lower limit $r \to +\infty$, we get

$$36M^2\left[\frac{M_2^2}{\Gamma^2(\nu(1-\mu)+\mu)} + \frac{b^{\mu+2(1-\nu)(1-\mu)}[\delta_1 + \delta_2\operatorname{Tr}(Q) + 2Hb^{2H-1}\delta_3 + \delta_4]}{\mu\Gamma^2(\mu)}\right]$$
$$\times\left[1 + \frac{b^{2\mu-1}\|(L_0)^{-1}\|^2\|B\|^2(M^2\|\sigma\|^2 + M_1^2)}{(2\mu-1)\Gamma^2(\mu)}\right] \geq 1.$$

This contradicts (4.2). Hence, for positive r, $\Phi(B_r) \subseteq B_r$ for positive number r.

In fact, the operator Φ maps B_r into a compact subset of B_r. To prove this, we first show that the set $V_r(t) = \{(\Phi x)(t) : x \in B_r\}$ is precompact in X, for every fixed $t \in J$. This is trivial for $t = 0$, since $V_r(0) = \{x_0\}$. Let t, $0 < t \leq b$, be fixed.

For $0 < \epsilon < t$ and arbitrary $\delta > 0$, take

$$\begin{aligned}
&\big(\Phi^{\epsilon,\delta}x\big)(t)\\
&= \frac{\mu}{\Gamma(\nu(1-\mu))}\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)T\big(s^\mu\theta\big)\big[x_0 - g(x)\big]\,d\theta\,ds\\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)\big[T\big((t-s)^\mu\theta\big)\sigma - AT\big((t-s)^\mu\theta\big)\big]Bu(s)\,d\theta\,ds\\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta\big)f_1\big(s,x(s)\big)\,d\theta\,ds\\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta\big)f_2\big(s,x(s)\big)\,d\theta\,d\omega(s)\\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta\big)G\big(s,x(s)\big)\,d\theta\,dB^H(s)\\
&\quad + \mu\int_0^{t-\epsilon}\int_\delta^\infty\int_V \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta\big)h\big(s,x(s),v\big)\,d\theta\tilde{N}(ds,dv)\\
&= \frac{\mu T(\epsilon^\mu\delta)}{\Gamma(\nu(1-\mu))}\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)T\big(s^\mu\theta - \epsilon^\mu\delta\big)\big[x_0 - g(x)\big]\,d\theta\,ds\\
&\quad + \mu T\big(\epsilon^\mu\delta\big)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)\big[T\big((t-s)^\mu\theta - \epsilon^\mu\delta\big)\sigma\\
&\quad - AT\big((t-s)^\mu\theta - \epsilon^\mu\delta\big)\big]Bu(s)\,d\theta\,ds\\
&\quad + \mu T\big(\epsilon^\mu\delta\big)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta - \epsilon^\mu\delta\big)f_1\big(s,x(s)\big)\,d\theta\,ds\\
&\quad + \mu T\big(\epsilon^\mu\delta\big)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta - T\big(\epsilon^\mu\delta\big)\big)f_2\big(s,x(s)\big)\,d\theta\,d\omega(s)
\end{aligned}$$

$$
\begin{aligned}
&+ \mu T\bigl(\epsilon^{\mu}\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta-\epsilon^\mu\delta\bigr)G\bigl(s,x(s)\bigr)\,d\theta\,dB^H(s)\\
&+ \mu T\bigl(\epsilon^{\mu}\delta\bigr)\int_0^{t-\epsilon}\int_\delta^\infty\int_V \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta-\epsilon^\mu\delta\bigr)\\
&\times h\bigl(s,x(s),v\bigr)\,d\theta\tilde{N}(ds,dv).
\end{aligned}
$$

Since $T(\epsilon^\mu\delta)$, $\epsilon^\mu\delta > 0$ is a compact operator, the set $V_r^{\epsilon,\delta}(t) = \{(\Phi^{\epsilon,\delta}x)(t) : x \in B_r\}$ is a precompact set in X for every ϵ, $0 < \epsilon < t$, and for all $\delta > 0$. Moreover, for every $x \in B_r$, we have

$$
\begin{aligned}
&\bigl\|\Phi x - \Phi^{\epsilon,\delta}x\bigr\|_{\bar{C}}^2\\
&\quad= \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\bigl\|(\Phi x)(t)-\bigl(\Phi^{\epsilon,\delta}x\bigr)(t)\bigr\|^2\\
&\quad\le \frac{36\mu^2\sup_{t\in J}t^{2(1-\nu)(1-\mu)}}{\Gamma^2(\nu(1-\mu))}\\
&\quad\quad\times E\biggl\|\int_0^t\int_0^\delta \theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)T\bigl(s^\mu\theta\bigr)\bigl[x_0-g(x)\bigr]\,d\theta\,ds\biggr\|^2\\
&\quad\quad+ 36\|B\|^2\bigl\|(L_0)^{-1}\bigr\|^2\sup_{t\in J}t^{2(1-\nu)(1-\mu)}E\biggl\|\mu\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)\bigl[T\bigl((t-s)^\mu\theta\bigr)\sigma\\
&\quad\quad- AT\bigl((t-s)^\mu\theta\bigr)\bigr]\biggl\{S_{\nu,\mu}(b)\bigl[x_0-g(x)\bigr]+\int_0^b P_\mu(b-\zeta)f_1\bigl(\zeta,x(\zeta)\bigr)\,d\zeta\\
&\quad\quad+ \int_0^b P_\mu(b-\zeta)f_2\bigl(\zeta,x(\zeta)\bigr)\,d\omega(\zeta)+\int_0^b P_\mu(b-\zeta)G\bigl(\zeta,x(\zeta)\bigr)\,dB^H(\zeta)\\
&\quad\quad+ \int_0^b P_\mu(b-s)\int_V h\bigl(\zeta,x(\zeta),v\bigr)\tilde{N}(d\zeta,dv)\biggr\}\,d\theta\,ds\biggr\|^2\\
&\quad\quad+ 36\sup_{t\in J}t^{2(1-\nu)(1-\mu)}E\biggl\|\mu\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta\bigr)f_1\bigl(s,x(s)\bigr)\,d\theta\,ds\biggr\|^2\\
&\quad\quad+ 36\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\quad\quad\times E\biggl\|\mu\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta\bigr)f_2\bigl(s,x(s)\bigr)\,d\theta\,d\omega(s)\biggr\|^2\\
&\quad\quad+ 36\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\quad\quad\times E\biggl\|\mu\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta\bigr)G\bigl(s,x(s)\bigr)\,d\theta\,dB^H(s)\biggr\|^2\\
&\quad\quad+ 36\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\quad\quad\times E\biggl\|\mu\int_{t-\epsilon}^t\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\bigl((t-s)^\mu\theta\bigr)\,d\theta\int_V h\bigl(s,x(s),v\bigr)\tilde{N}(ds,dv)\biggr\|^2\\
&\quad\quad+ 36\|B\|^2\bigl\|(L_0)^{-1}\bigr\|^2\sup_{t\in J}t^{2(1-\nu)(1-\mu)}E\biggl\|\mu\int_0^t\int_0^\delta \theta(t-s)^{\mu-1}\Psi_\mu(\theta)\bigl[T\bigl((t-s)^\mu\theta\bigr)\sigma\\
&\quad\quad- AT\bigl((t-s)^\mu\theta\bigr)\bigr]\biggl\{S_{\nu,\mu}(b)\bigl[x_0-g(x)\bigr]+\int_0^b P_\mu(b-\zeta)f_1\bigl(\zeta,x(\zeta)\bigr)\,d\zeta
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^b P_\mu(b-\zeta)f_2\big(\zeta,x(\zeta)\big)\,d\omega(\zeta)+\int_0^b P_\mu(b-\zeta)G\big(\zeta,x(\zeta)\big)\,dB^H(\zeta)\\
&+\int_0^b P_\mu(b-s)\int_V h\big(\zeta,x(\zeta),v\big)\tilde{N}(d\zeta,dv)\Big\}\,d\theta\,ds\Big\|^2\\
&+36\sup_{t\in J}t^{2(1-\nu)(1-\mu)}E\Big\|\mu\int_0^t\int_0^\delta \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta\big)f_1\big(s,x(s)\big)\,d\theta\,ds\Big\|^2\\
&+36\sup_{t\in J}t^{2(1-\nu)(1-\mu)}E\Big\|\mu\int_0^t\int_0^\delta \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta\big)f_2\big(s,x(s)\big)\,d\theta\,d\omega(s)\Big\|^2\\
&+36\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\times E\Big\|\mu\int_0^t\int_0^\delta \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta\big)G\big(s,x(s)\big)\,d\theta\,dB^H(s)\Big\|^2\\
&+36\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\times E\Big\|\mu\int_0^t\int_0^\delta \theta(t-s)^{\mu-1}\Psi_\mu(\theta)T\big((t-s)^\mu\theta\big)\,d\theta\int_V h\big(s,x(s),v\big)\tilde{N}(ds,dv)\Big\|^2.
\end{aligned}
$$

We see that, for each $x\in B_r$, $\|\Phi x-\Phi^{\epsilon,\delta}x\|_{\tilde{C}}^2\to 0$ as $\epsilon\to 0^+$ and $\delta\to 0^+$. Therefore, there are precompact sets arbitrarily close to the set $V_r(t)$ and so $V_r(t)$ is precompact in X.

Next we prove that the family $\{\Phi x : x\in B_r\}$ is an equicontinuous family of functions. Let $x\in B_r$ and $t_1,t_2\in J$ such that $0<t_1<t_2$, then

$$
\begin{aligned}
&E\big\|(\Phi x)(t_2)-(\Phi x)(t_1)\big\|^2\\
&\quad\le 36E\big\|\big(S_{\nu,\mu}(t_2)-S_{\nu,\mu}(t_1)\big)\big[x_0-g(x)\big]\big\|^2\\
&\qquad+36\|B\|^2\big\|(L_0)^{-1}\big\|^2E\Big\|\int_{t_1}^{t_2}\big[P_\mu(t_2-s)\sigma-AP_\mu(t_2-s)\big]\Big\{S_{\nu,\mu}(b)\big[x_0-g(x)\big]\\
&\qquad+\int_0^b P_\mu(b-\zeta)f_1\big(\zeta,x(\zeta)\big)\,d\zeta+\int_0^b P_\mu(b-\zeta)f_2\big(\zeta,x(\zeta)\big)\,d\omega(\zeta)\\
&\qquad+\int_0^b P_\mu(b-\zeta)G\big(\zeta,x(\zeta)\big)\,dB^H(\zeta)+\int_0^b P_\mu(b-\zeta)\int_V h\big(\zeta,x(\zeta),v\big)\tilde{N}(d\zeta,dv)\Big\}\,ds\Big\|^2\\
&\qquad+36E\Big\|\int_{t_1}^{t_2}P_\mu(t_2-s)f_1\big(s,x(s)\big)\,ds\Big\|^2+36E\Big\|\int_{t_1}^{t_2}P_\mu(t_2-s)f_2\big(s,x(s)\big)\,d\omega(s)\Big\|^2\\
&\qquad+36E\Big\|\int_{t_1}^{t_2}P_\mu(t_2-s)G\big(s,x(s)\big)\,dB^H(s)\Big\|^2\\
&\qquad+36E\Big\|\int_{t_1}^{t_2}P_\mu(t_2-s)\int_V h\big(s,x(s),v\big)\tilde{N}(ds,dv)\Big\|^2\\
&\qquad+36\|B\|^2\big\|(L_0)^{-1}\big\|^2E\Big\|\int_0^{t_1}\big[P_\mu(t_2-s)\sigma-P_\mu(t_1-s)\sigma-AP_\mu(t_2-s)+AP_\mu(t_1-s)\big]\\
&\qquad\times\Big\{S_{\nu,\mu}(b)\big[x_0-g(x)\big]+\int_0^b P_\mu(b-\zeta)f_1\big(\zeta,x(\zeta)\big)\,d\zeta\\
&\qquad+\int_0^b P_\mu(b-\zeta)f_2\big(\zeta,x(\zeta)\big)\,d\omega(\zeta)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^b P_\mu(b-\zeta)G\big(\zeta,x(\zeta)\big)\,dB^H(\zeta)+\int_0^b P_\mu(b-\zeta)\int_V h\big(\zeta,x(\zeta),\nu\big)\tilde{N}(d\zeta,dv)\Big\}\,ds\Big\|^2 \\
&+36E\Big\|\int_0^{t_1}\big[P_\mu(t_2-s)-P_\mu(t_1-s)\big]f_1\big(s,x(s)\big)\,ds\Big\|^2 \\
&+36E\Big\|\int_0^{t_1}\big[P_\mu(t_2-s)-P_\mu(t_1-s)\big]f_2\big(s,x(s)\big)\,d\omega(s)\Big\|^2 \\
&+36E\Big\|\int_0^{t_1}\big[P_\mu(t_2-s)-P_\mu(t_1-s)\big]G\big(s,x(s)\big)\,dB^H(s)\Big\|^2 \\
&+36E\Big\|\int_0^{t_1}\big[P_\mu(t_2-s)-P_\mu(t_1-s)\big]\int_V h\big(s,x(s),\nu\big)\tilde{N}(ds,dv)\Big\|^2.
\end{aligned}
$$

From the above fact, we see that $E\|(\Phi x)(t_2)-(\Phi x)(t_1)\|^2$ tends to zero independently of $x\in B_r$ as $t_2\to t_1$. Thus, $\Phi(B_r)$ is both equicontinuous and bounded. By the Arzela–Ascoli theorem $\Phi(B_r)$ is precompact in X. Hence Φ is a completely continuous operator on X. From the Schauder fixed point theorem, Φ has a fixed point in B_r. Any fixed point of Φ is a mild solution of (1.1) on J. Therefore the system (1.1) is exact null controllable on J. □

5 Applications

Let us consider the nonlocal Hilfer fractional stochastic partial differential system with fractional Brownian motion and Poisson jump in the following form:

$$
\begin{cases}
D_{0+}^{\nu,\frac{4}{7}}(x(t,\xi))=\frac{\partial^2}{\partial\xi^2}x(t,\xi)+F_1(t,x(t,\xi))+F_2(t,x(t,\xi))\frac{d\omega(t)}{dt}\\
\qquad\qquad\qquad +F_3(t,x(t,\xi))\frac{dB^H(t)}{dt}\\
\qquad\qquad\qquad +\int_V F_4(t,x(t,\xi),\nu)\tilde{N}(dt,dv),\quad t\in J=(0,b],\xi\in\Pi,\\
x(t,\xi)=u(t,\xi),\quad t\in J,\xi\in\Gamma,\\
I_{0+}^{\frac{3(1-\nu)}{7}}x(0,\xi)+\sum_{i=1}^m c_i x(t_i,\xi)=x_0(\xi),\quad \xi\in\Pi,
\end{cases}
\tag{5.1}
$$

where $D_{0+}^{\nu,\frac{4}{7}}$ is the Hilfer fractional derivative, $0\le\nu\le1$, $\mu=\frac{4}{7}$, m is a positive integer, $0<t_1<t_2<\cdots<t_m<b$, Π is a bounded and open subset of R^n with sufficiently smooth boundary Γ, $\omega(t)$ is Wiener process, B^H is a fractional Brownian motion with Hurst parameter $H\in(\frac{1}{2},1)$ and $u\in L_2(\Gamma)$.

The functions $x(t)(\xi)=x(t,\xi)$, $f_1(t,x(t))(\xi)=F_1(t,x(t,\xi))$, $f_2(t,x(t))(\xi)=F_2(t,x(t,\xi))$, $G(t,x(t))(\xi)=F_3(t,x(t,\xi))$, $h(t,x(t),\nu)(\xi)=F_1(t,x(t,\xi),\nu)$, and $g(x)(\xi)=\sum_{i=1}^m c_i x(t_i,\xi)$.

Let $X=Y=K=L_2(\Pi)$, $U=L_2(\Gamma)$ $B_1=I$, the identity operator and $\sigma x=\frac{\partial^2 x}{\partial\xi^2}$ with domain $D(\sigma)=\{x\in L_2(\Pi):\frac{\partial^2 x}{\partial\xi^2}\in L_2(\Pi)\}$.

The operator θ is the trace operator such that $\theta x=x|_\Gamma$ is well defined and belongs to $H^{-1/2}(\Gamma)$ for each $x\in D(\sigma)$.

Define the operator $A:D(A)\subset X\to X$ given by $Ax=\Delta x$ with domain $D(A)=H_0^1(\Pi)\cup H^2(\Pi)$ where $H^k(\Pi)$, $H^\mu(\Gamma)$ and $H_0^1(\Pi)$ are the usual Sobolev spaces on Π, Γ. Then A can be written as

$$
Ax=\sum_{n=1}^\infty(-n^2)(x,x_n)x_n,\quad x\in D(A),
$$

where $x_n(y)=\sqrt{2}\sin ny$, $n=1,2,3,\ldots$, is the orthogonal set of eigenvectors of A.

It is well known that A generates a compact semigroup $\{T(t), t \geq 0\}$ in X and

$$T(t)x = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t}(x, x_n)x_n, \quad x \in X,$$

with

$$\|T(t)\| \leq e^{-t} \leq 1.$$

Moreover, the two operators $S_{\nu,\frac{4}{7}}(t)$ and $P_{\frac{4}{7}}(t)$ can be defined by

$$S_{\nu,\frac{4}{7}}(t)x = \frac{4}{7\Gamma(\frac{3\nu}{7})} \int_0^t \int_0^{\infty} \theta(t-s)^{\frac{3\nu}{7}-1} s^{\frac{3}{7}} \xi_{\frac{4}{7}}(\theta) T\bigl(t^{\frac{4}{7}}\theta\bigr)x \, d\theta \, ds,$$

$$P_{\frac{4}{7}}(t)x = \frac{4}{7} \int_0^{\infty} \theta(t-s)^{\frac{3}{7}} \xi_{\frac{4}{7}}(\theta) T\bigl((t-s)^{\frac{4}{7}}\theta\bigr)x \, d\theta.$$

Clearly,

$$\|P_{\frac{4}{7}}(t)\| \leq \frac{t^{\frac{3}{7}}}{\Gamma(\frac{4}{7})}, \qquad \|S_{\nu,\frac{4}{7}}(t)\| \leq \frac{t^{\frac{3}{7}(\nu-1)}}{\Gamma(\frac{3}{7}\nu + \frac{4}{7})}.$$

Define the fractional Brownian motion in Y by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta^H(t) e_n,$$

where $H \in (\frac{1}{2}, 1)$ and $\{\beta_n^H\}_{n\in N}$ is a sequence of one-dimensional fractional Brownian motions that are mutually independent.

Also, we define the linear operator $B : L_2(\Gamma) \to L_2(\Pi)$ by $Bu = \nu_u$, where ν_u is the unique solution to the Dirichlet boundary value problem,

$$\Delta \nu_u = 0 \quad \text{in } \Pi,$$

$$\nu_u = u \quad \text{in } \Gamma.$$

We consider the fractional linear system

$$\begin{cases} D_{0+}^{\nu,\frac{4}{7}}(x(t,\xi)) = \frac{\partial^2}{\partial \xi^2} x(t,\xi) + F_1(t,\xi) + F_2(t,\xi) \frac{d\omega(t)}{dt} \\ \qquad\qquad + F_3(t,\xi) \frac{dB^H(t)}{dt}, \quad t \in J = (0,b], \xi \in \Pi, \\ x(t,\xi) = u(t,\xi), \quad t \in J, \xi \in \Gamma, \\ I_{0+}^{\frac{3(1-\nu)}{7}} x(0,\xi) = x_0(\xi), \quad \xi \in \Pi. \end{cases} \tag{5.2}$$

The system (5.2) has exact null controllability if there is a $\gamma > 0$, such that

$$\int_0^b \bigl\| B^* \bigl[P_{\frac{4}{7}}(b-s)\sigma - A P_{\frac{4}{7}}(b-s) \bigr]^* y \bigr\|^2 \, ds \geq \gamma \biggl[\bigl\| S^*_{\nu,\frac{4}{7}}(b) y \bigr\|^2 + \int_0^b \bigl\| P^*_{\frac{4}{7}}(b-s) y \bigr\|^2 \, ds \biggr],$$

or equivalently

$$\int_0^b \left\| B\left[P_{\frac{4}{7}}(b-s)\sigma - AP_{\frac{4}{7}}(b-s)\right]y \right\|^2 ds \geq \gamma \left[\left\| S_{\nu,\frac{4}{7}}(b)y \right\|^2 + \int_0^b \left\| P_{\frac{4}{7}}(b-s)y \right\|^2 ds \right].$$

Hence, the linear fractional system (5.2) is exactly null controllable on J. So the hypothesis (H10) is satisfied. Hence, all the hypotheses of Theorem 4.1 are satisfied and

$$\begin{aligned} &36M^2\left[\frac{M_2^2}{\Gamma^2(\nu(1-\mu)+\mu)} + \frac{b^{\mu+2(1-\nu)(1-\mu)}[\delta_1 + \delta_2 \operatorname{Tr}(Q) + 2Hb^{2H-1}\delta_3 + \delta_4]}{\mu\Gamma^2(\mu)}\right] \\ &\quad \times \left[1 + \frac{b^{2\mu-1}\|(L_0)^{-1}\|^2\|B\|^2(M^2\|\sigma\|^2 + M_1^2)}{(2\mu-1)\Gamma^2(\mu)}\right] < 1. \end{aligned}$$

Then the system (5.1) is exactly null controllable on J.

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