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Oscillation and nonoscillation for Caputo–Hadamard impulsive fractional differential inclusions

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Abstract

In this paper, the concept of upper and lower solutions method combined with the fixed point theorem is used to investigate the existence of oscillatory and nonoscillatory solutions for a class of initial value problem for Caputo–Hadamard impulsive fractional differential inclusions.

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1 Introduction

Fractional differential equations and integrals are valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, numerous applications have been addressed in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. For examples and details, we refer the reader to the monographs [2, 4, 5, 16, 19, 21], and a series of recent research articles; see [23–27] and the references therein. Recently, many researchers studied different fractional problems involving the Caputo and Hadamard derivatives; see, for example, [3, 6, 7]. Some classes of fractional differential equations on unbounded domains have been considered in [13]. Sufficient conditions for the oscillation of solutions of ordinary and fractional differential equations are given in [15, 22]. On the other hand, oscillation and nonoscillation solutions of impulsive equations have been discussed in [11, 12, 14].

The method of upper and lower solutions has been successfully applied to the study of the existence of solutions for ordinary and fractional differential equations and inclusions. See the monograph [20] and the paper [1, 10], and the references therein.

This paper deals with the existence of oscillatory and nonoscillatory solutions for the following class of initial value problems for the Caputo–Hadamard impulsive fractional differential inclusion:

$$ {}^{\mathrm{Hc}}D_{t_k}^{\alpha} y(t) \in F\bigl(t, y(t)\bigr), \quad \text{a.e. } t \in J = (t_k, t_{k+1}), \tag{1}$$

$$ y\bigl(t_k^+\bigr) = I_k\bigl(y\bigl(t_k^-\bigr)\bigr), \quad k = 1, \ldots, \tag{2}$$

$$y(1) = y_*, \tag{3}$$

where ${}^{\mathrm{Hc}}D^{\alpha}_{t_k}$ is the Caputo–Hadamard fractional derivative of order $0 < \alpha \leq 1$, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $y_* \in \mathbb{R}$, $I_k \in C(\mathbb{R},\mathbb{R})$, $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} < \cdots < \infty$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1,\ldots$.

This paper initiates the study of oscillatory and nonoscillatory solutions for impulsive fractional differential inclusions involving the Caputo–Hadamard fractional derivative.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Let $C(J,\mathbb{R})$ be the space of all continuous functions from J into $\mathbb{R}$.

$$\|y\|_\infty = \sup_{t\in J}\bigl|y(t)\bigr|.$$

Let $BC(J,\mathbb{R})$ be the Banach space of all continuous and bounded functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup_{t\in J}\bigl|y(t)\bigr|,$$

and let $L^1(J,\mathbb{R})$ be the Banach space of Lebesgue integrable functions $y : J \longrightarrow \mathbb{R}$ with the norm

$$\|y\|_{L^1} = \int_1^T \bigl|y(t)\bigr|\,dt.$$

By $L^\infty(J,\mathbb{R})$ we denote the Banach space of measurable functions $y : J \longrightarrow \mathbb{R}$ which are essentially bounded, with the norm

$$\|y\|_{L^\infty} = \inf\bigl\{c > 0 : \bigl|y(t)\bigr| \leq c, \text{for a.e. } t \in J\bigr\}.$$

Denote by $AC(J,\mathbb{R})$ the space of absolutely continuous functions from J into $\mathbb{R}$.

For a given Banach space $(X, \|\cdot\|)$, we set

$$\begin{aligned}
P_{\mathrm{cl}}(X) &= \bigl\{Y \in \mathcal{P}(X) : Y \text{ closed}\bigr\},\\
P_{\mathrm{b}}(X) &= \bigl\{Y \in \mathcal{P}(X) : Y \text{ bounded}\bigr\},\\
P_{\mathrm{cp}}(X) &= \bigl\{Y \in \mathcal{P}(X) : Y \text{ compact}\bigr\},\\
P_{\mathrm{cp,cv}}(X) &= \bigl\{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\bigr\}.
\end{aligned}$$

A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for all $B \in P_{\mathrm{b}}(X)$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\}$).

G is called upper semicontinuous (u.s.c.) on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denote by Fix G. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \to d\bigl(y, G(t)\bigr) = \inf\bigl\{|y-z| : z \in G(t)\bigr\}$$

is measurable.

Lemma 2.1 ([17]) *Let G be a completely continuous multivalued map with nonempty compact values, then G is u.s.c. if and only if G has a closed graph.*

Definition 2.2 A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if:

(1) $t \to F(t,u)$ is measurable for each $u \in \mathbb{R}$;
(2) $u \to F(t,u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in C(J,\mathbb{R})$, define the set of selection of F by

$$S_{F\circ y} = \bigl\{v \in L^1\bigl([1,T],\mathbb{R}\bigr) : v(t) \in F\bigl(t, y(t)\bigr) \text{ a.e. } t \in [1,T]\bigr\}.$$

Let (X,d) be a metric space induced from the normed space $(X,|\cdot|)$. The function $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\Bigl\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\Bigr\}$$

is known as the Hausdorff–Pompeiu metric. For more details on multivalued maps see the books of Hu and Papageorgiou [17].

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. Let $\delta = t\frac{d}{dt}$, and set

$$AC^n_\delta(J,\mathbb{R}) = \bigl\{y : J \longrightarrow \mathbb{R}, \delta^{n-1}y(t) \in AC(J,\mathbb{R})\bigr\}.$$

Definition 2.3 ([19]) The Hadamard fractional integral of order $r > 0$ for a function $h \in L^1([1,+\infty),\mathbb{R})$ is defined as

$${}^{\mathrm{H}}I^r h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log\frac{t}{s}\right)^{r-1} \frac{h(s)}{s}\, ds,$$

provided the integral exists for a.e. $t > 1$.

Example 2.4 Let $q > 0$. Then

$${}^{\mathrm{H}}I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q}; \quad \text{for a.e. } t \in [1,+\infty).$$

Definition 2.5 ([19]) The Hadamard fractional derivative of order $r > 0$ applied to the function $h \in AC^n_\delta([1,+\infty),\mathbb{R})$ is defined as

$$\left({}^{\mathrm{H}}D^q_1 h\right)(t) = \delta^n\left({}^{\mathrm{H}}I^{n-r}_1 h\right)(t),$$

where $n-1 < r < n, n = [r] + 1$, and $[r]$ is the integer part of r.

Definition 2.6 ([18]) For a given function $h \in AC^n_\delta([a,b],\mathbb{R})$, such that $0 < a < b$, the Caputo–Hadamard fractional derivative of order $r > 0$ is defined as follows:

$${}^{\mathrm{Hc}}D^r y(t) = {}^{\mathrm{H}}D^r\left[y(s) - \sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log\frac{s}{a}\right)^k\right](t),$$

where $\operatorname{Re}(\alpha) \geq 0$ and $n = [\operatorname{Re}(\alpha)] + 1$.

Lemma 2.7 ([18]) *Let $y \in AC^n_\delta([a,b],\mathbb{R})$ or $C^n_\delta([a,b],\mathbb{R})$ and $\alpha \in \mathbb{C}$. Then*

$${}^{\mathrm{H}}I^r\left({}^{\mathrm{Hc}}D^r y\right)(t) = y(t) - \sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log\frac{t}{a}\right)^k.$$

3 Main results

we consider the space,

$$\begin{aligned} PC(J,\mathbb{R}) = \bigl\{&y : J \to \mathbb{R}, y \in C\bigl((t_k,t_{k+1}],\mathbb{R}\bigr), k = 0,\ldots, \\ &\text{and there exist}, y\bigl(t_k^+\bigr) \text{ and } y\bigl(t_k^-\bigr), k = 1,\ldots, \text{with } y\bigl(t_k^-\bigr) = y(t_k)\bigr\}. \end{aligned}$$

This set is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t\in J}\bigl|y(t)\bigr|.$$

Let us start by defining what we mean by a solution of problem (1)–(3).

Definition 3.1 A function $y \in PC \cap AC((t_k,t_{k+1}),\mathbb{R})$, $k = 0,\ldots,$ is said to be a solution of (1)–(3) if y satisfies the inclusion ${}^{\mathrm{Hc}}D^\alpha_{t_k} y(t) \in F(t,y(t))$ a.e. on (t_k,t_{k+1}) and conditions $y(t_k^+) = I_k(y(t_k^-))$, $k = 1,\ldots$, $y(1) = y_*$.

The following concept of lower and upper solutions was introduced by Benchohra and Boucherif [8, 9] for initial initial value problems for impulsive differential inclusions of first order. This will the basic tool in the approach that follows.

Definition 3.2 A function $u \in PC \cap AC((t_k,t_{k+1}),\mathbb{R})$, $k = 0,\ldots,$ is said to be a lower solution of (1)–(3) if there exists $v_1 \in L^1(J,\mathbb{R})$ such that $v_1(t) \in F(t,u(t))$ a.e. $t \in J$, ${}^{\mathrm{Hc}}D^\alpha_{t_k} u(t) \leq F(t,u(t))$ on (t_k,t_{k+1}) and $u(t_k^+) \leq I_k(u(t_k))$, $k = 1,\ldots$. Similarly, a function $v \in PC \cap AC((t_k,t_{k+1}),\mathbb{R})$, $k = 0,\ldots,$ is said to be an upper solution of (1)–(3) if there exists $v_2 \in L^1(J,\mathbb{R})$ such that $v_2(t) \in F(t,v(t))$ a.e. $t \in J$, ${}^{\mathrm{Hc}}D^\alpha_{t_k} v(t) \geq F(t,v(t))$ on (t_k,t_{k+1}) and $v(t_k^+) \geq I_k(v(t_k))$, $k = 1,\ldots$.

For the study of this problem we first list the following hypotheses:

(H1) $F : J \times \mathbb{R} \longrightarrow P_{\mathrm{cp,cv}}(\mathbb{R})$ is a Carathéodory multivalued map.

(H2) For all $r > 0$ there exists a function $h_r \in L^\infty(J, \mathbb{R}^+)$ with

$$\left|F(t,y)\right| \le h_r(t) \quad \text{for a.e. } t \in J \text{ and all } |y| \le r.$$

(H3) There exist u and $v \in PC((t_k, t_{k+1}), \mathbb{R})$, $k = 0, \ldots,$ lower and upper solutions for the problem (1)–(3) such that $u \le v$.

(H4)

$$u\left(t_k^+\right) \le \min_{y \in [u(t_k^-), v(t_k^-)]} I_k(y) \le \max_{y \in [u(t_k^-), v(t_k^-)]} I_k(y) \le v\left(t_k^+\right), \quad k = 1, \ldots.$$

(H5) There exists $l \in L^1(J, \mathbb{R}^+)$ such that

$$H_d\left(F(t,y), F(t,\bar{y})\right) \le l(t)|y - \bar{y}|; \quad \text{for every } y, \bar{y} \in \mathbb{R},$$

and

$$d\left(0, F(t,0)\right) \le l(t); \quad \text{a.e. } t \in J.$$

Theorem 3.3 *Assume that hypotheses* (H1)–(H4) *hold. Then the problem* (1)–(3) *has at least one solution y such that*

$$u(t) \le y(t) \le v(t) \quad \textit{for all } t \in J.$$

Proof The proof will be given in several steps.

Step 1: Consider the following problem:

$${}^{\mathrm{Hc}}D_{t_0}^{\alpha}(t) \in F\left(t, y(t)\right), \quad t \in J_1 := [t_0, t_1], \tag{4}$$

$$y(1) = y_*. \tag{5}$$

Transform the problem (4)–(5) into a fixed point problem. Consider the modified problem

$${}^{\mathrm{Hc}}D_{t_0}^{\alpha}(t) \in F\left(t, (\tau y)(t)\right), \quad t \in J_1, \tag{6}$$

$$y(1) = y_*, \tag{7}$$

where $\tau : C(J_1, \mathbb{R}) \longrightarrow C(J_1, \mathbb{R})$ be the truncation operator defined by

$$(\tau y)(t) = \begin{cases} u(t), & y(t) < u(t), \\ y(t), & u(t) \le y(t) \le v(t), \\ v(t), & y(t) > v(t). \end{cases}$$

A solution to (6)–(7) is a fixed point of the operator $G : C([t_0, t_1], \mathbb{R}) \longrightarrow P_{\mathrm{cp,cv}}(C([t_0, t_1], \mathbb{R}))$ defined by

$$G(y) = \left\{h \in C\bigl([t_0, t_1], \mathbb{R}\bigr) : h(t) = y_* + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}\right\},$$

where $g \in \tilde{S}^1_{F,\tau y}$ and

$$\begin{aligned}
&\tilde{S}^1_{F,\tau y} = \bigl\{g \in S^1_{F,\tau y} : g(t) \geq v_1(t) \text{ on } A_1 \text{ and } g(t) \leq v_2(t) \text{ on } A_2\bigr\},\\
&S^1_{F,\tau y} = \bigl\{g \in L^1(J_1, \mathbb{R}) : g(t) \in F\bigl(t, (\tau y)(t)\bigr) \text{ for a.e. } t \in J_1\bigr\},\\
&A_1 = \bigl\{t \in J_1 : y(t) < u(t) \leq v(t)\bigr\}, \qquad A_2 = \bigl\{t \in J_1 : u(t) \leq v(t) < y(t)\bigr\}.
\end{aligned}$$

Remark 3.4

(i) For each $y \in C([t_0, t_1], \mathbb{R})$, the set $\tilde{S}^1_{F,\tau y}$ is nonempty. In fact, (H_1) implies there exists $g_3 \in S^1_{F,\tau y}$, so we set

$$g = v_1\chi_{A_1} + v_2\chi_{A_2} + v_3\chi_{A_3},$$

where

$$A_3 = \bigl\{t \in J_1 : \alpha(t) \leq y(t) \leq \beta(t)\bigr\}.$$

Then, by decomposability, $g \in \tilde{S}^1_{F,\tau y}$.

(ii) By the definition of τ it is clear that for all $r > 0$ there exists a function $h_r \in L^\infty(J_1, \mathbb{R}^+)$ with

$$\bigl|F(t, (\tau y)(t)\bigr| \leq h_r(t) \quad \text{for a.e. } t \in J_1 \text{ and all } \bigl\|\tau(y)\bigr\|_\infty \leq r.$$

We shall show that G satisfies the assumptions of the nonlinear alternative of Leray–Schauder type. The proof will be given in several steps.

Claim 1 *A priori bounds on solutions.*

Let $y \in \lambda G(y)$ for some $\lambda \in (0, 1)$. Then there exists $g \in \tilde{S}^1_{F,\tau y}$ such that for some $\lambda \in (0, 1)$ we have, for each $t \in J_1$,

$$y(t) = \lambda\left[y_* + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}\right].$$

This implies by (H2) that for each $t \in J_1$ we have

$$\begin{aligned}
\bigl|y(t)\bigr| &\leq |y_*| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log\frac{t}{s}\right)^{\alpha-1} \bigl|g(s)\bigr|\frac{ds}{s}\\
&\leq |y_*| + \frac{(\log\frac{t_1}{t_0})^\alpha}{\Gamma(\alpha+1)} \|h_{r_1}\|_{L^\infty} := M.
\end{aligned}$$

Set

$$U = \{y \in C([t_0,t_1],\mathbb{R}) : \|y\|_\infty < M+1\}.$$

From the choice of U there is no $y \in \partial U$ such that $y = \lambda G(y)$ for some $\lambda \in (0,1)$. We first show that $G : \overline{U} \to P_{cp,cv}(C([t_0,t_1],\mathbb{R}))$ is compact.

Claim 2 *$G(y)$ is convex for each $y \in C([t_0,t_1],\mathbb{R})$.*

Indeed, if h_1, h_2 belong to $N(y)$, then there exist $g_1, g_2 \in \tilde{S}^1_{F,\tau y}$ such that for each $t \in J_1$

$$h_i = y_* + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \left(\log \frac{t}{s}\right)^{\alpha-1} g_i(s) \frac{ds}{s}, \quad i = 1,2.$$

Let $0 \le d \le 1$. Then for each $t \in J_1$ we have

$$\bigl(dh_1 + (1-d)h_2\bigr)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left[dg_1(s) + (1-d)g_2(s)\frac{ds}{s}\right].$$

Since $\tilde{S}^1_{F_1,\tau y}$ is convex (because $F(\cdot,(\tau y)(\cdot))$ has convex values),

$$dh_1 + (1-d)h_2 \in G(y).$$

Claim 3 *G maps bounded sets into sets in $C([t_0,t_1],\mathbb{R})$.*

Indeed, it is enough to show that for each $q > 0$ there exists a positive constant ℓ_q such that for each $y \in B_q = \{y \in C([t_0,t_1],\mathbb{R}) : \|y\|_\infty \le q\}$ one has $\|G(y)\|_{\mathcal{P}} \le \ell_q$.

Let $y \in B_q$ and $h \in N(y)$ then there exists $g \in \tilde{S}^1_{F,\tau y}$ such that for each $t \in J_1$ we have

$$h(t) = y_* + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}.$$

By (H2) we have for each $t \in J_1$

$$\begin{aligned} \bigl|h(t)\bigr| &\le |y_*| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \left(\log \frac{t}{s}\right)^{\alpha-1} \bigl|g(s)\bigr| \frac{ds}{s} \\ &\le |y_*| + \frac{(\log \frac{t_1}{t_0})^\alpha}{\Gamma(\alpha+1)} \|h_q\|_{L^\infty} := \ell_q. \end{aligned}$$

Claim 4 *G maps bounded set into equicontinuous sets of $C([t_0,t_1],\mathbb{R})$.*

Let $u_1, u_2 \in J_1$, $u_1 < u_2$ and B_q be a bounded set of $C([t_0,t_1],\mathbb{R})$ as in Step 2. Let $y \in B_q$ and $h \in G(y)$ then there exists $g \in \tilde{S}^1_{F,\tau y}$ such that for each $t \in J_1$ we have

$$h(t) = y_* + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}.$$

Then

$$\begin{aligned}\left|h(u_2)-h(u_1)\right| &= \left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{u_2}\left(\log\frac{u_2}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{u_1}\left(\log\frac{u_1}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}\right| \\ &\le \frac{(\log\frac{u_2}{u_1})^{\alpha}}{\Gamma(\alpha+1)}\|h_q\|_{L^\infty}.\end{aligned}$$

As $u_2 \longrightarrow u_1$ the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzela–Ascoli theorem we can conclude that $G : \overline{U} \to P_{\mathrm{cp,cv}}(C([t_0,t_1],\mathbb{R}))$ is a compact multivalued map.

Claim 5 *N is upper semicontinuous map.*

Let $y_n \to y^*$, $h_n \in G(y_n)$ and $h_n \to h^*$. We need to show that $h^* \in G(y^*)$. $h_n \in G(y_n)$ means that there exists $g_n \in \widetilde{S}^1_{\tau(y)}$ such that, for each $t \in J$,

$$h_n(t) = y_* + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} g_n(s)\frac{ds}{s}.$$

We must show that there exists $g^* \in \widetilde{S}^1_{\tau(y^*)}$ such that, for each $t \in J$,

$$h^*(t) = y_* + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} g^*(s)\frac{ds}{s}.$$

Since $F(t,\cdot)$ is upper semicontinuous, for every $\epsilon > 0$, there exists a natural number $n_0(\epsilon)$ such that, for every $n \geq n_0$, we have

$$g_n(t) \in F\bigl(t,\tau y_n(t)\bigr) \subset F\bigl(t,y^*(t)\bigr) + \epsilon B(0,1), \quad \text{a.e. } t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, there exists a subsequence $g_{n_m}(\cdot)$ such that

$$g_{n_m}(\cdot) \to v_*(\cdot) \quad \text{as } m \to \infty,$$

and

$$g^*(t) \in F\bigl(t,\tau y^*(t)\bigr), \quad \text{a.e. } t \in J.$$

For every $w \in F(t,\tau y^*(t))$, we have

$$\left|g_{n_m}(t) - g^*(t)\right| \leq \left|g_{n_m}(t) - w\right| + \left|w - g^*(t)\right|.$$

Then

$$\left|g_{n_m}(t) - g^*(t)\right| \leq d\bigl(g_{n_m}(t), F\bigl(t,\tau y^*(t)\bigr)\bigr).$$

We obtain an analogous relation by interchanging the roles of g_{n_m} and g^*, and it follows that

$$\begin{aligned}\left|g_{n_m}(t)-g^*(t)\right| &\leq H_d\big(F\big(t,\tau y_n(t)\big),F\big(t,\tau y^*(t)\big)\big)\\ &\leq l(t)\left\|y_n-y^*\right\|_\infty.\end{aligned}$$

Then

$$\begin{aligned}\left|h_{n_m}(t)-h^*(t)\right| &\leq \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left|g_{n_m}(s)-g^*(s)\right|\frac{ds}{s}\\ &\leq \frac{(\log\frac{t_1}{t_0})^\alpha}{\Gamma(\alpha+1)}(\log T)^\alpha\int_{t_0}^{t_1}l(s)\,ds\left\|y_{n_m}-y^*\right\|_\infty.\end{aligned}$$

Thus

$$\left\|h_{n_m}-h^*\right\|_\infty \leq \frac{(\log\frac{t_1}{t_0})^\alpha}{\Gamma(\alpha+1)}(\log T)^r\int_1^T l(s)\,ds\left\|y_{n_m}-y^*\right\|_\infty \longrightarrow 0, \quad \text{as } m\to\infty.$$

Hence, Lemma 2.1 implies that G is upper semicontinuous. As a consequence of the nonlinear alternative of Leray–Schauder type, we deduce that G has a fixed point y in U which is a solution of the problem (6)–(7).

Claim 5 *Every solution y of (6)–(7) satisfies*

$$u(t)\leq y(t)\leq v(t); \quad \textit{for all } t\in J_1.$$

Let y be a solution of (6)–(7). We prove that

$$u(t)\leq y(t); \quad \text{for all } t\in J_1.$$

Suppose not. Then there exist τ_1,τ_2 with $\tau_1<\tau_2$ such that $u(\tau_1)=y(\tau_1)$ and

$$u(t)>y(t); \quad \text{for all } t\in(\tau_1,\tau_2).$$

In view of the definition of τ one has

$${}^{\mathrm{Hc}}D^\alpha y(t)\in F\big(t,u(t)\big); \quad \text{for all } t\in(\tau_1,\tau_2).$$

An integration on $(\tau_1,t]$, with $t\in(\tau_1,\tau_2)$ and there exists $g(\cdot)\in F(\cdot,u(\cdot))$ yields

$$y(t)-y(\tau_1)=\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}.$$

Since u is a lower solution to (4)–(5),

$$u(t)-u(\tau_1)\leq\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}; \quad t\in(\tau_1,\tau_2).$$

It follows from $y(\tau_1) = u(\tau_1)$ that

$$u(t) \leq y(t); \quad \text{for all } t \in (\tau_1, \tau_2),$$

which is a contradiction, since $u(t) > y(t)$ for all $t \in (\tau_1, \tau_2)$. Consequently

$$u(t) \leq y(t); \quad \text{for all } t \in J_1.$$

Analogously, we can prove that

$$y(t) \leq v(t) \quad \text{for all } t \in J_1.$$

This shows that

$$u(t) \leq y(t) \leq v(t) \quad \text{for all } t \in J_1.$$

Consequently, the problem (4)–(5) has a solution y satisfying $u \leq y \leq v$. Denote this solution by y_0.

Step 2: Consider the following problem:

$$^{\mathrm{Hc}}D^{\alpha}_{t_1} y(t) \in F\big(t, y(t)\big), \quad t \in J_2 := [t_1, t_2], \tag{8}$$

$$y\big(t_1^+\big) = I_1\big(y_0\big(t_1^-\big)\big). \tag{9}$$

Consider the modified problem

$$^{\mathrm{Hc}}D^{\alpha}_{t_1} y(t) \in F_1\big(t, y(t)\big), \quad \text{a.e. } t \in J_2, \tag{10}$$

$$y\big(t_1^+\big) = I_1\big(y_0\big(t_1^-\big)\big). \tag{11}$$

A solution to (10)–(11) is a fixed point of the operator $G_1 : C([t_1, t_2], \mathbb{R}) \longrightarrow P_{cp,cv}(C([t_1, t_2], \mathbb{R}))$ defined by

$$G_1(y) = \left\{ h \in C\big([t_1, t_2], \mathbb{R}\big) : h(t) = \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} + I_1\big(y_0\big(t_1^-\big)\big) \right\},$$

where $g \in \widetilde{S}^1_{\tau(y)}$. Since $y_0(t_1) \in [u(t_1^-), v(t_1^-)]$, (H4) implies that

$$u\big(t_1^+\big) \leq I_1\big(y_0\big(t_1^-\big)\big) \leq v\big(t_1^+\big),$$

that is

$$u\big(t_1^+\big) \leq y\big(t_1^+\big) \leq v\big(t_1^+\big).$$

Claim 1 *A priori bounds on solutions.*

Let $y \in \lambda G_1(y)$ for some $\lambda \in (0,1)$. Then there exists $g \in \tilde{S}^1_{F,\tau y}$ such that for some $\lambda \in (0,1)$ we have, for each $t \in J_2$,

$$y(t) = \lambda\left[y_* + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} + I_1\big(y_0\big(t_1^-\big)\big)\right].$$

This implies by (H2) that for each $t \in J_1$ we have

$$\begin{aligned}\big|y(t)\big| &\le |y_*| + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\big|g(s)\big|\frac{ds}{s} \\ &\le |y_*| + \frac{(\log\frac{t_2}{t_1})^{\alpha}}{\Gamma(\alpha+1)}\|h_q\|_{L^\infty} + v\big(t_1^+\big) := M.\end{aligned}$$

Set

$$U = \big\{y \in C\big([t_1,t_2],\mathbb{R}\big) : \|y\|_\infty < M+1\big\}.$$

From the choice of U there is no $y \in \partial U$ such that $y = \lambda G_1(y)$ for some $\lambda \in (0,1)$. Using the same reasoning as that used for problem (4)–(5), we can conclude the existence of at least one solution y to (10)–(11).

Claim 5 *Every solution y of* (10)–(11) *satisfies*

$$u(t) \le y(t) \le v(t); \quad \textit{for all } t \in J_1.$$

Let y be a solution of (10)–(11). We prove that

$$u(t) \le y(t); \quad \text{for all } t \in J_2.$$

Suppose not. Then there exist τ_3, τ_4 with $\tau_3 < \tau_4$ such that $u(\tau_3) = y(\tau_4)$ and

$$u(t) > y(t); \quad \text{for all } t \in (\tau_3,\tau_4).$$

In view of the definition of τ one has

$${}^{\mathrm{Hc}}D^{\alpha} y(t) \in F\big(t,u(t)\big); \quad \text{for all } t \in (\tau_3,\tau_4).$$

An integration on $(\tau_3,t]$, with $t \in (\tau_3,\tau_4)$ and there exists $g \in F(t,u(t))$ yields

$$y(t) - y(\tau_3) = \frac{1}{\Gamma(\alpha)}\int_{\tau_3}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}.$$

Since u is a lower solution to (4)–(5),

$$u(t) - u(\tau_3) \le \frac{1}{\Gamma(\alpha)}\int_{\tau_3}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s}; \quad t \in (\tau_3,\tau_4).$$

It follows from $y(\tau_3) = u(\tau_3)$ that

$$u(t) \leq y(t); \quad \text{for all } t \in (\tau_3, \tau_4),$$

which is a contradiction, since $u(t) > y(t)$ for all $t \in (\tau_3, \tau_4)$. Consequently

$$u(t) \leq y(t); \quad \text{for all } t \in J_2.$$

Analogously, we can prove that

$$y(t) \leq v(t) \quad \text{for all } t \in J_2.$$

This shows that

$$u(t) \leq y(t) \leq v(t) \quad \text{for all } t \in J_2.$$

Denote this solution by y_1.

Step 3: We continue this process and take into account that $y_m := y|_{[t_{m-1},t_m]}$ is a solution to the problem

$$^{\mathrm{Hc}}D^{\alpha}_{t_{m-1}} y(t) \in F\big(t, y(t)\big), \quad \text{a.e. } t \in J_m := [t_{m-1}, t_m], \tag{12}$$

$$y\big(t_m^+\big) = I_m\big(y_{m-1}\big(t_{m-1}^-\big)\big). \tag{13}$$

Consider the following modified problem:

$$^{\mathrm{Hc}}D^{r}_{t_{m-1}} y(t) \in F_1\big(t, y(t)\big), \quad \text{a.e. } t \in J_m, \tag{14}$$

$$y\big(t_m^+\big) = I_m\big(y_{m-1}\big(t_{m-1}^-\big)\big). \tag{15}$$

A solution to (14)–(15) is a fixed point of the operator

$$G_m : C\big([t_{m-1}, t_m], \mathbb{R}\big) \longrightarrow P_{\mathrm{cp,cv}}\big(C\big([t_{m-1}, t_m], \mathbb{R}\big)\big)$$

defined by

$$G_m(y) = \left\{ h \in C\big([t_{m-1}, t_m], \mathbb{R}\big) : h(t) = \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s} + I_m\big(y\big(t_{m-1}^-\big)\big) \right\}.$$

Using the same reasoning as that used for problems (4)–(5) and (8)–(9) we can conclude the existence of at least one solution y to (12)–(13). Denote this solution by y_{m-1}.

The solution y of the problem (1)–(3) is then defined by

$$y(t) = \begin{cases} y_0(t), & t \in [t_0, t_1], \\ y_2(t), & t \in (t_1, t_2], \\ \vdots & \\ y_{m-1}(t), & t \in (t_{m-1}, t_m], \\ \vdots & \end{cases}$$

The proof is complete. □

3.1 Nonoscillation and oscillation of solutions

The following theorem gives sufficient conditions to ensure the nonoscillation of solutions of problem (1)–(3).

Theorem 3.5 *Let u and v be lower and upper solutions, respectively, of (1)–(3) with $u \leq v$ and assume that*

(H5) *u is eventually positive nondecreasing, or v is eventually negative nonincreasing.*

Then every solution y of (1)–(3) such that $y \in [u, v]$ is nonoscillatory.

Proof Assume that u is eventually positive. Thus there exists $T_u > t_0$ such that

$$u(t) > 0 \quad \text{for all } t > T_u.$$

Hence $y(t) > 0$ for all $t > T_u$, and $t \neq t_k$, $k = 1, \ldots$. For some $k \in N$ and $t > t_u$, we have $y(t_k^+) = I_k(y(t_k))$. From (H4) we get $y(t_k^+) > u(t_k^+)$. Since for each $h > 0$, $u(t_k + h) \geq u(t_k) > 0$, then $I_k(y(t_k)) > 0$ for all $t_k > T_u$, $k = 1, \ldots$, which means that y is nonoscillatory. Analogously, if v is eventually negative, then there exists $T_v > t_0$ such that

$$y(t) < 0 \quad \text{for all } t > T_v,$$

which means that y is nonoscillatory. This completes the proof. □

The following theorem discusses the oscillation of solutions to problem (1)–(3).

Theorem 3.6 *Let u and v be lower and upper solutions, respectively, of (1)–(3), and assume that the sequences $u(t_k)$ and $v(t_k)$, $k = 1, \ldots$, are oscillatory. Then every solution y of (1)–(3) such that $y \in [u, v]$ is oscillatory.*

Proof Suppose on the contrary that y is a nonoscillatory solution of (1)–(3). Then there exists $T_y > 0$ such that $y(t) > 0$ for all $t > T_y$, or $y(t) < 0$ for all $t > T_y$. In the case that $y(t) > 0$ for all $t > T_y$ we have $v(t_k) > 0$ for all $t_k > T_y$, $k = 1, \ldots$, which is a contradiction since $v(t_k)$ is an oscillatory upper solution. Analogously in the case $y(t) < 0$ for all $t > T_y$ we have $u(t_k) < 0$ for all $t_k > T_y$, $k = 1, \ldots$, which is also a contradiction, since $u(t_k)$ is an oscillatory lower solution. □

3.2 An example

We consider the following impulsive fractional differential equation:

$$^{\mathrm{Hc}}D^{\alpha} y(t) \in F\bigl(t, y(t)\bigr), \quad \text{a.e. } t \in J = (t_k, t_{k+1}), 0 < \alpha < 1, k = 1, \ldots, \tag{16}$$

$$y\bigl(t_k^+\bigr) = I_k\bigl(y\bigl(t_k^-\bigr)\bigr), \quad k = 1, \ldots, \tag{17}$$

$$y(1) = y_*, \tag{18}$$

where

$$F(t, y) = \bigl\{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\bigr\},$$

$f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$. We assume that for each $t \in J$, $f_1(t, \cdot)$ is lower semicontinuous (i.e., the set $\{y \in \mathbb{R} : f_1(t,y) > \delta\}$ is open for each $\delta \in \mathbb{R}$), and assume that for each $t \in J$, $f_2(t, \cdot)$ is upper semicontinuous (i.e., the set the set $\{y \in \mathbb{R} : f_2(t,y) < \delta\}$ is open for each $\delta \in \mathbb{R}$). Assume that there are $z \in L^\infty([0,T], \mathbb{R}^+)$ such that

$$\max\bigl(\bigl|f_1(t,y)\bigr|, \bigl|f_2(t,y)\bigr|\bigr) \le z(t), \quad t \in J, \text{and all } y \in \mathbb{R}.$$

It is clear that F is compact and convex-valued, and it is upper semicontinuous. Assume that there exist $g_1(\cdot), g_2(\cdot) \in L^1(J, \mathbb{R})$ such that

$$g_1(t) \le \max\bigl(\bigl|f_1(t,y)\bigr|, \bigl|f_2(t,y)\bigr|\bigr) \le g_2(t) \quad \text{for all } t \in J, \text{and } y \in \mathbb{R},$$

and for each $t \in J$

$$\int_1^t g_1(s)\frac{ds}{s} \le I_k\biggl(\int_1^t g_1(s)\frac{ds}{s}\biggr), \quad k \in \mathbb{N},$$
$$\int_1^t g_2(s)\frac{ds}{s} \ge I_k\biggl(\int_1^t g_2(s)\frac{ds}{s}\biggr), \quad k \in \mathbb{N}.$$

Consider the functions

$$u(t) := \int_1^t g_1(s)\frac{ds}{s}, \qquad v(t) := \int_1^t g_2(s)\frac{ds}{s}.$$

Clearly, u and v are lower and upper solutions of the problem (16)–(18), respectively; that is,

$${}^{\mathrm{Hc}}D^\alpha u(t) \le f\bigl(t, u(t)\bigr) \quad \text{for all } t \in J \text{ and all } y \in \mathbb{R},$$

and

$${}^{\mathrm{Hc}}D^\alpha v(t) \ge f\bigl(t, v(t)\bigr) \quad \text{for all } t \in J \text{ and all } y \in \mathbb{R}.$$

Since all the conditions of Theorem 3.3 are satisfied, the problem (16)–(18) has at least one solution y on J with $u \le y \le v$. If $g_1(t) > 0$ then u is positive and nondecreasing, thus $y(t)$ is nonoscillatory. If $g_2(t) < 0$ then v is negative and nonincreasing, thus $y(t)$ is nonoscillatory. If the sequences $u(t_k)$ and $v(t_k)$ are both oscillatory, then $y(t)$ is oscillatory.

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