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Symmetry reductions of the (3 + 1)-dimensional modified Zakharov–Kuznetsov equation

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Abstract
This paper is concerned with the symmetry reductions of the (3 + 1)-dimensional modified Zakharov–Kuznetsov equation of ion-acoustic waves in a magnetized plasma. The direct symmetry method is applied to determine the symmetry and the corresponding vector field. Then, the considered equation is reduced to lower-dimensional equations with the aid of the obtained symmetry. At last, some exact solutions of the modified Zakharov–Kuznetsov equation are found in terms of the lower-dimensional equations.

Keywords: Zakharov–Kuznetsov equation; Direct symmetry method; Symmetry reduction

1 Introduction

The Zakharov–Kuznetsov (ZK) equation [1]

$$u_t + uu_x + \nabla^2 u_x = 0 \tag{1}$$

was first proposed by Zakharov and Kuznetsov to describe the evolution of weakly nonlinear ion-acoustic waves in a plasma consisting of hot isothermal electrons and cold ions in the presence of a uniform magnetic field in the x direction. Equation (1) also appears in many other scientific fields including geochemistry, optical fiber, and solid state physics [2–5]. In [6], Shivamoggi provided a detailed discussion of the analytical properties of Eq. (1). Nawaz et al. [7] found appropriate solutions for the ZK equations with fully nonlinear dispersion by the homotopy analysis method.

In 1999, Munro and Parkes considered a more realistic situation where the electrons are non-isothermal [8]. With an appropriately modified form of the electron number density given in [9], they showed that the reductive perturbation can lead to the following modified Zakharov–Kuznetsov (mZK) equation:

$$16(u_t - ku_x) + 30u^{\frac{1}{2}} u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0, \tag{2}$$

where k is a positive constant. Later, in [10] and [11], Munro and Parkes addressed the stability of solitary wave solutions and that of obliquely propagating solitary wave solutions to the mZK equation, respectively. In 2016, by using an extended direct algebraic method, Seadaway presented traveling wave solutions to the mZK equation and analyzed the stability for the electric fields and the electric field potentials [12].

It is noted that the mZK equation is a high dimensional nonlinear evolution equation and, thus, the study of its reduction problem is of theoretical interest. The Lie-group method, originally proposed by Sophus Lie, is a classical method to determine the symmetry reduction of partial differential equations (PDEs) [13–16]. During the past several decades, there have been many extensions of the Lie-group method such as the nonclassical Lie group method [17], the CK direct method [18], the direct symmetry method [19], and so on [20–24]. Among them, the direct symmetry method is an effective approach for seeking symmetry reductions. In [25] and [26], the method was used to investigate the Gardner–KP equation and the $(2+1)$-dimensional Jaulent–Miodek equation, respectively. To our knowledge, there is no result concerning the application of the direct symmetry method to the mZK equation partly due to its high dimension and nonlinear term $u^{\frac{1}{2}}u_x$, which motivates the present work.

Based on the above discussion, this paper considers the problem of seeking symmetry reductions of the mZK equation. In Sect. 2, with the help of the direct symmetry method, the symmetry and the corresponding vector field of the mZK equation are determined. In Sect. 3, by solving the symmetry equation, similarity transformations are constructed, which are applied to reduce the mZK equation to $(2 + 1)$-dimensional or even $(1 + 1)$-dimensional equations. In Sect. 4, some exact solutions including trigonometric function solutions, hyperbolic function solutions, and Weierstrass function solutions of the mZK equation are presented in terms of the lower-dimensional equations. Finally, the conclusion is provided in Sect. 5.

2 Symmetry analysis

For an arbitrary nonlinear evolution equation

$$\Phi(x,t,u,u_x,u_t,\dots) = 0, \tag{3}$$

where $u_x = \frac{\partial u}{\partial x}$. The function $\sigma(x,t,u,u_x,u_t,\dots)$ is called a symmetry [27] of Eq. (3) if it satisfies the following equation for an arbitrary solution $u(x,t)$:

$$\varphi'(u)\sigma = 0, \tag{4}$$

where

$$\varphi'(u)\sigma = \frac{\partial\varphi}{\partial u}\sigma + \frac{\partial\varphi}{\partial u_x}\sigma_x + \frac{\partial\varphi}{\partial u_t}\sigma_t + \frac{\partial\varphi}{\partial u_{xx}}\sigma_{xx} + \cdots .$$

Note that Eq. (4) is a linear PDE of the symmetry σ. Therefore the linear combination of symmetry σ is also a symmetry of Eq. (3).

According to Eq. (4), the symmetry of mZK equation must satisfy

$$16(\sigma_t - k\sigma_x) + 15u^{-\frac{1}{2}}\sigma u_x + 30u^{\frac{1}{2}}\sigma_x + \sigma_{xxx} + \sigma_{xyy} + \sigma_{xzz} = 0. \tag{5}$$

Here, we set

$$\begin{aligned}\sigma(x,y,z,t,u) = {}& a(x,y,z,t)u_t + b(x,y,z,t)u_x \\ & + c(x,y,z,t)u_y + d(x,y,z,t)u_z \\ & + e(x,y,z,t)u + g(x,y,z,t),\end{aligned} \tag{6}$$

where a, b, c, d, e, and g are functions to be determined later. With the help of Maple, one can expand Eq. (5) by means of Eqs. (2) and (6). Then, taking the coefficients of u and those of the derivatives of u to zero yields the following twenty-one determining equation concerning a, b, c, d, e, and g:

$$\begin{aligned}
&u_{xxxxx} : -\frac{3}{16}a_x = 0, \\
&u_{xxxxy} : -\frac{1}{8}a_y = 0, \\
&u_{xxxxz} : -\frac{1}{8}a_z = 0, \\
&u_{xxx} : 3b_x - a_t = 0, \\
&u_{xxy} : 2b_y = 0, \\
&u_{xxz} : 2b_z = 0, \\
&u_{xyy} : b_x - a_t + 2c_y = 0, \\
&u_{xyz} : 2c_z + 2d_y = 0, \\
&u_{xzz} : b_x - a_t + 2d_z = 0, \\
&u_{xx} : 3b_{xx} = 0, \\
&u_{xy} : c_{zz} + c_{yy} + 2e_y = 0, \\
&u_{xz} : d_{zz} + d_{yy} + 2e_z = 0, \\
&u^{\frac{1}{2}}u_x : 15e + 30b_x - 30a_t = 0, \\
&u^{\frac{1}{2}}u_y : 30c_x = 0, \\
&u^{\frac{1}{2}}u_z : 30d_x = 0, \\
&u^{-\frac{1}{2}}u_x : 15g = 0, \\
&u_x : 16b_t - 16kb_x + 16ka_t + e_{yy} + e_{zz} = 0, \\
&u_y : 16c_t = 0, \\
&u_z : 16d_t = 0, \\
&u^{\frac{3}{2}} : 30e_x = 0, \\
&u : 16e_t = 0.
\end{aligned}$$

Solving the above equations yields

$$
\begin{aligned}
a &= \delta_0 t + \delta_1, \\
b &= \frac{1}{3}\delta_0 x - \frac{2}{3}k\delta_0 t + \delta_2, \\
c &= \frac{1}{3}\delta_0 y + \delta_3 z + \delta_4, \\
d &= \frac{1}{3}\delta_0 z - \delta_3 y + \delta_5, \\
e &= \frac{4}{3}\delta_0, \\
g &= 0,
\end{aligned}
$$

where δ_0, δ_1, δ_2, δ_3, δ_4, and δ_5 are arbitrary constants. Hence we obtain a general symmetry of the (3 + 1)-dimensional nonlinear mZK equation

$$
\begin{aligned}
\sigma = {}& (\delta_0 t + \delta_1)u_t + \left(\frac{1}{3}\delta_0 x - \frac{2}{3}k\delta_0 t + \delta_2\right)u_x + \left(\frac{1}{3}\delta_0 y + \delta_3 z + \delta_4\right)u_y \\
& + \left(\frac{1}{3}\delta_0 z - \delta_3 y + \delta_5\right)u_z + \frac{4}{3}\delta_0 u.
\end{aligned} \tag{7}
$$

The corresponding vector field of the above symmetry can be expressed as

$$
\begin{aligned}
\mathcal{V} = {}& (\delta_0 t + \delta_1)\frac{\partial}{\partial t} + \left(\frac{1}{3}\delta_0 x - \frac{2}{3}k\delta_0 t + \delta_2\right)\frac{\partial}{\partial x} + \left(\frac{1}{3}\delta_0 y + \delta_3 z + \delta_4\right)\frac{\partial}{\partial y} \\
& + \left(\frac{1}{3}\delta_0 z - \delta_3 y + \delta_5\right)\frac{\partial}{\partial z} - \frac{4}{3}\delta_0 u\frac{\partial}{\partial u},
\end{aligned} \tag{8}
$$

which has the following infinitesimal generators:

$$
\begin{aligned}
\mathcal{V}_1 &= t\frac{\partial}{\partial t} + \frac{1}{3}x\frac{\partial}{\partial x} - \frac{2}{3}kt\frac{\partial}{\partial x} + \frac{1}{3}y\frac{\partial}{\partial y} + \frac{1}{3}z\frac{\partial}{\partial z} - \frac{4}{3}u\frac{\partial}{\partial u}, \\
\mathcal{V}_2 &= \frac{\partial}{\partial t}, \\
\mathcal{V}_3 &= \frac{\partial}{\partial x}, \\
\mathcal{V}_4 &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \\
\mathcal{V}_5 &= \frac{\partial}{\partial y}, \\
\mathcal{V}_6 &= \frac{\partial}{\partial z}.
\end{aligned}
$$

The commutation relation of these infinitesimal generators is given in Table 1.

The adjoint representation can be defined by Lie series

$$
\mathrm{Ad}\big(\exp(\varepsilon\mathcal{V}_i)\mathcal{V}_j\big) = \mathcal{V}_j - \varepsilon[\mathcal{V}_i, \mathcal{V}_j] + \frac{\varepsilon^2}{2!}\big[\mathcal{V}_i, [\mathcal{V}_i, \mathcal{V}_j]\big] - \cdots, \tag{9}
$$

Table 1 Commutation relation of the Lie algebra of Eq. (2)

	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	$\mathcal{V}_6$
$\mathcal{V}_1$	0	$\frac{2}{3}k\mathcal{V}_3 - \mathcal{V}_2$	$-\frac{1}{3}\mathcal{V}_3$	0	$-\frac{1}{3}\mathcal{V}_5$	$-\frac{1}{3}\mathcal{V}_6$
$\mathcal{V}_2$	$\mathcal{V}_2 - \frac{2}{3}k\mathcal{V}_3$	0	0	0	0	0
$\mathcal{V}_3$	$\frac{1}{3}\mathcal{V}_3$	0	0	0	0	0
$\mathcal{V}_4$	0	0	0	0	$\mathcal{V}_6$	$-\mathcal{V}_5$
$\mathcal{V}_5$	$\frac{1}{3}\mathcal{V}_5$	0	0	$-\mathcal{V}_6$	0	0
$\mathcal{V}_6$	$\frac{1}{3}\mathcal{V}_6$	0	0	$\mathcal{V}_5$	0	0

Table 2 Adjoint representation of the Lie algebra of Eq. (2)

	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	$\mathcal{V}_6$
$\mathcal{V}_1$	$\mathcal{V}_1$	Δ_1	$e^{\frac{\varepsilon}{3}}\mathcal{V}_3$	$\mathcal{V}_4$	$e^{\frac{\varepsilon}{3}}\mathcal{V}_5$	$e^{\frac{\varepsilon}{3}}\mathcal{V}_6$
$\mathcal{V}_2$	$\mathcal{V}_1 - \varepsilon\mathcal{V}_2 + \frac{2k}{3}\varepsilon\mathcal{V}_3$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	$\mathcal{V}_6$
$\mathcal{V}_3$	$\mathcal{V}_1 - \frac{\varepsilon}{3}\mathcal{V}_3$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	$\mathcal{V}_6$
$\mathcal{V}_4$	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	Δ_2	Δ_3
$\mathcal{V}_5$	$\mathcal{V}_1 - \frac{1}{3}\varepsilon\mathcal{V}_5$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4 + \varepsilon\mathcal{V}_6$	$\mathcal{V}_5$	$\mathcal{V}_6$
$\mathcal{V}_6$	$\mathcal{V}_1 - \frac{1}{3}\varepsilon\mathcal{V}_6$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4 - \varepsilon\mathcal{V}_5$	$\mathcal{V}_5$	$\mathcal{V}_6$

where ε is the parameter. The adjoint representation of the Lie algebra is given in Table 2, where $\Delta_1 = e^{\varepsilon}\mathcal{V}_2 - ke^{\frac{\varepsilon}{3}}(e^{\frac{2\varepsilon}{3}} - 1)\mathcal{V}_3$, $\Delta_2 = \cos(\varepsilon)\mathcal{V}_5 - \sin(\varepsilon)\mathcal{V}_6$, and $\Delta_3 = \cos(\varepsilon)\mathcal{V}_6 + \sin(\varepsilon)\mathcal{V}_5$.

3 Symmetry reductions

In this section we apply the obtained symmetry to deduce symmetry reductions of Eq. (2). We first solve the symmetry equation $\sigma = 0$ to obtain similarity variables and then substitute them into the original mZK equation (2) to determine the corresponding reduction equations. To obtain the similarity variables ζ, η, ω, and $f(\zeta,\eta,\omega)$ of Eq. (2), we have to solve the characteristic equations of $\sigma = 0$

$$\frac{dt}{\delta_0 t + \delta_1} = \frac{dx}{\frac{1}{3}\delta_0 x - \frac{2}{3}k\delta_0 t + \delta_2} = \frac{dy}{\frac{1}{3}\delta_0 y + \delta_3 z + \delta_4} = \frac{dz}{\frac{1}{3}\delta_0 z - \delta_3 y + \delta_5} = \frac{du}{-\frac{4}{3}\delta_0 u}. \tag{10}$$

In terms of different choices of parameters δ_0, δ_1, δ_2, δ_3, δ_4, and δ_5, we can get various reduced equations of (2). In the following, let us discuss six concrete cases.

Case I. $\delta_0 = \delta_1 = \delta_3 = 0$, $\delta_2 \neq 0$, $\delta_4 \neq 0$, $\delta_5 \neq 0$

By solving system (10), one can get similarity variables as follows:

$$\begin{aligned}
&\zeta = \delta_4 x - \delta_2 y,\\
&\eta = \delta_5 x - \delta_2 z,\\
&\omega = t,\\
&u = f(\zeta,\eta,\omega).
\end{aligned}$$

Using the above similarity variables, mZK equation (2) can be reduced to

$$\begin{aligned}
&16(f_\omega - k\delta_4 f_\zeta - k\delta_5 f_\eta) + 30f^{\frac{1}{2}}(\delta_4 f_\zeta + \delta_5 f_\eta) + (\delta_4^3 + \delta_2^2\delta_4)f_{\zeta\zeta\zeta}\\
&\quad + (3\delta_4^2\delta_5 + \delta_2^2\delta_5)f_{\zeta\zeta\eta} + (3\delta_4\delta_5^2 + \delta_2^2\delta_4)f_{\zeta\eta\eta} + (\delta_5^3 + \delta_2^2\delta_5)f_{\eta\eta\eta} = 0,
\end{aligned} \tag{11}$$

which is a $(2 + 1)$-dimensional PDE.

Case II. $\delta_0 = \delta_3 = 0$, $\delta_1 \neq 0$, $\delta_2 \neq 0$, $\delta_4 \neq 0$, $\delta_5 \neq 0$
In such a case, the similarity variables of Eq. (2) are given as

$$\begin{aligned} &\zeta = \delta_1 x - \delta_2 t, \\ &\eta = \delta_1 y - \delta_4 t, \\ &\omega = \delta_1 z - \delta_5 t, \\ &u = f(\zeta, \eta, \omega). \end{aligned}$$

Substituting the above similarity variables into Eq. (2), one can obtain

$$-16(\delta_2 + k\delta_1)f_\zeta - 16\delta_4 f_\eta - 16\delta_5 f_\omega + 30\delta_1 f^{\frac{1}{2}} f_\zeta + \delta_1^3(f_{\zeta\zeta\zeta} + f_{\zeta\eta\eta} + f_{\zeta\omega\omega}) = 0. \tag{12}$$

Case III. $\delta_0 = \delta_1 = \delta_2 = \delta_4 = \delta_5 = 0$, $\delta_3 \neq 0$
In such a case, the similarity variables of Eq. (2) are

$$\begin{aligned} &\zeta = y^2 + z^2, \\ &\eta = x, \\ &\omega = t, \\ &u = f(\zeta, \eta, \omega). \end{aligned}$$

Substituting the above similarity variables into Eq. (2), the mZK equation can be reduced to

$$16f_\omega - 16kf_\eta + 30f^{\frac{1}{2}} f_\eta + f_{\eta\eta\eta} + 4\zeta f_{\zeta\zeta\eta} + 4f_{\zeta\eta} = 0. \tag{13}$$

Case IV. $\delta_0 = \delta_4 = \delta_5 = 0$, $\delta_1 \neq 0$, $\delta_2 \neq 0$, $\delta_3 \neq 0$
Solving Eq. (10), we obtain

$$\begin{aligned} &\zeta = y^2 + z^2, \\ &\eta = \delta_3 x - \delta_2 \arctan\left(\frac{y}{z}\right), \\ &\omega = \delta_3 t - \delta_1 \arctan\left(\frac{y}{z}\right), \\ &u = f(\zeta, \eta, \omega). \end{aligned}$$

Hence Eq. (2) is reduced to a (2 + 1)-dimensional variable-coefficient PDE

$$\begin{aligned} &4\zeta^2 f_{\zeta\zeta\eta} + \left(\delta_3^2\zeta + \delta_2^2\right) f_{\eta\eta\eta} + 30\zeta f^{\frac{1}{2}} f_\eta - 16k\zeta f_\eta + 16\zeta f_\omega + 4\zeta f_{\zeta\eta} \\ &\quad + 2\delta_1\delta_2 f_{\eta\eta\omega} + \delta_1^2 f_{\eta\omega\omega} = 0. \end{aligned} \tag{14}$$

Case V. $\delta_2 = \delta_3 = \delta_4 = \delta_5 = 0$, $\delta_1 \neq 0$, $\delta_0 \neq 0$

In such a case, the similarity variables of Eq. (2) are

$$\begin{aligned}
\zeta &= \frac{\delta_0 x + \delta_0 kt + 3k\delta_1}{(\delta_0 t + \delta_1)^{\frac{1}{3}}}, \\
\eta &= \frac{y}{(\delta_0 t + \delta_1)^{\frac{1}{3}}}, \\
\omega &= \frac{z}{(\delta_0 t + \delta_1)^{\frac{1}{3}}}, \\
u &= (\delta_0 t + \delta_1)^{-\frac{4}{3}} f(\zeta, \eta, \omega).
\end{aligned}$$

Substituting the above similarity variables into Eq. (2) yields

$$90f^{\frac{1}{2}}f_\zeta - 16\zeta f_\zeta - 16\eta f_\eta - 16\omega f_\omega + 3\delta_0^2 f_{\zeta\zeta\zeta} + 3f_{\zeta\omega\omega} + 3f_{\zeta\eta\eta} - 64f = 0. \tag{15}$$

Case VI. $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = 0$, $\delta_0 \neq 0$

The similarity variables of Eq. (2) are given by

$$\begin{aligned}
\zeta &= \frac{z}{y}, \\
\eta &= \frac{t}{y^3}, \\
\omega &= \frac{kt + x}{y}, \\
u &= f(\zeta, \eta, \omega) y^{-4}.
\end{aligned}$$

Thus, Eq. (2) is reduced to

$$\begin{aligned}
&16f_\eta + 30f_\omega + 30f^{\frac{1}{2}}f_\omega + (1+\omega)^2 f_{\omega\omega\omega} + 2\zeta\omega f_{\zeta\omega\omega} + \left(1+\zeta^2\right) f_{\zeta\zeta\omega} \\
&\quad + 12\omega f_{\omega\omega} + 12\zeta f_{\zeta\omega} + 6\eta\omega f_{\eta\omega\omega} + 42\eta f_{\eta\omega} + 9\eta^2 f_{\eta\eta\omega} = 0.
\end{aligned} \tag{16}$$

Remark 1 In addition to PDEs with constant coefficients, the direct symmetry method can also be applied to investigate variable-coefficient PDEs. For instance, one can apply the direct symmetry method to variable-coefficient reduced equation (13) to further reduce the mZK equation.

In fact, it is not difficult to see that the symmetry of Eq. (13) satisfies

$$16(\sigma_\omega - k\sigma_\eta) + 15f^{-\frac{1}{2}}\sigma f_\eta + 30f^{\frac{1}{2}}\sigma_\eta + \sigma_{\eta\eta\eta} + 4\zeta\sigma_{\zeta\zeta\eta} + 4\sigma_{\zeta\eta} = 0, \tag{17}$$

where we set

$$\sigma(\zeta, \eta, \omega, f) = a(\zeta, \eta, \omega) f_\zeta + b(\zeta, \eta, \omega) f_\eta + c(\zeta, \eta, \omega) f_\omega + e(\zeta, \eta, \omega) f + g(\zeta, \eta, \omega), \tag{18}$$

where a, b, c, e, and g are functions to be found later. To ensure that the expansion of Eq. (17) is true for an arbitrary solution f, we must take the coefficients of f and its derivatives to be zero. Hence we have

$$c_\zeta = c_\eta = 0.$$

According to the same procedure, it leads to

$$a_\eta = a_\omega = 0, \qquad b_\zeta = 0, \qquad e_\eta = e_\omega = 0, \qquad b_{\eta\eta} = 0, \qquad g = 0. \tag{19}$$

Hence, Eq. (17) is reduced to

$$\begin{aligned}
&(128\zeta e_{\zeta\zeta} + 512b_\omega + 512kc_\omega + 128e_\zeta - 512kb_\eta)f^{\frac{3}{2}}f_\eta \\
&\quad + (96b_\eta - 32c_\omega)f^{\frac{3}{2}}f_{\eta\eta\eta} + (960b_\eta - 960c_\omega + 480e)f^2 f_\eta \\
&\quad + (-128a - 128\zeta c_\omega + 256\zeta a_\zeta + 128\zeta b_\eta)f^{\frac{3}{2}}f_{\zeta\zeta\eta} \\
&\quad + (128a_\zeta + 128b_\eta + 256\zeta e_\zeta - 128c_\omega + 128\zeta a_{\zeta\zeta})f^{\frac{3}{2}}f_{\zeta\eta} = 0.
\end{aligned} \tag{20}$$

From Eqs. (19)–(20), we obtain

$$\begin{aligned}
&a = \frac{2}{3}\lambda_1\zeta, \qquad b = \frac{1}{3}\lambda_1\eta - \frac{2}{3}k\lambda_1\omega + \lambda_2, \\
&c = \lambda_1\omega + \lambda_3, \qquad e = \frac{4}{3}\lambda_1, \qquad g = 0,
\end{aligned} \tag{21}$$

where λ_1, λ_2, and λ_3 are arbitrary constants. Thus we get a symmetry of Eq. (13) as follows:

$$\sigma = \frac{2}{3}\lambda_1\zeta f_\zeta + \left(\frac{1}{3}\lambda_1\eta - \frac{2}{3}k\lambda_1\omega + \lambda_2\right)f_\eta + (\lambda_1\omega + \lambda_3)f_\omega + \frac{4}{3}\lambda_1 f. \tag{22}$$

The corresponding characteristic equation of $\sigma = 0$ is

$$\frac{d\zeta}{\frac{2}{3}\lambda_1\zeta} = \frac{d\eta}{\frac{1}{3}\lambda_1\eta - \frac{2}{3}k\lambda_1\omega + \lambda_2} = \frac{d\omega}{\lambda_1\omega + \lambda_3} = \frac{df}{-\frac{4}{3}\lambda_1 f}. \tag{23}$$

From Eq. (23) we can obtain the following similarity variables ϕ, φ, and F:

$$\begin{aligned}
&\phi = (\lambda_1\omega + \lambda_3)/\zeta^{\frac{3}{2}}\lambda_1, \qquad \varphi = \big[(k\omega + \eta)\lambda_1 + 3k\lambda_3 + 3\lambda_2\big]/\zeta^{\frac{1}{2}}\lambda_1, \\
&F(\phi, \varphi) = f\zeta^2.
\end{aligned} \tag{24}$$

By using the obtained similarity variables, we obtain the further reduced equation of Eq. (13)

$$\begin{aligned}
&11\varphi F_{\varphi\varphi} + (\varphi^2 + 1)F_{\varphi\varphi\varphi} + 6\phi\varphi F_{\phi\varphi\varphi} + 16F_\phi + 25F_\varphi + 30F_\varphi\sqrt{F} \\
&\quad + 9\phi^2 F_{\phi\phi\varphi} + 39\phi F_{\phi\varphi} = 0,
\end{aligned} \tag{25}$$

which is a $(1+1)$-dimensional nonlinear PDE. It is easy to find that the symmetry of Eq. (25) satisfies the following equation:

$$\begin{aligned} &11\varphi\sigma_{\varphi\varphi} + (\varphi^2+1)\sigma_{\varphi\varphi\varphi} + 6\phi\varphi\sigma_{\phi\varphi\varphi} + 16\sigma_\phi + 25\sigma_\varphi + 30\sigma_\varphi\sqrt{F} \\ &\quad + 15F_\varphi F^{-\frac{1}{2}}\sigma + 9\phi^2\sigma_{\phi\phi\varphi} + 39\phi\sigma_{\phi\varphi} = 0. \end{aligned} \tag{26}$$

Using the same direct symmetry method, we can reduce Eq. (25) to an ordinary differential equation (ODE). Here we omit them.

Remark 2 It is shown that $(2+1)$-dimensional Eq. (13) can be reduced to $(1+1)$-dimensional partial differential equation Eq. (25). Similarly, Eqs. (14)–(16) can be discussed by the same method. In theory, Eq. (25) can be further reduced to an ODE. This problem will be discussed in our future work.

4 Discussion of the solutions of mZK equation

One of the main functions for finding symmetry reductions is to use them to seek exact solutions. There are many effective direct methods that can be used to solve the obtained reduced equations such as the tanh method [28], the homogeneous balance method [29], the Horota bilinear method [30], the Darboux transformation method [31], and so on (see [32–39] for reference). Here, we use the traveling wave transformation to transform reduced equations (11) and (12) to ODEs for obtaining exact solutions. Let

$$\theta = l\zeta + m\eta + n\omega, \tag{27}$$

where l, m, and n are nonzero constants. Then Eq. (12) can be transformed into an ODE as follows:

$$A_1f' + A_2f^{\frac{1}{2}}f' + A_3f''' = 0, \tag{28}$$

where $f' = df/d\theta$, $\theta = l(\delta_1x - \delta_2t) + m(\delta_1y - \delta_4t) + n(\delta_1z - \delta_5t)$, $A_1 = -16l(\delta_2 + k\delta_1) - 16m\delta_4 - 16n\delta_5$, $A_2 = 30\delta_1 l$, and $A_3 = \delta_1^3(l^3 + lm^2 + \ln^2)$. The same procedure can be followed, then reduced equation (11) is also transformed into Eq. (28), where

$$\theta = l(\delta_4x - \delta_2y) + m(\delta_5x - \delta_2z) + nt, \tag{29}$$

$$A_1 = 16(n - k\delta_4l - k\delta_5m), \tag{30}$$

$$A_2 = 30\delta_4l + 30\delta_5m, \tag{31}$$

$$\begin{aligned} A_3 &= (\delta_4^3 + \delta_2^2\delta_4)l^3 + (3\delta_4^2\delta_5 + \delta_2^2\delta_5)l^2m + (3\delta_4\delta_5^2 + \delta_2^2\delta_4)lm^2 \\ &\quad + (\delta_5^3 + \delta_2^2\delta_5)m^3. \end{aligned} \tag{32}$$

Integrating Eq. (28) with respect to the independent variable θ yields that

$$A_1f + \frac{2}{3}A_2f^{\frac{3}{2}} + A_3f'' + A_4 = 0, \tag{33}$$

where A_4 is the integral scalar. Multiplying Eq. (33) by f' and then integrating both sides, we have

$$A_1 f^2 + \frac{8}{15} A_2 f^{\frac{5}{2}} + A_3 f'^2 + 2A_4 f + A_5 = 0,$$

where A_5 is the integral scalar. Using the transformation $f^{\frac{1}{2}} = g$, we obtain

$$A_1 g^2 + \frac{8}{15} A_2 g^3 + 4A_3 g'^2 + 2A_4 + A_5/g^2 = 0.$$

By solving the above equation, one can obtain the following solutions which are expressed in the form of trigonometric functions, hyperbolic functions, and Weierstrass function:

$$g_1(\theta) = -\frac{15}{8}\frac{A_1}{A_2}\sec^2\left(\frac{\sqrt{A_1A_3}}{4A_3}\theta\right), \qquad A_4 = A_5 = 0, \qquad A_1A_3 > 0,$$
$$g_2(\theta) = -\frac{15}{8}\frac{A_1}{A_2}\csc^2\left(\frac{\sqrt{A_1A_3}\theta}{4A_3}\theta\right), \qquad A_4 = A_5 = 0, \qquad A_1A_3 > 0,$$
$$g_3(\theta) = -\frac{15}{8}\frac{A_1}{A_2}\operatorname{sech}^2\left(\frac{\sqrt{-A_1A_3}}{4A_3}\theta\right), \qquad A_4 = A_5 = 0, \qquad A_1A_3 < 0,$$
$$g_4(\theta) = \frac{15}{8}\frac{A_1}{A_2}\operatorname{csch}^2\left(\frac{\sqrt{-A_1A_3}}{4A_3}\theta\right), \qquad A_4 = A_5 = 0, \qquad A_1A_3 < 0,$$
$$g_5(\theta) = \frac{1}{\sqrt[3]{-A_2}}\rho\left(\frac{\sqrt{30}}{30}\frac{\sqrt{A_3(-A_2)^{\frac{2}{3}}}}{A_3}\theta, 0, 15A_4\right), \qquad A_1 = A_5 = 0,$$

where θ is given by Eq. (29), and ρ is the Weierstrass P function. Reverting back to the original variables, one can obtain the corresponding exact solutions of mZK equation (2)

$$u_1(x,y,z,t) = \frac{225}{64}\frac{A_1^2}{A_2^2}\sec^4\left(\frac{\sqrt{A_1A_3}}{4A_3}\theta\right), \qquad A_1A_3 > 0,$$
$$u_2(x,y,z,t) = \frac{225}{64}\frac{A_1^2}{A_2^2}\csc^4\left(\frac{\sqrt{A_1A_3}}{4A_3}\theta\right), \qquad A_1A_3 > 0,$$
$$u_3(x,y,z,t) = \frac{225}{64}\frac{A_1^2}{A_2^2}\operatorname{sech}^4\left(\frac{\sqrt{-A_1A_3}}{4A_3}\theta\right), \qquad A_1A_3 < 0,$$
$$u_4(x,y,z,t) = \frac{225}{64}\frac{A_1^2}{A_2^2}\operatorname{csch}^4\left(\frac{\sqrt{-A_1A_3}}{4A_3}\theta\right), \qquad A_1A_3 < 0,$$
$$u_5(x,y,z,t) = \frac{1}{\sqrt[3]{A_2^2}}\rho^2\left(\frac{\sqrt{30}}{30}\frac{\sqrt{A_3(-A_2)^{\frac{2}{3}}}}{A_3}\theta, 0, 15A_4\right), \qquad A_1 = 0,$$

where θ, A_1, A_2, and A_3 are given by Eqs. (29)–(32).

Choosing $l = m = n = \delta_2 = \delta_4 = \delta_5 = 1$ in $u_3(x,y,z,t)$, we have the following solitary wave solution:

$$u(x,y,z,t) = \frac{1}{4}\operatorname{sech}^4\left(\frac{\sqrt{3}}{6}(2x - y - z + t)\right. \tag{34}$$

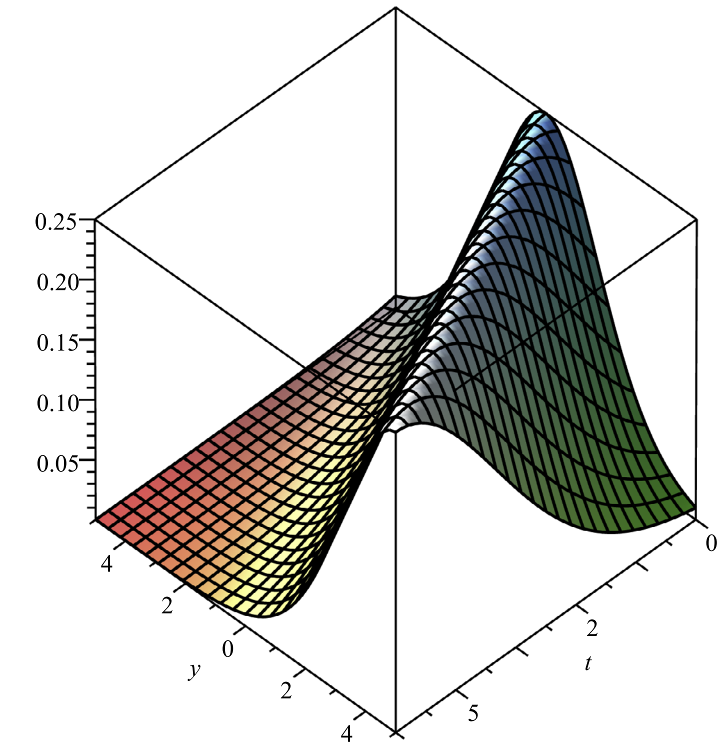

Figure 1 The solution of the mZK with $k = 1$, $z = 2x$

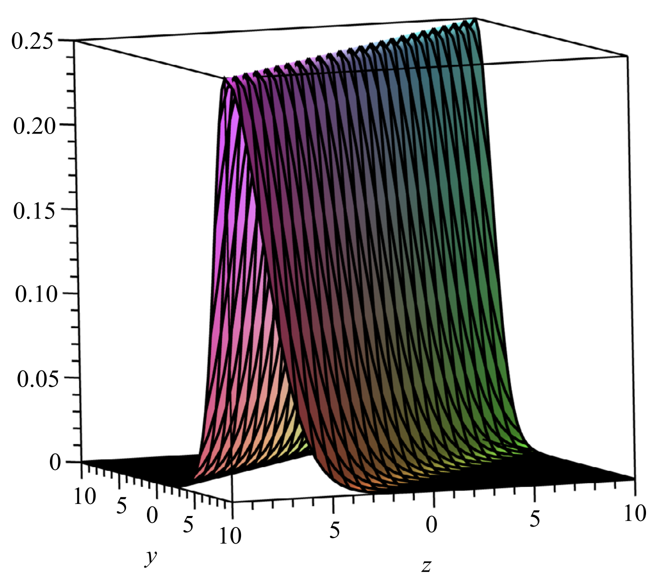

Figure 2 The solution of the mZK with $k = 1$, $t = -2x$

of the mZK equation with $k = 1$. The evolutions of solution (34) are given in Fig. 1, Fig. 2, and Fig. 3, respectively.

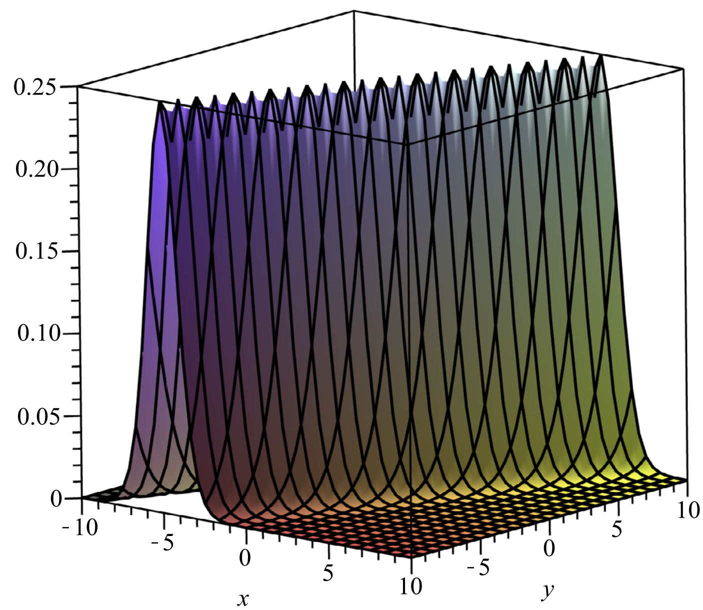

Figure 3 The solution of the mZK with $k = 1$, $t = z$

At the end of this section, let us consider the obtained reduced equation (15). By using (27), it can be transformed into the following ODE:

$$90lf^{\frac{1}{2}}f' - 16\theta f' + 3\big(\delta_0^2 l^3 + lm^2 + n^2 l\big)f''' - 64f = 0, \tag{35}$$

where

$$\begin{aligned} &f' = df/d\theta, \\ &\theta = \big(l(\delta_0 x + \delta_0 kt + 3k\delta_1) + my + nz\big)/(\delta_0 t + \delta_1)^{1/3}. \end{aligned} \tag{36}$$

It is not difficult to find that Eq. (35) has the following exact solution:

$$f = A\theta^{-4},$$

where A, l, m, and n satisfy the relationship $A^{1/2}l + \delta_0^2 l^3 + lm^2 + n^2 l = 0$. Therefore the mZK equation (2) possesses an exact solution as follows:

$$u_6(x, y, z, t) = (\delta_0 t + \delta_1)^{-\frac{4}{3}} A\theta^{-4},$$

where θ is given in Eq. (36).

5 Conclusions

By implementing the direct symmetry method, we have determined the symmetry σ and the corresponding vector field $\mathcal{V}$ of the (3 + 1)-dimensional mZK equation. In view of the compatibility of $\sigma = 0$ and the mZK equation, we have got six (2 + 1)-dimensional symmetry reduction equations. Then, in terms of the obtained lower dimensional reduced equations, we have found exact solutions of the mZK equation including trigonometric function solutions, hyperbolic function solutions, and Weierstrass function solutions. Time delays and stochastic disturbances are often unavoidable in practical systems [40, 41]. Future study will focus on stochastic Zakharov-Kuznetsov equations with time delays.

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Competing interests
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Authors' contributions
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