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Periodic solutions for quaternion-valued fuzzy cellular neural networks with time-varying delays

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Abstract

In this paper, quaternion-valued fuzzy cellular neural networks (QVFCNNs) with time-varying delays are considered. First, we decompose QVFCNNs into their equivalent real-valued systems according to Hamilton's multiplication rules. Then, we establish the existence and global exponential stability of periodic solutions of QVFCNNs by using the Schauder fixed point theorem and by constructing an appropriate Lyapunov function. Our results are completely new and supplementary to the known results. Finally, we give a numerical example to illustrate the effectiveness of our results.

Keywords: Quaternion-valued fuzzy cellular neural networks; Schauder fixed point theorem; Periodic solution; Global exponential stability; Time-varying delays

1 Introduction

Quaternion was invented by the Irish mathematician W. R. Hamilton in 1843 [1]. The skew field of quaternion is denoted by

$$\mathbb{H} := \{q = q_0 + iq_1 + jq_2 + kq_3\},$$

where q_0, q_1, q_2, q_3 are real numbers and the elements i, j , and k obey Hamilton's multiplication rules:

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \quad i^2 = j^2 = k^2 = ijk = -1.$$

Quaternion multiplication does not conform to the law of commutation, so the study on quaternion is much more difficult than that on plurality. Fortunately, over the past 20 years, especially in algebra area, quaternion has been a hot topic for the effective applications in the real world. Also, a new class of differential equations, named quaternion differential equations, has been already applied successfully to the fields such as quantum mechanics [2, 3], robotic manipulation [4], fluid mechanics [5], differential geometry [6], communications problems, signal processing [7–9], and neural networks [10–13]. In particular, in recent years, the applications of quaternion-valued neural networks (QVNNs),

which are described by quaternion-valued differential equations, have been widely investigated. One practical application by QVNNs is the 3D geometrical affine transformation, especially spatial rotation, which can be represented based on QVNNs efficiently and compactly [14, 15]. Other practical applications of QVNNs are image impression, color night vision [16], etc.

On the one hand, it is well known that neural networks have been extensively studied for their wide application in image processing, pattern recognition, optimization solvers, artificial intelligence, fixed point calculations, and other engineering fields [17–20]. These applications rely heavily on their dynamics. Many scholars tried to shed some light on the information about the dynamics of QVNNs. For example, authors of [21] studied the global μ -stability criteria for quaternion-valued neural networks with unbounded time-varying delays; from the view of matrix measure, based on Halanay inequality instead of Lyapunov function, authors of [22] derived some sufficient conditions to guarantee the global exponential stability for QVNNs; authors of [23] investigated the periodicity of QVNNs by a continuation theorem of coincidence degree theory; authors of [24] studied the existence of pseudo almost periodic solutions of QVNNs. However, as we all know, up to now, there have been few results about the dynamics of quaternion-valued neural networks.

On the other hand, fuzzy cellular neural networks (FCNNs) proposed by Yang and Yang in 1996 [25] have been successfully applied in many fields such as physics, chemistry, biology, economics, sociology, medicine, meteorology, and so on [26]. In recent years, many researchers have done a lot of research on the solutions of general fuzzy differential equations and the dynamics of FCNNs (see [27–47] and the references therein). As is known, periodic oscillations and almost periodic oscillations are important dynamical behaviors of non-autonomous neural networks. However, as far as we know, no scholars have studied the periodic solutions of quaternion-valued fuzzy cellular neural networks (QVFCNNs).

Considering the discussion above, in this paper, we are committed to studying the following QVFCNN with time-varying delays:

$$\begin{aligned} \dot{x}_p(t) = & -a_p(t)x_p(t) + \sum_{q=1}^n b_{pq}(t)f_q(x_q(t - \tau_{pq}(t))) + \sum_{q=1}^n d_{pq}(t)\mu_q(t) \\ & + \bigwedge_{q=1}^n \alpha_{pq}(t) \int_{t-\eta_{pq}(t)}^t g_q(s, x) ds + \bigvee_{q=1}^n \beta_{pq}(t) \int_{t-\xi_{pq}(t)}^t g_q(s, x) ds \\ & + \bigwedge_{q=1}^n T_{pq}(t)\mu_q(t) + \bigvee_{q=1}^n S_{pq}(t)\mu_q(t) + I_p(t), \end{aligned} \quad (1)$$

where $p = 1, 2, \dots, n$, n is the number of neurons in layers; $x_p(t) \in \mathbb{H}$ is the state of the p th neuron at time t , and $\mu_q(t) \in \mathbb{H}$ is the deviations of the q th neuron at time t ; $a_p(t) > 0$ represents the rate at which the p th neuron will reset its potential to the resting state in isolation when it is disconnected from the network and the external inputs at time t , $\alpha_{pq}(t) \in \mathbb{H}$, $\beta_{pq}(t) \in \mathbb{H}$, $T_{pq}(t) \in \mathbb{H}$, and $S_{pq}(t) \in \mathbb{H}$ are the elements of fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feed forward MIN template, and fuzzy feed forward MAX template, respectively; $b_{pq}(t) \in \mathbb{H}$ and $d_{pq}(t) \in \mathbb{H}$ are the elements of feedback template and feed forward template, $\bigwedge$, $\bigvee$ denote the fuzzy AND and fuzzy OR operations, respectively; $f_q(\cdot)$ and $g_q(\cdot) : \mathbb{H} \rightarrow \mathbb{H}$ are the activation functions; $\tau_{pq}(t) \geq 0$,

$\eta_{pq}(t) \geq 0$, and $\xi_{pq}(t) \geq 0$ correspond to transmission delays at time t ; $I_p(t) \in \mathbb{H}$ denotes the input of the p th neuron at time t .

Our main purpose of this paper is by using the Schauder fixed point theorem and constructing an appropriate Lyapunov function to study the existence, the uniqueness, and the global exponential stability of periodic solutions of (1). Our results are completely new and when (1) degenerates into real-valued system, our results remain new and supplementary to the known results obtained in [23, 32, 45–47], and our method used in this paper is different from those used in [23, 32, 45–47].

The rest of this paper is organized as follows. In Sect. 2, we introduce some symbols and definitions and present some preliminary results needed later. In Sect. 3, we establish some sufficient conditions for the existence and global exponential stability of periodic solutions of (1). In Sect. 4, an example is given to illustrate the effectiveness of the results.

2 Preliminaries

In this section, we introduce some definitions, notions, lemmas and decompose (1) into an equivalent real-valued system.

Definition 1 Let $h = h^R + ih^I + jh^J + h^K : \mathbb{R} \rightarrow \mathbb{H}$, where $h^l : \mathbb{R} \rightarrow \mathbb{H}$, $l \in \{R, I, J, K\} := \Upsilon$. h is said to be ω -anti-periodic if for every $l \in \Upsilon$, $h^l : \mathbb{R} \rightarrow \mathbb{R}$ is ω -anti-periodic on $\mathbb{R}$.

Lemma 1 ([48] Schauder fixed point theorem) *Let E be a normed space, and let C be a convex, bounded, and closed subset of E . If $T : C \rightarrow C$ is continuous with $\overline{T(C)}$ compact, then T has at least one fixed point.*

Throughout this paper, we assume the following.

(H₁) Let $x_p = x_p^R + ix_p^I + jx_p^J + kx_p^K$, $x_p^R, x_p^I, x_p^J, x_p^K : \mathbb{R} \rightarrow \mathbb{R}$ and assume $f_p(x_p)$ and $g_p(x_p)$ can be expressed as

$$\begin{aligned} f_p(x_p) &= f_p^R(x_p^R, x_p^I, x_p^J, x_p^K) + if_p^I(x_p^R, x_p^I, x_p^J, x_p^K) \\ &\quad + jf_p^J(x_p^R, x_p^I, x_p^J, x_p^K) + kf_p^K(x_p^R, x_p^I, x_p^J, x_p^K), \\ g_p(x_p) &= g_p^R(x_p^R, x_p^I, x_p^J, x_p^K) + ig_p^I(x_p^R, x_p^I, x_p^J, x_p^K) \\ &\quad + jg_p^J(x_p^R, x_p^I, x_p^J, x_p^K) + kg_p^K(x_p^R, x_p^I, x_p^J, x_p^K), \end{aligned}$$

where $f_p^l, g_p^l : \mathbb{R}^4 \rightarrow \mathbb{R}$, $p = 1, 2, \dots, n$, $l \in \Upsilon$.

(H₂) Functions $a_p, \tau_{pq}, \eta_{pq}, \xi_{pq} \in C(\mathbb{R}, \mathbb{R}^+)$, $b_{pq}, d_{pq}, \alpha_{pq}, \beta_{pq}, T_{pq}, S_{pq}, \mu_q, I_p \in C(\mathbb{R}, \mathbb{H})$ are ω -periodic functions, where $x_q^l \in \mathbb{R}$, $p, q = 1, 2, \dots, n$.

(H₃) Functions $f_q^l, g_q^l \in C(\mathbb{R}^4, \mathbb{R})$ and there exist positive constants L_f, L_g such that, for all $x_q^l, y_q^l \in \mathbb{R}$,

$$\begin{aligned} &|f_q^l(x_q^R, x_q^I, x_q^J, x_q^K) - f_q^l(y_q^R, y_q^I, y_q^J, y_q^K)| \\ &\leq L_f(|x_q^R - y_q^R| + |x_q^I - y_q^I| + |x_q^J - y_q^J| + |x_q^K - y_q^K|), \\ &|g_q^l(x_q^R, x_q^I, x_q^J, x_q^K) - g_q^l(y_q^R, y_q^I, y_q^J, y_q^K)| \\ &\leq L_g(|x_q^R - y_q^R| + |x_q^I - y_q^I| + |x_q^J - y_q^J| + |x_q^K - y_q^K|) \end{aligned}$$

and $f_q^l(0, 0, 0, 0) = 0, g_q^l(0, 0, 0, 0) = 0$, $q = 1, 2, \dots, n$, $l \in \Upsilon$.

For $p, q = 1, 2, \dots, n$, assume that

$$\begin{aligned} b_{pq} &= b_{pq}^R + ib_{pq}^I + jb_{pq}^J + kb_{pq}^K, & d_{pq} &= d_{pq}^R + id_{pq}^I + jd_{pq}^J + kd_{pq}^K, \\ \alpha_{pq} &= \alpha_{pq}^R + i\alpha_{pq}^I + j\alpha_{pq}^J + k\alpha_{pq}^K, & \beta_{pq} &= \beta_{pq}^R + i\beta_{pq}^I + j\beta_{pq}^J + k\beta_{pq}^K, \\ T_{pq} &= T_{pq}^R + iT_{pq}^I + jT_{pq}^J + kT_{pq}^K, & S_{pq} &= S_{pq}^R + iS_{pq}^I + jS_{pq}^J + kS_{pq}^K, \\ \mu_q &= \mu_q^R + i\mu_q^I + j\mu_q^J + k\mu_q^K, & I_p &= I_p^R + iI_p^I + jI_p^J + kI_p^K. \end{aligned}$$

We will adopt the following notations:

$$\begin{aligned} b_{pq} &= \sup_{t \in \mathbb{R}} \{ |b_{pq}^R(t)|, |b_{pq}^I(t)|, |b_{pq}^J(t)|, |b_{pq}^K(t)| \}, \\ d_{pq} &= \sup_{t \in \mathbb{R}} \{ |d_{pq}^R(t)|, |d_{pq}^I(t)|, |d_{pq}^J(t)|, |d_{pq}^K(t)| \}, \\ \alpha_{pq} &= \sup_{t \in \mathbb{R}} \{ |\alpha_{pq}^R(t)|, |\alpha_{pq}^I(t)|, |\alpha_{pq}^J(t)|, |\alpha_{pq}^K(t)| \}, \\ \beta_{pq} &= \sup_{t \in \mathbb{R}} \{ |\beta_{pq}^R(t)|, |\beta_{pq}^I(t)|, |\beta_{pq}^J(t)|, |\beta_{pq}^K(t)| \}, \\ T_{pq} &= \sup_{t \in \mathbb{R}} \{ |T_{pq}^R(t)|, |T_{pq}^I(t)|, |T_{pq}^J(t)|, |T_{pq}^K(t)| \}, \\ S_{pq} &= \sup_{t \in \mathbb{R}} \{ |S_{pq}^R(t)|, |S_{pq}^I(t)|, |S_{pq}^J(t)|, |S_{pq}^K(t)| \}, \\ \mu_q &= \sup_{t \in \mathbb{R}} \{ |\mu_q^R(t)|, |\mu_q^I(t)|, |\mu_q^J(t)|, |\mu_q^K(t)| \}, \\ I_p &= \sup_{t \in \mathbb{R}} \{ |I_p^R(t)|, |I_p^I(t)|, |I_p^J(t)|, |I_p^K(t)| \}, \\ \bar{a}_p &= \frac{1}{\omega} \int_0^\omega a_p(t) dt, & a_p^- &= \min_{t \in \mathbb{R}} \{ a_p(t) \}, \\ \tau_{pq} &= \sup_{t \in \mathbb{R}} \{ \tau_{pq}(t) \}, & \eta_{pq} &= \sup_{t \in \mathbb{R}} \{ \eta_{pq}(t) \}, \\ \xi_{pq} &= \sup_{t \in \mathbb{R}} \{ \xi_{pq}(t) \}, & \dot{\tau}_{pq} &= \sup_{t \in \mathbb{R}} \{ \dot{\tau}_{pq}(t) \}, \\ \rho^+ &= \max_{\substack{1 \leq q \leq n \\ l \in \Upsilon}} \{ \rho_{pl} \}, & \rho^- &= \min_{\substack{1 \leq q \leq n \\ l \in \Upsilon}} \{ \rho_{pl} \}. \end{aligned}$$

The initial value of system (1) is given by

$$\begin{aligned} x_p(t) &= \varphi_p(t) \in \mathbb{H}, \quad t \in [t_0 - \theta, t_0], \quad \theta = \max_{1 \leq p, q \leq n} \{ \tau_{pq}, \eta_{pq}, \xi_{pq} \}, \\ \varphi_p(t) &= \varphi_p^R(t) + i\varphi_p^I(t) + j\varphi_p^J(t) + k\varphi_p^K(t), \end{aligned}$$

where $\varphi_p^l \in C([t_0 - \theta, t_0], \mathbb{R})$, $l \in \Upsilon$.

Based on (H_1) and Hamilton's rules, one can obtain from (1) that

$$\dot{x}_p^R(t) = -a_p(t)x_p^R(t) + \sum_{q=1}^n (b_{pq}^R(t)\tilde{f}_q^R[t, x, \tau] - b_{pq}^I(t)\tilde{f}_q^I[t, x, \tau])$$

$$\begin{aligned}
& -b_{pq}^I(t)\tilde{f}_q^J[t, x, \tau] - b_{pq}^K(t)\tilde{f}_q^K[t, x, \tau] + \sum_{q=1}^n (d_{pq}^R(t)\mu_q^R(t) \\
& - d_{pq}^I(t)\mu_q^I(t) - d_{pq}^J(t)\mu_q^J(t) - d_{pq}^K(t)\mu_q^K(t)) \\
& + \bigwedge_{q=1}^n \left(\alpha_{pq}^R(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^R[s, x] ds \right. \\
& - \alpha_{pq}^I(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^I[s, x] ds - \alpha_{pq}^J(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^J[s, x] ds \\
& - \alpha_{pq}^K(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^K[s, x] ds \Big) + \bigvee_{q=1}^n \left(\beta_{pq}^R(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^R[s, x] ds \right. \\
& - \beta_{pq}^I(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^I[s, x] ds - \beta_{pq}^J(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^J[s, x] ds \\
& - \beta_{pq}^K(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^K[s, x] ds \Big) + \bigwedge_{q=1}^n (T_{pq}^R(t)\mu_q^R(t) - T_{pq}^I(t)\mu_q^I(t) \\
& - T_{pq}^J(t)\mu_q^J(t) - T_{pq}^K(t)\mu_q^K(t)) + \bigvee_{q=1}^n (S_{pq}^R(t)\mu_q^R(t) - S_{pq}^I(t)\mu_q^I(t) \\
& - S_{pq}^J(t)\mu_q^J(t) - S_{pq}^K(t)\mu_q^K(t)) + I_p^R(t) \triangleq F_{pR}(t, x(t)), \quad p = 1, 2, \dots, n, \\
\dot{x}_p^I(t) & = -a_p(t)x_p^I(t) + \sum_{q=1}^n (b_{pq}^I(t)\tilde{f}_q^R[t, x, \tau] + b_{pq}^R(t)\tilde{f}_q^I[t, x, \tau] \\
& - b_{pq}^K(t)\tilde{f}_q^J[t, x, \tau] + b_{pq}^J(t)\tilde{f}_q^K[t, x, \tau]) + \sum_{q=1}^n (d_{pq}^I(t)\mu_q^R(t) \\
& + d_{pq}^R(t)\mu_q^I(t) - d_{pq}^K(t)\mu_q^J(t) + d_{pq}^J(t)\mu_q^K(t)) \\
& + \bigwedge_{q=1}^n \left(\alpha_{pq}^I(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^R[s, x] ds \right. \\
& + \alpha_{pq}^R(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^I[s, x] ds - \alpha_{pq}^K(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^J[s, x] ds \\
& + \alpha_{pq}^J(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^K[s, x] ds \Big) + \bigvee_{q=1}^n \left(\beta_{pq}^I(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^R[s, x] ds \right. \\
& + \beta_{pq}^R(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^I[s, x] ds - \beta_{pq}^K(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^J[s, x] ds \\
& + \beta_{pq}^J(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^K[s, x] ds \Big) + \bigwedge_{q=1}^n (T_{pq}^I(t)\mu_q^R(t) + T_{pq}^R(t)\mu_q^I(t) \\
& - T_{pq}^K(t)\mu_q^J(t) + T_{pq}^J(t)\mu_q^K(t)) + \bigvee_{q=1}^n (S_{pq}^I(t)\mu_q^R(t) + S_{pq}^R(t)\mu_q^I(t) \\
& - S_{pq}^K(t)\mu_q^J(t) + S_{pq}^J(t)\mu_q^K(t)) + I_p^I(t) \triangleq F_{pI}(t, x(t)), \quad p = 1, 2, \dots, n,
\end{aligned}$$

$$\begin{aligned}
\dot{x}_p^J(t) = & -a_p(t)x_p^J(t) + \sum_{q=1}^n (b_{pq}^J(t)\tilde{f}_q^R[t, x, \tau] + b_{pq}^K(t)\tilde{f}_q^I[t, x, \tau]) \\
& + b_{pq}^R(t)\tilde{f}_q^J[t, x, \tau] - b_{pq}^I(t)\tilde{f}_q^K[t, x, \tau] + \sum_{q=1}^n (d_{pq}^J(t)\mu_q^R(t) \\
& + d_{pq}^K(t)\mu_q^I(t) + d_{pq}^R(t)\mu_q^J(t) - d_{pq}^I(t)\mu_q^K(t)) \\
& + \bigwedge_{q=1}^n \left(\alpha_{pq}^J(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^R[s, x] ds \right. \\
& + \alpha_{pq}^K(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^I[s, x] ds + \alpha_{pq}^R(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^J[s, x] ds \\
& \left. - \alpha_{pq}^I(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^K[s, x] ds \right) + \bigvee_{q=1}^n \left(\beta_{pq}^J(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^R[s, x] ds \right. \\
& + \beta_{pq}^K(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^I[s, x] ds + \beta_{pq}^R(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^J[s, x] ds \\
& \left. - \beta_{pq}^I(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^K[s, x] ds \right) + \bigwedge_{q=1}^n (T_{pq}^J(t)\mu_q^R(t) + T_{pq}^K(t)\mu_q^I(t) \\
& + T_{pq}^R(t)\mu_q^J(t) - T_{pq}^I(t)\mu_q^K(t)) + \bigvee_{q=1}^n (S_{pq}^J(t)\mu_q^R(t) + S_{pq}^K(t)\mu_q^I(t) \\
& + S_{pq}^R(t)\mu_q^J(t) - S_{pq}^I(t)\mu_q^K(t)) + I_p^J(t) \triangleq F_{pI}(t, x(t)), \quad p = 1, 2, \dots, n, \\
\dot{x}_p^K(t) = & -a_p(t)x_p^K(t) + \sum_{q=1}^n (b_{pq}^K(t)\tilde{f}_q^R[t, x, \tau] - b_{pq}^J(t)\tilde{f}_q^I[t, x, \tau]) \\
& + b_{pq}^I(t)\tilde{f}_q^J[t, x, \tau] + b_{pq}^R(t)\tilde{f}_q^K[t, x, \tau] + \sum_{q=1}^n (d_{pq}^K(t)\mu_q^R(t) \\
& - d_{pq}^J(t)\mu_q^I(t) + d_{pq}^I(t)\mu_q^J(t) + d_{pq}^R(t)\mu_q^K(t)) \\
& + \bigwedge_{q=1}^n \left(\alpha_{pq}^K(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^R[s, x] ds \right. \\
& - \alpha_{pq}^J(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^I[s, x] ds + \alpha_{pq}^I(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^J[s, x] ds \\
& + \alpha_{pq}^R(t) \int_{t-\eta_{pq}(t)}^t \tilde{g}_q^K[s, x] ds \left. \right) + \bigvee_{q=1}^n \left(\beta_{pq}^K(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^R[s, x] ds \right. \\
& - \beta_{pq}^J(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^I[s, x] ds + \beta_{pq}^I(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^J[s, x] ds \\
& + \beta_{pq}^R(t) \int_{t-\xi_{pq}(t)}^t \tilde{g}_q^K[s, x] ds \left. \right) + \bigwedge_{q=1}^n (T_{pq}^K(t)\mu_q^R(t) - T_{pq}^J(t)\mu_q^I(t) \\
& + T_{pq}^I(t)\mu_q^J(t) + T_{pq}^R(t)\mu_q^K(t)) + \bigvee_{q=1}^n (S_{pq}^K(t)\mu_q^R(t) - S_{pq}^J(t)\mu_q^I(t) \\
& + S_{pq}^I(t)\mu_q^J(t) + S_{pq}^R(t)\mu_q^K(t))
\end{aligned}$$

$$+ S_{pq}^I(t)\mu_q^I(t) + S_{pq}^R(t)\mu_q^K(t) + I_p^K(t) \triangleq F_{pK}(t, x(t)), \quad p = 1, 2, \dots, n,$$

where $x_p^R(t) + ix_p^I(t) + jx_p^J(t) + kx_p^K(t) = x_p(t)$, $\tilde{f}_q^l[t, x, \tau]$ and $\tilde{g}_q^l[s, x]$ denote $f_q^l(x_q^R(t - \tau_{pq}(t)), x_q^I(t - \tau_{pq}(t)), x_q^J(t - \tau_{pq}(t)), x_q^K(t - \tau_{pq}(t)))$, and $g_q^l(x_q^R(s), x_q^I(s), x_q^J(s), x_q^K(s))$, respectively, $p, q = 1, 2, \dots, n, l \in \Upsilon$. That is, (1) can be expressed in the following form:

$$\dot{x}_p^l(t) = F_{pl}(t, x(t)), \quad p = 1, 2, \dots, n, l \in \Upsilon, \quad (2)$$

with the initial value

$$x_p^l(s) = \varphi_p^l(s), \quad s \in [t_0 - \theta, t_0], p = 1, 2, \dots, n.$$

Remark 1 If $x = (x_1^R, \dots, x_n^R, x_1^I, \dots, x_n^I, x_1^J, \dots, x_n^J, x_1^K, \dots, x_n^K)^T$ is a solution to system (2), then $u = (x_1, x_2, \dots, x_n)^T$, where $x_p = x_p^R + ix_p^I + jx_p^J + kx_p^K, p = 1, 2, \dots, n$, is a solution to (1). Thus, to study the existence and stability of solutions of (1), we just need to study the existence and stability of solutions of system (2).

Definition 2 Let $x = (x_1^R, \dots, x_n^R, x_1^I, \dots, x_n^I, x_1^J, \dots, x_n^J, x_1^K, \dots, x_n^K)^T$ be a solution of (2) with the initial value $\varphi = (\varphi_1^R, \dots, \varphi_n^R, \varphi_1^I, \dots, \varphi_n^I, \varphi_1^J, \dots, \varphi_n^J, \varphi_1^K, \dots, \varphi_n^K)^T$ and $y = (y_1^R, \dots, y_n^R, y_1^I, \dots, y_n^I, y_1^J, \dots, y_n^J, y_1^K, \dots, y_n^K)^T$ be an arbitrary solution of system (2) with the initial value $\varphi = (\varphi_1^R, \dots, \varphi_n^R, \varphi_1^I, \dots, \varphi_n^I, \varphi_1^J, \dots, \varphi_n^J, \varphi_1^K, \dots, \varphi_n^K)^T$, respectively, where $\varphi, \psi \in C([t_0 - \theta, t_0], \mathbb{R}^{4n})$. If there exist constants $\lambda > 0$ and $M > 0$ such that

$$\|x(t) - y(t)\| \leq M\|\varphi - \psi\|e^{-\lambda(t-t_0)}, \quad t \geq t_0,$$

where

$$\|\varphi - \psi\| = \max_{l \in \Upsilon} \left\{ \sup_{1 \leq q \leq n} \sup_{s \in [t_0 - \theta, t_0]} \rho_{pl}^{-1} |\varphi_p^l(s) - \psi_p^l(s)| \right\}.$$

Then the solution x of system (2) is said to be globally exponentially stable.

Lemma 2 ([25]) Let $\alpha_{pq}^l, \beta_{pq}^l, x_q^R, x_q^I, x_q^J, x_q^K, y_q^R, y_q^I, y_q^J, y_q^K : \mathbb{R} \rightarrow \mathbb{R}$ and $g_q^r : \mathbb{R}^4 \rightarrow \mathbb{R}$ be continuous functions for $p, q = 1, 2, \dots, n, l, r \in \Upsilon$, then we have

$$\begin{aligned} & \left| \bigwedge_{q=1}^n \alpha_{pq}^l g_q^r(x_q^R, x_q^I, x_q^J, x_q^K) - \bigwedge_{q=1}^n \alpha_{pq}^l g_q^r(y_q^R, y_q^I, y_q^J, y_q^K) \right| \\ & \leq \sum_{q=1}^n |\alpha_{pq}^l| |g_q^r(x_q^R, x_q^I, x_q^J, x_q^K) - g_q^r(y_q^R, y_q^I, y_q^J, y_q^K)|, \\ & \left| \bigvee_{q=1}^n \beta_{pq}^l g_q^r(x_q^R, x_q^I, x_q^J, x_q^K) - \bigvee_{q=1}^n \beta_{pq}^l g_q^r(y_q^R, y_q^I, y_q^J, y_q^K) \right| \\ & \leq \sum_{q=1}^n |\beta_{pq}^l| |g_q^r(x_q^R, x_q^I, x_q^J, x_q^K) - g_q^r(y_q^R, y_q^I, y_q^J, y_q^K)|. \end{aligned}$$

3 Main results

In this section, we establish some sufficient conditions for the existence and exponential stability of periodic solutions of (1).

Set

$$\mathbb{X} = \{x = (x_1^R, x_1^I, x_1^J, x_1^K, \dots, x_n^R, x_n^I, x_n^J, x_n^K)^T \in C(\mathbb{R}, \mathbb{R}^{4n}), \\ x(t + \omega) = x(t), t \in \mathbb{R}\}.$$

For constant vector $\rho = (\rho_{1R}, \rho_{1I}, \rho_{1J}, \rho_{1K}, \dots, \rho_{nR}, \rho_{nI}, \rho_{nJ}, \rho_{nK})^T \in \mathbb{R}^{4n}$ with $\rho_{pl} > 0$ for $p = 1, 2, \dots, n, l \in \Upsilon$, we define the following norm:

$$\|x\|_{\mathbb{X}} = \max_{\substack{1 \leq p \leq n \\ l \in \Upsilon}} \left\{ \sup_{t \in [0, \omega]} \rho_{pl}^{-1} |x_p^l(t)| \right\}.$$

Then $\mathbb{X}$ is a Banach space when it is endowed with the norm $\|\cdot\|_{\mathbb{X}}$.

Obviously, if $x = (x_1^R, x_1^I, x_1^J, x_1^K, \dots, x_n^R, x_n^I, x_n^J, x_n^K)^T \in \mathbb{X}$ is a solution of system (2), then

$$\begin{aligned} (x_p^l(t) e^{\int_{t_0}^t a_p(s) ds})' &= \dot{x}_p^l(t) e^{\int_{t_0}^t a_p(s) ds} + a_p(t) x_p^l(t) e^{\int_{t_0}^t a_p(s) ds} \\ &= e^{\int_{t_0}^t a_p(s) ds} [F_{pl}(t, x(t)) + a_p(t) x_p^l(t)] \end{aligned} \quad (3)$$

for $p = 1, 2, \dots, n, l \in \Upsilon$. Integrating both sides of (3) over $[t, t + \omega]$, we can get

$$\begin{aligned} x_p^l(t) &= \int_t^{t+\omega} \frac{e^{\int_{t_0}^s a_p(u) du}}{e^{\int_{t_0}^{t+\omega} a_p(u) du} - e^{\int_{t_0}^t a_p(u) du}} [F_{pl}(s, x(s)) + a_p(s) x_p^l(s)] ds \\ &= \int_t^{t+\omega} \frac{e^{-\int_s^{t+\omega} a_p(u) du}}{1 - e^{-\omega a_p}} [F_{pl}(s, x(s)) + a_p(s) x_p^l(s)] ds \\ &= \int_t^{t+\omega} G_p(t, s) [F_{pl}(s, x(s)) + a_p(s) x_p^l(s)] ds, \end{aligned} \quad (4)$$

where $G_p(t, s) = \frac{e^{-\int_s^{t+\omega} a_p(u) du}}{1 - e^{-\omega a_p}}$, $t \leq s \leq t + \omega$, $p = 1, 2, \dots, n, l \in \Upsilon$.

Define an operator $\Phi : \mathbb{X} \rightarrow \mathbb{X}$ by

$$\Phi x = (\Phi x_1^R, \Phi x_1^I, \Phi x_1^J, \Phi x_1^K, \dots, \Phi x_n^R, \Phi x_n^I, \Phi x_n^J, \Phi x_n^K)^T, \quad (5)$$

where for $p = 1, 2, \dots, n, l \in \Upsilon$,

$$(\Phi x)_p^l(t) = \int_t^{t+\omega} G_p(t, s) [F_{pl}(s, x(s)) + a_p(s) x_p^l(s)] ds, \quad t \in \mathbb{R}. \quad (6)$$

From (4), it is easy to verify that $x \in \mathbb{X}$ is an ω -periodic solution of system (2) provided x is a fixed point of Φ in $\mathbb{X}$.

It is easy to see that if there exist positive numbers $\rho_{1R}, \rho_{1I}, \rho_{1J}, \rho_{1K}, \dots, \rho_{nR}, \rho_{nI}, \rho_{nJ}, \rho_{nK}$ such that

$$\gamma = \max_{\substack{1 \leq p \leq n \\ l, r \in \Upsilon}} \sup_{t \in \mathbb{R}} \left\{ \rho_{pl}^{-1} \int_t^{t+\omega} G_p(t, s) \left(\sum_{q=1}^n (|b_{pq}^R(s)| + |b_{pq}^I(s)| + |b_{pq}^J(s)|) \right) \right.$$

$$\begin{aligned}
& + |b_{pq}^K(s)| \rho_{qr} L_f + \sum_{q=1}^n (|\alpha_{pq}^R(s)| + |\alpha_{pq}^I(s)| + |\alpha_{pq}^J(s)| + |\alpha_{pq}^K(s)|) \rho_{qr} L_g \theta \\
& + \sum_{q=1}^n (|\beta_{pq}^R(s)| + |\beta_{pq}^I(s)| + |\beta_{pq}^J(s)| + |\beta_{pq}^K(s)|) \rho_{qr} L_g \theta \Big) ds \Big\} < \frac{1}{4}, \quad (7)
\end{aligned}$$

then there exists a sufficiently large $\kappa \geq 1$ such that $\gamma \leq \frac{1}{4}(1 - \iota\kappa^{-1})$, where

$$\iota = \max_{\substack{1 \leq p \leq n \\ l \in \Upsilon}} \left\{ \frac{\omega(\sum_{q=1}^n 4(d_{pq}\mu_q + T_{pq}\mu_q + S_{pq}\mu_q) + I_p)}{\rho_{pl}(1 - e^{-\omega\tilde{a}_p})} \right\}.$$

Lemma 3 *Under hypotheses (H_1) – (H_3) , suppose that there exist positive numbers $\rho_{1R}, \rho_{1I}, \rho_{1J}, \rho_{1K}, \dots, \rho_{nR}, \rho_{nI}, \rho_{nJ}, \rho_{nK}$ such that inequality (7) holds, then the mapping Φ is a self-mapping in the region $\mathbb{B} = \{x = (x_1^R, x_1^I, x_1^J, x_1^K, \dots, x_n^R, x_n^I, x_n^J, x_n^K)^T \in \mathbb{X} : \|x\| \leq \kappa\}$.*

Proof For $x \in \mathbb{B}$, from (6), we have

$$\begin{aligned}
& |\rho_{pR}^{-1}(\Phi x)_p^R(t)| \\
& = \left| \rho_{pR}^{-1} \int_t^{t+\omega} G_p(t, s) [F_{pR}(s, x(s)) + a_p(s)x_p^R(s)] ds \right| \\
& = \left| \rho_{pR}^{-1} \int_t^{t+\omega} G_p(t, s) \left\{ \sum_{q=1}^n (b_{pq}^R(s)\tilde{f}_q^R[s, x, \tau] - b_{pq}^I(s)\tilde{f}_q^I[s, x, \tau] \right. \right. \\
& \quad - b_{pq}^J(s)\tilde{f}_q^J[s, x, \tau] - b_{pq}^K(s)\tilde{f}_q^K[s, x, \tau]) + \sum_{q=1}^n (d_{pq}^R(s)\mu_q^R(s) \\
& \quad - d_{pq}^I(s)\mu_q^I(s) - d_{pq}^J(s)\mu_q^J(s) - d_{pq}^K(s)\mu_q^K(s)) \\
& \quad + \bigwedge_{q=1}^n \left(\alpha_{pq}^R(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^R[u, x] du - \alpha_{pq}^I(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^I[u, x] du \right. \\
& \quad - \alpha_{pq}^J(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^J[u, x] du - \alpha_{pq}^K(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^K[u, x] du \Big) \\
& \quad + \bigvee_{q=1}^n \left(\beta_{pq}^R(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^R[u, x] du - \beta_{pq}^I(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^I[u, x] du \right. \\
& \quad - \beta_{pq}^J(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^J[u, x] du - \beta_{pq}^K(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^K[u, x] du \Big) \\
& \quad + \bigwedge_{q=1}^n (T_{pq}^R(s)\mu_q^R(s) - T_{pq}^I(s)\mu_q^I(s) - T_{pq}^J(s)\mu_q^J(s) - T_{pq}^K(s)\mu_q^K(s)) \\
& \quad + \left. \bigvee_{q=1}^n (S_{pq}^R(s)\mu_q^R(s) - S_{pq}^I(s)\mu_q^I(s) - S_{pq}^J(s)\mu_q^J(s) - S_{pq}^K(s)\mu_q^K(s)) + I_p^R(s) \right\} ds \Big| \\
& \leq \rho_{pR}^{-1} \int_t^{t+\omega} G_p(t, s) \left\{ \sum_{q=1}^n (|b_{pq}^R(s)| |\tilde{f}_q^R[s, x, \tau] - \tilde{f}_q^R[0]| \right. \\
& \quad + |b_{pq}^I(s)| |\tilde{f}_q^I[s, x, \tau] - \tilde{f}_q^I[0]| + |b_{pq}^J(s)| |\tilde{f}_q^J[s, x, \tau] - \tilde{f}_q^J[0]|
\end{aligned}$$

$$\begin{aligned}
& + \left| b_{pq}^K(s) \right| \left| \tilde{f}_q^K[s, x, \tau] - \tilde{f}_q^K[0] \right| \\
& + \left| \bigwedge_{q=1}^n \alpha_{pq}^R(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^R[u, x] du - \bigwedge_{q=1}^n \alpha_{pq}^R(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^R[0] du \right| \\
& + \left| \bigwedge_{q=1}^n \alpha_{pq}^I(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^I[u, x] du - \bigwedge_{q=1}^n \alpha_{pq}^I(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^I[0] du \right| \\
& + \left| \bigwedge_{q=1}^n \alpha_{pq}^J(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^J[u, x] du - \bigwedge_{q=1}^n \alpha_{pq}^J(s) \int_{s-\eta_{pq}(t)}^s \tilde{g}_q^J[0] du \right| \\
& + \left| \bigwedge_{q=1}^n \alpha_{pq}^K(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^K[u, x] du - \bigwedge_{q=1}^n \alpha_{pq}^K(s) \int_{s-\eta_{pq}(s)}^s \tilde{g}_q^K[0] du \right| \\
& + \left| \bigvee_{q=1}^n \beta_{pq}^R(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^R[u, x] du - \bigvee_{q=1}^n \beta_{pq}^R(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^R[0] du \right| \\
& + \left| \bigvee_{q=1}^n \beta_{pq}^I(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^I[u, x] du - \bigvee_{q=1}^n \beta_{pq}^I(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^I[0] du \right| \\
& + \left| \bigvee_{q=1}^n \beta_{pq}^J(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^J[u, x] du - \bigvee_{q=1}^n \beta_{pq}^J(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^J[0] du \right| \\
& + \left| \bigvee_{q=1}^n \beta_{pq}^K(s) \int_{s-\xi_{pq}(s)}^s \tilde{g}_q^K[u, x] du - \bigvee_{q=1}^n \beta_{pq}^K(s) \int_{s-\xi_{pq}(s)}^t \tilde{g}_q^K[0] du \right| \Big\} ds \\
& + \frac{\omega}{\rho_{pR}(1 - e^{-\omega \bar{a}_p})} \left(\sum_{q=1}^n 4(d_{pq}\mu_q + T_{pq}\mu_q + S_{pq}\mu_q) + I_p \right) \\
& \leq \rho_{pR}^{-1} \int_t^{t+\omega} G_p(t, s) \left\{ \sum_{q=1}^n (|b_{pq}^R(s)| + |b_{pq}^I(s)| + |b_{pq}^J(s)| + |b_{pq}^K(s)|) L_f \right. \\
& \quad \times (|x_q^R(s - \tau_{pq}(s))| + |x_q^I(s - \tau_{pq}(s))| + |x_q^J(s - \tau_{pq}(s))| + |x_q^K(s - \tau_{pq}(s))|) \\
& \quad + \sum_{q=1}^n (|\alpha_{pq}^R(s)| + |\alpha_{pq}^I(s)| + |\alpha_{pq}^J(s)| + |\alpha_{pq}^K(s)|) L_g \int_{s-\eta_{pq}(s)}^s (|x_q^R(u)| \\
& \quad + |x_q^I(u)| + |x_q^J(u)| + |x_q^K(u)|) du + \sum_{q=1}^n (|\beta_{pq}^R(s)| + |\beta_{pq}^I(s)| + |\beta_{pq}^J(s)| \\
& \quad + |\beta_{pq}^K(s)|) L_g \int_{s-\xi_{pq}(s)}^s (|x_q^R(u)| + |x_q^I(u)| + |x_q^J(u)| + |x_q^K(u)|) du \Big\} ds \\
& + \frac{\omega(\sum_{q=1}^n 4(d_{pq}\mu_q + T_{pq}\mu_q + S_{pq}\mu_q) + I_p)}{\rho_{pR}(1 - e^{-\omega \bar{a}_p})} \\
& \leq \rho_{pR}^{-1} \int_t^{t+\omega} G_p(t, s) \left\{ \sum_{q=1}^n (|b_{pq}^R(s)| + |b_{pq}^I(s)| + |b_{pq}^J(s)| + |b_{pq}^K(s)|) (\rho_{qR} + \rho_{qI} \right. \\
& \quad + \rho_{qJ} + \rho_{qK}) L_f + \sum_{q=1}^n (|\alpha_{pq}^R(s)| + |\alpha_{pq}^I(s)| + |\alpha_{pq}^J(s)| + |\alpha_{pq}^K(s)|) (\rho_{qR} + \rho_{qI}
\end{aligned}$$

$$\begin{aligned}
& + \rho_{q_I} + \rho_{q_K})L_g\theta + \sum_{q=1}^n (|\beta_{pq}^R(s)| + |\beta_{pq}^I(s)| + |\beta_{pq}^J(s)| + |\beta_{pq}^K(s)|)(\rho_{q_R} + \rho_{q_I} \\
& + \rho_{q_J} + \rho_{q_K})L_g\theta \Bigg\} \|x\| ds + \frac{\omega(\sum_{q=1}^n 4(d_{pq}\mu_q + T_{pq}\mu_q + S_{pq}\mu_q) + I_p)}{\rho_{p_R}(1 - e^{-\omega\tilde{a}_p})} \\
& \leq 4\gamma\kappa + \iota \leq 4\kappa \times \frac{1}{4}(1 - \iota\kappa^{-1}) + \iota = \kappa, \quad p = 1, 2, \dots, n.
\end{aligned}$$

Similarly, from (6), we can obtain

$$|\rho_{p_l}^{-1}(\Phi x)_p^l(t)| \leq \kappa, \quad p = 1, 2, \dots, n, l = I, J, K.$$

Thus,

$$\|\Phi x\|_{\mathbb{X}} = \max_{\substack{1 \leq p \leq n \\ l \in \mathcal{I}}} \left\{ \sup_{t \in \mathbb{R}} \rho_{p_l}^{-1} |(\Phi x)_p^l(t)| \right\} \leq \kappa.$$

The proof is complete. $\square$

Lemma 4 Assume that (H_1) – (H_3) hold and suppose further that there exist positive numbers $\rho_{1_R}, \rho_{1_I}, \rho_{1_J}, \rho_{1_K}, \dots, \rho_{n_R}, \rho_{n_I}, \rho_{n_J}, \rho_{n_K}$ such that inequality (7) holds, then the mapping $\Phi : \mathbb{B} \rightarrow \mathbb{B}$ defined by (5) is completely continuous.

Proof It is obvious that Φ is continuous.

Next, we prove that Φ is compact. For any $M > 0$, let $S = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} \leq M\}$. Then, for $x \in S$, we have

$$\begin{aligned}
|f_q^l(x_q^R(s), x_q^I(s), x_q^J(s), x_q^K(s))| & \leq (\rho_{q_R} + \rho_{q_I} + \rho_{q_J} + \rho_{q_K})L_f M, \\
|g_q^l(x_q^R(s), x_q^I(s), x_q^J(s), x_q^K(s))| & \leq (\rho_{q_R} + \rho_{q_I} + \rho_{q_J} + \rho_{q_K})L_g M.
\end{aligned}$$

For $\forall x \in S$, we have

$$\begin{aligned}
& |\rho_{p_l}^{-1}(\Phi x)_p^l(t)| \\
& \leq \rho_{p_l}^{-1} \int_t^{t+\omega} G_p(t, s) \left\{ \sum_{q=1}^n (|b_{pq}^R(s)| + |b_{pq}^I(s)| + |b_{pq}^J(s)| + |b_{pq}^K(s)|)(\rho_{q_R} + \rho_{q_I} \right. \\
& \quad + \rho_{q_J} + \rho_{q_K})L_f M + \bigwedge_{q=1}^n (|\alpha_{pq}^R(s)| + |\alpha_{pq}^I(s)| + |\alpha_{pq}^J(s)| + |\alpha_{pq}^K(s)|)(\rho_{q_R} + \rho_{q_I} \\
& \quad + \rho_{q_J} + \rho_{q_K})L_g M\theta + \bigvee_{q=1}^n (|\beta_{pq}^R(s)| + |\beta_{pq}^I(s)| + |\beta_{pq}^J(s)| + |\beta_{pq}^K(s)|)(\rho_{q_R} + \rho_{q_I} \\
& \quad + \rho_{q_J} + \rho_{q_K})L_g M\theta \Bigg\} ds + \frac{\omega(\sum_{q=1}^n 4(d_{pq}\mu_q + T_{pq}\mu_q + S_{pq}\mu_q) + I_p)}{\rho_{p_l}(1 - e^{-\omega\tilde{a}_p})} \\
& \leq \rho_{p_l}^{-1} \int_t^{t+\omega} G_p(t, s) \left\{ \sum_{q=1}^n (|b_{pq}^R(s)| + |b_{pq}^I(s)| + |b_{pq}^J(s)| + |b_{pq}^K(s)|)(\rho_{q_R} + \rho_{q_I} \right.
\end{aligned}$$

$$\begin{aligned}
& + \rho_{q_J} + \rho_{q_K})L_f M + \sum_{q=1}^n (|\alpha_{pq}^R(s)| + |\alpha_{pq}^I(s)| + |\alpha_{pq}^J(s)| + |\alpha_{pq}^K(s)|)(\rho_{q_R} + \rho_{q_I}) \\
& + \rho_{q_J} + \rho_{q_K})L_g M\theta + \sum_{q=1}^n (|\beta_{pq}^R(s)| + |\beta_{pq}^I(s)| + |\beta_{pq}^J(s)| + |\beta_{pq}^K(s)|)(\rho_{q_R} + \rho_{q_I}) \\
& + \rho_{q_J} + \rho_{q_K})L_g M\theta \Bigg\} ds + \frac{\omega(\sum_{q=1}^n 4(d_{pq}\mu_q + T_{pq}\mu_q + S_{pq}\mu_q) + I_p)}{\rho_{p_l}(1 - e^{-\omega\bar{a}_p})} \\
& \leq \frac{\omega}{\rho_{p_l}(1 - e^{-\omega\bar{a}_p})} \sum_{q=1}^n [16\rho^+ M(b_{pq}L_f + \alpha_{pq}L_g\theta + \beta_{pq}L_g\theta) \\
& + 4\mu_q(d_{pq} + T_{pq} + S_{pq}) + I_p] := C_{pl}, \quad p = 1, 2, \dots, n, l \in \Upsilon.
\end{aligned}$$

Therefore,

$$\|\Phi x\|_{\mathbb{X}} = \max_{\substack{1 \leq p \leq n \\ l \in \Upsilon}} \left\{ \sup_{t \in [0, \omega]} \rho_{p_l}^{-1} |(\Phi x)_p^l(t)| \right\} \leq \max_{\substack{1 \leq p \leq n \\ l \in \Upsilon}} \{C_{pl}\}, \quad \forall x \in S,$$

which implies that ΦS is uniformly bounded.

On the other hand,

$$\begin{aligned}
[(\Phi x)_p^l(t)]' &= \left[\int_t^{t+\omega} G_p(t, s) [F_{p_l}(s, x(s)) + a_p(s)x_p^l(s)] ds \right]' \\
&= \int_t^{t+\omega} \frac{\partial G_p(t, s)}{\partial t} [F_{p_l}(s, x(s)) + a_p(s)x_p^l(s)] ds \\
&\quad + G_p(t, t+\omega) [F_{p_l}(t+\omega, x(t+\omega)) + a_p(t+\omega)x_p^l(t+\omega)] \\
&\quad - G_p(t, t) [F_{p_l}(t, x(t)) + a_p(t)x_p^l(t)] \\
&= -a_p(t)(\Phi x)_p^l(t) + F_{p_l}(t, x(t)) + a_p(t)x_p^l(t), \quad p = 1, 2, \dots, n, l \in \Upsilon.
\end{aligned}$$

Hence, for $x \in S$, we have

$$\begin{aligned}
|\rho_{p_l}^{-1} [(\Phi x)_p^l(t)]'| &= \rho_{p_l}^{-1} |-a_p(t)(\Phi x)_p^l(t) + F_{p_l}(t, x(t)) + a_p(t)x_p^l(t)| \\
&\leq \rho_{p_l}^{-1} |-a_p(t)(\Phi x)_p^l(t)| + \rho_{p_l}^{-1} |F_{p_l}(t, x(t)) + a_p(t)x_p^l(t)| \\
&\leq a_p(t)C + \rho_{p_l}^{-1} \sum_{q=1}^n [16\rho^+ M(b_{pq}L_f + \alpha_{pq}L_g + \beta_{pq}L_g) \\
&\quad + 4\mu_q(d_{pq} + T_{pq} + S_{pq}) + I_p] := H_{pl}, \quad p = 1, 2, \dots, n, l \in \Upsilon.
\end{aligned}$$

Therefore,

$$|(\Phi x)'(t)| \leq \max_{\substack{1 \leq p \leq n \\ l \in \Upsilon}} \left\{ \sup_{t \in [0, \omega]} \rho_{p_l}^{-1} |[(\Phi x)_p^l(t)]'| \right\} \leq \max_{\substack{1 \leq p \leq n \\ l \in \Upsilon}} \{H_{pl}\}, \quad \forall x \in S,$$

which implies that ΦS is equi-continuous.

Hence, by the Ascoli–Arzela theorem, we know that the operator Φ is compact, and so it is completely continuous. The proof is complete. $\square$

According to Lemma 1 and Remark 1, from Lemma 3 and Lemma 4, we have the following theorem.

Theorem 1 Assume that (H_1) – (H_3) hold. Suppose further that there exist positive numbers $\rho_{1_R}, \rho_{1_I}, \rho_{1_J}, \rho_{1_K}, \dots, \rho_{n_R}, \rho_{n_I}, \rho_{n_J}, \rho_{n_K}$ such that inequality (7) holds, then system (1) has at least one ω -periodic solution.

Next, we will show the exponential stability of periodic solutions.

Theorem 2 Assume that (H_1) – (H_3) hold. Furthermore, assume the following.

(H_4) Functions $\tau_{pq}(t)$ are continuous differential functions and $\dot{\tau}_{pq} < 1$, where $p, q = 1, 2, \dots, n$.

(H_5) There exists a positive constant $\lambda > 0$ satisfying

$$\max_{1 \leq p \leq n} \left\{ \lambda - a_p^- + \sum_{q=1}^n 16e^{\lambda\theta} \left(\frac{b_{qp}L_f}{1 - \dot{\tau}_{qp}} + (\alpha_{qp}\eta_{qp} + \beta_{qp}\xi_{qp})L_g \right) \right\} \leq 0.$$

Then system (1) has a unique ω -periodic solution and it is globally exponentially stable.

Proof From Theorem 1, we see that system (2) has an ω -periodic solution $x(t)$. Suppose that the initial value of $x(t)$ is $\varphi(t)$. Let $y(t)$ be an arbitrary solution of system (2) with the initial value $\psi(t)$. Set $z_p^l(t) = x_p^l(t) - y_p^l(t)$, by (2), for $l = R, p = 1, 2, \dots, n$, we have

$$\begin{aligned} & D^+(e^{\lambda(t-t_0)} |z_p^R(t)|) \\ &= e^{\lambda(t-t_0)} D^+ |z_p^R(t)| + \lambda e^{\lambda(t-t_0)} |z_p^R(t)| \\ &= \lambda e^{\lambda(t-t_0)} |z_p^R(t)| + e^{\lambda(t-t_0)} \operatorname{sgn}(z_p^R(t)) \dot{z}_p^R(t) \\ &\leq \lambda e^{\lambda(t-t_0)} |z_p^R(t)| + e^{\lambda(t-t_0)} \left\{ -a_p^- |z_p^R(t)| + \sum_{q=1}^n 4b_{pq}L_f (|z_p^R(t - \tau_{pq}(t))| \right. \\ &\quad + |z_p^I(t - \tau_{pq}(t))| + |z_p^J(t - \tau_{pq}(t))| + |z_p^K(t - \tau_{pq}(t))|) \\ &\quad + \bigwedge_{q=1}^n 4\alpha_{pq}L_g \int_{t-\eta_{pq}(t)}^t (|z_p^R(s)| + |z_p^I(s)| + |z_p^J(s)| + |z_p^K(s)|) ds \\ &\quad \left. + \bigvee_{q=1}^n 4\beta_{pq}L_g \int_{t-\xi_{pq}(t)}^t (|z_p^R(s)| + |z_p^I(s)| + |z_p^J(s)| + |z_p^K(s)|) ds \right\} \\ &\leq \lambda e^{\lambda(t-t_0)} |z_p^R(t)| + e^{\lambda(t-t_0)} \left\{ -a_p^- |z_p^R(t)| + \sum_{q=1}^n 4b_{pq}L_f (|z_p^R(t - \tau_{pq}(t))| \right. \\ &\quad + |z_p^I(t - \tau_{pq}(t))| + |z_p^J(t - \tau_{pq}(t))| + |z_p^K(t - \tau_{pq}(t))|) \\ &\quad + \sum_{q=1}^n 4\alpha_{pq}L_g \int_{t-\eta_{pq}(t)}^t (|z_p^R(s)| + |z_p^I(s)| + |z_p^J(s)| + |z_p^K(s)|) ds \\ &\quad \left. + \sum_{q=1}^n 4\beta_{pq}L_g \int_{t-\xi_{pq}(t)}^t (|z_p^R(s)| + |z_p^I(s)| + |z_p^J(s)| + |z_p^K(s)|) ds \right\}. \end{aligned}$$

Repeating the above process, for $p = 1, 2, \dots, n, l = I, J, K$, we can obtain that

$$\begin{aligned} & D^+(e^{\lambda(t-t_0)}|z_p^I(t)|) \\ & \leq \lambda e^{\lambda(t-t_0)}|z_p^R(t)| + e^{\lambda(t-t_0)} \left\{ -a_p^-|z_p^R(t)| + \sum_{q=1}^n 4b_{pq}L_f(|z_p^R(t-\tau_{pq}(t))| \right. \\ & \quad + |z_p^I(t-\tau_{pq}(t))| + |z_p^J(t-\tau_{pq}(t))| + |z_p^K(t-\tau_{pq}(t))|) \\ & \quad + \sum_{q=1}^n 4\alpha_{pq}L_g \int_{t-\eta_{pq}(t)}^t (|z_p^R(s)| + |z_p^I(s)| + |z_p^J(s)| + |z_p^K(s)|) ds \\ & \quad \left. + \sum_{q=1}^n 4\beta_{pq}L_g \int_{t-\xi_{pq}(t)}^t (|z_p^R(s)| + |z_p^I(s)| + |z_p^J(s)| + |z_p^K(s)|) ds \right\}. \end{aligned}$$

Define a Lyapunov function as follows:

$$V(t) = V_R(t) + V_I(t) + V_J(t) + V_K(t),$$

where, for $l \in \mathcal{I}$,

$$\begin{aligned} V_l(t) = & \sum_{p=1}^n \left\{ \rho_{pl}^{-1} e^{\lambda(t-t_0)} |z_p^I(t)| + \sum_{q=1}^n \frac{4b_{pq}L_f \rho_{pl}^{-1} e^{\lambda(\theta-t_0)}}{1 - \dot{\tau}_{pq}} \right. \\ & \times \int_{t-\tau_{pq}(t)}^t e^{\lambda s} (|z_q^R(s)| + |z_q^I(s)| + |z_q^J(s)| + |z_q^K(s)|) ds \\ & + \sum_{q=1}^n 4\alpha_{pq}L_g \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \int_{-\eta_{pq}}^0 \int_{t+u}^t e^{\lambda s} (|z_q^R(s)| + |z_q^I(s)| \\ & + |z_q^J(s)| + |z_q^K(s)|) ds du + \sum_{q=1}^n 4\beta_{pq}L_g \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \\ & \left. \times \int_{-\xi_{pq}}^0 \int_{t+u}^t e^{\lambda s} (|z_q^R(s)| + |z_q^I(s)| + |z_q^J(s)| + |z_q^K(s)|) ds du \right\}. \end{aligned}$$

Now we calculate the time derivative of $V(t)$ along the trajectories of system (2), we get

$$\begin{aligned} D^+ V_l(t) = & \sum_{p=1}^n \left\{ \rho_{pl}^{-1} D^+(e^{\lambda(t-t_0)}|z_p^I(t)|) + \sum_{q=1}^n \frac{4b_{pq}L_f \rho_{pl}^{-1} e^{\lambda(\theta-t_0)}}{1 - \dot{\tau}_{pq}} \right. \\ & \times e^{\lambda t} (|z_q^R(t)| + |z_q^I(t)| + |z_q^J(t)| + |z_q^K(t)|) \\ & - \sum_{q=1}^n \frac{4b_{pq}L_f \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} (1 - \dot{\tau}_{pq}(t))}{1 - \dot{\tau}_{pq}} e^{\lambda(t-\tau_{pq}(t))} (|z_q^R(t-\tau_{pq}(t))| \\ & + |z_q^I(t-\tau_{pq}(t))| + |z_q^J(t-\tau_{pq}(t))| + |z_q^K(t-\tau_{pq}(t))|) \\ & + \sum_{q=1}^n 4\alpha_{pq}L_g \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \int_{-\eta_{pq}}^0 e^{\lambda t} (|z_q^R(t)| + |z_q^I(t)| + |z_q^J(t)| \\ & \left. + |z_q^K(t)|) ds du \right\}. \end{aligned}$$

$$\begin{aligned}
& + |z_q^K(t)| \, du - \sum_{q=1}^n 4\alpha_{pq} L_g \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \int_{-\eta_{pq}}^0 e^{\lambda(t+u)} (|z_q^R(t+u)| \\
& + |z_q^I(t+u)| + |z_q^J(t+u)| + |z_q^K(t+u)|) \, du \\
& + \sum_{q=1}^n 4\beta_{pq} L_g \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \int_{-\xi_{pq}}^0 e^{\lambda t} (|z_q^R(t)| + |z_q^I(t)| + |z_q^J(t)| \\
& + |z_q^K(t)|) \, du - \sum_{q=1}^n 4\beta_{pq} L_g \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \int_{-\xi_{pq}}^0 e^{\lambda(t+u)} (|z_q^R(t+u)| \\
& + |z_q^I(t+u)| + |z_q^J(t+u)| + |z_q^K(t+u)|) \, du \Big\} \\
\leq & \sum_{p=1}^n \left\{ \lambda \rho_{pl}^{-1} e^{\lambda(t-t_0)} |z_p^I(t)| - a_p^- \rho_{pl}^{-1} e^{\lambda(t-t_0)} |z_p^I(t)| \right. \\
& + \sum_{q=1}^n \frac{4b_{pq} L_f \rho_{pl}^{-1} e^{\lambda(\theta-t_0)}}{1 - \dot{\tau}_{pq}} e^{\lambda t} (|z_q^R(t)| + |z_q^I(t)| + |z_q^J(t)| + |z_q^K(t)|) \\
& + \sum_{p=1}^n 4b_{pq} L_f \rho_{pl}^{-1} e^{\lambda(t-t_0)} \left(1 - \frac{e^{\lambda(\theta-\tau_{pq}(t))} (1 - \dot{\tau}_{pq}(t))}{1 - \dot{\tau}_{pq}} \right) \\
& \times (|z_q^R(t - \tau_{pq}(t))| + |z_q^I(t - \tau_{pq}(t))| + |z_q^J(t - \tau_{pq}(t))| \\
& + |z_q^K(t - \tau_{pq}(t))|) + \sum_{q=1}^n 4\alpha_{pq} L_g \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \int_{-\eta_{pq}}^0 e^{\lambda t} (|z_q^R(t)| \\
& + |z_q^I(t)| + |z_q^J(t)| + |z_q^K(t)|) \, du + \sum_{q=1}^n 4\alpha_{pq} L_g \rho_{pl}^{-1} e^{\lambda(t-t_0)} \\
& \times (1 - e^{\lambda(\theta-\eta_{pq})}) \int_{t-\eta_{pq}}^t (|z_q^R(s)| + |z_q^I(s)| + |z_q^J(s)| + |z_q^K(s)|) \, ds \\
& + \sum_{q=1}^n 4\beta_{pq} L_g \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \int_{-\xi_{pq}}^0 e^{\lambda t} (|z_q^R(t)| + |z_q^I(t)| + |z_q^J(t)| \\
& + |z_q^K(t)|) \, du + \sum_{q=1}^n 4\beta_{pq} L_g \rho_{pl}^{-1} e^{\lambda(t-t_0)} (1 - e^{\lambda(\theta-\xi_{pq})}) \\
& \times \int_{t-\xi_{pq}}^t (|z_q^R(s)| + |z_q^I(s)| + |z_q^J(s)| + |z_q^K(s)|) \, ds \Big\} \\
\leq & \sum_{p=1}^n \left\{ \lambda \rho_{pl}^{-1} e^{\lambda(t-t_0)} |z_p^I(t)| - a_p^- \rho_{pl}^{-1} e^{\lambda(t-t_0)} |z_p^I(t)| \right. \\
& + \sum_{q=1}^n \frac{4b_{pq} L_f \rho_{pl}^{-1} e^{\lambda(\theta-t_0)}}{1 - \dot{\tau}_{pq}} e^{\lambda t} (|z_q^R(t)| + |z_q^I(t)| + |z_q^J(t)| + |z_q^K(t)|) \\
& + \sum_{q=1}^n 4\alpha_{pq} L_g \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \eta_{pq} e^{\lambda t} (|z_q^R(t)| + |z_q^I(t)| + |z_q^J(t)|
\end{aligned}$$

$$\begin{aligned}
& + |z_q^K(t)| \Big) du + \sum_{q=1}^n 4\beta_{pq} L_g \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \xi_{pq} e^{\lambda t} (|z_q^R(t)| + |z_q^I(t)| \\
& + |z_q^J(t)| + |z_q^K(t)|) du \Big\} \\
& \leq \sum_{p=1}^n \left\{ (\lambda - a_p^-) \rho_{pl}^{-1} e^{\lambda(t-t_0)} |z_p^I(t)| + \sum_{q=1}^n \left(\frac{4b_{pq} L_f e^{\lambda \theta}}{1 - \tau_{pq}} \right. \right. \\
& \quad \left. \left. + 4\alpha_{pq} L_g e^{\lambda \theta} \eta_{pq} + 4\beta_{pq} L_g e^{\lambda \theta} \xi_{pq} \right) \right. \\
& \quad \left. \times \rho_{pl}^{-1} e^{\lambda(t-t_0)} (|z_q^R(t)| + |z_q^I(t)| + |z_q^J(t)| + |z_q^K(t)|) \right\}, \quad l \in \Upsilon.
\end{aligned}$$

From this and (H_5) , we have

$$\begin{aligned}
D^+ V(t) & = D^+ V_R(t) + D^+ V_I(t) + D^+ V_J(t) + D^+ V_K(t) \\
& \leq \sum_{p=1}^n \left\{ (\lambda - a_p^-) e^{\lambda(t-t_0)} (\rho_{pR}^{-1} |z_p^R(t)| + \rho_{pI}^{-1} |z_p^I(t)| + \rho_{pJ}^{-1} |z_p^J(t)| + \rho_{pK}^{-1} |z_p^K(t)|) \right. \\
& \quad + \sum_{q=1}^n \left(\frac{4b_{pq} L_f e^{\lambda \theta}}{1 - \tau_{pq}} + 4\alpha_{pq} L_g e^{\lambda \theta} \eta_{pq} + 4\beta_{pq} L_g e^{\lambda \theta} \xi_{pq} \right) e^{\lambda(t-t_0)} (\rho_{pR}^{-1} + \rho_{pI}^{-1} \\
& \quad \left. + \rho_{pJ}^{-1} + \rho_{pK}^{-1}) (|z_q^R(t)| + |z_q^I(t)| + |z_q^J(t)| + |z_q^K(t)|) \right\} \\
& \leq \sum_{p=1}^n \left\{ (\lambda - a_p^-) \frac{1}{\rho^-} e^{\lambda(t-t_0)} (|z_p^R(t)| + |z_p^I(t)| + |z_p^J(t)| + |z_p^K(t)|) \right. \\
& \quad + \sum_{q=1}^n \left(\frac{16b_{qp} L_f e^{\lambda \theta}}{1 - \tau_{qp}} + 16\alpha_{qp} L_g e^{\lambda \theta} \eta_{qp} + 16\beta_{qp} L_g e^{\lambda \theta} \xi_{qp} \right) \frac{1}{\rho^-} e^{\lambda(t-t_0)} (|z_p^R(t)| \\
& \quad \left. + |z_p^I(t)| + |z_p^J(t)| + |z_p^K(t)|) \right\} \\
& = \sum_{p=1}^n \left\{ \lambda - a_p^- + \sum_{q=1}^n \left(\frac{16b_{qp} L_f e^{\lambda \theta}}{1 - \tau_{qp}} + 16\alpha_{qp} L_g e^{\lambda \theta} \eta_{qp} + 16\beta_{qp} L_g e^{\lambda \theta} \xi_{qp} \right) \right\} \\
& \quad \times \frac{1}{\rho^-} e^{\lambda(t-t_0)} (|z_p^R(t)| + |z_p^I(t)| + |z_p^J(t)| + |z_p^K(t)|) \\
& \leq 0.
\end{aligned}$$

Hence, $V(t) \leq V(t_0)$ for $t \geq t_0$. By the expression of $V(t)$, we can obtain

$$\begin{aligned}
V(t) & \geq \sum_{p=1}^n e^{\lambda(t-t_0)} (\rho_{pR}^{-1} |z_p^R(t)| + \rho_{pI}^{-1} |z_p^I(t)| + \rho_{pJ}^{-1} |z_p^J(t)| + \rho_{pK}^{-1} |z_p^K(t)|) \\
& \geq 4ne^{\lambda(t-t_0)} \|z(t)\| = 4ne^{\lambda(t-t_0)} \|x(t) - y(t)\|
\end{aligned}$$

and

$$\begin{aligned}
V_l(t_0) &= \sum_{p=1}^n \left\{ \rho_{p_l}^{-1} |z_p^l(t_0)| + \sum_{q=1}^n \frac{4b_{pq} L_f \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)}}{1 - \dot{\tau}_{pq}} \right. \\
&\quad \times \int_{t_0-\tau_{pq}(t_0)}^{t_0} e^{\lambda s} (|z_q^R(s)| + |z_q^I(s)| + |z_q^J(s)| + |z_q^K(s)|) ds \\
&\quad + \sum_{q=1}^n 4\alpha_{pq} L_g \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)} \int_{-\eta_{pq}}^0 \int_{t_0+u}^{t_0} e^{\lambda s} (|z_q^R(s)| + |z_q^I(s)| + |z_q^J(s)| \\
&\quad + |z_q^K(s)|) ds du + \sum_{q=1}^n 4\beta_{pq} L_g \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)} \int_{-\xi_{pq}}^0 \int_{t_0+u}^{t_0} e^{\lambda s} (|z_q^R(s)| \\
&\quad + |z_q^I(s)| + |z_q^J(s)| + |z_q^K(s)|) ds du \Big\} \\
&= \sum_{p=1}^n \left\{ \rho_{p_l}^{-1} |z_p^l(t_0)| + \sum_{q=1}^n \frac{4b_{pq} L_f \rho_{p_R} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)}}{1 - \dot{\tau}_{pq}} \right. \\
&\quad \times \int_{t_0-\tau_{pq}(t_0)}^{t_0} e^{\lambda s} \rho_{p_R}^{-1} |z_q^R(s)| ds \\
&\quad + \sum_{q=1}^n \frac{4b_{pq} L_f \rho_{p_l} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)}}{1 - \dot{\tau}_{pq}} \int_{t_0-\tau_{pq}(t_0)}^{t_0} e^{\lambda s} \rho_{p_l}^{-1} |z_q^I(s)| ds \\
&\quad + \sum_{q=1}^n \frac{4b_{pq} L_f \rho_{p_l} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)}}{1 - \dot{\tau}_{pq}} \int_{t_0-\tau_{pq}(t_0)}^{t_0} e^{\lambda s} \rho_{p_l}^{-1} |z_q^J(s)| ds \\
&\quad + \sum_{q=1}^n \frac{4b_{pq} L_f \rho_{p_K} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)}}{1 - \dot{\tau}_{pq}} \int_{t_0-\tau_{pq}(t_0)}^{t_0} e^{\lambda s} \rho_{p_K}^{-1} |z_q^K(s)| ds \\
&\quad + \sum_{q=1}^n 4\alpha_{pq} L_g \rho_{p_R} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)} \int_{-\eta_{pq}}^0 \int_{t_0+u}^{t_0} e^{\lambda s} \rho_{p_R}^{-1} |z_q^R(s)| ds du \\
&\quad + \sum_{q=1}^n 4\alpha_{pq} L_g \rho_{p_l} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)} \int_{-\eta_{pq}}^0 \int_{t_0+u}^{t_0} e^{\lambda s} \rho_{p_l}^{-1} |z_q^I(s)| ds du \\
&\quad + \sum_{q=1}^n 4\alpha_{pq} L_g \rho_{p_l} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)} \int_{-\eta_{pq}}^0 \int_{t_0+u}^{t_0} e^{\lambda s} \rho_{p_l}^{-1} |z_q^J(s)| ds du \\
&\quad + \sum_{q=1}^n 4\alpha_{pq} L_g \rho_{p_K} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)} \int_{-\eta_{pq}}^0 \int_{t_0+u}^{t_0} e^{\lambda s} \rho_{p_K}^{-1} |z_q^K(s)| ds du \\
&\quad + \sum_{q=1}^n 4\beta_{pq} L_g \rho_{p_R} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)} \int_{-\xi_{pq}}^0 \int_{t_0+u}^{t_0} e^{\lambda s} \rho_{p_R}^{-1} |z_q^R(s)| ds du \\
&\quad + \sum_{q=1}^n 4\beta_{pq} L_g \rho_{p_l} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)} \int_{-\xi_{pq}}^0 \int_{t_0+u}^{t_0} e^{\lambda s} \rho_{p_l}^{-1} |z_q^I(s)| ds du \\
&\quad + \sum_{q=1}^n 4\beta_{pq} L_g \rho_{p_l} \rho_{p_l}^{-1} e^{\lambda(\theta-t_0)} \int_{-\xi_{pq}}^0 \int_{t_0+u}^{t_0} e^{\lambda s} \rho_{p_l}^{-1} |z_q^J(s)| ds du
\end{aligned}$$

$$\begin{aligned}
& + \sum_{q=1}^n 4\beta_{pq} L_g \rho_{pK} \rho_{pl}^{-1} e^{\lambda(\theta-t_0)} \int_{-\xi_{pq}}^0 \int_{t_0+u}^{t_0} e^{\lambda s} \rho_{pK}^{-1} |z_q^K(s)| ds du \Big\} \\
& \leq \sum_{p=1}^n \left\{ 1 + \sum_{q=1}^n \left(\frac{16\tau_{pq} b_{pq} L_f \rho_{pl}^{-1}}{1 - \tau_{pq}} + 8\eta_{pq}^2 \alpha_{pq} L_g \rho_{pl}^{-1} \right. \right. \\
& \quad \left. \left. + 8\xi_{pq}^2 \beta_{pq} L_g \rho_{pl}^{-1} \right) \rho^+ e^{\lambda\theta} \right\} \|\varphi - \psi\|.
\end{aligned}$$

Denote

$$\begin{aligned}
M = \max_{\substack{1 \leq p \leq n \\ l \in \Upsilon}} \Big\{ & 1 + \sum_{q=1}^n \left(\frac{16\tau_{pq} b_{pq} L_f \rho_{pl}^{-1}}{1 - \tau_{pq}} + 8\eta_{pq}^2 \alpha_{pq} L_g \rho_{pl}^{-1} \right. \\
& \left. + 8\xi_{pq}^2 \beta_{pq} L_g \rho_{pl}^{-1} \right) \rho^+ e^{\lambda\theta} \Big\} > 0.
\end{aligned}$$

Thus, we have

$$4ne^{\lambda(t-t_0)} \|x(t) - y(t)\| \leq V(t_0) \leq V(t) \leq 4nM \|\varphi - \psi\|, \quad t \geq t_0,$$

that is,

$$\|x(t) - y(t)\| \leq M \|\varphi - \psi\| e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

Therefore, the ω -periodic solution $x(t)$ of system (2) is globally exponentially stable. In view of Remark 1, we see that the ω -periodic solution of system (1) is also globally exponentially stable. The uniqueness of the ω -periodic solution follows from the global exponential stability. The proof is complete. $\square$

4 An illustrative example

In this section, an example is given to illustrate the effectiveness of our results obtained in this paper.

Example 1 Consider the following QVFCNN:

$$\begin{aligned}
\dot{x}_p(t) = & -a_p(t)x_p(t) + \sum_{q=1}^n b_{pq}(t)f_q(x_q(t - \tau_{pq}(t))) + \sum_{q=1}^n d_{pq}(t)\mu_q(t) \\
& + \bigwedge_{q=1}^n \alpha_{pq}(t) \int_{t-\eta_{pq}(t)}^t g_q(s, x) ds + \bigvee_{q=1}^n \beta_{pq}(t) \int_{t-\xi_{pq}(t)}^t g_q(s, x) ds \\
& + \bigwedge_{q=1}^n T_{pq}(t)\mu_q(t) + \bigvee_{q=1}^n S_{pq}(t)\mu_q(t) + I_p(t),
\end{aligned} \tag{8}$$

where

$$\dot{x}_p(t) = \dot{x}_p^R(t) + i\dot{x}_p^I(t) + j\dot{x}_p^J(t) + k\dot{x}_p^K(t) \in \mathbb{Q}, \quad p = 1, 2,$$

$$\begin{aligned}
f_q(x_q) &= \frac{1}{4} \sin\left(x_q^R + \frac{1}{2}x_q^I\right) + \frac{1}{2}j \sin(x_q^R + x_q^I + x_q^J + x_q^K) + \frac{1}{4}kx_q^J, \\
g_q(x_q) &= \frac{1}{4}j(x_q^R + x_q^I + x_q^J + x_q^K) + \frac{1}{4}k \sin(x_q^R + x_q^K), \\
\begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} &= \begin{bmatrix} \sin^2(20t) + 20 \\ -5 \cos^2(20t) + 25 \end{bmatrix}, \\
\begin{bmatrix} I_1(t) \\ I_2(t) \end{bmatrix} &= \begin{bmatrix} \sin 40t + i \sin 40t + j \cos^3(40t) + \frac{1}{3}k \cos 40t \\ \cos 40t + i \sin 40t + j \cos 40t + k \sin^5(40t) \end{bmatrix}, \\
\begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix} &= \frac{1}{4} \begin{bmatrix} \cos 40t + i \sin 80t + j \sin^2(20t) & \sin 40t + k \cos 40t \\ i \cos 40t + j \sin 80t - k \cos^3(40t) & \cos^2(20t) + j \cos^4(20t) + k \sin 40t \end{bmatrix}, \\
\begin{bmatrix} d_{11}(t) & d_{12}(t) \\ d_{21}(t) & d_{22}(t) \end{bmatrix} &= \begin{bmatrix} \sin 40t + \frac{1}{2}i \cos 40t + k \sin^2(20t) & \frac{2}{3} \sin 40t + j \cos^3(40t) - k \cos 40t \\ \cos 40t + j \cos^3(40t) + \frac{1}{2}k \sin 40t & \sin^3(40t) - j \cos 40t + k \sin 40t \end{bmatrix}, \\
\begin{bmatrix} \alpha_{11}(t) & \alpha_{12}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) \end{bmatrix} &= \frac{1}{4} \begin{bmatrix} \cos^5(20t) + i \sin 40t + k \cos 40t & i \sin 40t \\ i \sin 40t + \frac{1}{2}k \cos^2(20t) & \sin 40t + i \cos 40t + j \sin^3(40t) \end{bmatrix}, \\
\begin{bmatrix} \beta_{11}(t) & \beta_{12}(t) \\ \beta_{21}(t) & \beta_{22}(t) \end{bmatrix} &= \frac{1}{4} \begin{bmatrix} \cos^5(40t) + \frac{1}{2}i \sin 40t + k \cos 40t & i \sin^2(20t) \\ \frac{1}{\sqrt{2}}i \sin 40t + k \cos 40t & \sin 40t + i \cos 40t + j \sin^3(40t) \end{bmatrix}, \\
\begin{bmatrix} T_{11}(t) & T_{12}(t) \\ T_{21}(t) & T_{22}(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{3} \cos^2(20t) + i \cos 40t + k \frac{1}{3} & \frac{1}{2} \sin^2(20t) + j \sin^3(40t) \\ \frac{1}{2} \sin 40t + \frac{1}{4}k \cos 40t & \frac{1}{4} + \frac{1}{4}k \sin 40t \end{bmatrix}, \\
\begin{bmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{5} \cos 40t + k \sin^2(20t) & \frac{1}{2} \sin^2(20t) + j \cos^2(20t) \\ \frac{1}{2} \sin^4(20t) + i \cos^2(20t) & \frac{1}{4} + k \sin 40t \end{bmatrix}, \\
\begin{bmatrix} \mu_1(t) \\ \mu_2(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{3} \cos 40t + i \sin 40t + k \cos 40t \\ \frac{1}{2} \sin 40t + k \cos 40t \end{bmatrix}, \\
\begin{bmatrix} \tau_{11}(t) & \tau_{12}(t) \\ \tau_{21}(t) & \tau_{22}(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{40} \cos^2(20t) & \frac{1}{40} \sin^2(20t) \\ \frac{1}{40} \sin^2(20t) & \frac{1}{4} \end{bmatrix}, \\
\begin{bmatrix} \eta_{11}(t) & \eta_{12}(t) \\ \eta_{21}(t) & \eta_{22}(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} \sin^2(20t) & \frac{1}{4} \cos^2(20t) \\ \frac{1}{4} \sin^2(20t) & \frac{1}{4} \sin^2(20t) \end{bmatrix}, \\
\begin{bmatrix} \xi_{11}(t) & \xi_{12}(t) \\ \xi_{21}(t) & \xi_{22}(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{4} & \frac{1}{3} \cos^2(20t) \\ \frac{1}{2} \cos^2(20t) & \frac{1}{2} \sin^2(20t) \end{bmatrix}.
\end{aligned}$$

By computing, we have

$$a_1^- = a_2^- = 20, \quad L_f = \frac{1}{8}, \quad L_g = \frac{1}{4}, \quad \theta = \frac{1}{2}, \quad \omega = \frac{\pi}{20},$$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix},$$

$$\begin{bmatrix} \dot{\tau}_{11}(t) & \dot{\tau}_{12}(t) \\ \dot{\tau}_{21}(t) & \dot{\tau}_{22}(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \sin 40t & \frac{1}{2} \sin 40t \\ \frac{1}{2} \sin 40t & 0 \end{bmatrix},$$

$$\begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{40} & \frac{1}{40} \\ \frac{1}{40} & \frac{1}{4} \end{bmatrix}, \quad \begin{bmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix},$$

$$\begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} \dot{\tau}_{11} & \dot{\tau}_{12} \\ \dot{\tau}_{21} & \dot{\tau}_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Take $\lambda = 1$, then we have

$$\max_{1 \leq p \leq 2} \left\{ \frac{4\omega}{1 - e^{-\omega \bar{a}_p}} \left[\sum_{q=1}^n (b_{pq} L_f + \alpha_{pq} L_g \theta + \beta_{pq} L_g \theta) \right] \right\} = 0.246 < \frac{1}{4}$$

and

$$\max_{1 \leq p \leq 2} \left\{ \lambda - a_p^- + \sum_{q=1}^n 16e^{\lambda \theta} \left(\frac{b_{qp} L_f}{1 - \dot{\tau}_{qp}} + (\alpha_{qp} \eta_{qp} + \beta_{qp} \xi_{qp}) L_g \right) \right\} = -17.28 \leq 0.$$

Therefore, all the conditions of Theorem 2 are satisfied. According to Theorem 2, system (8) has a unique $\frac{\pi}{20}$ -periodic solution and it is globally exponentially stable (see Figs. 1–3).

Remark 2 Even when QVFCNN (8) degenerates into a real-valued system, papers [23, 32, 45–47] cannot be used to judge that (8) has a globally exponentially stable periodic solution.

5 Conclusions

In this paper, by using the Schauder fixed point theorem and constructing a suitable Lyapunov function, we investigated the existence and global exponential stability of periodic solutions for QVFCNNs with time-varying delays. To the best of our knowledge, this is the first paper to study the periodic solutions of QVFCNNs. Our results are new and supplement some previously known ones even when system (1) degenerates into a real-valued

system. Our methods used in this paper can be used to study other types of quaternion-valued neural networks such as Hopfield neural networks, BAM neural networks, Cohen–Grossberg neural networks, and so on.

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Authors' contributions

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