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On a nonlocal integral boundary value problem of nonlinear Langevin equation with different fractional orders

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Abstract

In this paper we develop the existence theory for a nonlinear Langevin equation involving Caputo fractional derivatives of different orders and Riemann–Liouville fractional integral supplemented with nonlocal multi-point and multi-strip boundary conditions. We make use of the modern methods of functional analysis to obtain the existence and uniqueness results for the given problem, which are well illustrated with the aid of examples. Our results are new and correspond to some new ones for specific choices of the parameters involved in the problem.

1 Introduction

Langevin equation is an important equation of mathematical physics that is used in modeling the phenomena occurring in fluctuating environment such as Brownian motion. The classical form of this equation was derived in terms of ordinary derivatives by Paul Langevin in 1908 [1]. Langevin equation is also known as a stochastic differential equation as it governs the fast motion of microscopic variables (degrees of freedom) of the dynamical systems. However, the classical Langevin equation failed to describe the complex systems, and its several generalizations were proposed to model the physical phenomena in disordered regions [2], anomalous diffusion processes in complex and viscoelastic environment [3, 4], etc.

Fractional calculus, dealing with arbitrary order differential and integral operators, received great attention during the last few decades. Many researchers focused on developing the theoretical aspects, methods of solution, and applications of fractional differential equations. The popularity of the subject results from the nonlocal nature of fractional order operators, which can account for the past history of the phenomena under investigation. Now one can observe the shift in the study of mathematical models from integer order differential operators to their fractional order analogue. For application details in physics, biology, chemistry, medical sciences, etc., we refer the reader to the works [5–10].

There has been shown a great interest in the study of nonlocal nonlinear fractional order single-valued and multivalued boundary value problems in recent years. The interest in this topic results from the applicability of nonlocal and integral conditions in mathemati-

cal modeling of many real world situations arising in applied and biological sciences. For details and examples, see [11–23] and the references cited therein.

Inspired by the wide application of tools of fractional calculus, the fractional analogue of Langevin equation was proposed by replacing the ordinary derivative by fractional order derivative in its classical form. Examples include motor single-file diffusion [24], Kramers–Fokker–Planck equation and Langevin equation [25], control system [26], etc. Lim et al. [27] introduced a new form of Langevin equation containing two different fractional orders, which has made it possible to investigate fractal processes in a more flexible environment. For some recent works on fractional Langevin equation, see the papers [28–36] and the references cited therein.

In this paper, we consider the following nonlocal integral boundary value problem for a nonlinear Langevin equation with different order Caputo fractional derivatives:

$$\begin{cases} {}^c D^\kappa ({}^c D^\zeta + \chi)x(t) = f(t, x(t), I^p x(t)), & t \in [0, 1], \\ x(0) = \sum_{j=1}^m v_j x(\sigma_j), \quad x'(0) = 0, \quad a_1 x(1) + a_2 x'(1) = \sum_{i=1}^n \rho_i \int_{\xi_i}^{\eta_i} x(s) ds, \end{cases} \quad (1.1)$$

where ${}^c D^\gamma$ denotes the Caputo fractional derivative of order $\gamma (= \zeta, \kappa)$, $p, \chi > 0$, $0 < \zeta \leq 1$, $1 < \kappa \leq 2$, $a_1, a_2, v_j, \rho_i \in \mathbb{R}$, $0 < \sigma_j < \xi_i < \eta_i < 1$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, and $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function.

We arrange the rest of the paper as follows. In Sect. 2, we present some definitions of fractional calculus and fixed point theorems related to our work. Section 3 contains the main results, while examples illustrating the main results are given in Sect. 4. Some interesting observations are presented in the concluding section.

2 Preliminaries

This section is devoted to some basic concepts of fractional calculus [6].

Definition 2.1 The Riemann–Liouville fractional integral I_a^ω of order $\omega \in \mathbb{R}$ ($\omega > 0$) for a locally integrable real-valued function u on the interval (a, b) with $a, b \in \mathbb{R}$ is defined as

$$I_a^\omega u(t) = \frac{1}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1} u(s) ds,$$

where Γ denotes the Euler gamma function.

Definition 2.2 Let $u^{(m-1)} \in L^1[a, b]$ with $a, b \in \mathbb{R}$. The Riemann–Liouville fractional derivative D_a^ω of order ω ($m-1 < \omega < m$, $m \in \mathbb{N}$) for u is defined as

$$D_a^\omega u(t) = \frac{d^m}{dt^m} I_a^{1-\omega} u(t) = \frac{1}{\Gamma(m-\omega)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\omega} u(s) ds.$$

The Caputo fractional derivative ${}^c D_a^\omega$ of order ω ($m-1 < \omega < m$, $m \in \mathbb{N}$) is defined by

$${}^c D_a^\omega u(t) = D_a^\omega \left[u(t) - u(a) - u'(a) \frac{(t-a)}{1!} - \dots - u^{(m-1)}(a) \frac{(t-a)^{m-1}}{(m-1)!} \right].$$

Remark 2.3 If $u \in C^m[a, b]$, then the Caputo fractional derivative ${}^c D_a^\omega$ of order ω ($m - 1 < \omega < m$, $m \in \mathbb{N}$) is defined as

$${}^c D_a^\omega[u](t) = I_a^{1-\omega} u^{(m)}(t) = \frac{1}{\Gamma(m-\omega)} \int_a^t (t-s)^{m-1-\omega} u^{(m)}(s) ds.$$

In the present work, we denote the Riemann–Liouville fractional integral I_a^ω and the Caputo fractional derivative ${}^c D_a^\omega$ with $a = 0$ by I^ω and ${}^c D^\omega$, respectively.

3 Main results

Before presenting the main results, we prove an auxiliary lemma related to the linear variant of problem (1.1). This lemma plays an important role in transforming the given problem to a fixed point problem.

3.1 Auxiliary lemma

Lemma 3.1 Let $g \in C([0, 1], \mathbb{R})$. Then the problem consisting of linear fractional differential equation

$${}^c D^\kappa ({}^c D^\zeta + \chi)x(t) = g(t), \quad 1 < \kappa \leq 2, 0 < \zeta < 1, \quad (3.1)$$

and the boundary conditions of (1.1) is equivalent to the integral equation

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} g(s) ds - \chi \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} x(s) ds \\ & - \mu_1(t) \sum_{j=1}^m v_j \int_0^{\sigma_j} \left(\frac{(\sigma_j-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} g(s) - \chi \frac{(\sigma_j-s)^{\zeta-1}}{\Gamma(\zeta)} x(s) \right) ds \\ & + \mu_2(t) \left[\sum_{i=1}^n \rho_i \int_{\xi_i}^{\eta_i} \int_0^s \left(\frac{(s-u)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} g(u) - \chi \frac{(s-u)^{\zeta-1}}{\Gamma(\zeta)} x(u) \right) du ds \right. \\ & - a_1 \int_0^1 \left(\frac{(1-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} g(s) - \chi \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} x(s) \right) ds \\ & \left. - a_2 \int_0^1 \left(\frac{(1-s)^{\zeta+\kappa-2}}{\Gamma(\zeta+\kappa-1)} g(s) - \chi \frac{(1-s)^{\zeta-2}}{\Gamma(\zeta-1)} x(s) \right) ds \right], \end{aligned} \quad (3.2)$$

where

$$\mu_1(t) = \frac{1}{\Delta} \left(B_1 - B_2 \left(\frac{t^{\zeta+1}}{\Gamma(\zeta+2)} \right) \right), \quad \mu_2(t) = \frac{1}{\Delta} \left(A_1 - A_2 \left(\frac{t^{\zeta+1}}{\Gamma(\zeta+2)} \right) \right), \quad (3.3)$$

$$\Delta = A_1 B_2 - A_2 B_1 \neq 0, \quad (3.4)$$

$$A_1 = - \sum_{j=1}^m v_j \frac{\sigma_j^{\zeta+1}}{\Gamma(\zeta+2)}, \quad A_2 = 1 - \sum_{j=1}^m v_j, \quad (3.5)$$

$$B_1 = \frac{a_1 + (\zeta+1)a_2}{\Gamma(\zeta+2)} - \sum_{i=1}^n \rho_i \frac{\eta_i^{\zeta+2} - \xi_i^{\zeta+2}}{\Gamma(\zeta+3)}, \quad B_2 = a_1 - \sum_{i=1}^n \rho_i (\eta_i - \xi_i). \quad (3.6)$$

Proof Applying the integral operator I^κ to both sides of (3.1) and then the operator I^ζ to the resulting equation, we get

$$x(t) = I^{\zeta+\kappa}g(t) - \chi I^\zeta x(t) + c_0 \frac{t^\zeta}{\Gamma(\zeta+1)} + c_1 \frac{t^{\zeta+1}}{\Gamma(\zeta+2)} + c_2, \quad (3.7)$$

$$x'(t) = I^{\zeta+\kappa-1}g(t) - \chi I^{\zeta-1}x(t) + c_0 \frac{t^{\zeta-1}}{\Gamma(\zeta)} + c_1 \frac{t^\zeta}{\Gamma(\zeta+1)}, \quad (3.8)$$

where c_0, c_1, c_2 are arbitrary constants.

Using the condition $x'(0) = 0$ in (3.8), we find that $c_0 = 0$. Inserting the value of c_0 in (3.7) and making use of the condition $x(0) = \sum_{j=1}^m v_j x(\sigma_j)$ together with the notations (3.5), we get

$$A_1 c_1 + A_2 c_2 = J_1, \quad (3.9)$$

where

$$J_1 = \sum_{j=1}^m v_j (I^{\zeta+\kappa}g(\sigma_j) - \chi I^\zeta x(\sigma_j)). \quad (3.10)$$

Combining the condition $a_1 x(1) + a_2 x'(1) = \sum_{i=1}^{n-2} \rho_i \int_{\xi_i}^{\eta_i} x(s) ds$ with (3.7) and (3.8), and using the notations (3.6) yields

$$B_1 c_1 + B_2 c_2 = J_2, \quad (3.11)$$

where

$$\begin{aligned} J_2 = & \sum_{i=1}^n \rho_i \int_{\xi_i}^{\eta_i} (I^{\zeta+\kappa}g(s) - \chi I^\zeta x(s)) ds - a_1 (I^{\zeta+\kappa}g(1) - \chi I^\zeta x(1)) \\ & - a_2 (I^{\zeta+\kappa-1}g(1) - \chi I^{\zeta-1}x(1)). \end{aligned} \quad (3.12)$$

Solving systems (3.9) and (3.11) for c_1 and c_2 , we find that

$$c_1 = \frac{B_2 J_1 - A_2 J_2}{\Delta}, \quad c_2 = \frac{A_1 J_2 - B_1 J_1}{\Delta},$$

where Δ is given by (3.4). Substituting the values of c_0, c_1 , and c_2 in (3.7) together with the notations (3.3), we obtain solution (3.2). By direct computation, one can obtain the converse of the lemma. This completes the proof. $\square$

Denote by X the Banach space of all continuous functions from $[0, 1] \rightarrow \mathbb{R}$ endowed with norm $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$. By Lemma 3.1, we transform problem (1.1) into a fixed point problem as $x = \mathcal{A}x$, where the operator $\mathcal{A} : X \rightarrow X$ is defined by

$$\begin{aligned} (\mathcal{A}x)(t) = & \int_0^t \frac{(t-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} f(s, x(s), I^p x(s)) ds - \chi \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} x(s) ds \\ & - \mu_1(t) \sum_{j=1}^m v_j \int_0^{\sigma_j} \left(\frac{(\sigma_j-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} f(s, x(s), I^p x(s)) - \chi \frac{(\sigma_j-s)^{\zeta-1}}{\Gamma(\zeta)} x(s) \right) ds \end{aligned}$$

$$\begin{aligned}
& + \mu_2(t) \left[\sum_{i=1}^n \rho_i \int_{\xi_i}^{\eta_i} \int_0^s \left(\frac{(s-u)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} f(u, x(u), I^p x(u)) \right. \right. \\
& \quad \left. \left. - \chi \frac{(s-u)^{\zeta-1}}{\Gamma(\zeta)} x(u) \right) du ds \right. \\
& \quad \left. - a_1 \int_0^1 \left(\frac{(1-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} f(s, x(s), I^p x(s)) - \chi \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} x(s) \right) ds \right. \\
& \quad \left. - a_2 \int_0^1 \left(\frac{(1-s)^{\zeta+\kappa-2}}{\Gamma(\zeta+\kappa-1)} f(s, x(s), I^p x(s)) - \chi \frac{(1-s)^{\zeta-2}}{\Gamma(\zeta-1)} x(s) \right) ds \right], \\
& t \in [0, 1].
\end{aligned} \tag{3.13}$$

For the sake of computational convenience, we set

$$\begin{aligned}
\Lambda_1 &= \frac{1}{\Gamma(\zeta+\kappa+1)} + \bar{\mu}_1 \sum_{j=1}^m |v_j| \frac{\sigma_j^{\zeta+\kappa}}{\Gamma(\zeta+\kappa+1)} \\
&+ \bar{\mu}_2 \left(\sum_{i=1}^n |\rho_i| \frac{(\eta_i^{\zeta+\kappa+1} - \xi_i^{\zeta+\kappa+1})}{\Gamma(\zeta+\kappa+2)} + \frac{|a_1|}{\Gamma(\zeta+\kappa+1)} + \frac{|a_2|}{\Gamma(\zeta+\kappa)} \right),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\Lambda_2 &= \chi \left[\frac{1}{\Gamma(\zeta+1)} + \bar{\mu}_1 \sum_{j=1}^m |v_j| \frac{\sigma_j^{\zeta}}{\Gamma(\zeta+1)} \right. \\
&\quad \left. + \bar{\mu}_2 \left(\sum_{i=1}^n |\rho_i| \frac{(\eta_i^{\zeta+1} - \xi_i^{\zeta+1})}{\Gamma(\zeta+2)} + \frac{|a_1|}{\Gamma(\zeta+1)} + \frac{|a_2|}{\Gamma(\zeta)} \right) \right],
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
\bar{\mu}_1 &= \max_{t \in [0,1]} |\mu_1(t)| = \frac{1}{|\Delta|} \left(|B_1| + \frac{|B_2|}{\Gamma(\zeta+2)} \right), \\
\bar{\mu}_2 &= \max_{t \in [0,1]} |\mu_2(t)| = \frac{1}{|\Delta|} \left(|A_1| + \frac{|A_2|}{\Gamma(\zeta+2)} \right).
\end{aligned}$$

3.2 Existence results

In this subsection we prove the existence results for the given problem; the first one relies on Krasnoselskii's fixed point theorem [37], while the second one is based on the Leray–Schauder nonlinear alternative [38].

Theorem 3.2 *Let $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function satisfying the conditions:*

(C₁) $|f(t, x, y) - f(t, x_1, y_1)| \leq l(|x - x_1| + |y - y_1|)$ for all $t \in [0, 1]$, $l > 0$, $x, y, x_1, y_1 \in \mathbb{R}$;

(C₂) $|f(t, x, y)| \leq k(t)$ for all $(t, x, y) \in [0, 1] \times \mathbb{R}^2$ and $k \in C([0, 1], \mathbb{R}^+)$.

Then the boundary value problem (1.1) has at least one solution on $[0, 1]$ if

$$\Lambda_3 < 1, \tag{3.16}$$

where $\Lambda_3 = l(1 + \frac{1}{\Gamma(\zeta+1)})(\Lambda_1 - \frac{1}{\Gamma(\zeta+\kappa+1)}) + \Lambda_2 - \frac{\chi}{\Gamma(\zeta+1)}$, Λ_1 and Λ_2 are respectively given by (3.14) and (3.15) with $\Lambda_2 < 1$.

Proof By assumption (C_2) , we fix $r \geq \Lambda_1 \|k\| / (1 - \Lambda_2)$ and define a closed ball $B_r = \{x \in X : \|x\| \leq r\}$. Introduce the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on B_r as

$$\begin{aligned} (\mathcal{A}_1 x)(t) &= \int_0^t \frac{(t-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} f(s, x(s), I^p x(s)) ds - \chi \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} x(s) ds, \\ (\mathcal{A}_2 x)(t) &= -\mu_1(t) \sum_{j=1}^m v_j \int_0^{\sigma_j} \left(\frac{(\sigma_j-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} f(s, x(s), I^p x(s)) - \chi \frac{(\sigma_j-s)^{\zeta-1}}{\Gamma(\zeta)} x(s) \right) ds \\ &\quad + \mu_2(t) \left[\sum_{i=1}^n \rho_i \int_{\xi_i}^{\eta_i} \int_0^s \left(\frac{(s-u)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} f(u, x(u), I^p x(u)) \right. \right. \\ &\quad \left. \left. - \chi \frac{(s-u)^{\zeta-1}}{\Gamma(\zeta)} x(u) \right) du ds \right. \\ &\quad \left. - a_1 \int_0^1 \left(\frac{(1-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} f(s, x(s), I^p x(s)) - \chi \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} x(s) \right) ds \right. \\ &\quad \left. - a_2 \int_0^1 \left(\frac{(1-s)^{\zeta+\kappa-2}}{\Gamma(\zeta+\kappa-1)} f(s, x(s), I^p x(s)) - \chi \frac{(1-s)^{\zeta-2}}{\Gamma(\zeta-1)} x(s) \right) ds \right], \end{aligned}$$

$t \in [0, 1]$. Observe that $\mathcal{A}x = \mathcal{A}_1 x + \mathcal{A}_2 x$, $x \in B_r$ on $[0, 1]$. In order to apply the conclusion of Krasnoselskii's fixed point theorem [37], we verify its hypothesis in three steps.

(i) For $x, y \in B_r$, we show that $\mathcal{A}_1 x + \mathcal{A}_2 y \in B_r$. For that, we have

$$\begin{aligned} &\|\mathcal{A}_1 x + \mathcal{A}_2 y\| \\ &= \sup_{t \in [0, 1]} |\mathcal{A}_1 x(t) + \mathcal{A}_2 y(t)| \\ &\leq \sup_{t \in [0, 1]} \left\{ \int_0^t \frac{(t-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |f(s, x(s), I^p x(s))| ds + \chi \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s)| ds \right. \\ &\quad + |\mu_1(t)| \sum_{j=1}^m |v_j| \int_0^{\sigma_j} \left(\frac{(\sigma_j-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |f(s, x(s), I^p x(s))| + \chi \frac{(\sigma_j-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s)| \right) ds \\ &\quad + |\mu_2(t)| \left[\sum_{i=1}^n |\rho_i| \int_{\xi_i}^{\eta_i} \int_0^s \left(\frac{(s-u)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |f(u, x(u), I^p x(u))| \right. \right. \\ &\quad \left. \left. + \chi \frac{(s-u)^{\zeta-1}}{\Gamma(\zeta)} |x(u)| \right) du ds \right. \\ &\quad + |a_1| \left[\int_0^1 \frac{(1-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |f(s, x(s), I^p x(s))| ds + \chi \int_0^1 \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s)| ds \right] \\ &\quad \left. + |a_2| \left[\int_0^1 \frac{(1-s)^{\zeta+\kappa-2}}{\Gamma(\zeta+\kappa-1)} |f(s, x(s), I^p x(s))| ds + \chi \int_0^1 \frac{(1-s)^{\zeta-2}}{\Gamma(\zeta-1)} |x(s)| ds \right] \right\} \\ &\leq \|k\| \sup_{t \in [0, 1]} \left\{ \int_0^t \frac{(t-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} ds + |\mu_1(t)| \sum_{j=1}^m |v_j| \int_0^{\sigma_j} \frac{(\sigma_j-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} ds \right. \\ &\quad + |\mu_2(t)| \left[\sum_{i=1}^n |\rho_i| \int_{\xi_i}^{\eta_i} \int_0^s \frac{(s-u)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} du ds + |a_1| \int_0^1 \frac{(1-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} ds \right. \\ &\quad \left. \left. + |a_2| \int_0^1 \frac{(1-s)^{\zeta+\kappa-2}}{\Gamma(\zeta+\kappa-1)} ds \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \chi \|x\| \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} ds + |\mu_1(t)| \sum_{j=1}^m |v_j| \int_0^{\sigma_j} \frac{(\sigma_j-s)^{\zeta-1}}{\Gamma(\zeta)} ds \right. \\
& + |\mu_2(t)| \left[\sum_{i=1}^n |\rho_i| \int_{\xi_i}^{\eta_i} \int_0^s \frac{(s-u)^{\zeta-1}}{\Gamma(\zeta)} du ds + |a_1| \int_0^1 \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} ds \right. \\
& \left. \left. + |a_2| \int_0^1 \frac{(1-s)^{\zeta-2}}{\Gamma(\zeta-1)} ds \right] \right\} \\
& \leq \|k\| \left\{ \frac{1}{\Gamma(\zeta + \kappa + 1)} + \bar{\mu}_1 \sum_{j=1}^m |v_j| \frac{\sigma_j^{\zeta+\kappa}}{\Gamma(\zeta + \kappa + 1)} \right. \\
& + \bar{\mu}_2 \left(\sum_{i=1}^n |\rho_i| \frac{(\eta_i^{\zeta+\kappa+1} - \xi_i^{\zeta+\kappa+1})}{\Gamma(\zeta + \kappa + 2)} + \frac{|a_1|}{\Gamma(\zeta + \kappa + 1)} + \frac{|a_2|}{\Gamma(\zeta + \kappa)} \right) \Bigg\} \\
& + \chi \|x\| \left\{ \frac{1}{\Gamma(\zeta + 1)} + \bar{\mu}_1 \sum_{j=1}^m |v_j| \frac{\sigma_j^{\zeta}}{\Gamma(\zeta + 1)} \right. \\
& + \bar{\mu}_2 \left(\sum_{i=1}^n |\rho_i| \frac{(\eta_i^{\zeta+1} - \xi_i^{\zeta+1})}{\Gamma(\zeta + 2)} + \frac{|a_1|}{\Gamma(\zeta + 1)} + \frac{|a_2|}{\Gamma(\zeta)} \right) \Bigg\} \\
& \leq \|k\| \Lambda_1 + \Lambda_2 r \leq r,
\end{aligned}$$

which shows that $\mathcal{A}_1 x + \mathcal{A}_2 y \in B_r$.

(ii) We establish that $\mathcal{A}_2$ is a contraction. For $x, y \in X$ and for each $t \in [0, 1]$, we obtain

$$\begin{aligned}
& \|\mathcal{A}_2 x - \mathcal{A}_2 y\| \\
& \leq \sup_{t \in [0,1]} \left\{ |\mu_1(t)| \sum_{j=1}^m |v_j| \int_0^{\sigma_j} \left(\frac{(\sigma_j-s)^{\zeta+\kappa-1}}{\Gamma(\zeta + \kappa)} |f(s, x(s), I^p x(s)) - f(s, y(s), I^p y(s))| \right. \right. \\
& + \chi \frac{(\sigma_j-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s) - y(s)| \Bigg) ds \\
& + |\mu_2(t)| \left[\sum_{i=1}^n |\rho_i| \int_{\xi_i}^{\eta_i} \int_0^s \left(\frac{(s-u)^{\zeta+\kappa-1}}{\Gamma(\zeta + \kappa)} |f(u, x(u), I^p x(u)) - f(u, y(u), I^p y(u))| \right. \right. \\
& + \chi \frac{(s-u)^{\zeta-1}}{\Gamma(\zeta)} |x(u) - y(u)| \Bigg) du ds \\
& + |a_1| \left[\int_0^1 \frac{(1-s)^{\zeta+\kappa-1}}{\Gamma(\zeta + \kappa)} |f(s, x(s), I^p x(s)) - f(s, y(s), I^p y(s))| ds \right. \\
& + \chi \int_0^1 \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s) - y(s)| ds \Bigg] \\
& + |a_2| \left[\int_0^1 \frac{(1-s)^{\zeta+\kappa-2}}{\Gamma(\zeta + \kappa - 1)} |f(s, x(s), I^p x(s)) - f(s, y(s), I^p y(s))| ds \right. \\
& + \chi \int_0^1 \frac{(1-s)^{\zeta-2}}{\Gamma(\zeta - 1)} |x(s) - y(s)| ds \Bigg] \Bigg\} \\
& \leq l \left(1 + \frac{1}{\Gamma(p+1)} \right) \left\{ \bar{\mu}_1 \sum_{j=1}^m |v_j| \frac{\sigma_j^{\zeta+\kappa}}{\Gamma(\zeta + \kappa + 1)} \right.
\end{aligned}$$

$$\begin{aligned}
& + \bar{\mu}_2 \left(\sum_{i=1}^n |\rho_i| \frac{(\eta_i^{\zeta+\kappa+1} - \xi_i^{\zeta+\kappa+1})}{\Gamma(\zeta + \kappa + 2)} + \frac{|a_1|}{\Gamma(\zeta + \kappa + 1)} + \frac{|a_2|}{\Gamma(\zeta + \kappa)} \right) \Bigg\} \|x - y\| \\
& + \chi \left[\bar{\mu}_1 \sum_{j=1}^m |v_j| \frac{\sigma_j^\zeta}{\Gamma(\zeta + 1)} + \bar{\mu}_2 \left(\sum_{i=1}^n |\rho_i| \frac{(\eta_i^{\zeta+1} - \xi_i^{\zeta+1})}{\Gamma(\zeta + 2)} + \frac{|a_1|}{\Gamma(\zeta + 1)} + \frac{|a_2|}{\Gamma(\zeta)} \right) \right] \\
& \times \|x - y\| \\
& \leq \Lambda_3 \|x - y\|,
\end{aligned}$$

which, in view of condition (3.16), implies that $\mathcal{A}_2$ is a contraction.

(iii) $\mathcal{A}_1$ is compact and continuous. Continuity of f implies that the operator $\mathcal{A}_1$ is continuous. Also, $\mathcal{A}_1$ is uniformly bounded on B_r as

$$\begin{aligned}
\|\mathcal{A}_1 x\| & \leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\zeta+\kappa-1}}{\Gamma(\zeta + \kappa)} |f(s, x(s), I^p x(s))| ds + \chi \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s)| ds \right\} \\
& \leq \frac{\|k\|}{\Gamma(\zeta + \kappa + 1)} + \frac{\chi r}{\Gamma(\zeta + 1)}.
\end{aligned}$$

Next we show the compactness of the operator $\mathcal{A}_1$.

In view of (C_1) , we define $\sup_{(t,x,y) \in [0,1] \times B_r \times B_r} |f(t, x, y)| = \bar{f}$. Then, for $0 < t_1 < t_2 < 1$, we find that

$$\begin{aligned}
& |\mathcal{A}_1 x(t_2) - \mathcal{A}_1 x(t_1)| \\
& \leq \left| \frac{1}{\Gamma(\zeta + \kappa)} \int_0^{t_1} ((t_2 - s)^{\zeta+\kappa-1} - (t_1 - s)^{\zeta+\kappa-1}) f(s, x(s), I^p x(s)) ds \right. \\
& \quad + \int_{t_1}^{t_2} (t_2 - s)^{\zeta+\kappa-1} f(s, x(s), I^p x(s)) ds \\
& \quad \left. + \frac{\chi}{\Gamma(\zeta)} \int_0^{t_1} ((t_2 - s)^{\zeta-1} - (t_1 - s)^{\zeta-1}) x(s) ds + \int_{t_1}^{t_2} (t_2 - s)^{\zeta-1} x(s) ds \right| \\
& \leq \frac{\bar{f}}{\Gamma(\zeta + \kappa + 1)} [2(t_2 - t_1)^{\zeta+\kappa} + |t_1^{\zeta+\kappa} - t_2^{\zeta+\kappa}|] \\
& \quad + \frac{\chi r}{\Gamma(\zeta + 1)} [2(t_2 - t_1)^\zeta + |t_1^\zeta - t_2^\zeta|] \rightarrow 0 \quad \text{as } t_2 - t_1 \rightarrow 0,
\end{aligned}$$

independently of $x \in B_r$. Thus, $\mathcal{A}_1$ is relatively compact on B_r . Hence, by the Arzela–Ascoli theorem, $\mathcal{A}_1$ is compact on B_r . In view of steps (i)–(iii), the hypothesis of Krasnoselskii's fixed point theorem [37] is satisfied. Thus, by the conclusion of Krasnoselskii's fixed point theorem, there exists a point $x \in B_r$ such that $x = \mathcal{A}_1 x + \mathcal{A}_2 x$, which corresponds to a solution of problem (1.1) on $[0, 1]$. This completes the proof. $\square$

Theorem 3.3 Let $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function, and let the following conditions hold:

- (C₃) There exist a function $p_1 \in C([0, 1], \mathbb{R}^+)$ and a nondecreasing function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $|f(t, x, I^p x)| \leq p_1(t) \psi(|x|)$ for all $t \in [0, 1]$, $x \in \mathbb{R}$;

(C₄) There exists a positive constant δ such that

$$\frac{\delta}{\|p_1\|\psi(\delta)\Lambda_1 + \delta\Lambda_2} > 1,$$

where Λ_1 and Λ_2 are respectively given by (3.14) and (3.15).

Then the boundary value problem (1.1) has at least one solution on $[0, 1]$.

Proof We split the proof into several steps. In step 1, we show that the operator $\mathcal{A} : X \rightarrow X$ defined by (3.13) maps bounded sets into bounded sets in X . For a positive number r , let $B_r = \{x \in X : \|x\| \leq r\}$ be a bounded set in X . Then

$$\begin{aligned} & |(\mathcal{A}x)(t)| \\ & \leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |f(s, x(s), I^p x(s))| ds + \chi \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s)| ds \right. \\ & \quad + |\mu_1(t)| \sum_{j=1}^m |v_j| \int_0^{\sigma_j} \left(\frac{(\sigma_j-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |f(s, x(s), I^p x(s))| + \chi \frac{(\sigma_j-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s)| \right) ds \\ & \quad + |\mu_2(t)| \left[\sum_{i=1}^n |\rho_i| \int_{\xi_i}^{\eta_i} \int_0^s \left(\frac{(s-u)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |f(u, x(u), I^p x(u))| \right. \right. \\ & \quad \left. \left. + \chi \frac{(s-u)^{\zeta-1}}{\Gamma(\zeta)} |x(u)| \right) du ds \right. \\ & \quad + |a_1| \left(\int_0^1 \frac{(1-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |f(s, x(s), I^p x(s))| ds + \chi \int_0^1 \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s)| ds \right) \\ & \quad \left. + |a_2| \left(\int_0^1 \frac{(1-s)^{\zeta+\kappa-2}}{\Gamma(\zeta+\kappa-1)} |f(s, x(s), I^p x(s))| ds + \chi \int_0^1 \frac{(1-s)^{\zeta-2}}{\Gamma(\zeta-1)} |x(s)| ds \right) \right\} \\ & \leq \|p_1\|\psi(\|x\|) \left[\frac{1}{\Gamma(\zeta+\kappa+1)} + \bar{\mu}_1 \sum_{j=1}^m |v_j| \frac{\sigma_j^{\zeta+\kappa}}{\Gamma(\zeta+\kappa+1)} \right. \\ & \quad \left. + \bar{\mu}_2 \left(\sum_{i=1}^n |\rho_i| \frac{(\eta_i^{\zeta+\kappa+1} - \xi_i^{\zeta+\kappa+1})}{\Gamma(\zeta+\kappa+2)} + \frac{|a_1|}{\Gamma(\zeta+\kappa+1)} + \frac{|a_2|}{\Gamma(\zeta+\kappa)} \right) \right] \\ & \quad + \chi \|x\| \left[\frac{1}{\Gamma(\zeta+1)} + \bar{\mu}_1 \sum_{j=1}^m |v_j| \frac{\sigma_j^{\zeta}}{\Gamma(\zeta+1)} \right. \\ & \quad \left. + \bar{\mu}_2 \left(\sum_{i=1}^n |\rho_i| \frac{(\eta_i^{\zeta+1} - \xi_i^{\zeta+1})}{\Gamma(\zeta+2)} + \frac{|a_1|}{\Gamma(\zeta+1)} + \frac{|a_2|}{\Gamma(\zeta)} \right) \right] \\ & \leq \|p_1\|\psi(\|x\|)\Lambda_1 + \Lambda_2\|x\|, \end{aligned}$$

which, on taking the norm for $t \in [0, 1]$, yields

$$\|\mathcal{A}x\| \leq \|p_1\|\psi(r)\Lambda_1 + \Lambda_2 r.$$

Next we show that $\mathcal{A}$ maps bounded sets into equicontinuous sets of X . Let $\tau_1, \tau_2 \in [0, 1]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we obtain

$$|\mathcal{A}x(\tau_2) - \mathcal{A}x(\tau_1)|$$

$$\begin{aligned}
 &\leq \frac{\|p_1\|\psi(r)}{\Gamma(\zeta + \kappa + 1)} [2(\tau_2 - \tau_1)^{\zeta + \kappa} + |\tau_1^{\zeta + \kappa} - \tau_2^{\zeta + \kappa}|] + \frac{\chi r}{\Gamma(\zeta + 1)} [2(\tau_2 - \tau_1)^\zeta + |\tau_1^\zeta - \tau_2^\zeta|] \\
 &\quad + \frac{|B_2||\tau_2^{\zeta + 1} - \tau_1^{\zeta + 1}|}{|\Delta|\Gamma(\zeta + 2)} \sum_{j=1}^m |v_j| \left[\frac{\|p_1\|\psi(r)\sigma_j^{\zeta + \kappa}}{\Gamma(\zeta + \kappa + 1)} + \frac{\chi r \sigma_j^\zeta}{\Gamma(\zeta + 1)} \right] \\
 &\quad + \frac{|A_2||\tau_2^{\zeta + 1} - \tau_1^{\zeta + 1}|}{|\Delta|\Gamma(\zeta + 2)} \left[\sum_{i=1}^n |\rho_i| \left(\frac{\|p_1\|\psi(r)(\eta_i^{\zeta + \kappa + 1} - \xi_i^{\zeta + \kappa + 1})}{\Gamma(\zeta + \kappa + 2)} + \frac{\chi r(\eta_i^{\zeta + 1} - \xi_i^{\zeta + 1})}{\Gamma(\zeta + 2)} \right) \right. \\
 &\quad \left. + |a_1| \left(\frac{\|p_1\|\psi(r)}{\Gamma(\zeta + \kappa + 1)} + \frac{\chi r}{\Gamma(\zeta + 1)} \right) + |a_2| \left(\frac{\|p_1\|\psi(r)}{\Gamma(\zeta + \kappa)} + \frac{\chi r}{\Gamma(\zeta)} \right) \right] \rightarrow 0
 \end{aligned}$$

as $\tau_2 - \tau_1 \rightarrow 0$ independently of $x \in B_r$. In view of the foregoing arguments, we deduce by the Arzela–Ascoli theorem that the operator $\mathcal{A} : X \rightarrow X$ is completely continuous.

Lastly, we establish that the set of all solutions x to equation $x = \theta \mathcal{A}x$ is bounded for $\theta \in [0, 1]$. As argued in proving that the operator $\mathcal{A}$ is bounded, one can obtain that

$$\frac{\|x\|}{\|p_1\|\psi(\|x\|)\Lambda_1 + \Lambda_2\|x\|} \leq 1.$$

On the other hand, we can find a positive number δ such that $\|x\| \neq \delta$ by assumption (C_4) . Let us consider a set $V = \{x \in X : \|x\| < \delta\}$. Notice that the operator $\mathcal{A} : \overline{V} \rightarrow X$ is continuous and completely continuous. From the definition of V , we cannot find any $x \in \partial V$ satisfying the equation $x = \theta \mathcal{A}(x)$ for some $\theta \in (0, 1)$. Thus we conclude by the Leray–Schauder nonlinear alternative [38] that there exists a fixed point $x \in \overline{V}$ for the operator $\mathcal{A}$, which is a solution of problem (1.1). This completes the proof. $\square$

3.3 Uniqueness result

Theorem 3.4 Assume that $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function satisfying condition (C_1) . Then there exists a unique solution for the boundary value problem (1.1) on $[0, 1]$ provided that

$$l \left(1 + \frac{1}{\Gamma(p+1)} \right) \Lambda_1 + \Lambda_2 < 1, \tag{3.17}$$

where Λ_1 and Λ_2 are respectively given by (3.14) and (3.15).

Proof Consider a closed ball $B_{\bar{r}} = \{x \in X : \|x\| \leq \bar{r}\}$, where

$$\bar{r} \geq \frac{M_0 \Lambda_1}{1 - (l(1 + \frac{1}{\Gamma(p+1)})\Lambda_1 + \Lambda_2)}, \quad M_0 = \sup_{t \in [0, 1]} |f(t, 0, 0)| < \infty,$$

and show that $\mathcal{A}B_{\bar{r}} \subset B_{\bar{r}}$ ($\mathcal{A} : X \rightarrow X$ is defined by (3.13)).

Letting $\widehat{f}_x(s) = f(s, x(s), I^p x(s))$, $x \in B_{\bar{r}}$ and using condition (C_1) , we obtain

$$\begin{aligned}
 |\widehat{f}_x(s)| &= |f(s, x(s), I^p x(s)) - f(s, 0, 0) + f(s, 0, 0)| \\
 &\leq |f(s, x(s), I^p x(s)) - f(s, 0, 0)| + |f(s, 0, 0)|
 \end{aligned}$$

$$\begin{aligned} &\leq l(|x(s)| + |I^p x(s)|) + |f(s, 0, 0)| \\ &\leq l\left(1 + \frac{1}{\Gamma(p+1)}\right)\|x\| + M_0 \leq l\left(1 + \frac{1}{\Gamma(p+1)}\right)\bar{r} + M_0. \end{aligned}$$

Then

$$\begin{aligned} |\mathcal{A}x(t)| &\leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |\widehat{f}_x(s)| ds + \chi \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s)| ds \right. \\ &\quad + |\mu_1(t)| \sum_{j=1}^m |\nu_j| \int_0^{\sigma_j} \left(\frac{(\sigma_j-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |\widehat{f}_x(s)| + \chi \frac{(\sigma_j-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s)| \right) ds \\ &\quad + |\mu_2(t)| \left[\sum_{i=1}^n |\rho_i| \int_{\xi_i}^{\eta_i} \int_0^s \left(\frac{(s-u)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |\widehat{f}_x(s)| + \chi \frac{(s-u)^{\zeta-1}}{\Gamma(\zeta)} |x(u)| \right) du ds \right. \\ &\quad + |a_1| \left(\int_0^1 \frac{(1-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |\widehat{f}_x(s)| ds + \chi \int_0^1 \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s)| ds \right) \\ &\quad \left. \left. + |a_2| \left(\int_0^1 \frac{(1-s)^{\zeta+\kappa-2}}{\Gamma(\zeta+\kappa-1)} |\widehat{f}_x(s)| ds + \chi \int_0^1 \frac{(1-s)^{\zeta-2}}{\Gamma(\zeta-1)} |x(s)| ds \right) \right] \right\} \\ &\leq \left(l \left(1 + \frac{1}{\Gamma(p+1)} \right) \Lambda_1 + \Lambda_2 \right) \bar{r} + M_0 \Lambda_1 \leq \bar{r}, \end{aligned}$$

which implies that $\|\mathcal{A}x\| \leq \bar{r}$. Thus $\mathcal{A}B_{\bar{r}} \subset B_{\bar{r}}$.

Now, for $x, y \in X$ and for each $t \in [0, 1]$, we obtain

$$\begin{aligned} &\|\mathcal{A}x - \mathcal{A}y\| \\ &\leq \sup_{t \in [0,1]} \left\{ \int_0^t \frac{(t-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |\widehat{f}_x(s) - \widehat{f}_y(s)| ds + \chi \int_0^t \frac{(t-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s) - y(s)| ds \right. \\ &\quad + |\mu_1(t)| \sum_{j=1}^m |\nu_j| \int_0^{\sigma_j} \left(\frac{(\sigma_j-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |\widehat{f}_x(s) - \widehat{f}_y(s)| + \chi \frac{(\sigma_j-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s) - y(s)| \right) ds \\ &\quad + |\mu_2(t)| \left[\sum_{i=1}^n |\rho_i| \int_{\xi_i}^{\eta_i} \int_0^s \left(\frac{(s-u)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |\widehat{f}_x(u) - \widehat{f}_y(u)| \right. \right. \\ &\quad \left. \left. + \chi \frac{(s-u)^{\zeta-1}}{\Gamma(\zeta)} |x(u) - y(u)| \right) du ds \right. \\ &\quad + |a_1| \left(\int_0^1 \frac{(1-s)^{\zeta+\kappa-1}}{\Gamma(\zeta+\kappa)} |\widehat{f}_x(s) - \widehat{f}_y(s)| ds + \chi \int_0^1 \frac{(1-s)^{\zeta-1}}{\Gamma(\zeta)} |x(s) - y(s)| ds \right) \\ &\quad \left. \left. + |a_2| \left(\int_0^1 \frac{(1-s)^{\zeta+\kappa-2}}{\Gamma(\zeta+\kappa-1)} |\widehat{f}_x(s) - \widehat{f}_y(s)| ds + \chi \int_0^1 \frac{(1-s)^{\zeta-2}}{\Gamma(\zeta-1)} |x(s) - y(s)| ds \right) \right] \right\} \\ &\leq l \left(1 + \frac{1}{\Gamma(p+1)} \right) \left\{ \frac{1}{\Gamma(\zeta+\kappa+1)} + \bar{\mu}_1 \sum_{j=1}^m |\nu_j| \frac{\sigma_j^{\zeta+\kappa}}{\Gamma(\zeta+\kappa+1)} \right. \\ &\quad \left. + \bar{\mu}_2 \left(\sum_{i=1}^n |\rho_i| \frac{(\eta_i^{\zeta+\kappa+1} - \xi_i^{\zeta+\kappa+1})}{\Gamma(\zeta+\kappa+2)} + \frac{|a_1|}{\Gamma(\zeta+\kappa+1)} + \frac{|a_2|}{\Gamma(\zeta+\kappa)} \right) \right\} \|x - y\| \end{aligned}$$

$$\begin{aligned}
& + \chi \left[\frac{1}{\Gamma(\zeta + 1)} + \bar{\mu}_1 \sum_{j=1}^m |\nu_j| \frac{\sigma_j^\zeta}{\Gamma(\zeta + 1)} \right. \\
& \left. + \bar{\mu}_2 \left(\sum_{i=1}^n |\rho_i| \frac{(\eta_i^{\zeta+1} - \xi_i^{\zeta+1})}{\Gamma(\zeta + 2)} + \frac{|a_1|}{\Gamma(\zeta + 1)} + \frac{|a_2|}{\Gamma(\zeta)} \right) \right] \|x - y\| \\
& \leq \left[l \left(1 + \frac{1}{\Gamma(p + 1)} \right) \Lambda_1 + \Lambda_2 \right] \|x - y\|,
\end{aligned}$$

which shows that $\mathcal{A}$ is a contraction in view of condition (3.17). Hence we deduce by the Banach contraction mapping principle that the operator $\mathcal{A}$ has a unique fixed point, which corresponds to a unique solution of problem (1.1) on $[0, 1]$. This completes the proof. $\square$

4 Examples

Consider the following Langevin equation with fixed fractional orders:

$${}^c D^{\frac{3}{2}} \left({}^c D^{\frac{1}{2}} + \frac{1}{4} \right) x(t) = f(t, x(t), I^{\frac{3}{4}} x(t)), \quad t \in [0, 1], \quad (4.1)$$

subject to the multi-point and multi-strip boundary conditions

$$x(0) = \sum_{j=1}^m \nu_j x(\sigma_j), \quad x'(0) = 0, \quad a_1 x(1) + a_2 x'(1) = \sum_{i=1}^n \rho_i \int_{\xi_i}^{\eta_i} x(s) ds, \quad (4.2)$$

where $\zeta = \frac{1}{2}$, $\kappa = \frac{3}{2}$, $p = \frac{3}{4}$, $\chi = \frac{1}{4}$, $m = 3$, $n = 3$, $a_1 = 1$, $a_2 = 1$, $\sigma_1 = \frac{1}{28}$, $\sigma_2 = \frac{1}{21}$, $\sigma_3 = \frac{1}{14}$, $\xi_1 = \frac{1}{9}$, $\xi_2 = \frac{3}{9}$, $\xi_3 = \frac{5}{9}$, $\eta_1 = \frac{2}{9}$, $\eta_2 = \frac{4}{9}$, $\eta_3 = \frac{6}{9}$, $\nu_1 = \frac{1}{15}$, $\nu_2 = \frac{1}{10}$, $\nu_3 = \frac{1}{5}$, $\rho_1 = \frac{1}{12}$, $\rho_2 = \frac{1}{6}$, $\rho_3 = \frac{1}{4}$, and $f(t, x, I^p x)$ will be fixed later. Using the given values, we find that $\Lambda_1 \approx 1.111762$, $\Lambda_2 \approx 0.511583$ (Λ_1 and Λ_2 are respectively given by (3.14) and (3.15)).

In order to illustrate Theorem 3.2, we take the following nonlinear function in (4.1):

$$f(t, x, I^{\frac{3}{4}} x) = \frac{1}{\sqrt{t^2 + 144}} \left(\tan^{-1} x + e^{-t} \int_0^t \frac{(t-s)^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} x ds + \frac{1}{3} \right), \quad 0 < t < 1, \quad (4.3)$$

which obviously satisfies the Lipschitz condition (C_1) with $l = 1/12$, that is,

$$|f(t, x, I^{\frac{3}{4}} x) - f(t, y, I^{\frac{3}{4}} y)| \leq \frac{1}{12} \left(1 + \frac{1}{\Gamma(7/4)} \right) |x - y|.$$

Moreover, $\Lambda_3 \approx 0.335938 < 1$. Clearly the hypothesis of Theorem 3.2 holds true, and consequently its conclusion applies to problem (4.1)–(4.2) with $f(t, x, I^{\frac{3}{4}} x)$ given by (4.3).

To demonstrate the application of Theorem 3.3, we consider

$$f(t, x, I^{\frac{3}{4}} x) = \frac{1}{\sqrt{25 + t^2}} \left(\sin x + \frac{|x|}{4(1 + |x|)} + I^{\frac{3}{4}} x + \frac{1}{4} \right), \quad t \in [0, 1] \quad (4.4)$$

in (4.1) and note that

$$|f(t, x, I^{\frac{3}{4}} x)| \leq \frac{1}{\sqrt{25 + t^2}} \left(\frac{1}{2} + \left(1 + \frac{1}{\Gamma(\frac{7}{4})} \right) \|x\| \right),$$

with $p_1(t) = \frac{1}{\sqrt{25+t^2}}$, $\|p_1\| = \frac{1}{5}$, and $\psi(\|x\|) = \frac{1}{2} + (1 + \frac{1}{\Gamma(\frac{3}{4})})\|x\|$. By condition (C_4) , we find that $M > 4.60723755$. Thus all the assumptions of Theorem 3.3 are satisfied. Hence, there exists at least one solution for problem (4.1)–(4.2) with $f(t, x, I^{\frac{3}{4}}x)$ given by (4.4) on $[0, 1]$.

Since $l(1 + \frac{1}{\Gamma(\frac{7}{4})})\Lambda_1 + \Lambda_2 \approx 0.705036 < 1$, therefore problem (4.1)–(4.2) with $f(t, x, I^{\frac{3}{4}}x)$ given by (4.3) has a unique solution on $[0, 1]$ by Theorem 3.4.

5 Conclusions

We have discussed the solvability of a nonlinear Langevin equation involving Caputo fractional derivatives of different orders with nonlinearity depending upon the unknown function together with its Riemann–Liouville fractional integral, subject to nonlocal multi-point and multi-strip boundary conditions. The results obtained in this paper are new and strengthen the literature on Langevin equation. Moreover, several results follow as special cases of the ones presented in this paper by fixing the parameters in the boundary conditions. For example, our results correspond to the boundary conditions:

- $x(0) = 0$, $x'(0) = 0$, $a_1x(1) + a_2x'(1) = 0$ if we take all $\rho_i = 0 = v_j$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, and $a_1 \neq 0 \neq a_2$;
- $x(0) = \sum_{j=1}^m v_j x(\sigma_j)$, $x'(0) = 0$, $a_1x(1) + a_2x'(1) = 0$ by fixing all $\rho_i = 0$, $i = 1, 2, \dots, n$;
- $x(0) = 0$, $x'(0) = 0$, $a_1x(1) + a_2x'(1) = \sum_{i=1}^n \rho_i \int_{\xi_i}^{\eta_i} x(s) ds$ by taking all $v_j = 0$, $j = 1, 2, \dots, m$;
- $x(0) = \sum_{j=1}^m v_j x(\sigma_j)$, $x'(0) = 0$, $x(1) = \sum_{i=1}^n \rho_i \int_{\xi_i}^{\eta_i} x(s) ds$ for $a_1 = 1$, $a_2 = 0$;
- $x(0) = \sum_{j=1}^m v_j x(\sigma_j)$, $x'(0) = 0$, $\sum_{i=1}^n \rho_i \int_{\xi_i}^{\eta_i} x(s) ds = 0$ when $a_1 = 0 = a_2$ and all ρ_i ($i = 1, 2, \dots, n$) and v_j ($j = 1, 2, \dots, m$) are not zero.

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Abbreviations

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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