Chen et al. *Advances in Difference Equations* (2019) 2019:119
https://doi.org/10.1186/s13662-019-2001-z

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Anti-periodic boundary value problems with Riesz–Caputo derivative

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Abstract

This paper is concerned with a class of anti-periodic boundary value problems for fractional differential equations with the Riesz–Caputo derivative, which can reflected both the past and the future nonlocal memory effects. By means of new fractional Gronwall inequalities and some fixed point theorems, we obtain some existence results of solutions under the Lipschitz condition, the sublinear growth condition, the nonlinear growth condition and the comparison condition. Three examples are given to illustrate the results.

MSC: 26A33; 34B15; 34A08

Keywords: Riesz–Caputo derivative; Fractional Gronwall inequality; Fixed point theorem; Boundary value problem

1 Introduction

Fractional calculus goes back to Newton and Leibniz in the seventeenth century. It is a generalization of ordinary differential equations and integration to arbitrary non-integer order [1]. Fractional differential equations have become a focus spot in the field of differential equation during recent years. Existence, uniqueness and stability of initial (boundary) value problems are the main topics of fractional differential equations; see [1–23] and the references therein. For example, Zhou et al. [5, 6] obtained some existence and uniqueness results of fractional differential equations. Baleanu et al. [7–9] investigated the existence of solutions for fractional differential equations including the Caputo-Fabrizio derivative. Li et al. [16] and Aguila–Camacho et al. [17] discussed stability of fractional dynamic systems by Lyapunov method, Wang and Li [18, 19] investigated Ulam–Hyers stability of some fractional equations, Li et al. [1] and Chen et al. [15] considered the anti-periodic boundary value problem for fractional differential equations with impulses. Some periodic or anti-periodic boundary value problems for fractional differential equations were also discussed by Baleanu et al. [20–23].

The present states of many processes started at the past states, also relying on its development in the future, for example, stock price option. Another example is the applications to anomalous diffusion problem where Riesz derivative implies the nonlocality and it is adopted to depict the diffusion concentration's dependence on path. As pointed out by [2], the complete theory of fractional differential equations should be developed only with the use of both left and right derivatives which holds memory effects. However, most of the present work as regards fractional differential equations were focused on Riemann–Liouville and Caputo fractional derivatives, which are one-sided fractional operators only reflected the past or future memory effect. Fortunately, the Riesz derivative is a two-sided fractional operator including both left and right derivatives, which can reflect both the past and the future memory effects. This feature is particularly for fractional modeling on a finite domain. Some recent applications of this derivative to anomalous diffusion were given in [24, 25].

There is no literature to research the fractional ordinary differential equations with the Riesz–Caputo derivative. Here, we discuss the following fractional boundary value problems (BVP for short):

$$\begin{cases} {}_0^{RC}D_T^{\gamma}y(\tau) = g(\tau, y(\tau)), \quad \tau \in J, J = [0,T], 1 < \gamma \leq 2, \\ y(0) + y(T) = 0, \qquad y'(0) + y'(T) = 0, \end{cases} \tag{1}$$

where ${}_0^{RC}D_T^{\gamma}$ is a Riesz–Caputo derivative and $g : J \times R \to R$ is a continuous function with respect to τ and y.

2 Preliminaries

Some definitions and preliminary facts will be introduced in this section. Let $\beta > 0$, and $n-1 < \beta \leq n$, $n \in N$ and $n = \lceil \nu \rceil$, and $\lceil \cdot \rceil$ the ceiling of a number.

Definition 2.1 According to the classical Riesz–Caputo definition in [2, 3], for $0 \leq \tau \leq T$,

$$\begin{aligned} {}_0^{RC}D_T^{\beta}z(\tau) &= \frac{1}{\Gamma(n-\beta)} \int_0^T \frac{z^{(n)}(u)}{|\tau-u|^{\beta+1-n}}\,du \\ &= \frac{1}{2}\big({}_0^{C}D_{\tau}^{\beta} + (-1)^n {}_{\tau}^{C}D_T^{\beta}\big)z(\tau), \end{aligned}$$

where ${}_0^{C}D_{\tau}^{\beta}$ is the left Caputo derivative and ${}_{\tau}^{C}D_T^{\beta}$ is the right Caputo derivative,

$${}_0^{C}D_{\tau}^{\beta}z(\tau) = \frac{1}{\Gamma(n-\beta)} \int_0^{\tau} \frac{z^{(n)}(u)}{(\tau-u)^{\beta+1-n}}\,du,$$

and

$${}_{\tau}^{C}D_T^{\beta}z(\tau) = \frac{(-1)^n}{\Gamma(n-\beta)} \int_{\tau}^{T} \frac{z^{(n)}(u)}{(u-\tau)^{\beta+1-n}}\,du.$$

In particular, if $1 < \beta \leq 2$ and $z(\tau) \in C^2(0,T)$, then

$${}_0^{RC}D_T^{\beta}z(\tau) = \frac{1}{2}\big({}_0^{C}D_{\tau}^{\beta} + {}_{\tau}^{C}D_T^{\beta}\big)z(\tau).$$

Definition 2.2 ([4]) The fractional left, right and Riemann–Liouville integrals of order β are defined as

$${}_0I_{\tau}^{\beta}z(\tau) = \frac{1}{\Gamma(\beta)} \int_0^{\tau} (\tau-u)^{\beta-1}z(u)\,du,$$

$$
\begin{aligned}
{}_{\tau}I_{T}^{\beta}z(\tau) &= \frac{1}{\Gamma(\beta)}\int_{\tau}^{T}(u-\tau)^{\beta-1}z(u)\,du, \\
{}_{0}I_{T}^{\beta}z(\tau) &= \frac{1}{\Gamma(\beta)}\int_{0}^{T}|u-\tau|^{\beta-1}z(u)\,du.
\end{aligned}
$$

Lemma 2.1 ([4]) *If* $z(\tau)\in C^{n}[0,T]$, *then*

$$
{}_{0}I_{\tau}^{\beta}\,{}_{0}^{\mathrm{C}}D_{\tau}^{\beta}z(\tau) = z(\tau) - \sum_{l=0}^{n-1}\frac{z^{(l)}(0)}{l!}(\tau-0)^{l}
$$

and

$$
{}_{\tau}I_{T}^{\beta}\,{}_{\tau}^{\mathrm{C}}D_{T}^{\beta}z(\tau) = (-1)^{n}\left[z(\tau) - \sum_{l=0}^{n-1}\frac{(-1)^{l}z^{(l)}(T)}{l!}(T-\tau)^{l}\right].
$$

From the above definitions and lemmas, we have

$$
\begin{aligned}
&{}_{0}I_{T}^{\beta}\,{}_{0}^{\mathrm{RC}}D_{T}^{\beta}z(\tau) \\
&\quad= \frac{1}{2}\left({}_{0}I_{\tau}^{\beta}\,{}_{0}^{\mathrm{C}}D_{\tau}^{\beta} + {}_{\tau}I_{T}^{\beta}\,{}_{0}^{\mathrm{C}}D_{\tau}^{\beta}\right)z(\tau) + (-1)^{n}\frac{1}{2}\left({}_{0}I_{\tau}^{\beta}\,{}_{\tau}^{\mathrm{C}}D_{T}^{\beta} + {}_{\tau}I_{T}^{\beta}\,{}_{\tau}^{\mathrm{C}}D_{T}^{\beta}\right)z(\tau) \\
&\quad= \frac{1}{2}\left({}_{0}I_{\tau}^{\beta}\,{}_{0}^{\mathrm{C}}D_{\tau}^{\beta} + (-1)^{n}{}_{\tau}I_{T}^{\beta}\,{}_{\tau}^{\mathrm{C}}D_{T}^{\beta}\right)z(\tau).
\end{aligned}
$$

In particular, if $1<\beta\leq 2$ and $z(\tau)\in C^{2}(0,T)$, then

$$
{}_{0}I_{T}^{\beta}\,{}_{0}^{\mathrm{RC}}D_{T}^{\beta}z(\tau) = z(\tau) - \frac{1}{2}\bigl(z(0)+z(T)\bigr) - \frac{1}{2}z'(0)\tau + \frac{1}{2}z'(T)(T-\tau). \tag{2}
$$

Lemma 2.2 *Assume that* $f\in C(J,R)$. *A function* $y\in C^{2}(J)$ *given by*

$$
\begin{aligned}
y(\tau) &= -\frac{T}{2\Gamma(\gamma-1)}\int_{0}^{T}(T-u)^{\gamma-2}f(u)\,du \\
&\quad+ \frac{1}{\Gamma(\gamma)}\int_{0}^{\tau}(\tau-u)^{(\gamma-1)}f(u)\,du + \frac{1}{\Gamma(\gamma)}\int_{\tau}^{T}(u-\tau)^{(\gamma-1)}f(u)\,du,
\end{aligned}
\tag{3}
$$

is a unique solution of the following anti-periodic boundary value problem:

$$
\begin{cases}
{}_{0}^{\mathrm{RC}}D_{T}^{\gamma}y(\tau) = f(\tau), & \tau\in J, 1<\gamma\leq 2, \\
y(0)+y(T)=0, & y'(0)+y'(T)=0.
\end{cases}
\tag{4}
$$

Proof From (2) and the first equality of (4), we have

$$
\begin{aligned}
y(\tau) &= \frac{1}{2}\bigl(y(0)+y(T)\bigr) + \frac{1}{2}y'(0)\tau - \frac{1}{2}y'(T)(T-\tau) \\
&\quad+ \frac{1}{\Gamma(\gamma)}\int_{0}^{T}|\tau-u|^{\gamma-1}f(u)\,du
\end{aligned}
$$

$$= \frac{1}{2}\big(y(0) + y(T)\big) + \frac{1}{2}y'(0)\tau - \frac{1}{2}y'(T)(T-\tau) \\ + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{\gamma-1}f(u)\,du + \frac{1}{\Gamma(\gamma)}\int_\tau^T (u-\tau)^{\gamma-1}f(u)\,du. \tag{5}$$

Then

$$y'(\tau) = \frac{1}{2}\big(y'(0) + y'(T)\big) \\ + \frac{1}{\Gamma(\gamma-1)}\int_0^\tau (\tau-u)^{\gamma-2}f(u)\,du - \frac{1}{\Gamma(\gamma-1)}\int_\tau^T (u-\tau)^{\gamma-2}f(u)\,du.$$

By the boundary conditions $y(0) + y(T) = 0$, $y'(0) + y'(T) = 0$, we find that

$$y(0) = \frac{T}{2\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}f(u)\,du - \frac{1}{\Gamma(\gamma)}\int_0^T (T-u)^{\gamma-1}f(u)\,du,$$

$$y(T) = -\frac{T}{2\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}f(u)\,du + \frac{1}{\Gamma(\gamma)}\int_0^T (T-u)^{\gamma-1}f(u)\,du,$$

$$y'(0) = -\frac{1}{\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}f(u)\,du,$$

$$y'(T) = \frac{1}{\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}f(u)\,du.$$

Substituting the values of $y(0)$, $y(T)$, $y'(0)$ and $y'(T)$ into (5), we obtain (3). □

We now extend our results to the generalized Gronwall inequalities which appeared in [26].

Lemma 2.3 ([26]) *Let $z \in C(J,R)$ satisfy the following inequality*:

$$\big|z(t)\big| \le a + b\int_0^t \big|z(u)\big|^{\lambda_1}\,du + c\int_0^T \big|z(u)\big|^{\lambda_2}\,du, \quad t \in J,$$

where $\lambda_1 \in [0,1]$, $\lambda_2 \in [0,1)$, $a,b,c \ge 0$ are constants. Then there exists a constant $M^* > 0$ such that

$$\big|z(t)\big| \le M^*.$$

Corollary 2.1 *Let $z \in C(J,R)$ satisfy the following inequality*:

$$\big|z(t)\big| \le a + b\int_0^t \big|z(u)\big|^{\lambda_1}\,du + c\int_t^T \big|z(u)\big|^{\lambda_2}\,du + d\int_0^T \big|z(u)\big|^{\lambda_3}\,du, \quad t \in J, \tag{6}$$

where $\lambda_1 \in [0,1]$, $\lambda_2, \lambda_3 \in [0,1)$, $a,b,c,d \ge 0$ are constants. Then there exists a constant $M^* > 0$ such that

$$\big|z(t)\big| \le M^*.$$

Proof Let $\lambda_4 = \max\{\lambda_2, \lambda_3\}$. From (6), we have

$$\begin{aligned}
|z(t)| &\le a + b\int_0^t |z(u)|^{\lambda_1}\,du + c\int_t^T |z(u)|^{\lambda_4}\,du + d\int_0^T |z(u)|^{\lambda_4}\,du \\
&\le a + b\int_0^t |z(u)|^{\lambda_1}\,du + (c+d)\int_0^T |z(u)|^{\lambda_4}\,du.
\end{aligned}$$

By Lemma 2.3, we can directly obtain the result. □

Lemma 2.4 *Let $z \in C(J,R)$ satisfy the following inequality*:

$$\begin{aligned}
|z(t)| &\le a + b\int_0^t (t-u)^{\gamma-1}|z(u)|^{\lambda}\,du + c\int_t^T (u-t)^{\gamma-1}|z(u)|^{\lambda}\,du \\
&\quad + d\int_0^T (T-u)^{\gamma-2}|z(u)|^{\lambda}\,du, \qquad (7)
\end{aligned}$$

where $\gamma \in (1,2)$, $\lambda \in [0, 1-\frac{1}{\zeta})$ for some $1 < \zeta < \frac{1}{2-\gamma}$, $a,b,c,d \ge 0$ are constants. Then there exists a constant $M^* > 0$ such that

$$|z(t)| \le M^*.$$

Proof Let

$$y(t) = \begin{cases} 1, & |z(t)| \le 1, \\ z(t), & |z(t)| > 1. \end{cases}$$

Combining condition (7) and the Hölder inequality, it follows

$$\begin{aligned}
|z(t)| &\le |y(t)| \\
&\le (a+1) + b\int_0^t (t-u)^{\gamma-1}|y(u)|^{\lambda}\,du \\
&\quad + c\int_t^T (u-t)^{\gamma-1}|y(u)|^{\lambda}\,du + d\int_0^T (T-u)^{\gamma-2}|y(u)|^{\lambda}\,du \\
&\le (a+1) + b\biggl(\int_0^t (t-u)^{\zeta(\gamma-1)}\,du\biggr)^{\frac{1}{\zeta}}\biggl(\int_0^t |y(u)|^{\frac{\lambda\zeta}{\zeta-1}}\,du\biggr)^{\frac{\zeta-1}{\zeta}} \\
&\quad + c\biggl(\int_t^T (u-t)^{\zeta(\gamma-1)}\,du\biggr)^{\frac{1}{\zeta}}\biggl(\int_t^T |y(u)|^{\frac{\lambda\zeta}{\zeta-1}}\,du\biggr)^{\frac{\zeta-1}{\zeta}} \\
&\quad + d\biggl(\int_0^T (T-u)^{\zeta(\gamma-2)}\,du\biggr)^{\frac{1}{\zeta}}\biggl(\int_0^t |y(u)|^{\frac{\lambda\zeta}{\zeta-1}}\,du\biggr)^{\frac{\zeta-1}{\zeta}} \\
&\le (a+1) + b\biggl(\frac{T^{\zeta(\gamma-1)+1}}{\zeta(\gamma-1)+1}\biggr)^{\frac{1}{\zeta}}\int_0^t |y(u)|^{\frac{\lambda\zeta}{\zeta-1}}\,du
\end{aligned}$$

$$+ c\left(\frac{T^{\zeta(\gamma-1)+1}}{\zeta(\gamma-1)+1}\right)^{\frac{1}{\zeta}} \int_t^T \left|y(u)\right|^{\frac{\lambda\zeta}{\zeta-1}} du$$
$$+ d\left(\frac{T^{\zeta(\gamma-2)+1}}{\zeta(\gamma-2)+1}\right)^{\frac{1}{\zeta}} \int_0^T \left|y(u)\right|^{\frac{\lambda\zeta}{\zeta-1}} du.$$

Since $\frac{\lambda\zeta}{\zeta-1} \in [0,1)$, we can complete the rest proof immediately by Corollary 2.1. □

Finally, we introduce three fixed point theorems.

Lemma 2.5 (Schaefer fixed point theorem [27]) *Let X be a convex subset of a normed linear space Ω and $0 \in X$. Let $F : X \to X$ be a completely continuous operator, and let*

$$\omega(F) = \left\{y \in X : y = \lambda F y \text{ for some } \lambda \in (0,1)\right\}.$$

Then either $\omega(F)$ is unbounded or F has a fixed point.

Lemma 2.6 (The Leray–Schauder fixed point theorem [28]) *Assume that U is a closed convex subset of a Banach space V, and W is a relatively open subset of U with $0 \in W$, and $T : \overline{W} \to U$ is a continuous compact map. Then either*

I. *T exists a fixed point in $\overline{W}$;*

or

II. *there exist $x \in \partial W$ and $\lambda \in (0,1)$ with $x = \lambda x$.*

Lemma 2.7 (The Schauder fixed point theorem [29]) *Let V be a Banach space with $W \subseteq V$ closed, bounded and convex, and $T : W \to W$ is completely continuous. Then T has a fixed point in W.*

3 Main results

Let $C(0,T)$ be the space of all real functions y defined on $J = [0,T]$ with the norm $\|y\| = \sup_{\tau\in J} |y(\tau)|$, then $C(0,T)$ is a Banach space. For measurable functions $z : J \to R$, define the norm $\|z\|_{L^\rho(J,R)} = (\int_J |z(\tau)|^\rho \, d\tau)^{\frac{1}{\rho}}$, $1 \le \rho < \infty$. We denote by $L^\rho(J,R)$ the Banach space of all Lebesgue measurable functions z with $\|z\|_{L^\rho} < \infty$.

We transform BVP (1) into a fixed point problem, define an integral operator $A : J \to J$ by

$$Ay(\tau) = -\frac{T}{2\Gamma(\gamma-1)} \int_0^T (T-u)^{\gamma-2} g\big(u, y(u)\big) \, du$$
$$+ \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau-u)^{(\gamma-1)} g\big(u, y(u)\big) \, du + \frac{1}{\Gamma(\gamma)} \int_\tau^T (u-\tau)^{(\gamma-1)} g\big(u, y(u)\big) \, du. \quad (8)$$

Theorem 3.1 *Assume that*

(H_1) *there exists a constant $L_1 \ge 0$ such that $|g(\tau,y) - g(\tau,z)| \le L_1|y-z|$, for each $\tau \in J$ and $y, z \in R$.*

Then BVP (1) has a unique solution on J provided that

$$\left(\frac{T^\gamma}{2\Gamma(\gamma)} + \frac{2T^\gamma}{\Gamma(\gamma+1)}\right) L_1 < 1. \quad (9)$$

Proof

$$
\begin{aligned}
\left|(Ay)(\tau)-(Az)(\tau)\right| &\le \frac{T}{2\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}\left|g\big(u,y(u)\big)-g\big(u,z(u)\big)\right|du \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{(\gamma-1)}\left|g\big(u,y(u)\big)-g\big(u,z(u)\big)\right|du \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_\tau^T (u-\tau)^{(\gamma-1)}\left|g\big(u,y(u)\big)-g\big(u,z(u)\big)\right|du \\
&\le \frac{TL_1\|y-z\|}{2\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}\,du + \frac{L_1\|y-z\|}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{(\gamma-1)}\,du \\
&\quad + \frac{L_1\|y-z\|}{\Gamma(\gamma)}\int_\tau^T (u-\tau)^{(\gamma-1)}\,du \\
&\le \left(\frac{T^\gamma}{2\Gamma(\gamma)}+\frac{\tau^\gamma}{\Gamma(\gamma+1)}+\frac{(T-\tau)^\gamma}{\Gamma(\gamma+1)}\right)L_1\|y-z\| \\
&\le \left(\frac{T^\gamma}{2\Gamma(\gamma)}+\frac{2T^\gamma}{\Gamma(\gamma+1)}\right)L_1\|y-z\|.
\end{aligned}
$$

Therefore, according to (9),

$$
\begin{aligned}
\|Ay-Az\| &\le \left(\frac{T^\gamma}{2\Gamma(\gamma)}+\frac{2T^\gamma}{\Gamma(\gamma+1)}\right)L_1\|y-z\| \\
&< \|y-z\|.
\end{aligned}
$$

Then A is a contraction, as a consequence of the Banach fixed point theorem, we deduce that A has a fixed point which is an unique solution of BVP (1). □

Theorem 3.2 *Assume that*

(H_2) *there exist a constant* $L_2>0$ *and* $\sigma\in(0,1-\frac{1}{\zeta})$ *for some* $\zeta(\gamma-2)+1>0$ *with* $\gamma>1$ *such that* $g(\tau,y)\le L_2(1+|y|^\sigma)$ *for each* $\tau\in J$ *and all* $y\in R$.

Then BVP (1) *has at least one solution on* J.

Proof We will use the Schaefer fixed-point theorem to prove A has a fixed point. For the sake of convenience, we subdivide the proof into several steps.

Step 1. A is continuous.

Let $\{y_n\}$ be a sequence such that $y_n\to y$ in $C(J)$. Then, for each $t\in J$, we have

$$
\begin{aligned}
\left|(Ay_n)(\tau)-(Ay)(\tau)\right| &\le \frac{T}{2\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}\left|g\big(u,y_n(u)\big)-g\big(u,y(u)\big)\right|du \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{(\gamma-1)}\left|g\big(u,y_n(u)\big)-g\big(u,y(u)\big)\right|du \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_\tau^T (u-\tau)^{(\gamma-1)}\left|g\big(u,y_n(u)\big)-g\big(u,y(u)\big)\right|du \\
&\le \left(\frac{T^\gamma}{2\Gamma(\gamma)}+\frac{\tau^\gamma}{\Gamma(\gamma+1)}+\frac{(T-\tau)^\gamma}{\Gamma(\gamma+1)}\right)\left\|g\big(\cdot,y_n(\cdot)\big)-g\big(\cdot,y(\cdot)\big)\right\| \\
&\le \left(\frac{T^\gamma}{2\Gamma(\gamma)}+\frac{2T^\gamma}{\Gamma(\gamma+1)}\right)\left\|g\big(\cdot,y_n(\cdot)\big)-g\big(\cdot,y(\cdot)\big)\right\|.
\end{aligned}
$$

Since g is continuous function, we have

$$\|Ay_n - Ay\| \to 0 \quad \text{as } n \to \infty.$$

Step 2. A maps bounded sets into bounded sets in $C(J)$.

For each $y \in \Omega_\eta = \{y \in C(J) : \|y\| \leq \eta\}$ and $\tau \in J$, we get

$$\begin{aligned} \bigl|(Ay)(\tau)\bigr| &\leq \frac{T}{2\Gamma(\gamma-1)} \int_0^T (T-u)^{\gamma-2} \bigl|g\bigl(u, y(u)\bigr)\bigr|\,du \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_0^\tau (\tau-u)^{(\gamma-1)} \bigl|g\bigl(u, y(u)\bigr)\bigr|\,du \\ &\quad + \frac{1}{\Gamma(\gamma)} \int_\tau^T (u-\tau)^{(\gamma-1)} \bigl|g\bigl(u, y(u)\bigr)\bigr|\,du \\ &\leq \biggl(\frac{T^\gamma}{2\Gamma(\gamma)} + \frac{\tau^\gamma}{\Gamma(\gamma+1)} + \frac{(T-\tau)^\gamma}{\Gamma(\gamma+1)}\biggr) L_2\bigl(1+\eta^\sigma\bigr) \\ &\leq \biggl(\frac{T^\gamma}{2\Gamma(\gamma)} + \frac{2T^\gamma}{\Gamma(\gamma+1)}\biggr) L_2\bigl(1+\eta^\sigma\bigr) := \iota. \end{aligned}$$

Then ι is a constant and

$$\|Ay\| \leq \iota,$$

which implies that A maps bounded sets into bounded sets.

Step 3. A maps bounded sets into equicontinuous sets in $C(J)$.

Let Ω_η be a bounded set of $C(J)$ as in Step 2, and let $y \in \Omega_\eta$. For each $\tau \in J$, we can estimate the derivative $(Ay)'(\tau)$:

$$\begin{aligned} \bigl|(Ay)'(\tau)\bigr| &\leq \frac{1}{\Gamma(\gamma-1)} \int_0^\tau (\tau-u)^{(\gamma-2)} \bigl|g\bigl(u, y(u)\bigr)\bigr|\,du \\ &\quad + \frac{1}{\Gamma(\gamma-1)} \int_\tau^T (u-\tau)^{(\gamma-2)} \bigl|g\bigl(u, y(u)\bigr)\bigr|\,du \\ &\leq \biggl(\frac{\tau^{\gamma-1}}{\Gamma(\gamma)} + \frac{(T-\tau)^{\gamma-1}}{\Gamma(\gamma)}\biggr) L_2\bigl(1+\eta^\sigma\bigr) \\ &\leq \frac{2T^{\gamma-1}L_2(1+\eta^\sigma)}{\Gamma(\gamma)} := \kappa. \end{aligned}$$

Hence, let $\tau', \tau'' \in J$, $\tau' < \tau''$, we have

$$\bigl|(Ay)\bigl(\tau''\bigr) - (Ay)\bigl(\tau'\bigr)\bigr| = \int_{\tau'}^{\tau''} \bigl|(Ay)'(u)\bigr|\,du \leq \kappa\bigl(\tau'' - \tau'\bigr).$$

So $A(\Omega_\eta)$ is equicontinuous in $C(J)$. As a consequence of Steps 1 to 3 together with the Arzela–Ascoli theorem, we can conclude that $A : \Omega_\eta \to \Omega_\eta$ is continuous and completely continuous.

Step 4. A priori bounds. Lastly, we prove that the set

$$\omega(F) = \bigl\{y \in X : y = \lambda F y \text{ for some } \lambda \in (0,1)\bigr\}$$

is bounded. Let $y = \lambda Fy$ for some $\lambda \in (0,1)$. For each $\tau \in J$, we have

$$\begin{aligned}
\left|y(\tau)\right| &\le \frac{T}{2\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}\left|g\left(u,y(u)\right)\right|du \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{(\gamma-1)}\left|g\left(u,y(u)\right)\right|du \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_\tau^T (u-\tau)^{(\gamma-1)}\left|g\left(u,y(u)\right)\right|du \\
&\le \left(\frac{T^\gamma}{2\Gamma(\gamma)} + \frac{2T^\gamma}{\Gamma(\gamma+1)}\right)L_2 + \frac{TL_2}{2\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}\left|y(u)\right|^\sigma du \\
&\quad + \frac{L_2}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{(\gamma-1)}\left|y(u)\right|^\sigma du + \frac{L_2}{\Gamma(\gamma)}\int_\tau^T (u-\tau)^{(\gamma-1)}\left|y(u)\right|^\sigma du.
\end{aligned}$$

According to Lemma 2.4, there exists a $M_1^* > 0$ such that

$$\|y\| \le M_1^*.$$

As a consequence of the Schaefer fixed-point theorem, we deduce that A has a fixed point which is a solution of BVP (1) by Lemma 2.5. □

Theorem 3.3 *Assume that*

(H_3) *there exist $\varphi \in C(J)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that $|g(\tau,y)| \le \varphi(\tau)\psi(|y|)$ for $\tau \in J$ and $y \in R$.*

Then BVP (1) has at least one solution on J provide that there exists a constant $M_2^* > 0$ such that

$$\left(\frac{T^\gamma}{2\Gamma(\gamma)} + \frac{2T^\gamma}{\Gamma(\gamma+1)}\right)\frac{\varphi^*\psi(M_2^*)}{M_2^*} < 1, \tag{10}$$

where $\varphi^* = \sup\{\varphi(\tau) : \tau \in J\}$.

Proof Let $V_{M_2^*} = \{y \in C(J) : \|y\| \le M_2^*\}$, then $V_{M_2^*}$ is a closed, bounded and convex set.

For any $y \in V_{M_2^*}$, applying the conditions (H_3) and (10), we have

$$\begin{aligned}
\left|(Ay)(\tau)\right| &\le \frac{T}{2\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}\left|g\left(u,y(u)\right)\right|du \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{(\gamma-1)}\left|g\left(u,y(u)\right)\right|du \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_\tau^T (u-\tau)^{(\gamma-1)}\left|g\left(u,y(u)\right)\right|du \\
&\le \frac{T}{2\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}\varphi(u)\psi\left(\left|y(u)\right|\right)du \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^\tau (\tau-u)^{(\gamma-1)}\varphi(u)\psi\left(\left|y(u)\right|\right)du \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_\tau^T (u-\tau)^{(\gamma-1)}\varphi(u)\psi\left(\left|y(u)\right|\right)du
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\frac{T^{\gamma}}{2\Gamma(\gamma)}+\frac{\tau^{\gamma}}{\Gamma(\gamma+1)}+\frac{(T-\tau)^{\gamma}}{\Gamma(\gamma+1)}\right)\varphi^{*}\psi\left(\|y\|\right)\\
&\le \left(\frac{T^{\gamma}}{2\Gamma(\gamma)}+\frac{2T^{\gamma}}{\Gamma(\gamma+1)}\right)\varphi^{*}\psi\left(M_2^{*}\right)\\
&< M_2^{*}.
\end{aligned}
$$

Then the operator $A : V_{M_2^*} \to C(J)$ is a continuous and completely continuous. From the choice of $V_{M_2^*}$, there is no $y \in \partial V_{M_2^*}$ such that $y = \lambda Ay$ for some $0 < \lambda < 1$. As a consequence of Lemma 2.6 (the Leray–Schauder fixed point theorem), we deduce that A has a fixed point y in $V_{M_2^*}$ which is a solution of BVP (1). □

Theorem 3.4 *Assume that*

(H$_4$) *there exist a $\nu \in (0, \gamma - 1)$ and a real function $\mu \in L^{\frac{1}{\nu}}(J, R^+)$ such that $|g(\tau, y)| \le \mu(\tau)$, for $\tau \in J$ and $y \in R$.*

Then BVP (1) *has at least one solution on J.*

Proof Let us fix

$$
r \ge \|\mu\|_{L^{\frac{1}{\nu}}}\left[\frac{T}{2\Gamma(\gamma-1)}\left(\frac{1-\nu}{\gamma-\nu-1}T^{\frac{\gamma-\nu-1}{1-\nu}}\right)^{1-\nu}+\frac{2}{\Gamma(\gamma)}\left(\frac{1-\nu}{\gamma-\nu}T^{\frac{\gamma-\nu}{1-\nu}}\right)^{1-\nu}\right], \tag{11}
$$

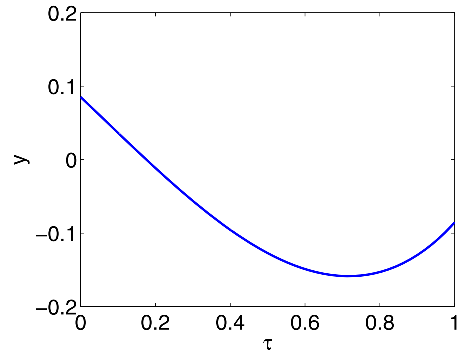
and consider $V_r = \{y \in C(J) : \|y\| \le r\}$. For any $y \in V_r$, applying condition (H$_4$), the Hölder inequality and (11), we have

$$
\begin{aligned}
\left|(Ay)(\tau)\right| &\le \frac{T}{2\Gamma(\gamma-1)}\int_0^T (T-u)^{\gamma-2}\mu(u)\,du+\frac{1}{\Gamma(\gamma)}\int_0^{\tau}(\tau-u)^{(\gamma-1)}\mu(u)\,du\\
&\quad+\frac{1}{\Gamma(\gamma)}\int_{\tau}^{T}(u-\tau)^{(\gamma-1)}\mu(u)\,du\\
&\le \frac{T}{2\Gamma(\gamma-1)}\left(\int_0^T (T-u)^{\frac{\gamma-2}{1-\nu}}\,du\right)^{1-\nu}\left(\int_0^T \left(\mu(u)\right)^{\frac{1}{\nu}}\,du\right)^{\nu}\\
&\quad+\frac{1}{\Gamma(\gamma)}\left(\int_0^{\tau}(\tau-u)^{\frac{\gamma-1}{1-\nu}}\,du\right)^{1-\nu}\left(\int_0^{\tau}\left(\mu(u)\right)^{\frac{1}{\nu}}\,du\right)^{\nu}\\
&\quad+\frac{1}{\Gamma(\gamma)}\left(\int_{\tau}^{T}(u-\tau)^{\frac{\gamma-1}{1-\nu}}\,du\right)^{1-\nu}\left(\int_{\tau}^{T}\left(\mu(u)\right)^{\frac{1}{\nu}}\,du\right)^{\nu}\\
&\le \|\mu\|_{L^{\frac{1}{\nu}}}\left[\frac{T}{2\Gamma(\gamma-1)}\left(\frac{1-\nu}{\gamma-\nu-1}T^{\frac{\gamma-\nu-1}{1-\nu}}\right)^{1-\nu}\right.\\
&\quad\left.+\frac{1}{\Gamma(\gamma)}\left(\frac{1-\nu}{\gamma-\nu}\tau^{\frac{\gamma-\nu}{1-\nu}}\right)^{1-\nu}+\frac{1}{\Gamma(\gamma)}\left(\frac{1-\nu}{\gamma-\nu}(T-\tau)^{\frac{\gamma-\nu}{1-\nu}}\right)^{1-\nu}\right]\\
&\le \|\mu\|_{L^{\frac{1}{\nu}}}\left[\frac{T}{2\Gamma(\gamma-1)}\left(\frac{1-\nu}{\gamma-\nu-1}T^{\frac{\gamma-\nu-1}{1-\nu}}\right)^{1-\nu}+\frac{2}{\Gamma(\gamma)}\left(\frac{1-\nu}{\gamma-\nu}T^{\frac{\gamma-\nu}{1-\nu}}\right)^{1-\nu}\right]\\
&\le r.
\end{aligned}
$$

Then $A : V_r \to V_r$.

The proof of that A is completely continuous is similar to that of Theorem 3.1, and we do not give the details.

Figure 1 The numerical solution of (12)



As a consequence of Lemma 2.7 (the Schauder fixed point theorem), it can be deduced that A has a fixed point in V_r, which implies that (1) has at least one solution on J. □

4 Example

The applications of our main results will be illustrated by the following examples.

Example 4.1 Consider the following BVP:

$$\begin{cases} {}_0^{\mathrm{RC}}D_T^{\frac{11}{6}} y(\tau) = \tau, \quad 0 \le \tau \le 1, \\ y(0) + y(1) = 0, \qquad y'(0) + y'(1) = 0. \end{cases} \tag{12}$$

Where $g(\tau, y) = \tau$, let $L_1 = 0$, then the conditions (H_1) and (9) are satisfied. According to Theorem 3.1, the unique solution $y(\tau)$ of BVP (12) exists. Numerical experiment has been implemented in a MATLAB code. The numerical solution of (12) is displayed in Fig. 1.

Example 4.2 Consider the following BVP:

$$\begin{cases} {}_0^{\mathrm{RC}}D^{\frac{3}{2}} y(\tau) = \frac{|y(\tau)|^{\frac{1}{3}}}{(1+e^\tau)(1+|y(\tau)|)}, \quad 0 \le \tau \le 1, \\ y(0) + y(1) = 0, \qquad y'(0) + y'(1) = 0. \end{cases} \tag{13}$$

Here $g(\tau, y) = \frac{|y(\tau)|^{\frac{1}{3}}}{(1+e^\tau)(1+|y(\tau)|)}$, $\gamma = \frac{3}{2}$ and $T = 1$.

Let $\sigma = \frac{1}{3}$, $\zeta = \frac{3}{2}$, then $\sigma \in (0, 1 - \frac{1}{\zeta})$, $\zeta(\gamma - 2) + 1 = \frac{1}{4} > 0$ and

$$g(\tau, y) = \frac{|y(\tau)|^{\frac{1}{3}}}{(1 + e^\tau)(1 + |y(\tau)|)} \le \frac{1 + |y(\tau)|^{\frac{1}{3}}}{2(1 + |y(\tau)|)} \le \frac{1}{2}\bigl(1 + \bigl|y(\tau)\bigr|^{\frac{1}{3}}\bigr),$$

which implies that the condition (H_2) is satisfied. According to Theorem 3.2, BVP (13) has at least one solution on $[0, 1]$.

Let $\varphi(\tau) = \frac{1}{(1+e^\tau)}$, $\psi(|y|) = |y|^{\frac{1}{3}}$. Obviously, $g(\tau, y) \le \frac{1}{(1+e^\tau)}|y|^{\frac{1}{3}} = \varphi(\tau)\psi(|y|)$ and $\varphi^* = \frac{1}{2}$ which implies that the condition (H_3) is satisfied. Let $M_2^* = 27$, then

$$\left(\frac{T^\gamma}{2\Gamma(\gamma)} + \frac{2T^\gamma}{\Gamma(\gamma + 1)}\right)\frac{\varphi^*\psi(M_2^*)}{M_2^*} = \left(\frac{1}{2\Gamma(\frac{3}{2})} + \frac{2}{\Gamma(\frac{5}{2})}\right)\frac{\frac{1}{2} \times 3}{27} = \frac{11}{54\sqrt{\pi}} < 1,$$

which implies that the inequity (10) is satisfied. According to Theorem 3.3, BVP (13) has at least one solution on $[0,1]$.

Example 4.3 Consider the following BVP:

$$\begin{cases} {}_0^{\mathrm{RC}}D^{\frac{3}{2}}y(\tau) = \frac{|y(\tau)|}{(1+\tau)^2(1+|y(\tau)|)}, & 0 \le \tau \le 1, \\ y(0) + y(1) = 0, & y'(0) + y'(1) = 0. \end{cases} \tag{14}$$

Here $g(\tau, y) = \frac{|y(\tau)|}{(1+\tau)^2(1+|y(\tau)|)}$, $\gamma = \frac{3}{2}$ and $T = 1$.

$$g(\tau, y) = \frac{|y(\tau)|}{(1+\tau)^2(1+|y(\tau)|)} \le \frac{1}{(1+\tau)^2} := \mu(\tau).$$

Let $\nu = \frac{1}{3}$, then $\nu \in (0, \gamma - 1)$ and $\mu(\tau) = \frac{1}{(1+\tau)^2} \in L^3([0,1], R^+)$, which implies that the condition (H_4) is satisfied. According to Theorem 3.4, BVP (14) has at least one solution on $[0,1]$.

Acknowledgements
The authors would like to thank the anonymous reviewers and the editor for their valuable comments and suggestions, which helped improve the manuscript.

Funding
The present investigation was supported in part by the NNSF-China (Grant No. 11471278), and the Applied Characteristic Discipline in Xiangnan University.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors made equal contributions. All authors read and approved the final manuscript.

Publisher's Note

Received: 9 April 2018 Accepted: 30 January 2019 Published online: 21 March 2019

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