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Existence and asymptotic stability of periodic solutions for impulsive delay evolution equations

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Abstract

This paper we devote to considering the periodic problem for the impulsive evolution equations with delay in Banach space. By using operator semigroup theory and the fixed point theorem, we establish some new existence theorems of periodic mild solutions for the equations. In addition, with the aid of an integral inequality with impulsive and delay, we present essential conditions on the nonlinear and impulse functions to guarantee that the equations have an asymptotically stable ω -periodic mild solution.

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1 Introduction

Let X be a real Banach space with norm $\|\cdot\|$, $\mathcal{L}(X)$ stand for the Banach space of all bounded linear operators from X to X equipped with its natural topology. Let $r > 0$ be a constant, we denote by $PC([-r, 0], X)$ the Banach space of piecewise continuous functions from $[-r, 0]$ to X with finite points of discontinuity where functions are left continuous and have the right limits, with the sup-norm $\|\phi\|_{Pr} = \sup_{s \in [-r, 0]} \|\phi(s)\|$.

In this paper, we consider the periodic problem for the impulsive delay evolution equation in Banach space X ,

$$\begin{cases} u'(t) + Au(t) = F(t, u(t), u_t), & t \in \mathbb{R}, t \neq t_i, \\ \Delta u(t_i) = I_i(u(t_i)), & i \in \mathbb{Z}, \end{cases} \quad (1.1)$$

where $A : D(A) \subset X \rightarrow X$ is a closed linear operator and $-A$ generates a C_0 -semigroup $T(t)$ ($t \geq 0$) in X ; $F : \mathbb{R} \times X \times PC([-r, 0], X) \rightarrow X$ is a nonlinear mapping which is ω -periodic in t ; $u_t \in PC([-r, 0], X)$ is the history function defined by $u_t(s) = u(t + s)$ for $s \in [-r, 0]$; $p \in \mathbb{N}$ denotes the number of impulsive points between 0 and ω , $0 < t_1 < t_2 < \cdots < t_p < \omega < t_{p+1}$ are given numbers satisfying $t_{p+i} = t_i + \omega$ ($i \in \mathbb{Z}$); $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ represents the jump of the function u at t_i , $u(t_i^+)$ and $u(t_i^-)$ are the right and left limits of $u(t)$ at t_i , respectively; $I_i : X \rightarrow X$ ($i \in \mathbb{Z}$) are continuous functions satisfying $I_{i+p} = I_i$.

The theory of partial differential equations with delays has an extensive physical background and leads to a realistic mathematical model, and it has undergone a rapid development in the last 50 years. The evolution equations with delay are more realistic than the equations without delay in describing numerous phenomena observed in nature, hence the numerous properties of their solutions have been studied; see [7, 22] and the references therein for more comments.

One of the important research directions related to the asymptotic behavior of the solutions for the evolution equations with delay is to find conditions for the existence and stability of ω -periodic solutions in the case that the nonlinear mapping is ω -periodic function in t . In the last few decades, the existence and asymptotic stability of periodic solutions have been investigated by some authors (see [5, 8–10, 15–17, 23] and the references therein). In [5], under the assumption that the solutions of the associated homotopy equations were uniformly bounded, Burton and Zhang obtained the existence of periodic solutions of an abstract delay evolution equation. In [23], Xiang and Ahmed showed an existence result of periodic solution to the delay evolution equations in Banach spaces under the assumption that the corresponding initial value problem had a prior estimate. In [15–17], Liu studied periodic solutions by using bounded solutions or ultimate bounded solutions for delay evolution equations in Banach spaces. In [8], Huy and Dang studied the existence, uniqueness and stability of periodic solutions to a partial functional differential equation in Banach space in the case that the nonlinear function satisfied Lipschitz-type condition. Specially, in [10], Li discussed the existence and asymptotic stability of periodic solutions to the evolution equation with multiple delays in a Hilbert space. By using the analytic semigroup theory and the integral inequality with delays, the author obtained the essential conditions on the nonlinearity F to guarantee that the equation has ω -periodic solutions or an asymptotically stable ω -periodic solution.

On the other hand, it is well known that an impulsive evolution equation has an extensive physical, chemical, biological, engineering background and realistic mathematical model, and hence has been emerging as an important area of investigation in the last few decades. Since the end of the previous century, the theory of impulsive evolution equation in Banach space has been largely developed (see [1, 2, 6, 12, 13, 24] and the references therein). We would like to mention that Liang et al. [12] studied the periodic solutions to a kind of impulsive evolution equation with delay in Banach spaces. In the case that the nonlinear function satisfied Lipschitz conditions, the authors found that the evolution equation has a periodic solution by using the ultimate boundedness of solutions and Horn's fixed point theorem. Recently, in [11, 14], Liang et al. studied nonautonomous evolutionary equations with time delay and being impulsive. With the nonlinear term being continuous and Lipschitzian, they proved the existence theorem for periodic mild solutions to the nonautonomous delay evolution equations.

Although there have been many meaningful results on the delay or impulsive evolution equation periodic problem in Banach space, to the best of our knowledge, these results have a relatively large limitation. First of all, the most popular approach is the use of ultimate boundedness of solutions and the compactness of Poincaré map realized through some compact embeddings. However, in some concrete applications, it is difficult to choose appropriate initial conditions to guarantee the boundedness of the solution. Secondly, we observe that the most popular condition imposed on the nonlinear term F is its Lipschitz-type condition. In fact, for equations arising in complicated reaction–diffusion

processes, the nonlinear function F represents the source of material or population, which depends on time in diversified manners in many contexts. Thus, we may not hope to have the Lipschitz-type condition of F . Finally, there are few papers studying the asymptotically stable of periodic solutions for the impulsive evolution equations with delay.

Motivated by the papers mentioned above, we consider the periodic problem for the impulsive delay evolution equation (1.1) in Banach space. By using a periodic extension and the fixed point theorem, we study the existence of ω -periodic mild solutions for Eq. (1.1). It is worth mentioning that the assumption of prior boundedness of solutions is not employed and the nonlinear term F satisfies some growth condition, which is weaker than Lipschitz-type condition. On the other hand, by means of an integral inequality with impulsive and delay, we present the asymptotic stability result for Eq. (1.1), which will make up the research in this area blank.

The rest of this paper is organized as follows. In Sect. 2, we collect some well-known definitions and notions, and then provide preliminary results which will be used throughout this paper. In Sect. 3, we apply the operator semigroup theory to find the ω -periodic mild solutions for Eq. (1.1) and in Sect. 4, by strengthening the condition, we obtain the global asymptotic stability theorems for Eq. (1.1). In the last section, we give an example to illustrate the applicability of abstract results obtained in Sects. 3 and 4.

2 Preliminaries

Throughout this paper, we assume that X is a Banach space with norm $\|\cdot\|$.

Now, we recall some notions and properties of operator semigroups, which are essential for us. For the detailed theory of operator semigroups, we refer to [18]. Assume that $A : D(A) \subset X \rightarrow X$ is a closed linear operator and $-A$ is the infinitesimal generator of a C_0 -semigroup $T(t)$ ($t \geq 0$) in X . Then there exist $M > 0$ and $\nu \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\nu t}, \quad t \geq 0. \quad (2.1)$$

Let

$$\nu_0 = \inf\{\nu \in \mathbb{R} \mid \text{there exists } M > 0 \text{ such that } \|T(t)\| \leq Me^{\nu t}, \forall t \geq 0\},$$

then ν_0 is called the growth exponent of the semigroup $T(t)$ ($t \geq 0$). If $\nu_0 < 0$, then $T(t)$ ($t \geq 0$) is called an exponentially stable C_0 -semigroup.

If C_0 -semigroup $T(t)$ is continuous in the uniform operator topology for every $t > 0$ in X , it is well known that ν_0 can also be determined by $\sigma(A)$ (the resolvent set of A),

$$\nu_0 = -\inf\{\operatorname{Re} \lambda \mid \lambda \in \sigma(A)\}, \quad (2.2)$$

where $-A$ is the infinitesimal generator of C_0 -semigroup $T(t)$ ($t \geq 0$). We know that $T(t)$ ($t \geq 0$) is continuous in the uniform operator topology for $t > 0$ if $T(t)$ ($t \geq 0$) is compact semigroup (see [21]).

Let J denote the infinite interval $[0, +\infty)$, from [18], it follows that when $x_0 \in D(A)$ and $h \in C^1(J, X)$, the initial value problem of the linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J, \\ u(0) = x_0, \end{cases} \quad (2.3)$$

has a unique classical solution $u \in C^1(J, X) \cap C(J, X_1)$, which can be expressed by

$$u(t) = T(t)x_0 + \int_0^t T(t-s)h(s) ds, \quad (2.4)$$

where $X_1 = D(A)$ is Banach space with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A \cdot\|$. Generally, for $x_0 \in X$ and $h \in C(J, X)$, the function u given by (2.4) belongs to $C(J, X)$ and it is called a mild solution of the linear evolution equation (2.3).

Let $\tilde{D} = \{t_1, t_2, \dots, t_p\} \subset [0, \omega]$, where $p \in \mathbb{N}$ denotes the number of impulsive points between $[0, \omega]$. Write

$$PC([0, \omega], X) := \left\{ u : [0, \omega] \rightarrow X \mid u \text{ is continuous at } t \in [0, \omega] \setminus \tilde{D}, \right. \\ \left. u \text{ is continuous from left and has right limits at } t \in \tilde{D} \right\}$$

and

$$PC_\omega(\mathbb{R}, X) := \left\{ u : \mathbb{R} \rightarrow X \mid u \text{ is continuous at } t \in \mathbb{R} \setminus \{t_i\}, u \text{ is continuous from} \right. \\ \left. \text{left and has right limits at } t_i, i \in \mathbb{Z}, u(t + \omega) = u(t) \text{ for } t \in \mathbb{R} \right\}.$$

It is clear that the restriction of $PC_\omega(\mathbb{R}, X)$ on $[0, \omega]$ is $PC([0, \omega], X)$. Set

$$\|u\|_{PC} = \max \left\{ \sup_{t \in [0, \omega]} \|u(t+0)\|, \sup_{t \in [0, \omega]} \|u(t-0)\| \right\}.$$

It can be seen that equipped with the norm $\|\cdot\|_{PC}$, $PC_\omega(\mathbb{R}, X)$ (or $PC([0, \omega], X)$) is a Banach space.

Given $h \in PC_\omega(\mathbb{R}, X)$, we consider the existence of ω -periodic mild solution for the linear impulsive evolution equation in X

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in \mathbb{R}, t \neq t_i, \\ \Delta u(t_i) = v_i, & i \in \mathbb{Z}, \end{cases} \quad (2.5)$$

where $v_i \in X$ satisfy $v_{i+p} = v_i$ ($i \in \mathbb{Z}$).

Lemma 2.1 *Let $-A$ generate an exponentially stable C_0 -semigroup $T(t)$ ($t \geq 0$) in X and v_0 be a growth index of the semigroup $T(t)$ ($t \geq 0$). Then the linear impulsive evolution equation (2.5) exists a unique ω -periodic mild solution $u := Ph \in PC_\omega(\mathbb{R}, X)$. Furthermore, the operator $P : PC_\omega(\mathbb{R}, X) \rightarrow PC_\omega(\mathbb{R}, X)$ is a bounded linear operator.*

Proof Firstly, for $x_0 \in X$, we consider the existence of mild solution for the initial value problem of the linear impulsive evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J \setminus \{t_1, t_2, \dots\}, \\ \Delta u(t_i) = v_i, & i = 1, 2, \dots, \\ u(0) = x_0. \end{cases} \quad (2.6)$$

Let $t_0 = 0$, $v_0 = \theta$ and $J_i = (t_{i-1}, t_i]$, $i = 1, 2, \dots$. If $u \in PC(J, X)$ is a mild solution of Eq. (2.6), then the restriction of u on J_i satisfies the initial value problem of linear evolution equation without impulse

$$\begin{cases} u'(t) + Au(t) = h(t), & t_{i-1} < t \leq t_i, \\ u(t_{i-1}^+) = u(t_{i-1}) + v_{i-1}, & i = 1, 2, \dots \end{cases} \quad (2.7)$$

Thus, the initial value problem (2.7) exists a unique mild solution u on J_i which can be expressed by

$$u(t) = T(t - t_{i-1})(u(t_{i-1}) + v_{i-1}) + \int_{t_{i-1}}^t T(t-s)h(s)ds, \quad t \in J_i. \quad (2.8)$$

Iterating successively in the above equation with $u(t_j)$, $j = i-1, i-2, \dots, 1$, we can verify that u satisfies

$$u(t) = T(t)x_0 + \int_0^t T(t-s)h(s)ds + \sum_{0 \leq t_i < t} T(t-t_i^+)v_i, \quad t \in J. \quad (2.9)$$

Inversely, we can see that the function $u \in PC(J, X)$ defined by (2.8) is a mild solution of the initial value problem (2.6).

Secondly, we demonstrate that the linear impulsive evolution equation (2.5) exists a unique ω -periodic mild solution. It is clear that the ω -periodic mild solution of Eq. (2.5) restricted on J is the mild solution of the initial value problem (2.6) with the initial value

$$x_0 := u(0) = u(\omega),$$

namely

$$(I - T(\omega))x_0 = \int_0^\omega T(\omega-s)h(s)ds + \sum_{i=1}^p T(\omega-t_i^+)v_i. \quad (2.10)$$

For any $v \in (0, |v_0|)$, there exists $M > 0$ such that

$$\|T(t)\| \leq Me^{-vt} \leq M, \quad t \geq 0. \quad (2.11)$$

In X , define an equivalent norm $|\cdot|$ by $|x| = \sup_{0 \leq s \leq t} \|e^{vs}T(t-s)x\|$, then $\|x\| \leq |x| \leq M\|x\|$. By $|T(t)|$ we denote the norm of $T(t)$ in $(X, |\cdot|)$, it is easy to obtain $|T(t)| \leq e^{-vt}$ for $t \geq 0$, which implies that $(I - T(\omega))$ has bounded inverse operator $(I - T(\omega))^{-1}$, and

$$|(I - T(\omega))^{-1}| \leq \frac{1}{1 - e^{-v\omega}}. \quad (2.12)$$

Therefore, there exists a unique initial value

$$x_0 = (I - T(\omega))^{-1} \left(\int_0^\omega T(\omega-s)h(s)ds + \sum_{i=1}^p T(\omega-t_i^+)v_i \right) := B(h), \quad (2.13)$$

such that the mild solution u of Eq. (2.6) given by (2.8) satisfies the periodic boundary condition $u(0) = u(\omega) = x_0$.

For $t \in J$, by the semigroup properties of $T(t)$, we have

$$\begin{aligned}
 u(t + \omega) &= T(t + \omega)u(0) + \int_0^{t+\omega} T(t + \omega - s)h(s) ds + \sum_{0 < t_i < t+\omega} T(t + \omega - t_i^+)v_i \\
 &= T(t + \omega)u(0) + \int_0^\omega T(t + \omega - s)h(s) ds + \int_\omega^{t+\omega} T(t + \omega - s)h(s) ds \\
 &\quad + \sum_{i=1}^p T(t + \omega - t_i^+)v_i + \sum_{\omega < t_i < t+\omega} T(t + \omega - t_i^+)v_i \\
 &= T(t) \left(T(\omega)u(0) + \int_0^\omega T(\omega - s)h(s) ds + \sum_{i=1}^p T(\omega - t_i^+)v_i \right) \\
 &\quad + \int_0^t T(t - s)h(s) ds + \sum_{0 < t_i < t} T(t - t_i^+)v_i \\
 &= T(t)u(0) + \int_0^t T(t - s)h(s) ds + \sum_{0 < t_i < t} T(t - t_i^+)v_i \\
 &= u(t).
 \end{aligned}$$

Therefore, the ω -periodic extension of u on \mathbb{R} is a unique ω -periodic mild solution of Eq. (2.5).

Finally, by (2.8) and (2.13), we obtain

$$\begin{aligned}
 u(t) &= T(t)x_0 + \int_0^t T(t - s)h(s) ds + \sum_{0 < t_i < t} T(t - t_i^+)v_i \\
 &= T(t)(I - T(\omega))^{-1} \left(\int_0^\omega T(\omega - s)h(s) ds + \sum_{i=1}^p T(\omega - t_i^+)v_i \right) \\
 &\quad + \int_0^t T(t - s)h(s) ds + \sum_{0 < t_i < t} T(t - t_i^+)v_i \\
 &= (I - T(\omega))^{-1} \int_{t-\omega}^t T(t - s)h(s) ds \\
 &\quad + (I - T(\omega))^{-1} \sum_{i=1}^p T(t + \omega - t_i)v_i + \sum_{0 < t_i < t} T(t - t_i)v_i \\
 &= (I - T(\omega))^{-1} \int_{t-\omega}^t T(t - s)h(s) ds \\
 &\quad + (I - T(\omega))^{-1} \sum_{t-\omega \leq t_i < t} T(t - t_i)v_i \\
 &:= Ph(t),
 \end{aligned}$$

thus, it is easy to prove that the solution operator $P : PC_\omega(\mathbb{R}, X) \rightarrow PC_\omega(\mathbb{R}, X)$ is a bounded linear operator. This completes the proof of Lemma 2.1. \square

3 Existence and uniqueness

In this section, we discuss the existence of ω -periodic mild solution to Eq. (1.1) for the case that the semigroup $T(t)$ ($t \geq 0$) generated by $-A$ is a compact semigroup, which implies that $T(t)$ is a compact operator for any $t > 0$.

Now, we are in a position to state and prove our main results of this section.

Theorem 3.1 *Let X be a Banach space, $-A$ generates an exponentially stable compact semigroup $T(t)$ ($t \geq 0$) in X . Assume that $F : \mathbb{R} \times X \times PC([-r, 0], X) \rightarrow X$ is continuous and $F(t, \cdot, \cdot)$ is ω -periodic in t , $I_k \in C(X, X)$ ($k \in \mathbb{Z}$) satisfies $I_{k+p} = I_k$, p is the number of impulsive points between $[0, \omega]$. If the following conditions are satisfied:*

(H1) *there exist nonnegative constants c_0, c_1, c_2 such that*

$$\|F(t, x, \phi)\| \leq c_0 + c_1 \|x\| + c_2 \|\phi\|_{Pr}, \quad t \in \mathbb{R}, x \in X, \phi \in PC([-r, 0], X),$$

(H2) *for every I_k , $I_k(\theta) = \theta$, and there exist positive constants a_k such that*

$$\|I_k(x) - I_k(y)\| \leq a_k \|x - y\|, \quad a_{k+p} = a_k, \quad x, y \in X, k \in \mathbb{Z},$$

(H3) $(c_1 + c_2) + \frac{1}{\omega} \sum_{k=1}^p a_k < \frac{|v_0|}{M}$,

then Eq. (1.1) has at least one ω -periodic mild solution u .

Proof Define an operator $Q : PC_\omega(\mathbb{R}, X) \rightarrow PC_\omega(\mathbb{R}, X)$ by

$$\begin{aligned} (Qu)(t) &= (I - T(\omega))^{-1} \int_{t-\omega}^t T(t-s)f(s, u(s), u_s) ds \\ &\quad + (I - T(\omega))^{-1} \sum_{t-\omega \leq t_k < t} T(t-t_k)I_k(u(t_k)), \quad t \in \mathbb{R}. \end{aligned} \quad (3.1)$$

From Lemma 2.1 and the definition of Q , we easily see that the ω -periodic mild solution of Eq. (1.1) is equivalent to the fixed point of Q . In the following, we will prove Q has a fixed point by applying the fixed point theorem.

It is not difficult to prove that Q is continuous on $PC_\omega(\mathbb{R}, X)$. In fact, let $\{u_n\} \subset PC_\omega(\mathbb{R}, X)$ be a sequence such that $u_n \rightarrow u \in PC_\omega(\mathbb{R}, X)$ as $n \rightarrow \infty$, hence, for every $t \in \mathbb{R}$, we have $u_n(t) \rightarrow u(t) \in X$ and $u_{n,t} \rightarrow u_t \in PC([-r, 0], X)$ as $n \rightarrow \infty$. From $F : \mathbb{R} \times X \times PC([-r, 0], X) \rightarrow X$ is continuous, and $I_k \in C(X, X)$ ($k \in \mathbb{Z}$), it follows that

$$F(t, u_n(t), u_{n,t}) \rightarrow F(t, u(t), u_t), \quad n \rightarrow \infty, \quad (3.2)$$

and

$$I_k(u_n(t_k)) \rightarrow I_k(u(t_k)), \quad n \rightarrow \infty. \quad (3.3)$$

By (3.1)–(3.3) and the Lebesgue dominated convergence theorem, for every $t \in \mathbb{R}$, we have

$$\begin{aligned} &\|(Qu_n)(t) - (Qu)(t)\| \\ &\leq \|(I - T(\omega))^{-1}\| \cdot \left\| \sum_{t-\omega \leq t_k < t} T(t-t_k)(I_k(u_n(t_k)) - I_k(u(t_k))) \right\| \end{aligned}$$

$$\begin{aligned}
& + \|(I - T(\omega))^{-1}\| \cdot \left\| \int_{t-\omega}^{\omega} T(t-s) \cdot (F(s, u_n(s), u_{n,s}) - F(s, u(s), u_s)) ds \right\| \\
& \leq C \left(\sum_{t-\omega \leq t_k < t} \|T(t-t_k)\| \cdot \|I_k(u_n(t_k)) - I_k(u(t_k))\| \right. \\
& \quad \left. + \int_{t-\omega}^t \|T(t-s)\| \cdot \|F(s, u_n(s), u_{n,s}) - F(s, u(s), u_s)\| ds \right) \\
& \rightarrow 0 \quad (n \in \infty),
\end{aligned}$$

which implies that $Q : PC_{\omega}(\mathbb{R}, X) \rightarrow PC_{\omega}(\mathbb{R}, X)$ is continuous, where $C = \|(I - T(\omega))^{-1}\|$, by the proof of Lemma 2.1, one can obtain

$$\|(I - T(\omega))^{-1}\| \leq |(I - T(\omega))^{-1}| \leq \frac{1}{1 - e^{v_0\omega}}.$$

For any $R > 0$, let

$$\overline{\mathcal{Q}}_R = \{u \in PC_{\omega}(\mathbb{R}, X) \mid \|u\|_{PC} \leq R\}. \quad (3.4)$$

Note that $\overline{\mathcal{Q}}_R$ is a closed ball in $PC_{\omega}(\mathbb{R}, X)$ with center θ and radius r . Now, we show that there is a positive constant R such that $Q(\overline{\mathcal{Q}}_R) \subset \overline{\mathcal{Q}}_R$. If this were not case, then, for any $R > 0$, there exist $u \in \overline{\mathcal{Q}}_R$ and $t \in \mathbb{R}$ such that $\|(Qu)(t)\| > R$. Thus, we see by (H1) and (H2) that

$$\begin{aligned}
R & < \|(Qu)(t)\| \\
& \leq \|(I - T(\omega))^{-1}\| \cdot \int_{t-\omega}^{\omega} \|T(t-s)\| \cdot \|F(s, u(s), u_s)\| ds \\
& \quad + \|(I - T(\omega))^{-1}\| \cdot \sum_{t-\omega \leq t_k < t} \|T(t-t_k)\| \cdot \|I_k(u(t_k))\| \\
& \leq \frac{1}{1 - e^{v_0\omega}} \cdot \int_{t-\omega}^t Me^{v_0(t-s)} (c_0 + c_1 \|u(s)\| + c_2 \|u_s\|_{Pr}) ds \\
& \quad + \frac{1}{1 - e^{v_0\omega}} \cdot \sum_{k=1}^p Me^{v_0(t-t_k)} a_k \|u(t_k)\| \\
& \leq -\frac{M}{v_0} (c_0 + (c_1 + c_2) \|u\|_{PC}) - \frac{M}{v_0\omega} \sum_{k=1}^p a_k \|u\|_{PC} \\
& \leq -\frac{M}{v_0} (c_0 + (c_1 + c_2) R) - \frac{M}{v_0\omega} \sum_{k=1}^p a_k R.
\end{aligned}$$

Dividing on both sides by R and taking the lower limit as $R \rightarrow \infty$, we have

$$(c_1 + c_2) + \frac{1}{\omega} \sum_{k=1}^p a_k \geq -\frac{v_0}{M}, \quad (3.5)$$

which contradicts (H3). Hence, there is a positive constant R such that $Q(\overline{\mathcal{Q}}_R) \subset \overline{\mathcal{Q}}_R$.

In order to show that the operator Q has a fixed point on $\overline{\mathcal{Q}}_R$, we also introduce the decomposition $Q = Q_1 + Q_2$, where

$$Q_1 u(t) := (I - T(\omega))^{-1} \int_{t-\omega}^t T(t-s) F(s, u(s), u_s) ds, \quad (3.6)$$

$$Q_2 u(t) := (I - T(\omega))^{-1} \sum_{t-\omega \leq t_k < t} T(t-t_k) I_k(u(t_k)). \quad (3.7)$$

Then we will prove that Q_1 is a compact operator and Q_2 is a contraction.

Firstly, we prove that Q_1 is a compact operator. Clearly, Q_1 is continuous and Q_1 maps $\overline{\mathcal{Q}}_R$ into a bounded set in $PC_\omega(\mathbb{R}, X)$. Now, we demonstrate that $Q_1(\overline{\mathcal{Q}}_R)$ is equicontinuous. For every $u \in \overline{\mathcal{Q}}_R$, by the periodicity of u , we only consider it on $[0, \omega]$. Setting $0 \leq t_1 < t_2 \leq \omega$, we get

$$\begin{aligned} & Q_1 u(t_2) - Q_1 u(t_1) \\ &= (I - T(\omega))^{-1} \int_{t_2-\omega}^{t_2} T(t_2-s) F(s, u(s), u_s) ds \\ &\quad - (I - T(\omega))^{-1} \int_{t_1-\omega}^{t_1} T(t_1-s) F(s, u(s), u_s) ds \\ &= (I - T(\omega))^{-1} \int_{t_2-\omega}^{t_1} (T(t_2-s) - T(t_1-s)) F(s, u(s), u_s) ds \\ &\quad - (I - T(\omega))^{-1} \int_{t_1-\omega}^{t_2-\omega} T(t_1-s) F(s, u(s), u_s) ds \\ &\quad + (I - T(\omega))^{-1} \int_{t_1}^{t_2} T(t_2-s) F(s, u(s), u_s) ds \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

It is clear that

$$\|Q_1 u(t_2) - Q_1 u(t_1)\| \leq \|I_1\| + \|I_2\| + \|I_3\|. \quad (3.8)$$

Thus, we only need to check $\|I_i\|$ tend to 0 independently of $u \in \overline{\mathcal{Q}}_R$ when $t_2 - t_1 \rightarrow 0$, $i = 1, 2, 3$.

From the condition (H1), it follows that there is a constant $M' > 0$ such that

$$\|F(t, u(t), u_s)\| \leq M', \quad u \in \overline{\mathcal{Q}}_R, t \in \mathbb{R}.$$

Combining this fact with the equicontinuity of the semigroup $T(t)$ ($t \geq 0$), we have

$$\begin{aligned} \|I_1\| &\leq C \cdot \int_{t_2-\omega}^{t_1} \|T(t_2-s) - T(t_1-s)\| \cdot \|F(s, u(s), u_s)\| ds \\ &\leq CM' \int_{t_2-\omega}^{t_1} \|T(t_2-s) - T(t_1-s)\| ds \\ &\rightarrow 0, \quad \text{as } t_2 - t_1 \rightarrow 0, \end{aligned}$$

$$\begin{aligned}
\|I_2\| &\leq C \cdot \int_{t_1-\omega}^{t_2-\omega} \|T(t_1-s)\| \cdot \|F(s, u(s), u_s)\| ds \\
&\leq CMM'(t_2-t_1) \\
&\rightarrow 0, \quad \text{as } t_2-t_1 \rightarrow 0, \\
\|I_3\| &\leq C \cdot \int_{t_1}^{t_2} \|T(t_2-s)\| \cdot \|F(s, u(s), u_s)\| ds \\
&\leq CMM'(t_2-t_1) ds \\
&\rightarrow 0, \quad \text{as } t_2-t_1 \rightarrow 0.
\end{aligned}$$

Hence, $\|Q_1 u(t_2) - Q_1 u(t_1)\|$ tends to 0 independently of $u \in \overline{\mathcal{D}}_R$ as $t_2 - t_1 \rightarrow 0$, which means that $Q_1(\overline{\mathcal{D}}_R)$ is equicontinuous.

It remains to show that $(Q_1 \overline{\mathcal{D}}_R)(t)$ is relatively compact in X for all $t \in \mathbb{R}$. To do this, we define a set $(Q_\varepsilon \overline{\mathcal{D}}_R)(t)$ by

$$(Q_\varepsilon \overline{\mathcal{D}}_R)(t) := \{(Q_\varepsilon u)(t) \mid u \in \overline{\mathcal{D}}_R, 0 < \varepsilon < \omega, t \in \mathbb{R}\}, \quad (3.9)$$

where

$$\begin{aligned}
(Q_\varepsilon u)(t) &= (I - T(\omega))^{-1} \int_{t-\omega}^{t-\varepsilon} T(t-s) F(s, u(s), u_s) ds \\
&= T(\varepsilon)(I - T(\omega))^{-1} \int_{t-\omega}^{t-\varepsilon} T(t-s-\varepsilon) F(s, u(s), u_s) ds.
\end{aligned}$$

Since the operator $T(\varepsilon)$ is compact in X , thus, the set $(Q_\varepsilon \overline{\mathcal{D}}_R)(t)$ is relatively compact in X . For any $u \in \overline{\mathcal{D}}_R$ and $t \in \mathbb{R}$, from the inequality

$$\begin{aligned}
\|Q_1 u(t) - Q_\varepsilon u(t)\| &\leq C \int_{t-\varepsilon}^t \|T(t-s) F(s, u(s), u_s)\| ds \\
&\leq C \int_{t-\varepsilon}^t \|T(t-s) F(s, u(s), u_s)\| ds \\
&\leq CMM'\varepsilon,
\end{aligned}$$

which implies that the set $(Q_1 \overline{\mathcal{D}}_R)(t)$ is relatively compact in X for all $t \in \mathbb{R}$.

Thus, the Arzela–Ascoli theorem guarantees that Q_1 is a compact operator.

Secondly, we prove that Q_2 is a contraction. Let $u, v \in \overline{\mathcal{D}}_R$, by the condition (H2), we have

$$\begin{aligned}
&\|Q_2 u(t) - Q_2 v(t)\| \\
&\leq \|(I - T(\omega))^{-1}\| \cdot \left\| \sum_{t-\omega \leq t_k < t} T(t-t_k)(I_k(u(t_k)) - I_k(v(t_k))) \right\| \\
&\leq \frac{1}{1 - e^{v_0 \omega}} \cdot \sum_{k=1}^p M e^{v_0(t-t_k)} a_k \|u(t_k) - v(t_k)\| \\
&\leq -\frac{M}{v_0 \omega} \sum_{k=1}^p a_k \|u - v\|_{PC},
\end{aligned}$$

therefore,

$$\|Q_2u - Q_2v\|_{PC} \leq -\frac{M}{v_0\omega} \sum_{k=1}^p a_k \|u - v\|_{PC}. \quad (3.10)$$

From the condition (H3), we can deduce Q_2 is a contraction.

Therefore, by the famous Sadovskii fixed point theorem [19], we know that Q has a fixed point $u \in \overline{\Omega}_R$, that is, Eq. (1.1) has an ω -periodic mild solution. The proof is completed. \square

Furthermore, we assume that F satisfies Lipschitz condition, namely,

(H1') *there are positive constants c_1, c_2 , such that for every $t \in \mathbb{R}, x_0, x_1 \in X$ and $\phi, \psi \in PC([-r, 0], X)$*

$$\|F(t, x, \phi) - F(t, y, \psi)\| \leq c_1 \|x - y\| + c_2 \|\phi - \psi\|_{Pr},$$

then we can obtain the following result.

Theorem 3.2 *Let X be a Banach space, $-A$ generates an exponentially stable compact semigroup $T(t)$ ($t \geq 0$) in X . Assume that $F: \mathbb{R} \times X \times PC([-r, 0], X) \rightarrow X$ is continuous and $F(t, \cdot, \cdot)$ is ω -periodic in t , $I_k \in C(X, X)$ ($k \in \mathbb{Z}$). If the conditions (H1'), (H2) and (H3) hold, then Eq. (1.1) has unique ω -periodic mild solution u .*

Proof From (H1') we easily see that (H1) holds. In fact, for any $t \in \mathbb{R}$, $x \in X$ and $\phi \in PC([-r, 0], X)$, by the condition (H1'),

$$\begin{aligned} \|F(t, x, \phi)\| &\leq \|F(t, x, \phi) - F(t, \theta, \theta)\| + \|F(t, \theta, \theta)\| \\ &\leq c_1 \|x\| + c_2 \|\phi\|_{Pr} + \|F(t, \theta, \theta)\|. \end{aligned}$$

From the continuity and periodicity of F , we can choose $c_0 = \max_{t \in [0, \omega]} \|F(t, \theta, \theta)\|$, thus, the condition (H1) holds. Hence by Theorem 3.1, Eq. (1.1) has ω -periodic mild solutions.

Let $u, v \in PC_\omega(\mathbb{R}, X)$ be the ω -periodic mild solutions of Eq. (1.1), then they are the fixed points of the operator Q which is defined by (3.1). Hence,

$$\begin{aligned} \|u(t) - v(t)\| &= \|Qu(t) - Qv(t)\| \\ &\leq \left\| (I - T(\omega))^{-1} \int_{t-\omega}^t T(t-s) (F(s, u(s), u_s) - F(s, v(s), v_s)) ds \right\| \\ &\quad + \left\| (I - T(\omega))^{-1} \sum_{t-\omega \leq t_k < t} T(t-t_k) (I_k(u(t_k)) - I_k(v(t_k))) \right\| \\ &\leq \frac{1}{1 - e^{v_0\omega}} \cdot \int_{t-\omega}^t M e^{v_0(t-s)} \cdot \|F(s, u(s), u_s) - F(s, v(s), v_s)\| ds \\ &\quad + \frac{1}{1 - e^{v_0\omega}} \sum_{t-\omega \leq t_k < t} M e^{v_0(t-t_k)} \cdot \|I_k(u(t_k)) - I_k(v(t_k))\| \end{aligned}$$

$$\begin{aligned}
&\leq -\frac{M}{v_0} \int_{t-\omega}^t c_1 \|u(s) - v(s)\| + c_2 \|u_s - v_s\|_{Pr} ds \\
&\quad - \frac{M}{v_0 \omega} \sum_{k=1}^p a_k \|u(t_k) - v(t_k)\| \\
&\leq \left(-\frac{M}{v_0} (c_1 + c_2) - \frac{M}{v_0 \omega} \sum_{k=1}^p a_k \right) \|u - v\|_{PC},
\end{aligned}$$

which implies that

$$\|u - v\|_{PC} = \|Qu - Qv\|_{PC} \leq \left(-\frac{M}{v_0} (c_1 + c_2) - \frac{M}{v_0 \omega} \sum_{k=1}^p a_k \right) \|u - v\|_{PC}.$$

From this and the condition (H3), it follows that $u_2 = u_1$. Thus, Eq. (1.1) has only one ω -periodic mild solution. \square

4 Asymptotic stability

In order to prove the asymptotic stability of ω -periodic solutions for Eq. (1.1), we need discuss the existence and uniqueness of the following initial value problem:

$$\begin{cases} u'(t) + Au(t) = F(t, u(t), u_t), & t \geq 0, t \neq t_i, \\ \Delta u(t_i) = I_i(u(t_i)), & i = 1, 2, \dots, \\ u_0 = \varphi, \end{cases} \quad (4.1)$$

where $F: J \times X \times PC([-r, 0], X) \rightarrow X$ is continuous and $\varphi \in PC([-r, 0], X)$.

Define

$$\begin{aligned}
PC([-r, \infty), X) &:= \{u: [-r, \infty) \rightarrow X \mid u \text{ is continuous at } t \in [-r, \infty) \setminus \{t_j\}, \\
&\quad u \text{ is continuous from left and has right limits at } t_j, j \in \mathbb{N}\}.
\end{aligned}$$

If there exists $u \in PC([-r, \infty), X)$ satisfying $u(t) = \varphi(t)$ for $-r \leq t \leq 0$ and

$$u(t) = T(t)u(0) + \int_0^t T(t-s)F(s, u(s), u_s) ds + \sum_{0 < t_k < t} T(t-t_k)I_k(u(t_k)), \quad t \geq 0, \quad (4.2)$$

then u is called a mild solution of the initial value problem (4.1).

In order to obtain the results as regards asymptotic stability, we need the following integral inequality of Gronwall–Bellman type with delay and being impulsive.

Lemma 4.1 Assume that $\phi \in PC([-r, \infty), J)$, and there exist constants $\alpha_1 \geq 0, \alpha_2 \geq 0$ and $\beta_i \geq 0$ ($i = 1, 2, \dots$) such that, for every $t \geq 0$, ϕ satisfies the integral inequality

$$\phi(t) \leq \phi(0) + \alpha_1 \int_0^t \phi(s) ds + \alpha_2 \int_0^t \sup_{\tau \in [-r, 0]} \phi(s + \tau) ds + \sum_{0 < t_k < t} \beta_k \phi(t_k). \quad (4.3)$$

Then $\phi(t) \leq \|\phi\|_{Pr} \cdot \prod_{0 < t_k < t} (1 + \beta_k) e^{(\alpha_1 + \alpha_2)t}$ for every $t \geq 0$.

Proof Define a function $\psi : [-r, \infty) \rightarrow \mathbb{R}$ as follows:

$$\psi(t) = \sup_{s \in [-r, t]} \phi(s), \quad t \in [-r, \infty).$$

Then $\psi \in PC([-r, \infty), J)$ and $\phi(t) \leq \psi(t)$ for $t \in [-r, \infty)$. Similar to the proof of [10, Lemma 4.1], we can get the following inequality:

$$\psi(t) \leq \|\phi\|_{\text{Pr}} + (\alpha_1 + \alpha_2) \int_0^t \psi(s) ds + \sum_{0 < t_k < t} \beta_k \psi(t_k), \quad t \geq 0.$$

By [20, Lemma 1, p. 12], $\psi(t) \leq \|\phi\|_{\text{Pr}} \prod_{0 < t_k < t} (1 + \beta_k) e^{(\alpha_1 + \alpha_2)t}$ for every $t \geq 0$. Therefore, $\phi(t) \leq \|\phi\|_{\text{Pr}} \prod_{0 < t_k < t} (1 + \beta_k) e^{(\alpha_1 + \alpha_2)t}$ for every $t \geq 0$. \square

For the initial value problem (4.1), we have the following result.

Theorem 4.1 *Let X be a Banach space, $-A$ generates a C_0 -semigroup $T(t)$ ($t \geq 0$) in X . Assume that $F : J \times X \times PC([-r, 0], X) \rightarrow X$ is continuous, $I_k \in C(X, X)$ ($k = 1, 2, \dots$), and $\varphi \in PC([-r, 0], X)$. If the conditions (H1') and (H2) hold, then the initial value problem (4.1) has a unique mild solution $u \in PC([-r, \infty), X)$.*

Proof For $t \in [-r, t_1]$, the initial value problem (4.1) is in the following form:

$$\begin{cases} u'(t) + Au(t) = F(t, u(t), u_t), & t \in [0, t_1], \\ u(t) = \varphi(t), & t \in [-r, 0]. \end{cases} \quad (4.4)$$

Write

$$PC([-r, t_1], X) := \{u \mid u : [-r, t_1] \rightarrow X \text{ with } u|_{[0, t_1]} \in C([0, t_1], X) \\ \text{and } u|_{[-r, 0]} \in PC([-r, 0], X)\},$$

then $PC([-r, t_1], X)$ is a Banach space under the norm

$$\|u\|_1 = \sup_{t \in [0, t_1]} \|u(t)\| + \|u|_{[-r, 0]}\|_{\text{Pr}}.$$

For any $\varphi \in PC([-r, 0], X)$, let $PC_\varphi([-r, t_1], X) = \{u \in PC([-r, t_1], X) \mid u|_{[-r, 0]} = \varphi\}$, then $PC_\varphi([-r, t_1], X)$ is a closed convex subset of $PC([-r, t_1], X)$. For each $\varphi \in PC([-r, 0], X)$ and $u \in PC_\varphi([-r, t_1], X)$, define an operator as follows:

$$(\tilde{Q}u)(t) = \begin{cases} T(t)u(0) + \int_0^t T(t-s)F(s, u(s), u_s) ds, & t \in [0, t_1], \\ \varphi(t), & t \in [-r, 0]. \end{cases} \quad (4.5)$$

It is easy to see that \tilde{Q} is well defined, $\tilde{Q}u \in PC_\varphi([-r, t_1], X)$, and the mild solution of Eq. (4.4) for φ is equivalent to the fixed point of \tilde{Q} in $PC_\varphi([-r, t_1], X)$.

Now, we prove that \tilde{Q} has a fixed point in $PC_\varphi([-r, t_1], X)$. From the condition (H1') and (4.5), it follows that for any $u, v \in PC_\varphi([-r, t_1], X)$, $n = 1, 2, \dots$,

$$\|(\tilde{Q}^n u)(t) - (\tilde{Q}^n v)(t)\| \leq \frac{(M_1(c_1 + c_2)t)^n}{n!} \|u - v\|_1,$$

where M_1 is the bound of $\|T(t)\|$ on $[0, t_1]$. By the contraction principle, one shows that \tilde{Q} has a unique fixed point u_1 in $PC_\varphi([-r, t_1], X)$, which means the initial value problem (4.4) has a mild solution and

$$u_1(t) = \begin{cases} T(t)u(0) + \int_0^t T(t-s)F(s, u(s), u_s) ds, \\ \varphi(t), \quad t \in [-r, 0]. \end{cases} \quad (4.6)$$

For $t \in [-r, t_2]$, the initial value problem (4.1) is in the following form:

$$\begin{cases} u'(t) + Au(t) = F(t, u(t), u_t), & t \in (t_1, t_2], \\ u(t_1^+) = I_1(u(t_1)) + u(t_1), \\ u(t) = u_1(t), & t \in [-r, t_1]. \end{cases} \quad (4.7)$$

Similar to the proof of (4.4), we can prove that the initial value problem (4.7) has a mild solution $u_2 \in PC_\varphi([-r, t_2], X)$,

$$\begin{aligned} u_2(t) &= \begin{cases} T(t-t_1)u(t_1^+) + \int_{t_1}^t T(t-s)F(s, u(s), u_s) ds, & t \in (t_1, t_2], \\ u_1(t), & t \in [-r, t_1] \end{cases} \\ &= \begin{cases} T(t)u(0) + \int_0^t T(t-s)F(s, u(s), u_s) ds + T(t-t_1)I_1(u(t_1)), & t \in [0, t_2], \\ \varphi(t), & t \in [-r, 0]. \end{cases} \end{aligned}$$

Doing this interval by interval, we see that there exists $u \in PC_\varphi([-r, \infty), X)$ satisfying $u(t) = \varphi(t)$ for $-r \leq t \leq 0$ and

$$u(t) = T(t)u(0) + \int_0^t T(t-s)F(s, u(s), u_s) ds + \sum_{0 < t_k < t} T(t-t_k)I_k(u(t_k)), \quad t \geq 0, \quad (4.8)$$

which is a mild solution of the initial value problem (4.1).

Next, we show the uniqueness. Let $u, v \in PC([-r, \infty), X)$ be the mild solutions of the initial value problem (4.1), hence they satisfy the initial value condition $u(t) = v(t) = \varphi(t)$ for $-r \leq t \leq 0$ and (4.2). By the condition (H1') and (H2), for every $t \geq 0$, one has

$$\begin{aligned} &\|u(t) - v(t)\| \\ &\leq \left\| \int_0^t T(t-s)(F(s, u(s), u_s) - F(s, v(s), v_s)) ds \right\| \\ &\quad + \left\| \sum_{0 < t_k < t} T(t-t_k)(I_k(u(t_k)) - I_k(v(t_k))) \right\| \\ &\leq \int_0^t \|T(t-s)\| \cdot \|F(s, u(s), u_s) - F(s, v(s), v_s)\| ds \end{aligned}$$

$$\begin{aligned}
& + \sum_{0 < t_k < t} \|T(t - t_k)\| \cdot \|I_k(u(t_k)) - I_k(v(t_k))\| \\
& \leq M \int_0^t c_1 \|u(s) - v(s)\| + c_2 \|u_s - v_s\|_{\text{Pr}} ds \\
& \quad + M \sum_{0 < t_k < t} a_k \|u(t_k) - v(t_k)\| \\
& \leq M \int_0^t c_1 \|u(s) - v(s)\| + c_2 \sup_{\tau \in [-r, 0]} \|u(s + \tau) - v(s + \tau)\| ds \\
& \quad + M \sum_{0 < t_k < t} a_k \|u(t_k) - v(t_k)\|.
\end{aligned}$$

From Lemma 4.1, it follows that $\|u(t) - v(t)\| = 0$ for every $t \geq 0$. Hence, $u \equiv v$. This completes the proof of Theorem 4.1. \square

Theorem 4.2 *Let X be a Banach space, $-A$ generates an exponentially stable compact semigroup $T(t)$ ($t \geq 0$) in X . Assume that $F: \mathbb{R} \times X \times PC([-r, 0], X) \rightarrow X$ is continuous and $F(t, \cdot, \cdot)$ is ω -periodic in t , $I_k \in C(X, X)$ ($k \in \mathbb{Z}$). If the conditions (H1'), (H2) and (H3') $(c_1 + c_2 e^{-\nu_0 r}) + \frac{1}{\omega} \sum_{k=1}^p a_k < \frac{|\nu_0|}{M}$, hold, then the unique ω -periodic mild solution of the periodic problem (1.1) is globally asymptotically stable.*

Proof From the condition (H3'), it follows that the condition (H3) holds. By Theorem 3.2, the periodic problem (1.1) has a unique ω -periodic mild solution $u^* \in PC_\omega(\mathbb{R}, X)$. For any $\phi \in PC([-r, 0], X)$, the initial value problem (4.1) has a unique global mild solution $u = u(t, \phi) \in PC([-r, \infty), X)$ by Theorem 4.1.

By the semigroup representation of the solutions, u^* and u satisfy the integral equation (4.2). Thus, by (4.2) and condition (H1'), (H2), for any $t \geq 0$, we have

$$\begin{aligned}
& \|u(t) - u^*(t)\| \\
& \leq \|T(t)(u(0) - u^*(0))\| \\
& \quad + \left\| \int_0^t T(t-s) (c_1 (u(s) - u^*(s)) + c_2 (u_s - u_s^*)) ds \right\| \\
& \quad + \left\| \sum_{0 < t_k < t} T(t - t_k) (I_k(u(t_k)) - I_k(u^*(t_k))) \right\| \\
& \leq M e^{\nu_0 t} \|u(0) - u^*(0)\| + \int_0^t M e^{\nu_0(t-s)} c_1 \|u(s) - u^*(s)\| ds \\
& \quad + \int_0^t M e^{\nu_0(t-s)} c_2 \sup_{\tau \in [-r, 0]} \|u(s + \tau) - u^*(s + \tau)\| ds \\
& \quad + \sum_{0 < t_k < t} M e^{\nu_0(t-t_k)} a_k \|u(t_k) - u^*(t_k)\| ds \\
& \leq M e^{\nu_0 t} \|u(0) - u^*(0)\| + e^{\nu_0 t} \int_0^t M c_1 e^{-\nu_0 s} \|u(s) - u^*(s)\|
\end{aligned}$$

$$\begin{aligned}
& + e^{\nu_0 t} \int_0^t M c_2 e^{-\nu_0 r} \sup_{\tau \in [-r, 0]} e^{-\nu_0(s+\tau)} \|u(s+\tau) - u^*(s+\tau)\| ds \\
& + e^{\nu_0 t} \sum_{0 < t_k < t} M a_k e^{-\nu_0 t_k} \|u(t_k) - u^*(t_k)\|.
\end{aligned}$$

Then

$$\begin{aligned}
& e^{-\nu_0 t} \|u(t) - u^*(t)\| \\
& \leq M \|u(0) - u^*(0)\| + \int_0^t M c_1 e^{-\nu_0 s} \|u(s) - u^*(s)\| \\
& \quad + \int_0^t M c_2 e^{-\nu_0 r} \sup_{\tau \in [-r, 0]} e^{-\nu_0(s+\tau)} \|u(s+\tau) - u^*(s+\tau)\| ds \\
& \quad + \sum_{0 < t_k < t} M a_k e^{-\nu_0 t_k} \|u(t_k) - u^*(t_k)\|,
\end{aligned}$$

for $t \in [-r, \infty)$, let $\phi(t) = e^{-\nu_0 t} \|u(t) - u^*(t)\|$, one can obtain

$$\phi(t) \leq M \phi(0) + M c_1 \int_0^t \phi(s) + M c_2 e^{-\nu_0 r} \int_0^t \sup_{\tau \in [-r, 0]} \phi(s+\tau) ds + \sum_{0 < t_k < t} M a_k \phi(t_k),$$

hence, from Lemma 4.1, it follows that

$$e^{-\nu_0 t} \|u(t) - u^*(t)\| = \phi(t) \leq C(\varphi) \prod_{0 < t_k < t} (1 + M a_k) e^{M(c_1 + c_2 e^{-\nu_0 r})t}, \quad (4.9)$$

where $C(\varphi) = \sup_{s \in [-\tau, 0]} \{e^{-\nu_0 s} \|\varphi(s) - u^*(s)\|_{Pr}\}$. Set $k = np + q$ ($q = 1, 2, \dots, p-1$, $n = 0, 1, 2, \dots$), by the periodicity of a_k , one can obtain

$$\sum_{0 < t_k < t} \ln(1 + M a_k) \leq (n+1) \sum_{k=1}^p \ln(1 + M a_k),$$

thus,

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{\sum_{0 < t_k < t} \ln(1 + M a_k)}{t} & \leq \lim_{n \rightarrow \infty} \frac{(n+1) \sum_{k=1}^p \ln(1 + M a_k)}{n\omega} \\
& = \frac{1}{\omega} \sum_{k=1}^p \ln(1 + M a_k) \\
& \leq \frac{M}{\omega} \sum_{k=1}^p a_k.
\end{aligned}$$

Therefore, from the assumption (H3), it follows that

$$\sigma := -\nu_0 - \lim_{t \rightarrow \infty} \frac{\sum_{0 < t_k < t} \ln(1 + M a_k)}{t} - M(c_1 + c_2 e^{-\nu_0 r}) > 0. \quad (4.10)$$

Combining (4.9) with (4.10), we can obtain

$$\|u(t) - u^*(t)\| \leq C(\varphi) e^{-\sigma t} \rightarrow 0 \quad (t \rightarrow \infty). \quad (4.11)$$

Thus, the ω -periodic solution u^* is globally asymptotically stable and it exponentially attracts every mild solution of the initial value problem. This completes the proof of Theorem 4.2. \square

5 Application

In this section, we present one example, which does not aim at generality, but indicates how our abstract results can be applied to concrete problems.

Let $\overline{\Omega} \in \mathbb{R}^n$ be a bounded domain with a C^2 -boundary $\partial\Omega$ for $n \in \mathbb{N}$. Let ∇^2 is a Laplace operator, and λ_1 is the smallest eigenvalue of operator $-\nabla^2$ under the Dirichlet boundary condition $u|_{\partial\Omega} = 0$. It is well known [3, Theorem 1.16], that $\lambda_1 > 0$.

Under the above assumptions, we discuss the existence, uniqueness and asymptotic stability of time 2π -periodic solutions of the semilinear parabolic boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla^2 u = \frac{\lambda_1}{4} \sin t \cdot u(x, t) + \int_{t-r}^t e^{\frac{4}{\lambda_1}(s-t)} \cdot u(x, s) ds, & x \in \Omega, t \neq t_k, \\ \Delta u(x, t_k) = \frac{\lambda_1 \pi}{p} (e^{\sin u(x, t_k)} - 1), & x \in \Omega, t_k = \frac{2k-1}{p} \pi, k \in \mathbb{Z}, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (5.1)$$

where $p > 0$ is an integer, $r > 0$ is a real number.

Let $X = L^2(\Omega)$ with the norm $\|\cdot\|_2$, then X is a Banach space. Define an operator $A : D(A) \subset X \rightarrow X$ by

$$D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad Au = -\nabla^2 u. \quad (5.2)$$

From [4], we know that $-A$ is a selfadjoint operator in X , and generates an exponentially stable analytic semigroup $T_p(t)$ ($t \geq 0$), which is contractive in X . From the operator A being compact resolvent in $L^2(\Omega)$, $T_p(t)$ ($t \geq 0$) is a compact semigroup (see [18]), which implies that the growth exponent of the semigroup $T(t)$ ($t \geq 0$) satisfies $\nu_0 = -\lambda_1$. Therefore, for every $t > 0$, $\|T(t)\|_2 \leq M := 1$ and $\|(I - T(2\pi))\| \leq \frac{1}{1 - e^{-2\lambda_1\pi}}$.

Now, we define $u(t) = u(\cdot, t)$, $I_k(u(t_k)) = (e^{\sin u(\cdot, t_k)} - 1)$, and since for $u \in PC_{2\pi}(\mathbb{R}, X)$,

$$\int_{t-r}^t e^{\frac{4}{\lambda_1}(s-t)} \cdot u(s) ds = \int_{-r}^0 e^{\frac{4}{\lambda_1}s} \cdot u(t+s) ds = \int_{-r}^0 e^{\frac{4}{\lambda_1}s} \cdot u_t(s) ds,$$

we define $F : \mathbb{R} \times X \times PC([-r, 0], X) \rightarrow X$ by

$$F(t, \xi, \phi) = \frac{\lambda_1}{4} \sin t \cdot \xi + \int_{-r}^0 e^{\frac{4}{\lambda_1}s} \phi(s) ds, \quad (5.3)$$

thus, it is easy to see that $t_{k+p} = t_k + 2\pi$, $I_k : X \rightarrow X$ ($k \in \mathbb{Z}$) are continuous functions satisfying $I_{k+p} = I_k$ and $F : \mathbb{R} \times X \times PC([-r, 0], X) \rightarrow X$ is continuous function which is 2π -periodic in t . Hence, the impulsive and delay parabolic boundary value problem (5.1) can be reformulated as the abstract evolution equation (1.1) in X .

From the definition of F and I_k , for every $t \in \mathbb{R}$, $\xi_1, \xi_2 \in X$ and $\phi_1, \phi_2 \in PC([-r, 0], X)$, we have

$$\|F(t, \xi_1, \phi_1) - F(t, \xi_2, \phi_2)\|_2 \leq \frac{\lambda_1}{4} \|\xi_1 - \xi_2\|_2 + \frac{\lambda_1}{4} (1 - e^{-\frac{4r}{\lambda_1}}) \|\phi_1 - \phi_2\|_{Pr},$$

$$\|I_k(\xi_1) - I_k(\xi_2)\|_2 \leq \frac{\lambda_1 \pi}{p} \|\xi_1 - \xi_2\|_2, \quad k \in \mathbb{Z},$$

which implies that the conditions (H1'), (H2) and (H3) hold. Thus, by Theorem 3.2, the parabolic boundary value problem (5.1) has only one time 2π -periodic mild solution. Moreover, if $0 < r < \frac{\lambda_1 \ln 4}{\lambda_1^2 + 4}$, then we can deduce that the condition (H3') holds. From Theorem 4.2, one can see that the unique 2π -periodic mild solution of problem (5.1) is globally asymptotically stable.

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Competing interests

QL and MW declare that they have no competing interests.

Authors' contributions

QL and MW contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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