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Meromorphic solutions to certain class of differential equations in an angular domain

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Abstract

In this paper, we study the admissible meromorphic solutions to the algebraic differential equation $f^n f' + P_{n-1}(f) = ue^v$ in an angular domain instead of the whole complex plane, where $P_{n-1}(f)$ is a differential polynomial in f of degree $\leq n-1$ with small function coefficients, u is a non-vanishing small function of f and v is an entire function. Herein, mainly, we are able to show that the equation does not admit any meromorphic solution f under some conditions unless $P_{n-1}(f) \equiv 0$. Using this result, we are able to extend or generalize a well-known result of Hayman.

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1 Introduction

In a recent paper, Liao and Ye considered meromorphic solutions f to

$$f^n f' + Q_d(z, f) = ue^v, \quad (1.1)$$

where $Q_d(z, f)$ denotes a differential polynomial in f of degree d with rational function coefficients. More precisely, they obtained the following result.

Theorem 1.1 ([1]) *Let $Q_d(z, f)$ be a differential polynomial in f of degree d with rational function coefficients. Suppose that u is a non-zero rational function and v is a non-constant polynomial. If $n \geq d+1$ and the differential equation (1.1) admits a meromorphic solution f with finitely many poles, then f has the following form:*

$$f = se^{v/(n+1)} \quad \text{and} \quad Q_d(z, f) \equiv 0,$$

where s is a rational function with $s''((n+1)s' + v's) = (n+1)u$.

It becomes natural for us to ask the following question: Does the conclusion of Theorem 1.1 remains valid if the condition “ u is a non-zero rational function and v is a non-constant polynomial” is replaced by a weaker one, such as “ u and v are entire functions” in the above theorem?

In this paper, we consider a slightly more general case of Eq. (1.1), where u and the coefficients of $Q_d(z, f)$ are meromorphic functions, not necessarily rational functions. Furthermore, v is an entire function. In this case, we will show that there exists an angular domain $\Omega(\alpha, \beta) = \{z : \alpha \leq \arg z \leq \beta\}$ such that the conclusion of Theorem 1.1 remains valid, where $\alpha, \beta \in [0, 2\pi]$ and $0 < \beta - \alpha \leq 2\pi$.

The technique developed here will be different from what has been employed in [1]. Now, we first introduce the Nevanlinna theory on an angular domain, which can be found in Goldberg–Ostrovskii [2] and the references therein.

Herein let f denote a non-constant meromorphic function on the closed angular domain $\overline{\Omega}(\alpha, \beta) = \Omega(\alpha, \beta) \cup \{0\} \cup \{\infty\}$, and let $k = \pi(\beta - \alpha)^{-1}$, $1 \leq r < \infty$. Following Goldberg–Ostrovskii [2, pp. 23–26], we introduce the following notations [3]:

$$\begin{aligned} A_{\alpha\beta}(r, f) &= \frac{k}{\pi} \int_1^r \left(\frac{1}{t^k} - \frac{t^k}{r^{2k}} \right) [\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|] \frac{dt}{t}; \\ B_{\alpha\beta}(r, f) &= \frac{2k}{\pi r^k} \int_\alpha^\beta \log^+ |f(re^{i\phi})| \sin k(\phi - \alpha) d\phi; \\ C_{\alpha\beta}(r, f) &= 2k \int_1^r c_{\alpha\beta}(t, f) \left(\frac{1}{t^k} + \frac{t^k}{r^{2k}} \right) \frac{dt}{t}, \end{aligned}$$

where

$$c_{\alpha\beta}(r, f) = \sum_{1 < \rho_n \leq r, \alpha \leq \varphi_n \leq \beta} \sin k(\varphi_n - \alpha),$$

and the $\rho_n e^{i\varphi_n}$ are poles of f counted according to their multiplicity. The function $C_{\alpha\beta}(r, f)$ is called the angular counting function of the poles of f on $\overline{\Omega}(\alpha, \beta)$ and the Nevanlinna angular characteristic function is defined as $S_{\alpha\beta}(r, f)$:

$$S_{\alpha\beta}(r, f) = A_{\alpha\beta}(r, f) + B_{\alpha\beta}(r, f) + C_{\alpha\beta}(r, f).$$

Similarly, for any finite value a , we define here $A_{\alpha\beta}(r, f_a)$, $B_{\alpha\beta}(r, f_a)$, $C_{\alpha\beta}(r, f_a)$ and $S_{\alpha\beta}(r, f_a)$, where $f_a = 1/(f - a)$. In what follows, when there is no danger of confusion, next we omit the subscript of all the above notations and, respectively, use the notations $A(r, f_a)$, $B(r, f_a)$, $C(r, f_a)$ and $S(r, f_a)$ instead of $A_{\alpha\beta}(r, f_a)$, $B_{\alpha\beta}(r, f_a)$, $C_{\alpha\beta}(r, f_a)$ and $S_{\alpha\beta}(r, f_a)$ for any finite value a .

In this paper, unless otherwise stated, we address a meromorphic function that is defined and meromorphic in the whole complex plane. It is assumed the reader is familiar with the basic theory of the Nevanlinna value distribution and its standard symbols and notations (see e.g., [4, 5]).

For the order $\rho(f) = \infty$, we use the following concept of a proximate order as introduced in [6].

Theorem 1.2 ([6]) *Let $B(r)$ be a positive and continuous function in $[0, +\infty)$, which satisfies $\limsup_{r \rightarrow \infty} \frac{\log B(r)}{\log r} = \infty$, then there exists a continuously differentiable function $\rho(r)$, which satisfies the following conditions.*

- (i) $\rho(r)$ is continuous and nondecreasing for $r > r_0$ ($r_0 > 0$) and tends to $+\infty$ as $r \rightarrow +\infty$.

(ii) The function $U(r) = r^{\rho(r)}$ ($r \geq r_0$) satisfies the condition

$$\lim_{r \rightarrow +\infty} \frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)}.$$

(iii)

$$\limsup_{r \rightarrow +\infty} \frac{\log B(r)}{\log U(r)} = 1.$$

We define $\rho(r)$ and $U(r)$ in Theorem 1.2 by the proximate order and type function of $B(r)$, respectively. For a transcendental meromorphic function f of infinite order, we define its proximate order and type function as the proximate order and type function of $T(r, f)$. We denote $M(\rho(r))$ by the set of all meromorphic functions f in the complex plane such that

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log U(r)} = 1.$$

For the sake of simplicity, we use the Landau symbols $O(\cdots)$ and $o(\cdots)$ as $r \rightarrow \infty$, where and in what follows, $Q(r, f)$ is such a quantity that if the order $\rho(f) < \infty$, then $Q(r, f) = O(1)$, as $r \rightarrow \infty$; if the order $\rho(f) = \infty$, then $Q(r, f) = O(\log U(r))$. It is not necessarily the same for every occurrence in the context.

Throughout the paper, given a meromorphic function f , we define

$$S_f := \{h \mid \text{meromorphic}, S(r, h) = Q(r, f)\},$$

as $r \rightarrow \infty$, possibly outside a set of r values of finite linear measure.

In addition, we need the following concept (see, e.g., [7] and [8]).

Definition 1.3 Let $R(z, \omega)$ be rational in ω with meromorphic coefficients. A meromorphic solution ω of $(\omega')^n = R(z, \omega)$ is called admissible, if $S(r, a) = Q(r, \omega)$ holds for all coefficients a of $R(z, \omega)$.

Of course, admissibility makes sense relative to any family of meromorphic functions, without any reference to differential equations.

Before proceeding further, we recall the following result. Recently, Liu–Lü–Yang considered the possible admissible solution to the following equation.

Theorem 1.4 ([9]) Let p , q and u be non-vanishing meromorphic functions in complex plane, v be an entire function. Then the differential equation

$$pff' - q = ue^v \tag{1.2}$$

has no transcendental meromorphic solutions in the complex plane.

Motivated by the preceding, as a continuation and further studies on some of the related results in the complex plane, we will state our main results in Sect. 3, which extend some results earlier; see, e.g., [10–13] and the references therein.

2 Preliminary lemmas

To prove our results, the following lemmas are needed.

Lemma 2.1 ([2, pp. 23–26] and [2, Theorem 3.1]) *Let f be a non-constant meromorphic function and $\Omega(\alpha, \beta)$ be a domain, where $0 < \beta - \alpha \leq 2\pi$. Then, for an arbitrary complex number $a \in \mathbb{C}$ and any integer $k \geq 0$, we have*

$$S\left(r, \frac{1}{f-a}\right) = S(r, f) + O(1),$$

$$S(r, f^{(k)}) \leq 2^k S(r, f) + Q(r, f),$$

and

$$A\left(r, \frac{f^{(k)}}{f}\right) + B\left(r, \frac{f^{(k)}}{f}\right) = Q(r, f).$$

Lemma 2.2 ([2, Theorem 3.3]) *Let f be a meromorphic function. Then, for arbitrary q distinct values $a_j \in \mathbb{C} \cup \{\infty\}$ ($1 \leq j \leq q$),*

$$(q-2)S(r, f) \leq \sum_{j=1}^q \overline{C}\left(r, \frac{1}{f-a_j}\right) + Q(r, f),$$

where $\overline{C}(r, \frac{1}{f-a_j})$ is the reduced form of $C(r, \frac{1}{f-a_j})$.

Lemma 2.3 ([2]) *Let f be a meromorphic function. By $D(r, f)$, we denote any of the characteristics $A(r, f)$, $B(r, f)$, and $C(r, f)$. Then*

$$D(r, f+g) \leq D(r, f) + D(r, g) + \log 2,$$

$$D(r, f \cdot g) \leq D(r, f) + D(r, g),$$

and

$$D(r, f^k) = kD(r, f)$$

for any positive integer k .

Lemma 2.4 *Let f be a meromorphic solution of*

$$f^n P_1(z, f) = P_2(z, f),$$

where $P_1(z, f)$ and $P_2(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda \mid \lambda \in I\}$ such that $S(r, a_\lambda) = Q(r, f)$ for all $r \in I$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is less than or equal to n , then

$$A(r, P(r, f)) + B(r, P(r, f)) = Q(r, f).$$

We omit the proof of Lemma 2.4, since it is similar to the proof of Clunie's theorem [4, p. 68, Lemma 3.3] and [14, 15] in the plane.

3 Main results

In this section, we give our main results as follows.

Theorem 3.1 *Let f be a transcendental meromorphic function, $P_{n-1}(f)$ be a differential polynomial in f such that its coefficients are in S_f and $\deg P_{n-1}(f) \leq n-1$. Assume further that v is a non-constant entire function of finite order and u ($u \in S_f$) is a meromorphic function. Then, for any positive integer n , and any ε ($0 < \varepsilon < \pi/2$), there exists a direction $\arg z = \theta$ such that the equation*

$$f^n f' + P_{n-1}(f) = ue^v, \quad (3.1)$$

does not admit any non-constant meromorphic solution f on the domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ with $C(r, f) = Q(r, f)$ unless $P_{n-1}(f) = 0$. Moreover, if (3.1) admits a meromorphic solution f on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ with $C(r, f) = Q(r, f)$, then (3.1) will become $f^n f' = ue^v$ and f must have the form $f = \psi \exp(v/(n+1))$ as the only possible admissible solution of (3.1), where ψ is in S_f .

Proof First of all, we suppose that $f \in M(\rho(r))$. In this case, we show that $f^n f' + P_{n-1}(f)$ cannot be S_f . Otherwise, from $C(r, f) = Q(r, f)$ and Lemma 2.4, we get $A(r, f') + B(r, f') = Q(r, f)$ and then $S(r, f') = Q(r, f)$. A contradiction $S(r, f) = Q(r, f)$ now follows by relying on Theorem 2.6.5 in [16, p. 91]. Thus, for any meromorphic function f under the condition: $C(r, f) = Q(r, f)$,

$$S(r, f^n f' + P_{n-1}(f)) \neq Q(r, f). \quad (3.2)$$

Therefore (3.2) shows that ue^v is not in S_f .

It is well known that a meromorphic function $f \in M(\rho(r))$ has at least one Borel direction $\arg z = \theta$ of order $\rho(r)$. In the following, we prove that the direction $\arg z = \theta$ satisfies the theorem. For any ε ($0 < \varepsilon < \pi/2$), then

$$\limsup_{r \rightarrow \infty} \frac{\log S(r, f)}{\log r^{\rho(r)}} = 1$$

holds on $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cup \{0\} \cup \{\infty\}$.

To prove the theorem, we first assume that $P_{n-1}(f) \not\equiv 0$. By examining carefully the proof of the Milloux estimate in [4, p. 55], we deduce from the assumption $C(r, f) = Q(r, f)$ and (3.1) that

$$S(r, e^v) \leq (n+1)S(r, f) + Q(r, f),$$

which shows that v and v' are in S_f .

Taking the logarithmic derivative on both sides of (3.1) yields

$$\frac{nf^{n-1}(f')^2 + f^n f'' + P'_{n-1}(f)}{f^n f' + P_{n-1}(f)} = \frac{u'}{u} + v',$$

which gives the equality

$$-\left(\frac{u'}{u} + v'\right)f^n f' + nf^{n-1}(f')^2 + f^n f'' = \left(\frac{u'}{u} + v'\right)P_{n-1}(f) - P'_{n-1}(f). \quad (3.3)$$

Now, we set

$$\varphi = -\left(\frac{u'}{u} + v'\right)ff' + n(f')^2 + ff'' \quad (3.4)$$

It follows by (3.3) that

$$f^{n-1}\varphi = \left(\frac{u'}{u} + v'\right)P_{n-1}(f) - P'_{n-1}(f) := P_{n-1}^*(f). \quad (3.5)$$

Clearly, $P_{n-1}^*(f)$ is a differential polynomial in f with $\deg P_{n-1}^*(f) \leq n-1$. First of all, let us show that $\varphi \not\equiv 0$. Assume the contrary, that is, $\varphi \equiv 0$. Then $P_{n-1}^*(f) \equiv 0$, and accordingly (3.5) gives $BP_{n-1}(f) \equiv ue^v$ with a constant B . Since f is a non-constant meromorphic function, (3.1) shows that $B \neq 1$, and

$$f^n f' = (B-1)P_{n-1}(f),$$

which together with Lemma 2.4 implies $A(r, f') + B(r, f') = Q(r, f)$. Thereby we have $S(r, f') = Q(r, f)$ from the assumption $C(r, f) = Q(r, f)$. This is a contradiction. Hence $\varphi \not\equiv 0$. Moreover, applying Lemma 2.4 to (3.5) again, we can derive $A(r, \varphi) + B(r, \varphi) = Q(r, f)$ and so $S(r, \varphi) = Q(r, f)$. That is $\varphi \in S_f$.

Note that (3.4) can be represented in the form

$$\frac{\varphi}{f^2} = -\left(\frac{u'}{u} + v'\right)\frac{f'}{f} + n\left(\frac{f'}{f}\right)^2 + \frac{f''}{f},$$

which and Lemma 2.1 yield $A(r, \frac{\varphi}{f^2}) + B(r, \frac{\varphi}{f^2}) = Q(r, f)$. Thus, we obtain

$$A\left(r, \frac{1}{f}\right) + B\left(r, \frac{1}{f}\right) = Q(r, f). \quad (3.6)$$

On the other hand, it is clear that (3.4) shows that

$$C_2\left(r, \frac{1}{f}\right) \leq C\left(r, \frac{1}{f}\right) + Q(r, f) \leq S(r, \varphi) + Q(r, f) = Q(r, f),$$

which implies that the zeros of f are mainly simple zeros on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$. This observation will be repeatedly used later on. Thus, by (3.6) and the definition of $S(r, f)$, we obtain

$$S(r, f) = C\left(r, \frac{1}{f}\right) + Q(r, f) = C_1\left(r, \frac{1}{f}\right) + Q(r, f), \quad (3.7)$$

where in $C_1(r, 1/f)$ only the simple zeros of f are to be considered on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$.

Without loss of generality we may assume that z_0 be a simple zero of f such that $u(z_0) \neq 0$. Then we find by (3.4) that

$$n(f')^2(z_0) = \varphi(z_0). \quad (3.8)$$

If φ is a constant, and set

$$h(z) = \frac{f'(z) - \sqrt{\frac{\varphi}{n}}}{f(z)}. \quad (3.9)$$

Obviously $h \not\equiv 0$ since f is transcendental. It follows by (3.6), (3.8), (3.9) and Lemma 2.1 that

$$A(r, h) + B(r, h) = Q(r, f). \quad (3.10)$$

From (3.7) and (3.9), we arrive at $C(r, h) = Q(r, f)$, which and (3.10) show that $h \in S_f$. Now, we rewrite (3.9) as

$$f'(z) = h(z)f(z) + \sqrt{\frac{\varphi}{n}}. \quad (3.11)$$

By (3.11) and (3.4), we obtain

$$\left[(n+1)h^2 + h' - h \left(\frac{u'}{u} + v' \right) \right] f^2 + \left[2nh - \left(\frac{u'}{u} + v' \right) \right] \sqrt{\frac{\varphi}{n}} f + h \sqrt{\frac{\varphi}{n}} = 0,$$

this and Lemma 2.3 (or [2, p. 33, Theorem 6.3]) imply $h\sqrt{\frac{\varphi}{n}} \equiv 0$, a contradiction. Thus $\varphi' \not\equiv 0$. Again, by taking the logarithmic derivative on both sides of (3.4), we conclude

$$\varphi' = -t'f' - t(f')^2 - tff'' + (2n+1)f'f'' + ff''', \quad (3.12)$$

where $t = \frac{u'}{u} + v'$. By (3.12) and (3.8), we see that a simple zero z_0 of $f(z)$ such that $u(z_0) \neq 0$, is a zero of $(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z)$.

Two cases will now be considered below, depending on whether or not $(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z)$ vanishes identically.

Case 1. $(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z) \not\equiv 0$. Set

$$g(z) = \frac{(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z)}{f(z)}.$$

Trivially, $g(z) \in S_f$. Thus, we have

$$f'' = \frac{g}{(2n+1)\varphi} f + \frac{t\varphi + n\varphi'}{(2n+1)\varphi} f' := s_1 f + s_2 f' \quad (3.13)$$

and

$$f''' = (s'_1 + s_1 s_2) f + (s_1 + s'_2 + s_2^2) f'. \quad (3.14)$$

It follows by (3.14), (3.13), (3.12) and (3.4) that

$$\begin{aligned} & \left((2n+1)s_1 - t' - ts_2 + s_1 + s'_2 + s_2^2 + t \frac{\varphi'}{\varphi} - s_2 \frac{\varphi'}{\varphi} \right) f' \\ & + \left(s'_1 + s_1 s_2 - ts_1 - s_1 \frac{\varphi'}{\varphi} \right) f = 0. \end{aligned} \quad (3.15)$$

In this case, (3.15) and (3.7) imply

$$s_1' + s_1 s_2 - t s_1 - s_1 \frac{\varphi'}{\varphi} \equiv 0.$$

Since $g \neq 0$, we know that $s_1 \neq 0$. Thus, we can rewrite the above equation as

$$\frac{s_1'}{s_1} + s_2 - t - \frac{\varphi'}{\varphi} \equiv 0,$$

hence we have $(2n+1)\log s_1 = 2n(\log u + v) + (3n+1)\log \varphi + L$ with a constant L , which implies $(ue^v)^{2n} e^{L\varphi^{3n+1}} = s_1^{2n+1}$, we see that ue^v is in S_f , this contradicts (3.2).

Case 2. $(2n+1)\varphi f''(z) - (t\varphi + n\varphi')f'(z) \equiv 0$. In this case, we conclude

$$f'' = \xi f' \quad (3.16)$$

with $\xi = \frac{n\varphi'}{(2n+1)\varphi} + \frac{t}{2n+1}$. Thus (3.16) gives

$$f''' = (\xi' + \xi^2)f'. \quad (3.17)$$

It follows by (3.17), (3.16), (3.12) and (3.4) that

$$(\xi' + \xi^2)f' = \left(t' - t \frac{\varphi'}{\varphi}\right)f' + \left(t + \frac{\varphi'}{\varphi}\right)\xi f'.$$

Thus, we have

$$\xi' - t' \equiv -\xi(\xi - t) + (\xi - t)\frac{\varphi'}{\varphi}. \quad (3.18)$$

If $\xi - t \equiv 0$, then, by the definitions of t and ξ , we see that $(ue^v)^2 = \eta\varphi$, η is a constant. So ue^v is in S_f , this contradicts (3.2). Hence $\xi - t \neq 0$. In this case, again, by (3.18), we obtain $(2n+1)\log(\xi - t) = n\log \varphi + \log u + v + \tau$ with a constant τ . This gives $ue^v \in S_f$, this also contradicts (3.2).

This completes the proof of our conclusion, namely $f^n f' + P_{n-1}(f) = ue^v$ does not admit any meromorphic solution f on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ with $C(r, f) = Q(r, f)$ unless $P_{n-1}(f) \equiv 0$.

Assume that $\rho(f) < +\infty$. Note v is a non-constant entire function, so we have $\rho(f) > 0$. Let σ satisfy $0 < \sigma < \rho(f)$. For a given angular domain $\Omega(\alpha, \beta)$, $\beta - \alpha = \frac{\pi}{\sigma}$. Without loss of generality, we may assume that f has at least one Borel direction in the angular domain $\Omega(\alpha + \varepsilon, \beta - \varepsilon)$ ($0 < \varepsilon < \pi/2$). Hence, there exists a finite complex number a such that

$$\limsup_{r \rightarrow \infty} \frac{\log n(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), f = a)}{\log r} > \sigma = \frac{\pi}{\beta - \alpha}.$$

The Hadamard factorization theorem and (3.1) give $f = \psi \exp(v/(n+1))$, where ψ is in S_f . Then the conclusion of the theorem follows. The proof is finished. \square

Corollary 3.2 *Let f be a transcendental meromorphic function, $P_{n-1}(f)$ ($P_{n-1}(0) \neq 0$) denote a differential polynomial in f with its coefficients are in S_f and $\deg P_{n-1}(f) \leq n-1$. Then, for any positive integer n , and any ε ($0 < \varepsilon < \pi/2$), there exists a direction $\arg z = \theta$ such that $f^n f' + P_{n-1}(f)$ has infinitely many zeros on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ with $C(r, f) = Q(r, f)$.*

Remark 3.3 Take $f(z) = e^z + e^{-z}$, and $P_0(f) = -2$. Obviously, $ff'' + P_0(f) = e^{2z} + e^{-2z}$ has infinitely many zeros.

Based on Corollary 3.2 and Remark 3.3, we present the following more general conjecture; see, e.g., [3] and [17].

Conjecture 3.4 Let f be a transcendental meromorphic function, and assume further that $P_{n-1}(f)$ is a differential polynomial in f with its coefficients are in S_f and $\deg P_{n-1}(f) \leq n-1$ such that $P_{n-1}(0) \neq 0$. Then, for any positive integers n, k , and any ε ($0 < \varepsilon < \pi/2$), there exists a direction $\arg z = \theta$ such that

$$C\left(r, \frac{1}{f^n f^{(k)} + P_{n-1}(f)}\right) \neq Q(r, f)$$

on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ with $C(r, f) = Q(r, f)$.

Remark 3.5 We will give some examples below to show that the restricted conditions in Corollary 3.2 are sharp.

Example 3.6 Take $f(z) = e^{e^z} - 1$, and $P_2(f) = 2e^z f^2 + 3e^z f + e^z$. Obviously, $e^z \in S_f$, and $f^2 f' + P_2(f) = e^z e^{3e^z}$ has no zeros on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$.

Example 3.7 Let $f(z) = e^{z^2} - 1$, and $P_1(f) = 2zf + 2z$. Obviously, $ff' + P_1(f) = 2ze^{2z^2}$ has only one zero on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$.

Examples 3.6 and 3.7 tell us that the conclusion of Corollary 3.2 is false, if we replace the restricted condition “ $\deg P_{n-1}(f) \leq n-1$ ” by “ $\deg P_n(f) \leq n$ ”.

Example 3.8 If we take $f(z) = z^2 e^z$, $P_2(f) = ff' - (\frac{2}{z} + 1)f^2$, then $2/z + 1 \in S_f$, and $f^3 f' + P_2(f) = (2+z)z^7 e^{4z}$ has only finitely many zeros on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$.

Example 3.8 shows that the restricted condition “ $P_{n-1}(0) \neq 0$ ” in Corollary 3.2 is necessary.

Corollary 3.2 may be false if the condition “ $C(r, f) = Q(r, f)$ ” is violated. There is no difficulty in showing that Example 3.9 below is a counterexample.

Example 3.9 We take $\alpha = -\pi$, $\beta = \pi$, $f(z) = \frac{e^z}{e^z - 1}$, $P_1(f) = \frac{3}{2}f'' + \frac{3}{2}f' + f - 1$.

Indeed, a calculation yields $C(r, 1/f) = 0$, $C(r, 1/(f-1)) = 0$. It follows by Lemma 2.2 that $C(r, f) = S(r, f) + Q(r, f)$. Further, we would like to mention that $f^2 f' + P_1(f) = -\frac{1}{(e^z - 1)^4}$ has no zeros on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$.

By noting that f in the above counterexample (Example 3.9) satisfies “ $\overline{C}(r, f) = S(r, f) + Q(r, f)$ ”, we can see that it would be interesting if one could characterize all the solutions on some $\Omega(\theta - \varepsilon, \theta + \varepsilon)$ to the equation

$$f^n f^{(k)} + P_{n-1}(f) = ue^v$$

with $\overline{C}(r, f) \neq S(r, f) + Q(r, f)$, where n, k are positive integers.

From Theorem 3.1, we also obtain the following result, which complements the corresponding results in [1].

Theorem 3.10 *Let $f \in M(\rho(r))$ of finite order with $C(r, f) = Q(r, f)$, $q_m(f) = b_m f^m + \cdots + b_1 f + b_0$ a polynomial of degree m with $b_j \in S_f$ ($j = 0, 1, \dots, m$), and let n be an integer with $n \geq m + 1$. Then, for any ε ($0 < \varepsilon < \pi/2$), there exists a direction $\arg z = \theta$ such that $f'f^n + q_m(f)$ assumes every function $\gamma \in S_f$ infinitely many times on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$, except for a possible function $b_0 = q_m(0)$. On the other hand, if $f'f^n + q_m(f)$ assumes $b_0 = q_m(0)$ finitely many times on $\Omega(\theta - \varepsilon, \theta + \varepsilon)$, then $q_m(z) \equiv b_0$.*

Proof This theorem can be proved in the same manner as that in the proof of Theorem 3.1, so it is omitted here. \square

4 Conclusions

Using the different and much simpler proofs, this paper provides two main results on $\Omega(\alpha, \beta)$, which extend the main results that were derived in [1]. To bring about our results from the more general hypotheses without complicated calculations will probably be the most interesting feature of this note. And then some examples show that the restrict conditions are necessary. Finally, one more general conjecture was posed in this note.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed in drafting this manuscript. All authors read and approved the final manuscript.

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