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Random exponential attractor for second order non-autonomous stochastic lattice dynamical systems with multiplicative white noise in weighted spaces

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Abstract

This paper is concerned with the random exponential attractor for second order non-autonomous stochastic lattice system with multiplicative white noise and unbounded nonlinearity. Firstly, we transfer the stochastic lattice system into a random lattice system without noise term whose solutions generate a continuous cocycle on a weighted space of infinite sequences. Then we present some sufficient conditions for the existence of a random exponential attractor for a continuous cocycle in the product weighted space of sequences, which improved the existing conditions. Finally, we prove the existence of a random exponential attractor for the considered system in weighted space of sequences.

MSC: 37L55; 60H15; 35B40

Keywords: Random exponential attractor; Stochastic lattice system; Multiplicative white noise; Cocycle; Weighted space of sequences

1 Introduction

It is well known that the dynamics of a random dynamical system can be determined by the random attractor. However, the speed of a random attractor attracting orbits is sometimes relatively slow and the dimension of a random attractor is maybe infinite [1]. To this end, Shirikyan and Zelik [2] introduced the random exponential attractor (a positively invariant measurable set with a finite fractal dimension and attracting any trajectory exponentially) for an autonomous random dynamical system. The random exponential attractor (if it exists) contains the random attractor with finite fractal dimension, which implies that the dynamics of a random dynamical system can be described by finite independent parameters. Recently, Zhou in [3] established the existence of a random exponential attractor for a continuous cocycle (non-autonomous random dynamical system) on a separable Banach space and the first order stochastic lattice system driven by linear multiplicative white noise.

In this paper, we consider the following second order non-autonomous stochastic lattice system with multiplicative white noise and initial conditions: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{cases} \ddot{u}_i + \gamma (A\dot{u})_i + \alpha \dot{u}_i + (Au)_i + \lambda_i u_i + f_i(u_i, t) = g_i(t) + cu_i \circ \dot{W}(t), & i \in \mathbb{Z}, \\ u_i(\tau) = u_{0,i\tau}, & \dot{u}_i(\tau) = u_{1,i\tau}, & i \in \mathbb{Z}, \end{cases} \quad (1)$$

where $i \in \mathbb{Z}$, $u = (u_i)_{i \in \mathbb{Z}}$, $\dot{u} = (\dot{u}_i)_{i \in \mathbb{Z}}$, $\alpha, c > 0$, $\gamma \geq 0$, $\lambda_i > 0$, $u_i, g_i(t), f_i(u_i, t) \in \mathbb{R}$, and A is a non-negative linear coupled operator; $W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, the Borel σ -algebra $\mathcal{F}$ on Ω is generated by the compact open topology, and $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$ [4]; “ $\circ$ ” in (1) denotes the Stratonovich sense in the stochastic term.

For the second order autonomous stochastic lattice system (1) ($f_i(u_i, t) = f_i(u_i)$, $g_i(t) = g_i$ independent of t) with multiplicative and additive white noise, the existence and upper semi-continuity of the random attractor have been studied by some authors; see [5–12]. But as is well known, there is no result concerning the dimension of random attractor and existence of random exponential attractors for second order autonomous and non-autonomous stochastic lattice system (1). Based on the ideas of [3, 5, 9], in this paper, we aim to prove that under certain conditions, the system (1) possesses a random exponential attractor in weighted spaces of sequences, which implies that the corresponding autonomous and non-autonomous stochastic systems (1) have the random attractors with finite fractal dimension.

2 Random exponential attractor for second order non-autonomous stochastic lattice system

Let $\rho: \mathbb{Z} \rightarrow (0, +\infty)$, $\rho_i = \rho(i)$ for $i \in \mathbb{Z}$ be a positive weight function. Write

$$l_\rho^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}} : u_i \in \mathbb{R}, \sum_{i \in \mathbb{Z}} \rho_i |u_i|^2 < \infty \right\}$$

with inner product $(u, v)_\rho = \sum_{i \in \mathbb{Z}} \rho_i u_i v_i$ and norm $\|v\|_\rho^2 = (v, v)_\rho$ for $u = (u_i)_{i \in \mathbb{Z}}$, $v = (v_i)_{i \in \mathbb{Z}} \in l_\rho^2$.

Consider the system (1), which can be written in the following vector form:

$$\begin{cases} \ddot{u} + \gamma A\dot{u} + \alpha \dot{u} + Au + \lambda u + f(u, t) = g(t) + cu \circ \dot{W}(t), & t > \tau, \\ u(\tau) = u_\tau, & \dot{u}(\tau) = u_{1\tau}, & \tau \in \mathbb{R}, \end{cases} \quad (2)$$

where $A\dot{u} = ((A\dot{u})_i)_{i \in \mathbb{Z}}$, $Au = (Au)_i)_{i \in \mathbb{Z}}$, $\lambda u = (\lambda_i u_i)_{i \in \mathbb{Z}}$, $f(u, t) = (f_i(u_i, t))_{i \in \mathbb{Z}}$, $g(t) = (g_i(t))_{i \in \mathbb{Z}}$, $u \circ \dot{W}(t) = (u_i \circ \dot{W}(t))_{i \in \mathbb{Z}}$. The operator A has a decomposition $A = \overline{D}D = D\overline{D}$, where D is a bounded linear operator defined by

$$(Du)_i = \sum_{l=-m_0}^{m_0} d_l u_{i+l}, \quad |d_l| \leq c_0 \text{ (constant)}, \forall u = (u_i)_{i \in \mathbb{Z}}, m_0 \in \mathbb{N},$$

$\overline{D}$ is the adjoint of D in $l_\rho^2 = l_{\rho(i) \equiv 1, \forall i \in \mathbb{Z}}^2$.

Let $i \in \mathbb{Z}$, $G_i(s, t) = \int_0^s f_i(r, t) dr$, $\overline{G}(u, t) = \sum_{i \in \mathbb{Z}} \rho_i G_i(u_i, t)$ and we make the following assumptions on $\rho_i, f_i, g_i, \lambda_i$:

(A1) $0 < \bar{\rho} \leq \rho_i = \rho(i) \leq a_0 < +\infty$, $c_1 \rho(i) \leq \rho(i \pm 1) \leq c_2 \rho(i)$, $|\rho(i \pm 1) - \rho(i)| \leq c_3 \rho(i)$,
 $\forall i \in \mathbb{Z}$, for some positive constants $\bar{\rho}$, a_0 , c_1 , c_2 , c_3 .

(A2) $\forall i \in \mathbb{Z}$, $t, s \in \mathbb{R}$, $f_i, f'_{i,s} \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, and there exist positive constants a_1, a_2, a_3 ,
 $a_4, \underline{\lambda}, \bar{\lambda} > 0$ and functions $\beta_i \in C^1(\mathbb{R}, \mathbb{R})$ such that

$$\begin{cases} f_i(0, t) = 0, & |f'_{i,s}(s, t)| \leq a_1 |s| (1 + |s|^{p-1}), \quad p > 1, \\ a_2 f_i(s, t) s \geq G_i(s, t) \geq a_3 |s|^{p+2} - \beta_i^2(t), \\ G'_{i,t}(s, t) \leq a_4 G_i(s, t), \quad 0 \leq a_4 \leq \frac{\varepsilon}{2a_2}, \\ \underline{\lambda} \leq \lambda_i \leq \bar{\lambda} < +\infty. \end{cases}$$

(A3) $\forall t \in \mathbb{R}$, $\beta(t) = (\beta_i(t))_{i \in \mathbb{Z}}$, $\beta'(t) = (\beta'_i(t))_{i \in \mathbb{Z}}$, $g(t) = (g_i(t))_{i \in \mathbb{Z}} \in \mathbf{G}$, where

$$\mathbf{G} = \left\{ g \in C_b(\mathbb{R}, l^2_\rho) : \forall \eta > 0, \exists I(\eta) \in \mathbb{N} \text{ such that } \sup_{t \in \mathbb{R}} \sum_{|i| > I(\eta)} \rho_i g_i^2(t) < \eta \right\},$$

and $C_b(\mathbb{R}, l^2_\rho)$ is the space of all continuous bounded functions from $\mathbb{R}$ into l^2_ρ .

(A4) c_3 in (A1) satisfies

$$0 \leq c_3 \leq \min \left\{ \frac{\varepsilon}{2c_0 c_4}, \frac{\alpha}{2c_0 c_4 c_2^{m_0} (2m_0 + 1)^2 (\delta + \gamma)}, \frac{2}{c_0 c_4} \right\},$$

where

$$c_4 = c_2^{m_0-1} + c_2^{m_0-2} + \cdots + c_2 + 1,$$

$$0 < \varepsilon = \begin{cases} \frac{\alpha \underline{\lambda}}{2\alpha^2 + 3\underline{\lambda}}, & \gamma = 0, \\ \min \left\{ \frac{1}{\gamma}, \frac{\alpha \underline{\lambda}}{2\alpha^2 + 3\underline{\lambda}} \right\}, & \gamma \neq 0. \end{cases}$$

For any $u, v \in l^2_\rho$, define new inner products on l^2_ρ by

$$(u, v)_{\lambda, \rho} = \sum_{i \in \mathbb{Z}} \rho_i \lambda_i u_i v_i, \quad \|u\|_{\lambda, \rho}^2 = \sum_{i \in \mathbb{Z}} \rho_i \lambda_i u_i^2,$$

$$(u, v)_{\delta, \lambda, \rho} = \delta (Du, Dv)_\rho + (u, v)_{\lambda, \rho},$$

$$\|u\|_{\delta, \lambda, \rho}^2 = \delta \|Du\|_\rho^2 + \|u\|_{\lambda, \rho}^2, \quad \delta = 1 - \varepsilon \gamma \in [0, 1],$$

then the norms $\|\cdot\|_\rho$, $\|\cdot\|_{\lambda, \rho}$ and $\|\cdot\|_{\delta, \lambda, \rho}$ are equivalent to each other because

$$\underline{\lambda} \|u\|_\rho^2 \leq \|u\|_{\lambda, \rho}^2 \leq \|u\|_{\delta, \lambda, \rho}^2 \leq \frac{\delta \bar{\lambda} c_0^2 c_2^{m_0} (2m_0 + 1)^2}{\underline{\lambda}} \|u\|_\rho^2.$$

Let $l^2_{\delta, \lambda, \rho} = (l^2_\rho, (\cdot, \cdot)_{\delta, \lambda, \rho}, \|\cdot\|_{\delta, \lambda, \rho})$, $H = l^2_{\delta, \lambda, \rho} \times l^2_\rho$, then H is a separable Hilbert space.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be defined in Sect. 1. Define a family of mappings $(\theta_t)_{t \in \mathbb{R}}$ on Ω : $\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for $\omega \in \Omega$, then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic metric dynamical system [4]. Let us consider the Ornstein–Uhlenbeck stationary process $z(\theta_t \omega) = -\alpha \int_{-\infty}^0 e^{\alpha s} (\theta_t \omega)(s) ds$ for $t \in \mathbb{R}$ and $\omega \in \Omega$, which solves the Itô stochastic equation $dz(\theta_t \omega) + \alpha z(\theta_t \omega) dt =$

$dW(t, \omega)$, where $W(t, \omega) = \omega(t)$. It is well known that the random variable $|z(\omega)|$ is tempered and there exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$, such that for every $\omega \in \Omega_0$, $t \mapsto z(\theta_t \omega)$ is continuous in t and

$$\lim_{t \rightarrow \pm\infty} \frac{|z(\theta_t \omega)|}{t} = \lim_{t \rightarrow \pm\infty} \frac{\int_0^t z(\theta_s \omega) ds}{t} = 0. \quad (3)$$

For convenience, we still write Ω_0 as Ω .

Introduce a variable transformation $v = u_t + \varepsilon u - cu z(\theta_t \omega)$, then the problem (2) is equivalent to the following random system:

$$\dot{\varphi} + L\varphi = F(\varphi, \theta_t \omega), \quad \varphi_\tau(\omega) = \begin{pmatrix} u_\tau \\ v_\tau(\omega) \end{pmatrix} = \begin{pmatrix} u_\tau \\ u_{1,\tau} + \varepsilon u_\tau - cu_\tau z(\theta_\tau \omega) \end{pmatrix}, \quad (4)$$

where

$$\varphi(t, \tau, \omega) = \begin{pmatrix} u \\ v \end{pmatrix}, \quad L = \begin{pmatrix} \varepsilon & -1 \\ \lambda + A - \varepsilon(\alpha - \varepsilon) - \gamma \varepsilon A & \alpha - \varepsilon + \gamma A \end{pmatrix} I, \quad (5)$$

$$F(\varphi, \theta_t \omega) = \begin{pmatrix} cu z(\theta_t \omega) \\ [2\varepsilon cz(\theta_t \omega) - c^2 z^2(\theta_t \omega)]u - cz(\theta_t \omega)v - \gamma cz(\theta_t \omega)Au - f(u, t) + g(t) \end{pmatrix}. \quad (6)$$

It follows from assumptions (A1)–(A4) and theory of ordinary differential equations that, for every $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $\varphi_\tau(\omega) = \begin{pmatrix} u_\tau \\ v_\tau(\omega) \end{pmatrix} \in H$, the problem (4) has a unique solution $\varphi(\cdot, \tau, \omega, \varphi_\tau) \in C([\tau, \tau + T], H)$ for any $T > 0$, where $\varphi(\tau, \tau, \omega, \varphi_\tau) = \varphi_\tau$ and $\varphi(t, \tau, \omega, \varphi_\tau)$ is continuous in $\varphi_\tau \in H$, which defines a continuous cocycle $\Phi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \rightarrow H$ by

$$(t, \tau, \omega, \varphi_\tau) \rightarrow \Phi(t, \tau, \omega, \varphi_\tau) = \Phi(t, \tau, \omega) \varphi_\tau = \varphi(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_\tau(\theta_{-\tau} \omega))$$

over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ with state space H .

2.1 Random exponential attractor

In this subsection, we present the definition and existence conditions of a random exponential attractor for the continuous cocycle $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$ in H . Let $\mathcal{D}(H)$ be the collection of all the tempered families of nonempty subsets of H [3]. A family $D = \{D(\tau, \omega) \subset H\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ of nonempty subsets of H is said to be tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if, for every $\epsilon > 0$, $\tau \in \mathbb{R}$, and a.e. $\omega \in \Omega$, $\lim_{t \rightarrow \infty} e^{-\epsilon|t|} \|D(\tau + t, \theta_t \omega)\|_H = 0$, where $\|D(\tau, \omega)\|_H = \sup_{\varphi \in D(\tau, \omega)} \|\varphi\|_H$.

Definition 2.1 ([3]) A family $\{\mathcal{A}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ of subsets of H is called a random exponential attractor in $\mathcal{D}(H)$ for a continuous cocycle $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$ on H over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ if, for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, (i) $\mathcal{A}(\tau, \omega)$ is compact subset in H and measurable in ω ; (ii) $\Phi(t, \tau - t, \theta_{-t} \omega) \mathcal{A}(\tau - t, \theta_{-t} \omega) \subseteq \mathcal{A}(\tau, \omega)$ for all $t \geq 0$; (iii) there exists a random variable ζ_ω ($< \infty$) such that $\sup_{\tau \in \mathbb{R}} \dim_f \mathcal{A}(\tau, \omega) \leq \zeta_\omega < \infty$, where $\dim_f \mathcal{A}(\tau, \omega) = \limsup_{\varepsilon \rightarrow 0^+} \frac{\ln N_\varepsilon(\mathcal{A}(\tau, \omega))}{-\ln \varepsilon}$ is the fractal dimension of $\mathcal{A}(\tau, \omega)$ and $N_\varepsilon(\mathcal{A}(\tau, \omega))$ is the minimal number of balls with radius ε covering $\mathcal{A}(\tau, \omega)$ in H ; (iv) there exists a constant $\tilde{a} > 0$ such that, for any $D \in \mathcal{D}(H)$, there exist random variables $t_D(\tau, \omega) \geq 0$,

$Q(\tau, \omega, D) > 0$ satisfying $d_h(\Phi(t, \tau - t, \theta_{-t}\omega)D(\tau - t, \theta_{-t}\omega), \mathcal{A}(\tau, \omega)) \leq Q(\tau, \omega, D)e^{-\tilde{a}t}$ for all $t \geq t_D(\tau, \omega)$, where d_h denotes the Hausdorff semidistance between two subsets.

Based on Theorem 2.8 in [13] and Theorems 2.1–2.4 in [3], by making some slightly revision in the proof, we have the following theorem.

Theorem 2.1 *Consider the continuous cocycle $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$. Assume that:*

- (B1) *There exist a family of uniformly (with respect to $\tau \in \mathbb{R}$) tempered closed measurable set $D_0 = \{D_0(\omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega} \in \mathcal{D}(H)$ satisfying that, for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D(\tau, \omega) \in \mathcal{D}(H)$, there exists a $t_D(\tau, \omega) \geq 0$ such that $\Phi(t, \tau - t, \omega)D(\tau - t, \theta_{-t}\omega) \subseteq D_0(\omega)$ for all $t \geq t_D(\tau, \omega)$. Particularly, there exists a $t_{D_0}(\omega) \geq 0$ (independent of τ) such that $\Phi(t, \tau - t, \omega)D_0(\theta_{-t}\omega) \subseteq D_0(\omega)$ for all $t \geq t_{D_0}(\omega)$. For any $\omega \in \Omega$, $\tau \in \mathbb{R}$, set*

$$\mathcal{X}(\tau, \omega) = \bigcup_{t \geq t_{D_0}(\omega)} \Phi(t, \tau - t, \theta_{-t}\omega)D_0(\theta_{-t}\omega) \subseteq D_0(\omega).$$

- (B2) *There exist some positive numbers $\hat{\lambda} > 0$, $\hat{\delta} > 0$, $k \in \mathbb{N}$ (independent of (τ, ω)), but $\hat{\delta}$, k maybe depend on $\hat{\lambda}$) and random variables $C_0(\omega)$, $C_1(\omega) \geq 0$ such that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and any $u, v \in \mathcal{X}(\tau, \omega) \subset H$, we have*

$$\begin{aligned} \|\Phi(t, \tau, \omega)u - \Phi(t, \tau, \omega)v\|_H &\leq e^{\int_0^t C_0(\theta_s\omega) ds} \|u - v\|_H, \quad \forall t \in \left[0, \frac{8 \ln 2}{\hat{\lambda}}\right], \\ \sum_{|i| > k} \rho_i \left(\Phi\left(\frac{8 \ln 2}{\hat{\lambda}}, \tau, \omega\right)u - \Phi\left(\frac{8 \ln 2}{\hat{\lambda}}, \tau, \omega\right)v \right)_i^2 \\ &\leq \left(e^{-8 \ln 2 + \int_0^{\frac{8 \ln 2}{\hat{\lambda}}} C_1(\theta_s\omega) ds} + \frac{\hat{\delta}}{2} e^{\int_0^{\frac{8 \ln 2}{\hat{\lambda}}} C_0(\theta_s\omega) ds} \right)^2 \|u - v\|_H^2 \end{aligned}$$

and

$$\begin{cases} 0 \leq \mathbf{E}[C_1(\omega)] \leq \frac{\hat{\lambda}}{16}, & 0 \leq \mathbf{E}[C_0^2(\omega)] < \infty, \\ 0 < \hat{\delta} \leq \min\left\{\frac{1}{16}, e^{-\frac{128 \ln^2 2}{\hat{\lambda}^2 \ln \frac{3}{2}}(\mathbf{E}[C_0^2(\omega)] + \hat{\lambda} \mathbf{E}[C_0(\omega)])}\right\}, \end{cases}$$

where $\mathbf{E}$ denotes the expectation.

(B3)

$$\begin{cases} \lim_{t \rightarrow 0^+} \sup_{u \in \mathcal{X}(\tau, \omega)} \|\Phi(t, \tau, \omega)u - u\|_H = 0, \\ \lim_{t \rightarrow 0^+} \sup_{u \in \mathcal{X}(\tau - t, \theta_{-t}\omega)} \|\Phi(0, \tau - t, \theta_{-t}\omega)u - u\|_H = 0. \end{cases}$$

Then $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$ has a random exponential attractor $\{\mathcal{A}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ such that, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$,

- (i) $\mathcal{A}(\tau, \omega) (\subseteq \overline{\mathcal{X}(\tau, \omega)})$ is a compact set of H and measurable in ω ;
- (ii) $\Phi(t, \tau, \omega)\mathcal{A}(\tau, \omega) \subseteq \mathcal{A}(\tau + t, \theta_t\omega)$;
- (iii) $\dim_f \mathcal{A}(\tau, \omega) \leq \frac{4(2k+1) \ln(\frac{2\sqrt{2(2k+1)}}{\hat{\delta}} + 1)}{\ln \frac{4}{3}} < \infty$;

- (iv) for every set $D \in \mathcal{D}(H)$, there exist a random variable $T_{\omega,D} \geq 0$ and a tempered random variable $b_{\omega,D} > 0$ such that

$$d_H(\Phi(t, \tau, \omega)D(\tau, \omega), \mathcal{A}(t + \tau, \theta_t \omega)) \leq b_{\omega,D} e^{-\frac{\lambda \ln \frac{4}{3}}{32 \ln 2} t}, \quad t \geq T_{\omega,D};$$

- (v) for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\lim_{t \rightarrow 0} d_H(\mathcal{A}(\tau + t, \theta_t \omega), \mathcal{A}(\tau, \omega)) = 0$.

Remark 2.1 The condition (B2) in Theorem 2.1 above is different from the corresponding condition in the well-known publication [3].

In the following, we will prove the existence of a random exponential attractor for Φ based on Theorem 2.1.

2.2 Estimations of bound and tail of solutions

Write $\|g\|_\rho = \sup_{r \in \mathbb{R}} \|g(r)\|_\rho$, $\|\beta\|_\rho = \sup_{r \in \mathbb{R}} \|\beta(r)\|_\rho$ and

$$M_0^2(\omega) = c_5 K_0(\omega), \quad K_0(\omega) = \int_{-\infty}^0 e^{\mu s + a_7 \int_s^0 (|c| |z(\theta_l \omega)| + |c|^2 |z(\theta_l \omega)|^2) dl} ds, \quad (7)$$

where

$$\begin{cases} a_7 = \max\left\{\frac{2a_6}{\alpha} + \frac{2\varepsilon}{\sqrt{\alpha}} + 2, \frac{1}{\sqrt{\alpha}} + \frac{8a_5^2}{\alpha^2}, \frac{a_1+1}{a_3}\right\}, \\ a_5 = |\gamma| c_0^2 c_2^{m_0} (2m_0 + 1)^2, \quad a_6 = \frac{p-1}{p^{p-1}}, \\ c_5 = \frac{8}{\alpha} \|g\|_\rho^2 + 2\|\beta\|_\rho^2, \quad \mu = \min\left\{\varepsilon, \frac{\varepsilon}{2a_2}, 1\right\}. \end{cases}$$

In the following part of this section, we always assume that conditions (A1)–(A4) and

$$a_7 \left(\frac{|c|}{\sqrt{\pi\alpha}} + \frac{|c|^2}{2\alpha} \right) \leq \frac{\mu}{2} \quad (8)$$

hold.

Lemma 2.1 For every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}(H)$, there exist $T_D(\tau, \omega) \geq 0$ and a tempered random variable $M_0(\omega)$ (independent of τ), such that the solution $\varphi(r, \tau - t, \theta_{-\tau} \omega, \varphi_{\tau-t}(\theta_{-\tau} \omega)) \in H$ ($r \geq \tau - t$) of (4) with $\varphi_{\tau-t}(\theta_{-\tau} \omega) \in D(\tau - t, \theta_{-\tau} \omega)$ satisfies

$$\varphi(\tau, \tau - t, \theta_{-\tau} \omega, \varphi_{\tau-t}(\theta_{-\tau} \omega)) \leq M_0(\omega), \quad \forall t \geq T(\tau, \omega, D).$$

Proof From (3) and (8), it is easy to see that $M_0(\omega)$ is a tempered random variable. Taking the inner product of (4) with $\varphi(r) = \varphi(r, \tau - t, \theta_{-\tau} \omega, \varphi_{\tau-t}(\theta_{-\tau} \omega))$ in H , we have

$$\frac{1}{2} \frac{d}{dt} \|\varphi\|_H^2 + (L(\varphi), \varphi)_H = (f(\varphi, \theta_{r-\tau} \omega), \varphi)_H. \quad (9)$$

The second term of (9) is

$$(L\varphi, \varphi)_H = \varepsilon \|u\|_{\delta, \lambda, \rho}^2 - \delta (Du, Dv)_\rho - (v, u)_{\lambda, \rho} + \lambda (u, v)_\rho + (1 - \gamma \varepsilon) (Au, v)_\rho$$

$$-\varepsilon(\alpha - \varepsilon)(u, v)_\rho + (\alpha - \varepsilon)(v, v)_\rho + \gamma(Av, v)_\rho. \quad (10)$$

From [14], we have

$$(Au, v)_\rho \geq (Du, Dv)_\rho - \frac{1}{2}c_0c_3c_4\|Du\|_\rho^2 - \frac{1}{2}c_0c_2^{m_0}c_3c_4(2m_0 + 1)^2\|v\|_\rho^2, \quad (11)$$

$$(Av, v)_\rho \geq \left(1 - \frac{1}{2}c_0c_3c_4\right)\|Dv\|_\rho^2 - \frac{1}{2}c_0c_2^{m_0}c_3c_4(2m_0 + 1)^2\|v\|_\rho^2. \quad (12)$$

By (A4), (10)–(12), we have

$$\begin{aligned} (L\varphi, \varphi)_H &\geq \frac{3\varepsilon}{4}\|u\|_{\delta, \lambda, \rho}^2 - \varepsilon(\alpha - \varepsilon)(u, v)_\rho + \left(\frac{3\alpha}{4} - \varepsilon\right)\|v\|_\rho^2 \\ &\quad + \left(\frac{\delta\varepsilon}{4} - \frac{\delta}{2}c_0c_3c_4\right)\|Du\|_\rho^2 + \gamma\left(1 - \frac{1}{2}c_0c_3c_4\right)\|Dv\|_\rho^2 \\ &\quad + \left(\frac{\alpha}{4} - \frac{\delta + \gamma}{2}c_0c_2^{m_0}c_3c_4(2m_0 + 1)^2\right)\|v\|_\rho^2 \\ &\geq \frac{\varepsilon}{2}\|u\|_{\delta, \lambda, \rho}^2 + \frac{\varepsilon}{2}\|v\|_\rho^2 + \frac{\alpha}{4}\|v\|_\rho^2 + \frac{\varepsilon}{4}\|u\|_{\delta, \lambda, \rho}^2 + \left(\frac{3\alpha}{4} - \varepsilon\right)\|v\|_\rho^2 \\ &\quad + \left(\frac{\alpha}{2} - \frac{3\varepsilon}{2}\right)\|v\|_\rho^2 - \frac{\varepsilon\alpha}{\sqrt{\lambda}}\|u\|_{\delta, \lambda, \rho}\|v\|_\rho. \end{aligned}$$

From (A4), $4\frac{\varepsilon}{4}(\frac{\alpha}{2} - \frac{3\varepsilon}{2}) \geq \frac{\varepsilon^2\alpha^2}{\lambda}$, we have

$$(L\varphi, \varphi)_H \geq \frac{\varepsilon}{2}\|\varphi(r)\|_H^2 + \frac{\alpha}{4}\|v\|_\rho^2.$$

The right-hand side of (9) is

$$(f(\varphi, \theta_{r-\tau}\omega), \varphi)_H = (czu, u)_{\delta, \lambda, \rho} + (-czv + 2\varepsilon czu - c^2z^2u + g(r) - \gamma cAz u - f(u, r), v)_\rho. \quad (13)$$

For each term of (13), we have

$$(czu, u)_{\delta, \lambda, \rho} - (czv, v)_\rho \leq |c||z|\|\varphi(r)\|_H^2, \quad (g, v)_\rho \leq \frac{4}{\alpha}\|g\|_\rho^2 + \frac{\alpha}{8}\|v\|_\rho^2, \quad (14)$$

$$(2\varepsilon czu, v)_\rho - (c^2z^2u, v)_\rho \leq \left(\frac{|c||\varepsilon||z|}{\sqrt{\lambda}} + \frac{|c|^2|z|^2}{2\sqrt{\lambda}}\right)\|\varphi(r)\|_H^2, \quad (15)$$

$$\begin{aligned} -(\gamma cAz u, v)_\rho &\leq \frac{4|\gamma cz|^2\|Au\|_\rho^2}{\alpha} + \frac{\alpha}{8}\|v\|_\rho^2 \\ &\leq \frac{4|\gamma cz|^2c_0^4c_2^{2m_0}(2m_0 + 1)^4}{\alpha\lambda}\|u\|_{\lambda, \rho}^2 + \frac{\alpha}{8}\|v\|_\rho^2 \\ &\leq \frac{4|cz|^2a_5^2}{\alpha\lambda}\|\varphi\|_H^2 + \frac{\alpha}{8}\|v\|_\rho^2, \end{aligned} \quad (16)$$

$$-(f(u, r), v)_\rho = -(f(u, r), \dot{u})_\rho - \varepsilon(f(u, r), u)_\rho + cz(f(u, r), u)_\rho. \quad (17)$$

By (A2) we have

$$\begin{aligned} (f(u, r), \dot{u})_\rho &= \sum_{i \in \mathbb{Z}} \rho_i f_i(u_i, r) \dot{u}_i = \frac{d}{dt} \left(\sum_{i \in \mathbb{Z}} \rho_i G_i(u_i, r) \right) - \sum_{i \in \mathbb{Z}} \rho_i G'_{i,r}(u_i, r) \\ &\geq \frac{d}{dt} \bar{G}(u, r) - a_4 \bar{G}(u, r) \geq \frac{d}{dt} \bar{G}(u, r) - \frac{\varepsilon}{2a_2} \bar{G}(u, r), \end{aligned} \quad (18)$$

$$\varepsilon (f(u, r), u)_\rho = \varepsilon \sum_{i \in \mathbb{Z}} \rho_i f_i(u_i, r) u_i \geq \frac{\varepsilon}{a_2} \sum_{i \in \mathbb{Z}} \rho_i G_i(u_i, r) = \frac{\varepsilon}{a_2} \bar{G}(u, r). \quad (19)$$

By the Young inequality, we obtain

$$a_1 |s| \leq |s|^p + a_1^{\frac{p}{p-1}} \times \frac{p-1}{p^{\frac{p}{p-1}}} = |s|^p + a_6, \quad p > 1. \quad (20)$$

By (A4) and (20),

$$|f'_i(s, r)| \leq a_1 |s| + a_1 |s|^p \leq a_6 + (1 + a_1) |s|^p, \quad (21)$$

from (21), we get

$$|f_i(s, r)| \leq a_6 |s| + (1 + a_1) |s|^{p+1}. \quad (22)$$

From (22), we have

$$\begin{aligned} cz(f(u, r), u)_\rho &= cz \sum_{i \in \mathbb{Z}} \rho_i f_i(u_i, r) u_i \leq |c| |z| \left| \sum_{i \in \mathbb{Z}} \rho_i f_i(u_i, r) u_i \right| \\ &\leq a_6 |c| |z| \|u\|_\rho^2 + \frac{(a_1 + 1)}{a_3} |c| |z| \sum_{i \in \mathbb{Z}} \rho_i (G_i(u_i, r) + \beta_i^2(r)) \\ &\leq a_6 |c| |z| \|u\|_\rho^2 + \frac{(a_1 + 1)}{a_3} |c| |z| \bar{G}(u, r) \\ &\quad + \frac{(a_1 + 1)}{a_3} |c| |z| \|\beta\|_\rho^2. \end{aligned} \quad (23)$$

Thus, by all the above inequalities, we have

$$\begin{aligned} &\frac{d}{dt} [\|\varphi(r)\|_H^2 + 2\bar{G}(u, r) + 2\|\beta\|_\rho^2] + \mu [\|\varphi(r)\|_H^2 + 2\bar{G}(u, r) + 2\|\beta\|_\rho^2] \\ &\leq \left(2|c| |z| + \frac{2|c| |\varepsilon| |z|}{\sqrt{\lambda}} + \frac{|c|^2 |z|^2}{\sqrt{\lambda}} + \frac{8|cz|^2 |a_5^2|}{\alpha \lambda} + \frac{2a_6 |c| |z|}{\lambda} \right) \|\varphi(r)\|_H^2 \\ &\quad + \frac{2(a_1 + 1)}{a_3} |c| |z| \bar{G}(u, r) + \frac{2(a_1 + 1)}{a_3} |c| |z| \|\beta\|_\rho^2 + \frac{8}{\alpha} \|g\|_\rho^2 + 2\|\beta\|_\rho^2. \end{aligned}$$

Then, for $r \geq \tau - t$, we have

$$\frac{d}{dt} y(r) \leq (-\mu + a_7 (|c| |z(\theta_{r-\tau} \omega)| + |c|^2 |z(\theta_{r-\tau} \omega)|^2)) y(r) + c_5, \quad (24)$$

where

$$y(r) = \|\varphi(r)\|_H^2 + 2\overline{G}(u, r) + 2|\beta|_\rho^2 \geq \|\varphi(r)\|_H^2.$$

By the Gronwall inequality applied to (24) on $[\tau - t, r]$, we obtain

$$\begin{aligned} \|\varphi(r)\|_H^2 &\leq e^{\int_{\tau-t}^r (-\mu + a_7(|c||z(\theta_{l-\tau}\omega)| + |c|^2|z(\theta_{l-\tau}\omega)|^2)) dl} y(\tau - t) \\ &\quad + c_5 \int_{\tau-t}^r e^{\int_s^r (-\mu + a_7(|c||z(\theta_{l-\tau}\omega)| + |c|^2|z(\theta_{l-\tau}\omega)|^2)) dl} ds, \end{aligned} \quad (25)$$

and let $r = \tau$, it follows that

$$\|\varphi(\tau)\|_H^2 \leq e^{\int_{-\tau}^0 (-\mu + a_7(|c||z(\theta_l\omega)| + |c|^2|z(\theta_l\omega)|^2)) dl} y(\tau - t) + \frac{1}{2} M_0^2(\omega). \quad (26)$$

By $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in D(\tau - t, \theta_{-\tau}\omega) \in \mathcal{D}(H)$ and (8), we have

$$\lim_{t \rightarrow +\infty} e^{\int_{-\tau}^0 (-\mu + a_7(|c||z(\theta_l\omega)| + |c|^2|z(\theta_l\omega)|^2)) dl} y(\tau - t) = 0.$$

Thus, the proof is completed. $\square$

Let

$$D_0 = \{D_0(\omega) = \{\varphi \in H : \|\varphi\|_H \leq M_0(\omega)\}\} \in D(H),$$

then D_0 is a tempered family. By Lemma 2.1, there exists $T_{D_0}(\omega) > 0$ (independent of τ) such that, for any $\omega \in \Omega$, $\tau \in \mathbb{R}$,

$$\varphi(r, \tau - t, \theta_{-\tau}\omega, D_0(\theta_{-\tau}\omega)) \in D_0(\omega), \quad \forall r \geq \tau - t, t \geq T_{D_0}(\omega) \geq 0.$$

Choosing a smooth increasing function $\eta \in C^1(\mathbb{R}^+, \mathbb{R})$ satisfies

$$\begin{cases} \eta(s) = 0, & s \in [0, 1); \\ 0 \leq \eta(s) \leq 1, & s \in [1, 2); \\ \eta(s) = 1, & s \in [2, +\infty); \\ |\eta'(s)| \leq C_0, & \forall s \in \mathbb{R}^+, C_0 > 0. \end{cases}$$

Lemma 2.2 For every $\tau \in \mathbb{R}$, $t \geq 0$, $\omega \in \Omega$, $I \in \mathbb{N}$ and $v > 0$, there exists $T_v(\omega) > 0$ (independent of τ) such that the solution $\varphi(r) = \varphi(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))$ ($r \geq \tau - t$) of (4) with $\varphi_{\tau-t}(\theta_{-\tau}\omega) \in D_0(\theta_{-\tau}\omega)$ satisfies

$$\sum_{|i| \geq 2I} |\varphi_i(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})|_H^2 \leq v + \left(\frac{c_6}{I} + \gamma_I\right) K_2(\omega), \quad t > T_v(\omega), I \in \mathbb{N}, \quad (27)$$

where

$$\begin{cases} |\varphi_i|_H^2 = \rho_i [\delta (Du_i)^2 + \lambda_i u_i^2 + v_i^2], & c_6 = \frac{c_5}{2}; \\ \gamma_I = \sup_{r \in \mathbb{R}} \sum_{i \geq I} \rho_i (\frac{8}{\alpha} g_i^2(r) + 4\beta_i^2(r) + 4\beta_i'^2(r)); \\ K_2(\omega) = K_0(\omega)(1 + K_1(\omega)); \\ K_1(\omega) = \int_{-\infty}^0 \xi_1(\theta_s \omega) e^{\mu s + a_7 \int_s^0 (|c||z(\theta_l \omega)| + |c|^2 |z(\theta_l \omega)|^2)} dl ds; \\ \xi_1(\omega) = 2c_7(\underline{\lambda} + \varepsilon + 2|cz(\omega)|) + 2c_8(\delta + \gamma_0); \\ c_7 = \frac{\delta m_0 C_0 c_0^2 (2m_0 + 1)^2 c_2^{m_0}}{\underline{\lambda}}, \quad c_8 = m_0 C_0 c_0 c_2^{\frac{m_0}{2}} (1 + (2m_0 + 1)^2). \end{cases} \quad (28)$$

Proof Let $I \in \mathbb{N}$ be a suitable large integer, and set $x_i = \eta(\frac{|i|}{I})u_i$, $y_i = \eta(\frac{|i|}{I})v_i$, $w = (x, y) = (x_i, y_i)_{i \in \mathbb{Z}}^T$. Taking the inner product $(\cdot, \cdot)_H$ of (4) with w , we have

$$(\dot{\varphi}, w)_H + (L\varphi, w)_H = (F(\varphi), w)_H. \quad (29)$$

By some computations, it follows that

$$(\dot{\varphi}, w)_H \geq \frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) |\varphi_i|_H^2 - \frac{c_7}{I} (\underline{\lambda} + \varepsilon + |cz(\theta_{r-\tau} \omega)|) \|\varphi\|_H^2, \quad (30)$$

$$(L\varphi, w)_H \geq \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) \left(\frac{\varepsilon}{2} |\varphi_i|_H^2 + \frac{\alpha}{4} \rho_i v_i^2 \right) - \frac{c_8(\delta + \gamma)}{I} \|\varphi\|_H^2, \quad (31)$$

$$\begin{aligned} & (cz(\theta_{r-\tau} \omega)u, x)_{\delta, \lambda, \rho} - (cz(\theta_{r-\tau} \omega)v, y)_{\rho} \\ & \leq |c| |z(\theta_{r-\tau} \omega)| \left(\sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) |\varphi_i|_H^2 + \frac{c_7}{I} \|\varphi\|_H^2 \right), \end{aligned} \quad (32)$$

$$(2\varepsilon cz(\theta_{r-\tau} \omega)u, y)_{\rho} \leq \frac{\varepsilon |c| |z(\theta_{r-\tau} \omega)|}{\sqrt{\underline{\lambda}}} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) |\varphi_i|_H^2, \quad (33)$$

$$(-c^2 z^2(\theta_{r-\tau} \omega)u, y)_{\rho} \leq \frac{c^2 z^2(\theta_{r-\tau} \omega)}{2\sqrt{\underline{\lambda}}} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) |\varphi_i|_H^2, \quad (34)$$

$$\begin{aligned} & (-\gamma c A z(\theta_{r-\tau} \omega)u, y)_{\rho} \leq \frac{4|cz|^2 a_5^2}{\underline{\lambda} \alpha} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) |\varphi_i|_H^2 \\ & \quad + \frac{\alpha}{8} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) \rho_i v_i^2, \end{aligned} \quad (35)$$

$$(g, y)_{\rho} \leq \frac{4}{\alpha} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) \rho_i g_i^2 + \frac{\alpha}{8} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) \rho_i v_i^2, \quad (36)$$

$$\begin{aligned} & (f(u, r), y)_{\rho} \\ & \geq \frac{d}{dt} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) \rho_i G_i(u_i, r) + \frac{\varepsilon}{2a_2} \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) \rho_i G_i(u_i, r) \\ & \quad - a_6 |c| |z| \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) \rho_i u_i^2 - \frac{a_1 + 1}{a_3} |c| |z| \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) \rho_i G_i(u_i, r) \\ & \quad - \frac{a_1 + 1}{a_3} |c| |z| \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) \rho_i \beta_i^2(r). \end{aligned} \quad (37)$$

Thus, by putting (30)–(37) into (29), we obtain, for $r \geq \tau - t$,

$$\begin{aligned} & \frac{d}{dt} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{I} \right) [|\varphi_i(r)|_H^2 + 2\rho_i G_i(u_i, r) + 2\rho_i \beta_i^2(r)] \\ & \leq (-\mu + a_7 (|c| |z(\theta_{r-\tau} \omega)| + |c|^2 |z(\theta_{r-\tau} \omega)|^2)) \\ & \quad \times \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{I} \right) [|\varphi_i(r)|_H^2 + 2\rho_i G_i(u_i, r) + 2\rho_i \beta_i^2(r)] \\ & \quad + \frac{\xi_1(\theta_{r-\tau} \omega)}{I} \|\varphi(r)\|_H^2 + \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{I} \right) \rho_i \left[\frac{8}{\alpha} g_i^2(r) + 4\beta_i^2(r) + 4\beta_i'^2(r) \right]. \end{aligned} \quad (38)$$

By applying the Gronwall inequality to (38) on $[\tau - t, \tau]$, we have

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{I} \right) [|\varphi_i(\tau)|_H^2 + 2\rho_i G_i(u_i, \tau) + 2\rho_i \beta_i^2(\tau)] \\ & \leq [\|\varphi_{\tau-t}\|_H^2 + 2G(u_{\tau-t}, \tau - t) + 2\|\beta(\tau - t)\|_\rho^2] \\ & \quad \times e^{\int_{\tau-t}^0 [-\mu + a_7 (|c| |z(\theta_l \omega)| + |c|^2 |z(\theta_l \omega)|^2)] dl} \\ & \quad + \sup_{r \in \mathbb{R}} \sum_{i \geq I} \rho_i \left(\frac{8}{\alpha} g_i^2(r) + 4\beta_i^2(r) + 4\beta_i'^2(r) \right) \\ & \quad \times \int_{\tau-t}^0 e^{\int_s^0 [-\mu + a_7 (|c| |z(\theta_l \omega)| + |c|^2 |z(\theta_l \omega)|^2)] dl} ds \\ & \quad + \frac{1}{I} \int_{\tau-t}^{\tau} \xi_1(\theta_{s-\tau} \omega) \|\varphi(s)\|_H^2 \\ & \quad \times e^{\int_s^{\tau} [-\mu + a_7 (|c| |z(\theta_{l-\tau} \omega)| + |c|^2 |z(\theta_{l-\tau} \omega)|^2)] dl} ds. \end{aligned} \quad (39)$$

Since $\xi_1(\omega)$ is tempered and $\xi_1(\theta_t \omega)$ is continuous in t , by Proposition 4.3.3 of [15], for the fixed number $\mu > 0$, there exists a tempered random variable $\sigma(\omega)$ such that $\xi_1(\theta_t \omega) \leq \sigma(\omega) e^{\frac{\mu}{3}|t|}$ for $t \in \mathbb{R}$. Thus, by (25),

$$\begin{aligned} & \int_{\tau-t}^{\tau} \xi_1(\theta_{s-\tau} \omega) \|\varphi(s)\|_H^2 e^{\int_s^{\tau} [-\mu + a_7 (|c| |z(\theta_{l-\tau} \omega)| + |c|^2 |z(\theta_{l-\tau} \omega)|^2)] dl} ds \\ & \leq y(\tau - t) e^{\int_{\tau-t}^{\tau} (-\mu + a_7 (|c| |z(\theta_{l-\tau} \omega)| + |c|^2 |z(\theta_{l-\tau} \omega)|^2)) dl} \int_{\tau-t}^{\tau} \xi_1(\theta_{s-\tau} \omega) ds \\ & \quad + \frac{1}{2} M_0^2(\omega) \int_{\tau-t}^{\tau} \xi_1(\theta_{s-\tau} \omega) e^{\int_s^{\tau} [-\mu + a_7 (|c| |z(\theta_{l-\tau} \omega)| + |c|^2 |z(\theta_{l-\tau} \omega)|^2)] dl} ds. \end{aligned} \quad (40)$$

From (A4) and (22)

$$\begin{aligned} & y(\tau - t, \tau - t, \theta_{-\tau} \omega, \varphi_{\tau-t}) \\ & = \|\varphi_{\tau-t}(\theta_{-\tau} \omega)\|_H^2 + 2 \sum_{i \in \mathbb{Z}} \rho_i f_i(u_{i, \tau-t}, \tau - t) u_{i, \tau-t} + 2|\beta|_\rho^2 \\ & \leq \|\varphi_{\tau-t}(\theta_{-\tau} \omega)\|_H^2 + 2a_2(a_6 \|u_{\tau-t}\|_\rho^2 + (1 + a_1) \|u_{\tau-t}\|_\rho^{p+2}) + 2|\beta|_\rho^2, \end{aligned}$$

thus, (40) can be written as

$$\begin{aligned} & \int_{\tau-t}^{\tau} \xi_1(\theta_{s-\tau}\omega) y(s) e^{\int_s^{\tau} [-\mu+a_7(|c||z(\theta_{l-\tau}\omega)|+|c|^2|z(\theta_{l-\tau}\omega)|^2)]} dl ds \\ & \leq \frac{\mu c_9}{3} \sigma(\omega) (M_0^2(\theta_{-t}\omega) + M_0^{p+2}(\theta_{-t}\omega) + 1) e^{-\frac{2\mu}{3}t+a_7 \int_{-t}^0 (|c||z(\theta_l\omega)|+|c|^2|z(\theta_l\omega)|^2)} dl \\ & \quad + \frac{1}{2} M_0^2(\omega) \int_{-\infty}^0 \xi_1(\theta_s\omega) e^{\int_s^0 [-\mu+a_7(|c||z(\theta_l\omega)|+|c|^2|z(\theta_l\omega)|^2)]} dl ds, \end{aligned}$$

where $c_9 = \max\{1 + \frac{2a_7a_6}{\frac{\lambda}{2}}, \frac{2a_7(a_1+1)}{\frac{\lambda}{2}+1}, 2\|\beta\|_{\rho}^2\}$, and

$$\begin{aligned} & \int_{-\infty}^0 \xi_1(\theta_s\omega) e^{\int_s^0 [-\mu+a_7(|c||z(\theta_l\omega)|+|c|^2|z(\theta_l\omega)|^2)]} dl ds \\ & \leq \sigma(\omega) \int_{-\infty}^0 e^{\int_s^0 [-\frac{2\mu}{3}+a_7(|c||z(\theta_l\omega)|+|c|^2|z(\theta_l\omega)|^2)]} dl ds < \infty. \end{aligned}$$

Thus, by (39) and $I \geq 1$, we have

$$\begin{aligned} & \sum_{i \in \mathbb{Z}} \eta\left(\frac{|i|}{I}\right) [|\varphi_i(\tau)|_H^2 + 2\rho_i G_i(u_{i,\tau}, \tau) + 2\rho_i \beta_i^2(\tau)] \\ & \leq c_9 (M_0^2(\theta_{-t}\omega) + M_0^{p+2}(\theta_{-t}\omega) + 1) e^{\int_{-t}^0 (-\mu+a_7(|c||z(\theta_l\omega)|+|c|^2|z(\theta_l\omega)|^2))} dl \\ & \quad + \frac{1}{I} \frac{3}{\mu} c_9 (M_0^2(\theta_{-t}\omega) + M_0^{p+2}(\theta_{-t}\omega) + 1) e^{-\frac{2\mu}{3}t+a_7 \int_{-t}^0 (|c||z(\theta_l\omega)|+|c|^2|z(\theta_l\omega)|^2)} dl \\ & \quad + \frac{1}{2I} M_0^2(\omega) \int_{-\infty}^0 \xi_1(\theta_s\omega) e^{\int_s^0 (-\mu+a_7(|c||z(\theta_l\omega)|+|c|^2|z(\theta_l\omega)|^2))} dl ds \\ & \quad + \gamma_I \int_{-\infty}^0 e^{\int_s^0 [-\mu+a_7(|c||z(\theta_l\omega)|+|c|^2|z(\theta_l\omega)|^2)]} dl ds \\ & \leq c_9 \left(1 + \frac{3}{\mu}\right) (M_0^2(\theta_{-t}\omega) + M_0^{p+2}(\theta_{-t}\omega) + 1) e^{-\frac{2\mu}{3}t+a_7 \int_{-t}^0 (|c||z(\theta_l\omega)|+|c|^2|z(\theta_l\omega)|^2)} dl \\ & \quad + \left(\frac{c_6}{I} + \gamma_I\right) K_2(\omega). \end{aligned} \quad (41)$$

By (3), we have

$$\lim_{t \rightarrow +\infty} \left(\frac{2\mu}{3} - \frac{a_7}{t} \int_{-t}^0 (|c||z(\theta_l\omega)| + |c|^2|z(\theta_l\omega)|^2) ds \right) \geq \frac{\mu}{6} > 0.$$

Since $M_0^2(\omega) + M_0^{p+2}(\omega) + 1$ is tempered, we have

$$(M_0^2(\theta_{-t}\omega) + M_0^{p+2}(\theta_{-t}\omega) + 1) e^{-\frac{2\mu}{3}t+a_7 \int_{-t}^0 (|c||z(\theta_l\omega)|+|c|^2|z(\theta_l\omega)|^2)} dl \rightarrow 0 \quad (42)$$

as $t \rightarrow +\infty$. Hence, by (41)–(42), for any $\nu > 0$, $I \in \mathbb{N}$, there exists $T_\nu(\omega) > 0$, such that

$$\sum_{|i| > 2I} |\varphi_i(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t})|_H^2 \leq \nu + \left(\frac{c_6}{I} + \gamma_I\right) K_2(\omega), \quad t \geq T_\nu(\omega), I \in \mathbb{N}.$$

This completes the proof. $\square$

It follows directly from Lemmas 2.1 and 2.2 and Theorem 3.6 in [1], Theorem 3.3 in [10] that

Theorem 2.2 *The cocycle Φ possesses a random attractor $R = \{R(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ in $D(H)$ with the properties: for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, (i) $R(\tau, \omega) \subseteq D_0(\omega)$; (ii) $R(\tau, \omega)$ is compact in H and measurable in ω ; (iii) $\Phi(t, \tau, \omega, R(\tau, \omega)) = R(t + \tau, \theta_t \omega)$, $\forall t \geq 0$; (iv) for every $B = \{B(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega} \in D(H)$, $\lim_{t \rightarrow +\infty} d_H(\Phi(t, \tau - t, \theta_{-\tau} \omega)B(\tau - t, \theta_{-\tau} \omega), R(\tau, \omega)) = 0$.*

2.3 Existence of random exponential attractor

In this subsection, we prove the existence of a random exponential attractor for Φ based on Theorem 2.1. Obviously, the family of tempered closed random subsets $D_0 = \{D_0(\omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ satisfies condition (B1) in Theorem 2.1.

Taking a small enough positive constant $v_0 > 0$ such that

$$\frac{32a_1^2 v_0}{\alpha \underline{\lambda}^2 \bar{\rho}} + \frac{a_1^2 2^{p+4} v_0^p}{\alpha \underline{\lambda}^{p+1} \bar{\rho}^p} \leq \frac{\varepsilon}{2}. \quad (43)$$

For any $\omega \in \Omega$, set

$$T_{v_0}(\omega) = \min \left\{ t : c_9 \left(1 + \frac{3}{\mu} \right) M(\theta_{-t} \omega) e^{-\frac{2\mu}{3}t + a_7 \int_{-t}^0 (|c||z(\theta_l \omega)| + |c|^2 |z(\theta_l \omega)|^2) dl} \leq v_0 \right\} < +\infty,$$

where $M(\omega) = M_0^2(\omega) + M_0^{p+2}(\omega) + 1$.

For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, define a tempered bounded random subset $\mathcal{X}_1(\tau, \omega)$ of H as

$$\mathcal{X}_1(\tau, \omega) = \bigcup_{s \geq \max\{T_{D_0}(\omega), T_{v_0}(\omega)\}} \varphi(\tau, \tau - s, \theta_{-\tau} \omega, D_0(\theta_{-s} \omega)) \subseteq D_0(\omega) \subset H. \quad (44)$$

Now let us show that $\{\mathcal{X}_1(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ satisfies (B2) of Theorem 2.1.

For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq 0$, and $\varphi_{j, \tau-t}(\theta_{-\tau} \omega) \in \mathcal{X}_1(\tau - t, \theta_{-t} \omega) \subseteq D_0(\theta_{-t} \omega)$, $j = 1, 2$. Let $\varphi_j(r) = \varphi(r, \tau - t, \theta_{-\tau} \omega, \varphi_{j, \tau-t}(\theta_{-\tau} \omega)) = (u_j(r), v_j(r))$, $j = 1, 2$, $\psi(r) = \varphi_1(r) - \varphi_2(r) = (\xi(r), \zeta(r))$, $r \geq \tau - t$, then

$$\begin{cases} \dot{\psi} + L\psi = F(\varphi_1, \theta_{r-\tau} \omega) - F(\varphi_2, \theta_{r-\tau} \omega), \\ \psi_\tau(\omega) = (\xi_\tau, \zeta_\tau) = (u_{1\tau} - u_{2\tau}, v_{1\tau} - v_{2\tau}), \quad r \geq \tau - t, \end{cases} \quad (45)$$

where

$$\begin{aligned} & F(\varphi_1, \theta_{r-\tau} \omega) - F(\varphi_2, \theta_{r-\tau} \omega) \\ &= \begin{pmatrix} cz(\theta_{r-\tau} \omega) \xi \\ (2\varepsilon - cz)cz\xi - \gamma cAz\xi - cz\zeta - f(u_1, t) + f(u_2, t) \end{pmatrix}. \end{aligned}$$

By Lemma 2.2 and (44), we have

$$\|\varphi_1(r)\|_H \leq M_0(\theta_{r-\tau} \omega), \quad \|\varphi_2(r)\|_H \leq M_0(\theta_{r-\tau} \omega), \quad \forall r \geq \tau - t \quad (46)$$

and

$$\sum_{|i| \geq 2I} |\varphi_i(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))|_H^2 \leq \nu_0 + \left(\frac{c_6}{I} + \gamma_I\right) K_2(\theta_{r-\tau}\omega), \quad I \in \mathbb{N}, r \geq \tau - t. \quad (47)$$

Lemma 2.3 For every $\tau \in \mathbb{R}$, $t \geq 0$, $\omega \in \Omega$ and $\varphi_{j, \tau-t}(\theta_{-\tau}\omega) \in \mathcal{X}_1(\tau - t, \theta_{-\tau}\omega)$, $j = 1, 2$, there exist random variables $C_2(\omega) > 0$, $C_3(\omega) > 0$, such that

$$\begin{aligned} & \|\varphi(\tau, \tau - t, \theta_{-\tau}\omega) \varphi_{1, \tau-t}(\theta_{-\tau}\omega) - \varphi(\tau, \tau - t, \theta_{-\tau}\omega) \varphi_{2, \tau-t}(\theta_{-\tau}\omega)\|_H \\ & \leq e^{\int_{-\tau}^0 C_2(\theta_s\omega) ds} \|\varphi_{1, \tau-t}(\theta_{-\tau}\omega) - \varphi_{2, \tau-t}(\theta_{-\tau}\omega)\|_H \end{aligned} \quad (48)$$

and

$$\begin{aligned} & \sum_{|i| > 4I} |(\varphi(r, \tau - t, \theta_{-\tau}\omega, \varphi_{1, \tau-t}(\theta_{-\tau}\omega)) - \varphi(r, \tau - t, \theta_{-\tau}\omega, \varphi_{2, \tau-t}(\theta_{-\tau}\omega)))_i|_H^2 \\ & \leq \left(e^{-\frac{\varepsilon}{4}t + \int_{-\tau}^0 C_3(\theta_s\omega) ds} + \frac{\hat{\delta}_I}{2} e^{\int_{-\tau}^0 C_2(\theta_s\omega) ds} \right)^2 \|\psi(\tau - t)\|_H^2, \quad \forall I \in \mathbb{N}, \end{aligned} \quad (49)$$

where

$$\hat{\delta}_I = \frac{2}{\sqrt[4]{\varepsilon}} \sqrt{\frac{1}{I} + \gamma_I + \frac{1}{I^p} + \gamma_I^p}.$$

Proof Taking the inner product $(\cdot, \cdot)_H$ of (45) with $\psi(r)$, we find that

$$\begin{aligned} & \frac{d\|\psi(r)\|_H^2}{dt} \\ & \leq \left(-\varepsilon + 2\left(1 + \frac{\varepsilon}{\sqrt{\underline{\lambda}}}\right) |c| |z(\theta_{r-\tau}\omega)| + \left(\frac{8a_5^2}{\underline{\lambda}\alpha} + \frac{1}{\sqrt{\underline{\lambda}}}\right) |c|^2 |z(\theta_{r-\tau}\omega)|^2 \right) \|\psi(r)\|_H^2 \\ & \quad + \frac{2}{\alpha} \|f(u_2, r) - f(u_1, r)\|_\rho^2. \end{aligned} \quad (50)$$

By (A2) and (46), we have

$$\|f(u_2, r) - f(u_1, r)\|_\rho^2 = \sum_{i \in \mathbb{Z}} \rho_i |f_i(u_{2,i}, r) - f_i(u_{1,i}, r)|^2 \leq \frac{C_0(\theta_{r-\tau}\omega)}{\underline{\lambda}} \|\psi(r)\|_H^2,$$

where

$$c_{10} = a_1^2 \left(\frac{16}{\underline{\lambda}} + \frac{2^{2(p-1)}}{\underline{\lambda}^{p-1}} \right), \quad C_0(\omega) = c_{10} M_0^2(\omega) (1 + M_0^{2(p-1)}(\omega)). \quad (51)$$

Thus,

$$\frac{d\|\psi(r)\|_H^2}{dt} \leq (-\varepsilon + 2C_1(\theta_{r-\tau}\omega)) \|\psi(r)\|_H^2, \quad (52)$$

where

$$C_1(\omega) = \left(1 + \frac{\varepsilon}{\sqrt{\underline{\lambda}}} \right) |c| |z(\omega)| + \left(\frac{4a_5^2}{\underline{\lambda}\alpha} + \frac{1}{2\sqrt{\underline{\lambda}}} \right) |c|^2 |z(\omega)|^2 + \frac{C_0(\omega)}{\alpha \underline{\lambda}}. \quad (53)$$

By applying the Gronwall inequality to (52) on $[\tau - t, \tau]$, we have

$$\|\psi(\tau)\|_H^2 \leq e^{\int_{-\tau}^0 (-\varepsilon + 2C_1(\theta_s \omega)) ds} \|\psi(\tau - t)\|_H^2 \leq e^{2 \int_{-\tau}^0 C_1(\theta_s \omega) ds} \|\psi(\tau - t)\|_H^2. \quad (54)$$

Let $I \in \mathbb{N}$ and M be a suitable large integer, and set

$$\omega = (\omega_i)_{i \in \mathbb{Z}} = \left(\eta \left(\frac{|i|}{M} \right) \xi_i, \eta \left(\frac{|i|}{M} \right) \zeta_i \right) = (p_i, q_i).$$

Taking the inner product $(\cdot, \cdot)_H$ of (45) with ω , we find that, for $r \geq \tau - t$,

$$(\dot{\psi}, \omega)_H + (L\psi, \omega)_H = (F(\varphi_1(r), \theta_{r-\tau} \omega) - F(\varphi_2(r), \theta_{r-\tau} \omega), \omega)_H. \quad (55)$$

Similar to (30)–(35), and by (A2), we obtain, for $r \geq \tau - t$,

$$\begin{aligned} & 2(f(u_2, r) - f(u_1, r), q)_\rho \\ & \leq \frac{\alpha}{4} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) \rho_i |\zeta_i|^2 + \frac{16a_1^2}{\alpha} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) \rho_i (|u_{1,i}|^2 + |u_{2,i}|^2) |\xi_i|^2 \\ & \quad + \frac{16a_1^2}{\alpha} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) \rho_i (|u_{1,i}|^{2p} + |u_{2,i}|^{2p}) |\xi_i|^2. \end{aligned}$$

By (47),

$$\begin{aligned} \sum_{|i| \geq 2I} [|u_{1,i}|^2 + |u_{2,i}|^2] & \leq \frac{2}{\underline{\lambda} \bar{\rho}} v_0 + \frac{2}{\underline{\lambda} \bar{\rho}} \left(\frac{c_6}{I} + \gamma_I \right) K_2(\theta_{r-\tau} \omega), \\ \sum_{|i| \geq 2I} [|u_{1,i}|^{2p} + |u_{2,i}|^{2p}] & \leq \frac{2^p}{\underline{\lambda}^p \bar{\rho}^p} v_0^p + \frac{2^p}{\underline{\lambda}^p \bar{\rho}^p} \left(\frac{c_6}{I} + \gamma_I \right)^p K_2^p(\theta_{r-\tau} \omega). \end{aligned}$$

Then we have, for $M \geq 2I$,

$$\begin{aligned} & 2(f(u_2, r) - f(u_1, r), q)_\rho \\ & \leq \frac{\alpha}{4} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) \rho_i |\zeta_i|^2 + \left(\frac{32a_1^2 v_0}{\alpha \underline{\lambda}^2 \bar{\rho}} + \frac{a_1^2 2^{p+4} v_0^p}{\alpha \underline{\lambda}^{p+1} \bar{\rho}^p} \right) \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) |\psi_i|_H^2 \\ & \quad + \frac{32a_1^2}{\alpha \underline{\lambda} \bar{\rho}} \left(\frac{c_6}{I} + \gamma_I \right) K_2(\theta_{r-\tau} \omega) \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) \rho_i |\xi_i|^2 \\ & \quad + \frac{a_1^2 2^{p+4}}{\alpha \underline{\lambda}^p \bar{\rho}^p} \left(\frac{c_6}{I} + \gamma_I \right)^p K_2^p(\theta_{r-\tau} \omega) \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) \rho_i |\xi_i|^2, \end{aligned} \quad (56)$$

where $|\psi_i|_H^2 = \rho_i [\delta(D\xi)_i^2 + \lambda_i \xi_i^2 + \zeta_i^2]$. Thus, by (43) and the above inequality, we have, for $M \geq 2I$,

$$\begin{aligned} & \frac{d}{dt} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) |\psi_i(r)|_H^2 + \frac{\varepsilon}{2} \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) |\psi_i(r)|_H^2 \\ & \leq \left(\left(2 + \frac{2\varepsilon}{\sqrt{\underline{\lambda}}} \right) |c| |z(\theta_{r-\tau} \omega)| + \left(\frac{8a_5^2}{\underline{\lambda} \alpha} + \frac{1}{\sqrt{\underline{\lambda}}} \right) |c|^2 |z(\theta_{r-\tau} \omega)|^2 \right) \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) |\psi_i(r)|_H^2 \end{aligned}$$

$$\begin{aligned}
& + c_{11} \left(\frac{1}{I} + \gamma_I + \frac{1}{I^p} + \gamma_I^p \right) \cdot [\xi_1(\theta_{r-\tau}\omega) + K_2^p(\theta_{r-\tau}\omega)] \\
& \times e^{2 \int_r^{\tau-t} C_2(\theta_{s-\tau}\omega) ds} \|\psi(\tau-t)\|_H^2,
\end{aligned} \quad (57)$$

where

$$c_{11} = \max \left\{ \frac{1}{2}, \frac{32a_1^2}{\alpha \lambda^2 \bar{\rho}}, \frac{32c_6 a_1^2}{\alpha \lambda^2 \bar{\rho}}, \frac{2^{2p+3} c_6 a_1^2}{\alpha \lambda^{p+1} \bar{\rho}^p}, \frac{2^{2p+3} a_1^2}{\alpha \lambda^{p+1} \bar{\rho}^p} \right\}.$$

By applying the Gronwall inequality to (57) on $[\tau-t, \tau]$ ($t \geq 0$), we obtain

$$\begin{aligned}
& \sum_{i \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) |\psi_i(\tau)|_H^2 \\
& \leq \|\psi(\tau-t)\|_E^2 e^{\int_{-t}^0 [-\frac{\varepsilon}{2} + 2(1 + \frac{\varepsilon}{\sqrt{\lambda}})|c||z(\theta_s\omega)| + (\frac{8a_5^2}{2\alpha} + \frac{1}{\sqrt{\lambda}})|c|^2|z(\theta_s\omega)|^2] ds} \\
& \quad + \left(\frac{1}{I} + \gamma_I + \frac{1}{I^p} + \gamma_I^p \right) \|\psi(\tau-t)\|_H^2 \\
& \quad \times e^{\int_{-t}^0 2[C_1(\theta_s\omega) + (1 + \frac{\varepsilon}{\sqrt{\lambda}})|c||z(\theta_s\omega)| + (\frac{4a_5^2}{2\alpha} + \frac{1}{2\sqrt{\lambda}})|c|^2|z(\theta_s\omega)|^2] ds} \\
& \quad \times \int_{-t}^0 c_{11} (\xi_1(\theta_r\omega) + K_2^p(\theta_r\omega)) e^{\frac{\varepsilon}{2}r} dr.
\end{aligned} \quad (58)$$

Since $\sqrt{x} \leq e^x$ for all $x \geq 0$, it follows that

$$\begin{aligned}
& \int_{-t}^0 c_{11} (\xi_1(\theta_r\omega) + K_2^p(\theta_r\omega)) e^{\frac{\varepsilon}{2}r} dr \\
& \leq \left[\int_{-t}^0 c_{11}^2 (\xi_1(\theta_r\omega) + K_2^p(\theta_r\omega))^2 dr \right]^{\frac{1}{2}} \left[\int_{-t}^0 e^{\varepsilon r} dr \right]^{\frac{1}{2}} \\
& \leq \frac{1}{\sqrt{\varepsilon}} e^{\int_{-t}^0 c_{11}^2 (\xi_1(\theta_r\omega) + K_2^p(\theta_r\omega))^2 dr}.
\end{aligned}$$

By $M \geq 2I$ and (58),

$$\begin{aligned}
& \sum_{|i| > 4I} |\psi_i(\tau)|_H^2 \\
& \leq \sum_{|i| \in \mathbb{Z}} \eta \left(\frac{|i|}{M} \right) |\psi_i(\tau)|_H^2 \\
& \leq \left(e^{\int_{-t}^0 2(-\frac{\varepsilon}{4} + C_3(\theta_s\omega)) ds} + \hat{\delta}_I^2 e^{2 \int_{-t}^0 C_2(\theta_s\omega) ds} \right) \|\psi(\tau-t)\|_H^2,
\end{aligned}$$

where

$$C_2(\omega) = C_1(\omega) + C_3(\omega) + c_{11}^2 (\xi_1^2(\omega) + K_2^{2p}(\omega)), \quad (59)$$

$$C_3(\omega) = \left(1 + \frac{\varepsilon}{\sqrt{\lambda}} \right) |c||z(\omega)| + \left(\frac{4a_5^2}{2\alpha} + \frac{1}{2\sqrt{\lambda}} \right) |c|^2 |z(\omega)|^2, \quad (60)$$

thus, (49) holds. By (54) and $C_2(\omega) \geq C_1(\omega)$, it follows that (48) holds. The proof is completed. $\square$

Lemma 2.4 *Let the coefficient c of the random term in (2) satisfy*

$$|c|^2 < \min \left\{ \frac{\mu^2 \pi \alpha}{64^2 a_7^2}, \frac{\alpha \mu}{32 a_7}, \frac{\alpha^2}{64 p a_7}, \frac{\alpha^3}{32^2 p^2 a_7^2} \right\}, \quad \mu = \min \left\{ \varepsilon, \frac{\varepsilon}{2 a_2}, 1 \right\}, \quad (61)$$

then

$$0 \leq \mathbf{E}[C_3(\omega)] \leq \frac{\varepsilon}{64}, \quad 0 \leq \mathbf{E}[C_2(\omega)], \mathbf{E}[C_2^2(\omega)] < \infty. \quad (62)$$

Proof From [13, 16], the Ornstein–Uhlenbeck stationary process $z(\theta_t \omega)$ satisfies

$$\mathbf{E}[|z(\theta_s \omega)|^r] = \frac{\Gamma(\frac{1+r}{2})}{\sqrt{\pi} \alpha^{\frac{r}{2}}} \quad (\forall r \geq 0), \quad (63)$$

$$\begin{cases} \mathbf{E}[e^{\zeta \int_{\tau}^{\tau+t} |z(\theta_s \omega)|^2 ds}] \leq e^{\frac{\zeta}{\alpha} t}, & 0 \leq 2\zeta \leq \alpha^2, \tau \in \mathbb{R}, t \geq 0, \\ \mathbf{E}[e^{\zeta \int_{\tau}^{\tau+t} |z(\theta_s \omega)| ds}] \leq e^{\frac{\zeta}{\sqrt{\alpha}} t}, & 0 \leq \zeta^2 \leq \alpha^3, \tau \in \mathbb{R}, t \geq 0, \end{cases} \quad (64)$$

where $\Gamma(\cdot)$ is the Gamma function. It is easy to see that (61) implies (8). By (3), (59) and (60), we obtain

$$\mathbf{E}[C_3(\omega)] \leq \frac{a_7 |c|}{2\sqrt{\pi} \alpha} + \frac{a_7 |c|^2}{4\alpha} \leq \frac{1}{16} \frac{\varepsilon}{4}. \quad (65)$$

By (7), (61) and (63)–(64),

$$\begin{aligned} \mathbf{E}[M_0^{4p}(\omega)] &\leq c_5^{2p} \left(\int_{-\infty}^0 e^{\frac{p\mu}{2p-1}s} ds \right)^{2p-1} \int_{-\infty}^0 e^{p\mu s} \mathbf{E} e^{2pa_7 \int_s^0 (|c||z(\theta_l \omega)| + |c|^2 |z|^2(\theta_l \omega)) dl} ds \\ &\leq c_5^{2p} \left(\frac{2p-1}{p\mu} \right)^{2p-1} \int_{-\infty}^0 e^{p\mu s} \left(\mathbf{E} e^{4pa_7 |c| \int_s^0 |z(\theta_l \omega)| dl} + \mathbf{E} e^{4pa_7 |c|^2 \int_s^0 |z|^2(\theta_l \omega) dl} \right) ds \\ &\leq \frac{c_5^{2p}}{p} \left(\frac{2p-1}{p\mu} \right)^{2p-1} \left(\frac{1}{\mu - \frac{4|c|a_7}{\sqrt{\alpha}}} + \frac{1}{\mu - \frac{4|c|^2 a_7}{\alpha}} \right) = k_1. \end{aligned} \quad (66)$$

Then, by (51), (53) and (66),

$$\begin{aligned} \mathbf{E}[C_1^2(\omega)] &\leq 3\mathbf{E} \left(\left(1 + \frac{\varepsilon}{\sqrt{\lambda}} \right)^2 |c|^2 |z(\omega)|^2 + \left(\frac{4a_5^2}{\lambda \alpha} + \frac{1}{2\sqrt{\lambda}} \right)^4 |c|^4 |z(\omega)|^4 \right) \\ &\quad + 3\mathbf{E} \left(\frac{2c_{10}^2}{\alpha^2 \lambda^2} (M_0^4(\omega) + M_0^{4p}(\omega)) \right) \\ &\leq 3 \left(\left(1 + \frac{\varepsilon}{\sqrt{\lambda}} \right)^2 \frac{|c|^2}{2\alpha} + \left(\frac{4a_6^2}{\lambda \alpha} + \frac{1}{2\sqrt{\lambda}} \right)^4 \frac{\Gamma(\frac{5}{2})|c|^4}{\alpha^2 \sqrt{\pi}} \right) \\ &\quad + 3 \left(\frac{2c_{10}^2}{\alpha^2 \lambda^2} (M_0^4(\omega) + M_0^{4p}(\omega)) \right) = k_2. \end{aligned} \quad (67)$$

Similar to (66), we have

$$\mathbf{E}[K_0^{4p}(\omega)] = \mathbf{E} \left(\int_{-\infty}^0 e^{\mu s + a_7 \int_s^0 (|c||z(\theta_l \omega)| + |c|^2 |z|^2(\theta_l \omega)) dl} ds \right)^{4p}$$

$$\begin{aligned}
&\leq \left(\frac{4p-1}{2p\mu}\right)^{4p-1} \int_{-\infty}^0 e^{2p\mu s} (\mathbf{E} e^{8a_7 p|c| \int_s^0 |z(\theta_l \omega)| dl} + \mathbf{E} e^{8a_7 p c^2 \int_s^0 |z|^2(\theta_l \omega) dl}) ds \\
&\leq \frac{1}{2p} \left(\frac{4p-1}{2p\mu}\right)^{4p-1} \left(\frac{1}{\mu - \frac{4|c|a_7}{\sqrt{\alpha}}} + \frac{1}{\mu - \frac{4|c|^2 a_7}{\alpha}} \right) = k_3.
\end{aligned} \quad (68)$$

From (28),

$$\mathbf{E}[\xi_1^4(\omega)] \leq \mathbf{E}[c_{12}(1 + |z(\omega)|^4)] \leq c_{12} \left(1 + \frac{\Gamma(\frac{5}{2})}{\alpha^2 \sqrt{\pi}}\right) = k_4, \quad (69)$$

$$\mathbf{E}[\xi_1^{16p}(\omega)] \leq \mathbf{E}[c_{13}(1 + |z(\omega)|^{16p})] \leq c_{13} \left(1 + \frac{\Gamma(\frac{16p+1}{2})}{\alpha^{8p} \sqrt{\pi}}\right) = k_5. \quad (70)$$

Thus, by (28) and (70),

$$\begin{aligned}
\mathbf{E}[K_1^{8p}(\omega)] &= \mathbf{E} \left(\int_{-\infty}^0 \xi_1(\theta_s \omega) e^{\mu s + a_7 \int_s^0 (|c||z(\theta_l \omega)| + |c|^2 |z(\theta_l \omega)|^2) dl} ds \right)^{8p} \\
&\leq \left(\int_{-\infty}^0 e^{\frac{4p}{8p-1} \mu s} ds \right)^{8p-1} \mathbf{E} \int_{-\infty}^0 \xi_1^{8p}(\theta_s \omega) e^{4p\mu s + 8pa_7 \int_s^0 (|c||z(\theta_l \omega)| + |c|^2 |z(\theta_l \omega)|^2) dl} ds \\
&\leq \left(\frac{8p-1}{4p\mu}\right)^{8p-1} \times \int_{-\infty}^0 e^{4p\mu s} (\mathbf{E} e^{32pa_7|c| \int_s^0 |z(\theta_l \omega)| dl} + \mathbf{E} e^{32pa_7|c|^2 \int_s^0 |z(\theta_l \omega)|^2 dl}) ds \\
&\quad + \left(\frac{8p-1}{4p\mu}\right)^{8p-1} \int_{-\infty}^0 e^{4p\mu s} \mathbf{E} \xi_1^{16p}(\theta_s \omega) ds \\
&\leq \frac{1}{4p} \left(\frac{8p-1}{4p\mu}\right)^{8p-1} \left(\frac{1}{\mu - \frac{8a_7|c|}{\sqrt{\alpha}}} + \frac{1}{\mu - \frac{8a_7|c|^2}{\alpha}} + \frac{k_5}{\mu} \right) = k_6,
\end{aligned} \quad (71)$$

$$\mathbf{E}[K_2^{4p}(\omega)] \leq 2^{4p-1} (\mathbf{E}[K_0^{4p}(\omega)] + \mathbf{E}[K_0^{8p}(\omega)] + \mathbf{E}[K_1^{8p}(\omega)]) = k_7. \quad (72)$$

By (59), (67), (69) and (72), we have

$$0 \leq \mathbf{E}[C_2^2(\omega)] \leq c_{14}(1 + k_2 + k_4 + k_7) < \infty.$$

Since $0 \leq \mathbf{E}[C_2(\omega)] \leq \frac{1}{2}[1 + \mathbf{E}(C_2^2(\omega))]$, it follows that $0 \leq \mathbf{E}[C_2(\omega)] < \infty$. The proof is completed. $\square$

Lemma 2.5 For any $\tau \in \mathbb{R}$, $\omega \in \Omega$,

$$\begin{cases} \lim_{t \rightarrow 0^+} \sup_{\varphi \in \mathcal{X}_1(\tau, \omega)} \|\Phi(t, \tau, \omega)\varphi - \varphi\|_H = 0, \\ \lim_{t \rightarrow 0^+} \sup_{\varphi \in \mathcal{X}_1(\tau-t, \theta_{-t}\omega)} \|\Phi(0, \tau-t, \theta_{-t}\omega)\varphi - \varphi\|_H = 0. \end{cases}$$

Proof By (5), (6) and (44), it follows that, for $\varphi \in \mathcal{X}_1(\tau, \omega)$ and $t \geq 0$,

$$\begin{aligned}
&\|F(\varphi(r), \theta_{r-\tau}\omega)\|_H^2 \\
&\leq |c|^2 |z(\theta_{r-\tau}\omega)|^2 M_0^2(\theta_{r-\tau}\omega) + \frac{5(2c\varepsilon z(\theta_{r-\tau}\omega) - |c|^2 |z(\theta_{r-\tau}\omega)|^2)^2}{\sqrt{\lambda}} M_0^2(\theta_{r-\tau}\omega) \\
&\quad + 5|c|^2 |z(\theta_{r-\tau}\omega)|^2 M_0^2(\theta_{r-\tau}\omega) + 5|c|^2 a_5^2 |z(\theta_{r-\tau}\omega)|^2 M_0^2(\theta_{r-\tau}\omega)
\end{aligned}$$

$$+ 5a_1^2(1 + M_0^{p-1}(\theta_{r-\tau}\omega))^2 M_0^4(\theta_{r-\tau}\omega) + 5\|g\|_\rho^2 \quad (73)$$

and

$$\begin{aligned} \|L\varphi(r)\|_H^2 &\leq \|\varepsilon u - v\|_{\delta, \lambda, \rho}^2 + 5\lambda^2 \|u\|_\rho^2 + 5|\varepsilon|(\alpha - \varepsilon)\|u\|_\rho^2 \\ &\quad + 5\delta \|Au\|_\rho^2 + 5(\alpha - \varepsilon)^2 \|v\|_\rho^2 + 5\gamma \|Av\|_\rho^2 \\ &\leq c_{15} M_0^2(\theta_{r-\tau}\omega), \end{aligned} \quad (74)$$

therefore,

$$\begin{aligned} &\|\Phi(t, \tau, \omega)\varphi - \varphi\|_H^2 \\ &\leq t \int_\tau^{\tau+t} \|F(\varphi(r), \theta_{r-\tau}\omega) - L\varphi(r)\|_H^2 dr \\ &\leq 2t \int_0^t \left[\frac{5(2c\varepsilon z(\theta_r\omega) - |c|^2 |z(\theta_r\omega)|^2)^2}{\sqrt{\lambda}} + c_{16}|c|^2 |z(\theta_r\omega)|^2 \right] M_0^2(\theta_r\omega) dr \\ &\quad + 2t \int_0^t [5a_1^2(1 + M_0^{p-1}(\theta_r\omega))^2 M_0^4(\theta_r\omega) + 5\|g\|_\rho^2 + c_{15} M_0^2(\theta_r\omega)] dr \\ &\rightarrow 0 \quad \text{as } t \rightarrow 0, \end{aligned} \quad (75)$$

where $c_{16} = 6 + 5a_5^2$, thus, $\lim_{t \rightarrow 0^+} \sup_{\varphi \in \mathcal{X}_1(\tau, \omega)} \|\Phi(t, \tau, \omega)\varphi - \varphi\|_H = 0$.

Similarly, $\lim_{t \rightarrow 0^+} \sup_{\varphi \in \mathcal{X}_1(\tau-t, \theta_{-t}\omega)} \|\Phi(0, \tau-t, \theta_{-t}\omega)\varphi - \varphi\|_H = 0$. The proof is completed. $\square$

From Theorem 2.1 and Lemmas 2.3–2.5, we have the following result.

Theorem 2.3 Assume that (A1)–(A4) and (43), (61) hold. Then the continuous cocycle $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$ possesses a random exponential attractor $\{\mathcal{K}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ with properties: for any $\tau \in \mathbb{R}$, $\omega \in \Omega$,

- (i) $\mathcal{R}(\tau, \omega) \subseteq \mathcal{K}(\tau, \omega) \subseteq \overline{\mathcal{X}_1(\tau, \omega)}$ and $\mathcal{K}(\tau, \omega)$ is a compact set of H and measurable in ω .
- (ii) $\Phi(t, \tau, \omega)\mathcal{K}(\tau, \omega) \subseteq \mathcal{K}(t + \tau, \theta_t\omega)$ for all $t \geq 0$.
- (iii) There exists $I_0 = \min\{I : 0 < \hat{\delta}_I \leq \tilde{\gamma}\} + 1 \in \mathbb{N}$, where

$$\tilde{\gamma} = \min \left\{ \frac{1}{16}, e^{-\frac{2}{\ln 2} \left(\frac{32 \ln 2}{\varepsilon} \right)^2 (\mathbb{E}[C_2^2(\omega)] + \frac{\varepsilon}{4} \mathbb{E}[C_2(\omega)])} \right\},$$

such that

$$\dim_f \mathcal{R}(\tau, \omega) \leq \dim_f \mathcal{K}(\tau, \omega) \leq \frac{4(8I_0 + 1) \ln \left(\frac{2\sqrt{16I_0 + 2}}{\hat{\delta}_{I_0}} + 1 \right)}{\ln \frac{4}{3}} < \infty.$$

- (iv) For every set $D \in \mathcal{D}(H)$, there exist a random variable $T_{\omega, D} \geq 0$ and a tempered random variable $b_{\omega, D} > 0$ such that

$$d_h(\Phi(t, \tau, \omega)D(\tau, \omega), \mathcal{K}(t + \tau, \theta_t\omega)) \leq b_{\omega, D} e^{-\frac{\varepsilon \ln \frac{4}{3}}{128 \ln 2} t}, \quad t \geq T_{\omega, D}.$$

(v) For any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\lim_{t \rightarrow 0} d_h(\mathcal{K}(\tau + t, \theta_t \omega), \mathcal{K}(\tau, \omega)) = 0$.

Proof Taking $t = \frac{32 \ln 2}{\varepsilon}$ in (48) and (49). From (62),

$$0 < e^{-\frac{2}{\ln \frac{3}{2}} \left(\frac{32 \ln 2}{\varepsilon} \right)^2 (E[C_2^2(\omega)] + \frac{\varepsilon}{4} E[C_2(\omega)])} < \infty. \quad (76)$$

By $\lim_{I \rightarrow +\infty} \frac{1}{I} = \lim_{I \rightarrow +\infty} \frac{1}{I^{p+1}} = 0$ and $g(t)$, $\beta(t)$, $\beta'(t) \in \mathbf{G}$, it follows that $\lim_{I \rightarrow +\infty} \gamma_I = 0$ and

$$0 < \hat{\delta}_I = \frac{2}{\sqrt[p]{\varepsilon}} \sqrt{\frac{1}{I} + \gamma_I + \frac{1}{I^{p+1}} + \gamma_I^{p+1}} \rightarrow 0 \quad \text{as } I \rightarrow +\infty. \quad (77)$$

Hence there exists a finite integer $I_0 \in \mathbb{N}$, such that

$$0 < \hat{\delta}_{I_0} \leq \min \left\{ \frac{1}{16}, e^{-\frac{2}{\ln \frac{3}{2}} \left(\frac{32 \ln 2}{\varepsilon} \right)^2 (E[C_2^2(\omega)] + \frac{\varepsilon}{4} E[C_2(\omega)])} \right\}. \quad (78)$$

Finally, by Theorem 2.1, the statements in Theorem 2.3 hold. The proof is completed. $\square$

Acknowledgements

The authors would like to express their sincere thanks to the anonymous referees for their time and comments.

Funding

This work is supported by the National Natural Science Foundation of China under Grant Nos. 11471290, 11871437.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper and typed, read, and approved the final manuscript.

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Received: 10 April 2018 Accepted: 21 January 2019 Published online: 31 January 2019

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