Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence results of nonlocal boundary value problem for a nonlinear fractional differential coupled system involving fractional order impulses

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Abstract

In this paper, we study the nonlocal boundary value problem for a nonlinear fractional differential coupled system with fractional order impulses. Applying Nonlinear Alternative of Leray–Schauder, we obtain some new existence results for this system. As application, an interesting example is given to illustrate the effectiveness of our main result.

MSC: 34B10; 34B15; 34B37

Keywords: Fractional differential coupled system; Nonlocal boundary value conditions; Impulses; Fixed point theorem

1 Introduction

In describing some phenomena and processes of many fields such as physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, capacitor theory, electrical circuits, biology, control theory, fitting of experimental data, and so on, the fractional order calculus is an excellent and more accurate tool than the integral order calculus. For example, in physics, we use Newtons' law $\eta\varepsilon'(t) = \sigma(t)$ to describe the mechanics of viscous fluids, where $\sigma(t)$ and $\varepsilon(t)$ denote stress and strain at time t, respectively, and η is the viscosity of the material. However, we need to employ Nuttings' law [1] $\eta D^k_{0^+}\varepsilon'(t) = \sigma(t)$ ($k \in (n-1, n)$, $n \in \mathbb{N}$) to deal with the mechanics of viscous fluids involving some possible interpolation properties. As a consequence, the subject of fractional differential equations is gaining much importance and attention. There have been many papers focused on boundary value problems of fractional ordinary differential equations (see [1–31]). Especially, the nonlocal boundary value problems have been widely studied by many scholars because of their extensive applications in, e.g., blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so forth. The nonlocal boundary value problems of fractional-order differential equations constitute a class of very interesting and important problems. Such boundary value problems have been investigated in [8–14, 24, 25, 30].

In addition, the theory of impulse differential equations has seen significant development in recent years and played a very important role in modern applied mathematical models of real processes arising in phenomena studied in physics, population dynamics, chemical technology and biotechnology. Recently, some scholars have begun to study the boundary value problems for impulsive fractional differential equations (see [1, 15–26, 30, 32]). As is well known, the study on fractional differential coupled systems is more complicated and challenged than the study on a single fractional differential equation. Recently, some scholars began to investigate fractional differential coupled systems and obtained some good results (see [8, 12, 13, 24, 26, 31]). However, there are few papers on the impulsive fractional order coupled systems with nonlocal boundary conditions and impulses. Therefore, in this paper, we consider the following four-point boundary value problem for nonlinear fractional differential coupling system with fractional order impulses of the form

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}x(t) = f(t, x(t), {}^{C}D^{p}_{0^+}y(t)), \quad t \in J, t \neq t_k, \\ {}^{C}D^{\beta}_{0^+}y(t) = g(t, {}^{C}D^{q}_{0^+}x(t), y(t)), \quad t \in J, t \neq t_k, \\ {}^{C}D^{\gamma_1}_{0^+}x(t_k^+) - {}^{C}D^{\gamma_1}_{0^+}x(t_k^-) = J_{1k}(x(t_k)), \quad k = 1, \dots, n, \\ {}^{C}D^{\gamma_2}_{0^+}y(t_k^+) - {}^{C}D^{\gamma_2}_{0^+}y(t_k^-) = J_{2k}(y(t_k)), \quad k = 1, \dots, n, \\ x(0) = y(0) = 0, \qquad {}^{\mathrm{LR}}D^{\delta_1}_{0^+}x(z) = x(1), \qquad {}^{\mathrm{LR}}D^{\delta_2}_{0^+}y(w) = y(1), \end{cases} \tag{1.1}$$

where $J = [0, 1]$, $1 < \alpha$, $\beta < 2$, $0 < p$, $q, \gamma_1, \gamma_2, \delta_1, \delta_2, z, w < 1$, ${}^{C}D^{\alpha}_{0^+}$, ${}^{C}D^{\beta}_{0^+}$, ${}^{C}D^{p}_{0^+}$, ${}^{C}D^{q}_{0^+}$, ${}^{C}D^{\gamma_1}_{0^+}$, and ${}^{C}D^{\gamma_2}_{0^+}$ are the Caputo fractional derivatives; ${}^{\mathrm{LR}}D^{\delta_1}_{0^+}$ and ${}^{\mathrm{LR}}D^{\delta_2}_{0^+}$ are the Riemann–Liouville fractional derivatives; $f, g \in C(J \times R^2, R)$, $J_{1k}, J_{2k} \in C(R, R)$, and $\{t_k\}$ satisfies $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$, ${}^{C}D^{\gamma_1}_{0^+}x(t_k^+)$, ${}^{C}D^{\gamma_1}_{0^+}x(t_k^-)$, ${}^{C}D^{\gamma_2}_{0^+}y(t_k^+)$, ${}^{C}D^{\gamma_2}_{0^+}y(t_k^-)$ all exist, ${}^{C}D^{\gamma_1}_{0^+}x(t_k^-) = {}^{C}D^{\gamma_1}_{0^+}x(t_k)$, ${}^{C}D^{\gamma_2}_{0^+}y(t_k^-) = {}^{C}D^{\gamma_1}_{0^+}y(t_k)$, $k = 1, 2, \dots, n$.

The rest of this paper is organized as follows. In Sect. 2, we recall some definitions and lemmas of the Caputo and Riemann–Liouville fractional calculus. In Sect. 3, we shall prove the existence of solutions for system (1.1). In Sect. 4, some examples are given to demonstrate the application of our main results. Finally, conclusions are given in Sect. 5 to simply recall our studies and results obtained.

2 Preliminaries

Let $C(J)$ be the Banach space of continuous functions from J to $\mathbb{R}$ with the norm $\|\psi\|_C = \sup_{t\in J} |\psi(t)|$. Define the function space $\mathrm{PC}(J)$ by

$$\begin{aligned} \mathrm{PC}(J) = \bigl\{ &\psi(t) : \psi(t), {}^{C}D^{p}_{0^+}\psi(t), {}^{C}D^{q}_{0^+}\psi(t), {}^{\mathrm{LR}}D^{\delta_1}_{0^+}\psi(t), {}^{\mathrm{LR}}D^{\delta_2}_{0^+}\psi(t) \in C(J), {}^{C}D^{\gamma_1}_{0^+}\psi\bigl(t_k^+\bigr), \\ &{}^{C}D^{\gamma_1}_{0^+}\psi\bigl(t_k^-\bigr), {}^{C}D^{\gamma_2}_{0^+}\psi\bigl(t_k^+\bigr) \text{ and } {}^{C}D^{\gamma_2}_{0^+}\psi\bigl(t_k^-\bigr) \text{ all exist, and satisfy} \\ &{}^{C}D^{\gamma_1}_{0^+}\psi\bigl(t_k^-\bigr) = {}^{C}D^{\gamma_1}_{0^+}\psi(t_k), {}^{C}D^{\gamma_2}_{0^+}\psi\bigl(t_k^-\bigr) = {}^{C}D^{\gamma_2}_{0^+}\psi(t_k), \\ &0 < p, q, \delta_1, \delta_2, \gamma_1, \gamma_2 < 1, 1 \le k \le n \bigr\}. \end{aligned}$$

Obviously, $\mathrm{PC}(J)$ is a real Banach space equipped with the norm

$$\|\psi\|_{\mathrm{PC}} = \max\bigl\{\|\psi\|_C, \bigl\|{}^{c}D^{p}_{0^+}\psi\bigr\|_C, \bigl\|{}^{c}D^{q}_{0^+}\psi\bigr\|_C\bigr\}, \quad \forall \psi \in \mathrm{PC}(J).$$

Let $X = \mathrm{PC}(J) \times \mathrm{PC}(J)$. It is easily to verify that X is a Banach space with the norm $\|(u, v)\| = \max\{\|u\|_{\mathrm{PC}}, \|v\|_{\mathrm{PC}}\}$, $(u, v) \in X$.

For the readers' convenience, we introduce some necessary definitions and lemmas. These definitions and properties can be found in the literature.

Definition 2.1 ([32, 33]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : (a, \infty) \to R$ is defined by

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,ds,$$

provided that the right-hand side is pointwise defined on (a, ∞).

Definition 2.2 ([33]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (a, \infty) \to R$ is defined by

$${}^{\mathrm{LR}}D_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\,ds,$$

where $n-1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on (a, ∞).

Definition 2.3 ([32, 33]) If $f \in C^n((a, \infty), R)$ and $\alpha > 0$, then the Caputo fractional derivative of order α is defined as

$${}^{C}D_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds,$$

where $n-1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on (a, ∞).

Lemma 2.1 ([33]) *If $u \in C^n[0,1]$, and $\delta > 0$, then*

$$I_{0^+}^{\delta}\,{}^{C}D_{0^+}^{\delta} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k,$$

where $n = -[-\delta]$ and $[-\delta]$ denotes the integer part of the real number $-\delta$.

Lemma 2.2 ([32, 33]) *If $\alpha, \beta > 0$, $t \in [a,b]$ and $u(t) \in L[a,b]$, then*

$${}^{C}D_{a^+}^{\alpha} I_{a^+}^{\alpha} u(t) = u(t), \qquad I_{a^+}^{\alpha} I_{a^+}^{\beta} u(t) = I_{a^+}^{\alpha+\beta} u(t).$$

Lemma 2.3 (see [34], pp. 36–39) *Let $\alpha > 0$ and suppose n denotes the smallest integer greater than or equal to α. Then the following assertions hold*:

(i) *If $\lambda > -1$, $\lambda \neq \alpha - i$, $i = 1, 2, \ldots, n+1$, then for $t \in [a,b]$,*

$${}^{\mathrm{LR}}D_{a^+}^{\alpha} (t-a)^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} (t-a)^{\lambda-\alpha}.$$

(ii) ${}^{\mathrm{LR}}D_{a^+}^{\alpha} (t-a)^{\alpha-i} = 0$, $i = 1, 2, \ldots, n$.

(iii) ${}^{\mathrm{LR}}D_{a^+}^{\beta} I_{a^+}^{\alpha} u(t) = I_{a^+}^{\alpha} u(t)$, *for all $t \in [a,b]$, $\alpha \geq \beta \geq 0$.*

Lemma 2.4 (Nonlinear Alternative Of Leray–Schauder [35]) *Let X be a Banach space, C be a nonempty convex subset of X, Ω be an open subset of C with $\theta \in \Omega$. Suppose that $T : \overline{\Omega} \to C$ is a completely continuous mapping. Then either*

(i) *the mapping T has a fixed point in $\overline{\Omega}$, or*
(ii) *there exists a $u \in \partial\Omega$ and $\lambda \in (0,1)$ with $u = \lambda Tu$.*

Lemma 2.5 *Let $h_1 \in C(J)$. If $\Delta_1 \triangleq \frac{z^{1-\delta_1}}{\Gamma(2-\delta_1)} \neq 1$, then a function $x \in \mathrm{PC}(J)$ is a solution of the boundary value problem*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = h_1(t), \quad 1 < \alpha < 2, t \in J, t \neq t_k, \\ {}^{C}D_{0^+}^{\gamma_1}x(t_k^+) - {}^{C}D_{0^+}^{\gamma_1}x(t_k^-) = J_{1k}(x(t_k)), \quad 0 < \gamma_1 < 1, k = 1,2,\dots,n, \\ x(0) = 0, \qquad {}^{\mathrm{LR}}D_{0^+}^{\delta_1}x(z) = x(1), \qquad 0 < \delta_1 < 1, \qquad 0 < z < 1, \end{cases} \tag{2.1}$$

if and only if $x \in \mathrm{PC}(J)$ is a solution of the integral equation

$$\begin{aligned} x(t) = {} & I_{0^+}^{\alpha}h_1(t) + \left(\frac{I_{0^+}^{\alpha-\delta_1}h_1(z) - I_{0^+}^{\alpha}h_1(1) + C_1}{1-\Delta_1} + \sum_{i=1}^{k}\frac{\Gamma(2-\gamma_1)}{t_i^{1-\gamma_1}}J_{1i}\bigl(x(t_i)\bigr)\right)t \\ & - \Gamma(2-\gamma_1)t_k^{\gamma_1}J_{1k}\bigl(x(t_k)\bigr), \quad t \in (t_k, t_{k+1}], k = 0,1,\dots,n, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} C_1 = {} & \Delta_1\Gamma(2-\gamma_1)\sum_{i=1}^{j}\frac{J_{1i}(x(t_i))}{t_i^{1-\gamma_1}} - \frac{z^{-\delta_1}t_j^{\gamma_1}\Gamma(2-\gamma_1)}{\Gamma(1-\delta_1)}J_{1j}\bigl(x(t_j)\bigr) + t_n^{\gamma_1}\Gamma(2-\gamma_1)J_{1n}\bigl(x(t_n)\bigr) \\ & - \Gamma(2-\gamma_1)\sum_{i=1}^{n}\frac{J_{1i}(x(t_i))}{t_i^{1-\gamma_1}}, \quad t_j < z \leq t_{j+1}, j \in \{0,1,2,\dots,n\}. \end{aligned}$$

Proof When $t \in [0, t_1]$, from Lemma 2.1, we have

$$x(t) = I_{0^+}^{\alpha}h_1(t) + u_{10} + u_{11}t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h_1(s)\,ds + u_{10} + u_{11}t. \tag{2.3}$$

By $x(0) = 0$, we get $u_{10} = 0$. And it follows from (2.3) that

$${}^{C}D_{0^+}^{\gamma_1}x(t) = \frac{u_{11}t^{1-\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{1}{\Gamma(\alpha-\gamma_1)}\int_0^t (t-s)^{\alpha-\gamma_1-1}h_1(s)\,ds, \tag{2.4}$$

and

$${}^{C}D_{0^+}^{\gamma_1}x\bigl(t_1^-\bigr) = {}^{C}D_{0^+}^{\gamma_1}x(t_1) = \frac{u_{11}t_1^{1-\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{1}{\Gamma(\alpha-\gamma_1)}\int_0^{t_1} (t_1-s)^{\alpha-\gamma_1-1}h_1(s)\,ds. \tag{2.5}$$

When $t \in (t_1, t_2]$, we similarly have

$$x(t) = I_{0^+}^{\alpha}h_1(t) + u_{20} + u_{21}t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h_1(s)\,ds + u_{20} + u_{21}t, \tag{2.6}$$

$${}^{C}D_{0^+}^{\gamma_1}x(t) = \frac{u_{21}t^{1-\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{1}{\Gamma(\alpha-\gamma_1)}\int_0^t (t-s)^{\alpha-\gamma_1-1}h_1(s)\,ds, \tag{2.7}$$

and

$${}^{C}D_{0^+}^{\gamma_1}x\bigl(t_1^+\bigr) = \frac{u_{21}t_1^{1-\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{1}{\Gamma(\alpha-\gamma_1)}\int_0^{t_1} (t_1-s)^{\alpha-\gamma_1-1}h_1(s)\,ds. \tag{2.8}$$

By (2.5), (2.8) and ${}^{C}D_{0^+}^{\gamma_1}x(t_k^+) - {}^{C}D_{0^+}^{\gamma_1}x(t_k^-) = J_{1k}(x(t_k))$, we obtain

$$u_{21} - u_{11} = \frac{\Gamma(2-\gamma_1)}{t_1^{1-\gamma_1}} J_{11}\bigl(x(t_1)\bigr). \tag{2.9}$$

In view of the continuity of x at t_1, we have

$$u_{20} = -(u_{21} - u_{11})t_1 = -\Gamma(2-\gamma_1)t_1^{\gamma_1} J_{11}\bigl(x(t_1)\bigr). \tag{2.10}$$

When $t \in (t_k, t_{k+1}]$, $k = 2, 3, \ldots, n$, repeating the above calculation, we get

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h_1(s)\,ds + u_{k+1,0} + u_{k+1,1}t, \tag{2.11}$$

and

$$u_{k+1,1} - u_{k1} = \frac{\Gamma(2-\gamma_1)}{t_k^{1-\gamma_1}} J_{1k}\bigl(x(t_k)\bigr), \qquad u_{k+1,0} = -\Gamma(2-\gamma_1)t_k^{\gamma_1} J_{1k}\bigl(x(t_k)\bigr). \tag{2.12}$$

From (2.11) and (2.12), we have

$$x(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h_1(s)\,ds - \Gamma(2-\gamma_1)t_n^{\gamma_1} J_{1n}\bigl(x(t_n)\bigr) + u_{n+1,1}. \tag{2.13}$$

Equations (2.9) and (2.12) give

$$u_{k+1,1} = u_{11} + \sum_{i=1}^{k} \frac{\Gamma(2-\gamma_1)}{t_i^{1-\gamma_1}} J_{1i}\bigl(x(t_i)\bigr), \quad k = 1, 2, \ldots, n. \tag{2.14}$$

Denoting $t_0 = 0$, $t_{n+1} = 1$, and noticing $0 < z < 1$, we know that there exists $j \in \{0, 1, \ldots, n\}$ such that $z \in (t_j, t_{j+1}]$ and

$$x(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} h_1(s)\,ds + u_{j+1,0} + u_{j+1,1}z = I_{0^+}^{\alpha} h_1(z) + u_{j+1,0} + u_{j+1,1}z. \tag{2.15}$$

Applying Lemmas 2.2–2.3 and (2.15), we obtain

$${}^{\mathrm{LR}}D_{0^+}^{\delta_1}x(z) = \frac{1}{\Gamma(\alpha-\delta_1)} \int_0^z (z-s)^{\alpha-\delta_1-1} h_1(s)\,ds + \frac{u_{j+1,1}\Gamma(2)z^{1-\delta_1}}{\Gamma(2-\delta_1)} + \frac{u_{j+1,0}z^{-\delta_1}}{\Gamma(1-\delta_1)}. \tag{2.16}$$

Using ${}^{\mathrm{LR}}D_{0^+}^{\delta_1}x(z) = x(1)$, (2.12), (2.13), (2.14) and (2.16), we derive

$$\begin{aligned} u_{11} = {} & \frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1)} \int_0^z (z-s)^{\alpha-\delta_1-1} h_1(s)\,ds + \frac{\Delta_1 \Gamma(2-\gamma_1)}{1-\Delta_1} \sum_{i=1}^{j} \frac{J_{1i}(x(t_i))}{t_i^{1-\gamma_1}} \\ & - \frac{z^{-\delta_1} t_j^{\gamma_1} \Gamma(2-\gamma_1)}{(1-\Delta_1)\Gamma(1-\delta_1)} J_{1j}\bigl(x(t_j)\bigr) - \frac{1}{(1-\Delta_1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h_1(s)\,ds \\ & + \frac{t_n^{\gamma_1}\Gamma(2-\gamma_1)}{1-\Delta_1} J_{1n}\bigl(x(t_n)\bigr) - \frac{\Gamma(2-\gamma_1)}{1-\Delta_1} \sum_{i=1}^{n} \frac{J_{1i}(x(t_i))}{t_i^{1-\gamma_1}} \end{aligned}$$

$$= \frac{1}{1-\Delta_1}\Bigg[I_{0^+}^{\alpha-\delta_1}h_1(z) - I_{0^+}^{\alpha}h_1(1) + \Delta_1\Gamma(2-\gamma_1)\sum_{i=1}^{j}\frac{J_{1i}(x(t_i))}{t_i^{1-\gamma_1}} - \frac{z^{-\delta_1}t_j^{\gamma_1}\Gamma(2-\gamma_1)}{\Gamma(1-\delta_1)}$$

$$\times J_{1j}\big(x(t_j)\big) + t_n^{\gamma_1}\Gamma(2-\gamma_1)J_{1n}\big(x(t_n)\big) - \Gamma(2-\gamma_1)\sum_{i=1}^{n}\frac{J_{1i}(x(t_i))}{t_i^{1-\gamma_1}}\Bigg]$$

$$= \frac{A_1}{1-\Delta_1}. \tag{2.17}$$

Thus, for $(t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, n$, we have

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h_1(s)\,ds + u_{k+1,0} + u_{k+1,1}t$$

$$= I_{0^+}^{\alpha}h_1(t) - \Gamma(2-\gamma_1)t_k^{\gamma_1}J_{1k}\big(x(t_k)\big) + \left(u_{11} + \sum_{i=1}^{k}\frac{\Gamma(2-\gamma_1)}{t_i^{1-\gamma_1}}J_{1i}\big(x(t_i)\big)\right)t. \tag{2.18}$$

Substituting (2.17) into (2.18), one can easily obtain (2.2). The proof is completed. □

Similarly, we conclude the following lemma.

Lemma 2.6 *Let $h_2 \in \mathrm{PC}(J)$. If $\Delta_2 \triangleq \frac{w^{1-\delta_2}}{\Gamma(2-\delta_2)} \neq 1$, then a function $y \in \mathrm{PC}(J)$ is a solution of the boundary value problem*

$$\begin{cases} {}^{C}D_{0^+}^{\beta}y(t) = h_2(t), \quad 1 < \beta < 2, t \in J, t \neq t_k, \\ {}^{C}D_{0^+}^{\gamma_2}y(t_k^+) - {}^{C}D_{0^+}^{\gamma_2}y(t_k^-) = J_{2k}(y(t_k)), \quad 0 < \gamma_2 < 1, k = 1, 2, \ldots, n, \\ y(0) = 0, \qquad {}^{\mathrm{LR}}D_{0^+}^{\delta_2}y(w) = y(1), \qquad 0 < \delta_2 < 1, \qquad 0 < w < 1, \end{cases} \tag{2.19}$$

if and only if $y \in \mathrm{PC}(J)$ is a solution of the integral equation

$$x(t) = I_{0^+}^{\beta}h_2(t) + \left(\frac{I_{0^+}^{\beta-\delta_2}h_2(w) - I_{0^+}^{\beta}h_2(1) + C_2}{1-\Delta_2} + \sum_{i=1}^{k}\frac{\Gamma(2-\gamma_2)}{t_i^{1-\gamma_2}}J_{2i}\big(y(t_i)\big)\right)t$$

$$- \Gamma(2-\gamma_2)t_k^{\gamma_2}J_{2k}\big(y(t_k)\big), \quad t \in (t_k, t_{k+1}], k = 0, 1, \ldots, n, \tag{2.20}$$

where

$$C_2 = \Delta_2\Gamma(2-\gamma_2)\sum_{i=1}^{l}\frac{J_{2i}(y(t_i))}{t_i^{1-\gamma_2}} - \frac{w^{-\delta_2}t_l^{\gamma_2}\Gamma(2-\gamma_2)}{\Gamma(1-\delta_2)}J_{2l}\big(y(t_l)\big) + t_n^{\gamma_2}\Gamma(2-\gamma_2)J_{2n}\big(y(t_n)\big)$$

$$- \Gamma(2-\gamma_2)\sum_{i=1}^{n}\frac{J_{2i}(y(t_i))}{t_i^{1-\gamma_2}}, \quad t_l < w \leq t_{l+1}, l \in \{0, 1, 2, \ldots, n\}.$$

3 Main results

In this section, we shall investigate the existence of solution for system (1.1) by employing the nonlinear alternative of Leray–Schauder.

Theorem 3.1 *If the following conditions (H_1)–(H_6) hold, then the boundary value problem (1.1) has at least a pair of solutions. The conditions are:*

(H_1) *The functions $f, g \in C(J \times R^2, R)$, and $J_{1k}, J_{2k} \in C(R, R)$, $k = 1, 2, \ldots, n$.*

(H_2) *For all $u_i, v_i \in R$ $(i = 1,2)$, $t \in R$, there exist some constants $L_i, \hat{L}_i > 0$ $(i = 1,2)$ such that*

$$\begin{aligned}
&\bigl|f(t,u_1,v_1) - f(t,u_2,v_2)\bigr| \le L_1|u_1 - u_2| + L_2|v_1 - v_2|,\\
&\bigl|g(t,u_1,v_1) - g(t,u_2,v_2)\bigr| \le \hat{L}_1|u_1 - u_2| + \hat{L}_2|v_1 - v_2|.
\end{aligned}$$

(H_3) $N \triangleq \sup_{t\in[0,1]} |f(t,0,0)|$ *and* $\hat{N} \triangleq \sup_{t\in[0,1]} |g(t,0,0)|$ *all exist.*

(H_4) $0 < \Delta_1 = \frac{z^{1-\delta_1}}{\Gamma(2-\delta_1)} < 1$, $0 < \Delta_2 = \frac{w^{1-\delta_2}}{\Gamma(2-\delta_2)} < 1$.

(H_5) *For any $u, v \in R$, there exist some constants $M_k, \hat{M}_k > 0$, $k = 1,2,\dots,n$, such that*

$$\bigl|J_{1k}(u)\bigr| \le M_k|u|, \qquad \bigl|J_{2k}(v)\bigr| \le \hat{M}_k|v|.$$

(H_6) $\kappa_1 \triangleq \mathcal{M}_1 + \mathcal{N}_1 < 1$ *and* $\kappa_2 \triangleq \mathcal{M}_2 + \mathcal{N}_2 < 1$, *where*

$$\begin{aligned}
\mathcal{M}_1 &= (L_1 + L_2)\biggl(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1+1)} + \frac{1}{(1-\Delta_1)\Gamma(\alpha+1)}\biggr),\\
\mathcal{N}_1 &= \frac{\Gamma(2-\gamma_1)}{1-\Delta_1}\biggl(\frac{2}{t_1^{1-\gamma_1}} + \frac{1}{z^{\delta_1}\Gamma(1-\delta_1)} + 2 - \Delta_1\biggr)\sum_{i=1}^{n} M_i,\\
\mathcal{M}_2 &= (\hat{L}_1 + \hat{L}_2)\biggl(\frac{1}{\Gamma(\beta+1)} + \frac{1}{(1-\Delta_2)\Gamma(\beta-\delta_2+1)} + \frac{1}{(1-\Delta_2)\Gamma(\beta+1)}\biggr),\\
\mathcal{N}_2 &= \frac{\Gamma(2-\gamma_2)}{1-\Delta_2}\biggl(\frac{2}{t_1^{1-\gamma_2}} + \frac{1}{w^{\delta_2}\Gamma(1-\delta_2)} + 2 - \Delta_2\biggr)\sum_{i=1}^{n} \hat{M}_i.
\end{aligned}$$

Proof Let $\Omega = \{(x,y) \in X : \|(x,y)\| < r\}$, where $X = \mathrm{PC}(J) \times \mathrm{PC}(J)$ and $r \ge \max\{\frac{N\mathcal{M}_1}{1-\kappa_1}, \frac{\hat{N}\mathcal{M}_2}{1-\kappa_2}\}$. Then $\overline{\Omega} = \{(x,y) \in X : \|(x,y)\| \le r\}$, $\partial\Omega = \{(x,y) \in X : \|(x,y)\| = r\}$. According to Lemmas 2.5–2.6, we define the operator $T : \overline{\Omega} \to X$ as follows:

$$T(x,y)(t) = \bigl(T_1(x,y)(t), T_2(x,y)(t)\bigr)^T, \quad \forall (x,y) \in X, t \in J, \tag{3.1}$$

where

$$\begin{aligned}
T_1(x,y)(t) &= I_{0^+}^{\alpha} f\bigl(t, x(t), {}^{C}D_{0^+}^{p} y(t)\bigr)\\
&\quad + \frac{I_{0^+}^{\alpha-\delta_1} f(z, x(z), {}^{C}D_{0^+}^{p} y(z)) - I_{0^+}^{\alpha} f(1, x(1), {}^{C}D_{0^+}^{p} y(1))}{1-\Delta_1} t\\
&\quad + \Biggl(\frac{C_1}{1-\Delta_1} + \sum_{i=1}^{k} \frac{\Gamma(2-\gamma_1)}{t_i^{1-\gamma_1}} J_{1i}\bigl(x(t_i)\bigr)\Biggr) t - \Gamma(2-\gamma_1) t_k^{\gamma_1} J_{1k}\bigl(x(t_k)\bigr),\\
&\quad t \in (t_k, t_{k+1}], k = 0,1,\dots,n,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
T_2(x,y)(t) &= I_{0^+}^{\beta} g\bigl(t, {}^{C}D_{0^+}^{q} x(t), y(t)\bigr)\\
&\quad + \frac{I_{0^+}^{\beta-\delta_2} g(w, {}^{C}D_{0^+}^{q} x(w), y(w)) - I_{0^+}^{\beta} g(1, {}^{C}D_{0^+}^{q} x(1), y(1))}{1-\Delta_2} t\\
&\quad + \Biggl(\frac{C_2}{1-\Delta_2} + \sum_{i=1}^{k} \frac{\Gamma(2-\gamma_2)}{t_i^{1-\gamma_2}} J_{2i}\bigl(y(t_i)\bigr)\Biggr) t - \Gamma(2-\gamma_2) t_k^{\gamma_2} J_{2k}\bigl(y(t_k)\bigr),\\
&\quad t \in (t_k, t_{k+1}], k = 0,1,\dots,n,
\end{aligned} \tag{3.3}$$

$$\begin{aligned} C_1 ={}& \Delta_1 \Gamma(2-\gamma_1)\sum_{i=1}^{j}\frac{J_{1i}(x(t_i))}{t_i^{1-\gamma_1}} - \frac{z^{-\delta_1}t_j^{\gamma_1}\Gamma(2-\gamma_1)}{\Gamma(1-\delta_1)}J_{1j}\big(x(t_j)\big) \\ &+ t_n^{\gamma_1}\Gamma(2-\gamma_1)J_{1n}\big(x(t_n)\big) - \Gamma(2-\gamma_1)\sum_{i=1}^{n}\frac{J_{1i}(x(t_i))}{t_i^{1-\gamma_1}}, \\ & t_j < z \le t_{j+1}, j \in \{0,1,2,\dots,n\}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} C_2 ={}& \Delta_2 \Gamma(2-\gamma_2)\sum_{i=1}^{l}\frac{J_{2i}(y(t_i))}{t_i^{1-\gamma_2}} - \frac{w^{-\delta_2}t_l^{\gamma_2}\Gamma(2-\gamma_2)}{\Gamma(1-\delta_2)}J_{2l}\big(y(t_l)\big) \\ &+ t_n^{\gamma_2}\Gamma(2-\gamma_2)J_{2n}\big(y(t_n)\big) - \Gamma(2-\gamma_2)\sum_{i=1}^{n}\frac{J_{2i}(y(t_i))}{t_i^{1-\gamma_2}}, \\ & t_l < w \le t_{l+1}, l \in \{0,1,2,\dots,n\}. \end{aligned} \tag{3.5}$$

Thus, the existence of solution for system (1.1) is equivalent to the existence of a fixed point for the operator T defined by (3.1)–(3.5). Now we shall apply Lemma 2.4 to prove that T has a fixed point $(x^*(t), y^*(t)) \in \overline{\Omega}$. Firstly, we need to show that $T : \overline{\Omega} \to X$ is completely continuous. In fact, for all $(x,y) \in \overline{\Omega}$, $t \in J = [0,1]$, from conditions (H_1)–(H_5), we have

$$\begin{aligned} &\big|T_1(x,y)(t)\big| \\ &\quad\le I_{0^+}^{\alpha}\big|f\big(t,x(t),{}^{C}D_{0^+}^{p}y(t)\big)\big| \\ &\qquad+ \frac{I_{0^+}^{\alpha-\delta_1}|f(z,x(z),{}^{C}D_{0^+}^{p}y(z))| + I_{0^+}^{\alpha}|f(1,x(1),{}^{C}D_{0^+}^{p}y(1))|}{1-\Delta_1} \\ &\qquad+ \frac{|C_1|}{1-\Delta_1} + \sum_{i=1}^{k}\frac{\Gamma(2-\gamma_1)}{t_i^{1-\gamma_1}}\big|J_{1i}\big(x(t_i)\big)\big| + \Gamma(2-\gamma_1)t_k^{\gamma_1}\big|J_{1k}\big(x(t_k)\big)\big| \\ &\quad\le I_{0^+}^{\alpha}\big|f\big(t,x(t),{}^{C}D_{0^+}^{p}y(t)\big) - f(t,0,0)\big| + I_{0^+}^{\alpha}\big|f(t,0,0)\big| \\ &\qquad+ \frac{1}{1-\Delta_1}\big[I_{0^+}^{\alpha-\delta_1}\big|f\big(z,x(z),{}^{C}D_{0^+}^{p}y(z)\big) - f(z,0,0)\big| + I_{0^+}^{\alpha-\delta_1}\big|f(z,0,0)\big|\big] \\ &\qquad+ \frac{1}{1-\Delta_1}\big[I_{0^+}^{\alpha}\big|f\big(1,x(1),{}^{C}D_{0^+}^{p}y(1)\big) - f(1,0,0)\big| + I_{0^+}^{\alpha}\big|f(1,0,0)\big|\big] \\ &\qquad+ \frac{1}{1-\Delta_1}\Bigg[\Delta_1\Gamma(2-\gamma_1)\sum_{i=1}^{j}\frac{|J_{1i}(x(t_i))|}{t_i^{1-\gamma_1}} + \frac{z^{-\delta_1}t_j^{\gamma_1}\Gamma(2-\gamma_1)}{\Gamma(1-\delta_1)}\big|J_{1j}\big(x(t_j)\big)\big| \\ &\qquad+ t_n^{\gamma_1}\Gamma(2-\gamma_1)\big|J_{1n}\big(x(t_n)\big)\big| + \Gamma(2-\gamma_1)\sum_{i=1}^{n}\frac{|J_{1i}(x(t_i))|}{t_i^{1-\gamma_1}}\Bigg] \\ &\qquad+ \sum_{i=1}^{k}\frac{\Gamma(2-\gamma_1)}{t_i^{1-\gamma_1}}\big|J_{1i}\big(x(t_i)\big)\big| + \Gamma(2-\gamma_1)t_k^{\gamma_1}\big|J_{1k}\big(x(t_k)\big)\big| \\ &\quad\le I_{0^+}^{\alpha}\big(L_1\big|x(t)\big| + L_2\big|{}^{C}D_{0^+}^{p}y(t)\big|\big) + I_{0^+}^{\alpha}\sup_{0\le t\le 1}\big|f(t,0,0)\big| \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1-\Delta_1}\Big[I_{0^+}^{\alpha-\delta_1}\big(L_1\big|x(z)\big|+L_2\big|{}^CD_{0^+}^{p}y(z)\big)\big|\big)+I_{0^+}^{\alpha-\delta_1}\sup_{0\le t\le 1}\big|f(t,0,0)\big|\Big]\\
&+\frac{1}{1-\Delta_1}\Big[I_{0^+}^{\alpha}\big(L_1\big|x(1)\big|+L_2\big|{}^CD_{0^+}^{p}y(1)\big|\big)+I_{0^+}^{\alpha}\sup_{0\le t\le 1}\big|f(t,0,0)\big|\Big]\\
&+\frac{1}{1-\Delta_1}\Bigg[\Delta_1\Gamma(2-\gamma_1)\sum_{i=1}^{j}\frac{M_i|x(t_i)|}{t_i^{1-\gamma_1}}+\frac{z^{-\delta_1}t_j^{\gamma_1}\Gamma(2-\gamma_1)}{\Gamma(1-\delta_1)}M_j\big|x(t_j)\big|\\
&+t_n^{\gamma_1}\Gamma(2-\gamma_1)M_n\big|x(t_n)\big|+\Gamma(2-\gamma_1)\sum_{i=1}^{n}\frac{M_i|x(t_i)|}{t_i^{1-\gamma_1}}\Bigg]\\
&+\sum_{i=1}^{k}\frac{\Gamma(2-\gamma_1)}{t_i^{1-\gamma_1}}M_i\big|x(t_i)\big|+\Gamma(2-\gamma_1)t_k^{\gamma_1}M_k\big|x(t_k)\big|\\
\le{}&\frac{1}{\Gamma(\alpha+1)}\big(L_1\|x\|_{\mathrm{PC}}+L_2\|y\|_{\mathrm{PC}}+N\big)\\
&+\frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1+1)}\big(L_1\|x\|_{\mathrm{PC}}+L_2\|y\|_{\mathrm{PC}}+N\big)\\
&+\frac{1}{(1-\Delta_1)\Gamma(\alpha+1)}\big(L_1\|x\|_{\mathrm{PC}}+L_2\|y\|_{\mathrm{PC}}+N\big)\\
&+\frac{\Gamma(2-\gamma_1)}{1-\Delta_1}\bigg[\frac{2}{t_1^{1-\gamma_1}}+\frac{1}{z^{\delta_1}\Gamma(1-\delta_1)}+2-\Delta_1\bigg]\sum_{i=1}^{n}M_i\cdot\|x\|_{\mathrm{PC}}\\
\le{}&\Bigg[(L_1+L_2)\bigg(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1+1)}+\frac{1}{(1-\Delta_1)\Gamma(\alpha+1)}\bigg)\\
&+\frac{\Gamma(2-\gamma_1)}{1-\Delta_1}\bigg(\frac{2}{t_1^{1-\gamma_1}}+\frac{1}{z^{\delta_1}\Gamma(1-\delta_1)}+2-\Delta_1\bigg)\sum_{i=1}^{n}M_i\Bigg]\|x\|_{\mathrm{PC}}\\
&+N(L_1+L_2)\bigg(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1+1)}+\frac{1}{(1-\Delta_1)\Gamma(\alpha+1)}\bigg)\\
\le{}&(\mathcal{M}_1+\mathcal{N}_1)r+N\mathcal{M}_1=\kappa_1 r+N\mathcal{M}_1\le r.
\end{aligned}
\tag{3.6}
$$

Similarly, we also have

$$
\big|T_2(x,y)(t)\big|\le\kappa_2 r+\hat{N}\mathcal{M}_2\le r.
\tag{3.7}
$$

Estimates (3.6) and (3.7) indicate that T is uniformly bounded and $T(\overline{\Omega})\subset\overline{\Omega}$.

Next, we show that operator T is equicontinuous, that is, for any $\epsilon>0$, $\tau_2,\tau_1\in J=[0,1]$, $(x,y)\in\overline{\Omega}$, there exists $\delta=\delta(\epsilon)>0$ such that, when $|\tau_2-\tau_1|<\delta$, we have $\|T(x,y)(\tau_2)-T(x,y)(\tau_1)\|<\epsilon$. Indeed, for any $\tau_1,\tau_2\in[0,1]$, without loss of generality, let $\tau_1<\tau_2$ and $|\tau_2-\tau_1|<\xi$, where $\xi=\min_{0\le i\le n}\{t_{i+1}-t_i\}$, $t_0=0$, $t_{n+1}=1$. Similar to (3.6), we have

$$
\begin{aligned}
&\big|T_1(x,y)(\tau_2)-T_1(x,y)(\tau_1)\big|\\
&\quad\le\big|I_{0^+}^{\alpha}f\big(\tau_2,x(\tau_2),{}^CD_{0^+}^{p}y(\tau_2)\big)-I_{0^+}^{\alpha}f\big(\tau_1,x(\tau_1),{}^CD_{0^+}^{p}y(\tau_1)\big)\big|\\
&\qquad+\frac{I_{0^+}^{\alpha-\delta_1}|f(z,x(z),{}^CD_{0^+}^{p}y(z))|+I_{0^+}^{\alpha}|f(1,x(1),{}^CD_{0^+}^{p}y(1))|}{1-\Delta_1}|\tau_2-\tau_1|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left[\frac{|C_1|}{1-\Delta_1} + \sum_{i=1}^{k} \frac{\Gamma(2-\gamma_1)}{t_i^{1-\gamma_1}} \bigl| J_{1i}\bigl(x(t_i)\bigr)\bigr| \right] |\tau_2 - \tau_1| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \bigl[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\bigr] \bigl| f\bigl(s, x(s), {}^C D_{0^+}^{p} y(s)\bigr)\bigr| \, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} \bigl| f\bigl(s, x(s), {}^C D_{0^+}^{p} y(s)\bigr)\bigr| \, ds \\
&\quad + \frac{I_{0^+}^{\alpha-\delta_1} |f(z, x(z), {}^C D_{0^+}^{p} y(z))| + I_{0^+}^{\alpha} |f(1, x(1), {}^C D_{0^+}^{p} y(1))|}{1-\Delta_1} |\tau_2 - \tau_1| \\
&\quad + \left[\frac{|C_1|}{1-\Delta_1} + \sum_{i=1}^{k} \frac{\Gamma(2-\gamma_1)}{t_i^{1-\gamma_1}} \bigl| J_{1i}\bigl(x(t_i)\bigr)\bigr| \right] |\tau_2 - \tau_1| \\
&\le \frac{1}{\Gamma(\alpha+1)} \bigl(L_1 \|x\|_{\mathrm{PC}} + L_2 \|y\|_{\mathrm{PC}} + N\bigr) \bigl(\tau_2^{\alpha} - \tau_1^{\alpha} - (\tau_2 - \tau_1)^{\alpha}\bigr) \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \bigl(L_1 \|x\|_{\mathrm{PC}} + L_2 \|y\|_{\mathrm{PC}} + N\bigr) (\tau_2 - \tau_1)^{\alpha} \\
&\quad + \frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1+1)} \bigl(L_1 \|x\|_{\mathrm{PC}} + L_2 \|y\|_{\mathrm{PC}} + N\bigr) |\tau_2 - \tau_1| \\
&\quad + \frac{1}{(1-\Delta_1)\Gamma(\alpha+1)} \bigl(L_1 \|x\|_{\mathrm{PC}} + L_2 \|y\|_{\mathrm{PC}} + N\bigr) |\tau_2 - \tau_1| \\
&\quad + \frac{\Gamma(2-\gamma_1)}{1-\Delta_1} \left[\frac{2}{t_1^{1-\gamma_1}} + \frac{1}{z^{\delta_1}\Gamma(1-\delta_1)} + 1 \right] \sum_{i=1}^{n} M_i \cdot \|x\|_{\mathrm{PC}} \cdot |\tau_2 - \tau_1| \\
&\le \frac{1}{\Gamma(\alpha+1)} (L_1 r + L_2 r + N) \bigl(\tau_2^{\alpha} - \tau_1^{\alpha}\bigr) \\
&\quad + \frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1+1)} (L_1 r + L_2 r + N) |\tau_2 - \tau_1| \\
&\quad + \frac{1}{(1-\Delta_1)\Gamma(\alpha+1)} (L_1 r + L_2 r + N) |\tau_2 - \tau_1| \\
&\quad + \frac{\Gamma(2-\gamma_1)}{1-\Delta_1} \left[\frac{2}{t_1^{1-\gamma_1}} + \frac{1}{z^{\delta_1}\Gamma(1-\delta_1)} + 1 \right] \sum_{i=1}^{n} M_i \cdot r \cdot |\tau_2 - \tau_1| \\
&= \frac{1}{\Gamma(\alpha)} (L_1 r + L_2 r + N) \eta^{\alpha-1} |\tau_2 - \tau_1| \\
&\quad + \frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1+1)} (L_1 r + L_2 r + N) |\tau_2 - \tau_1| \\
&\quad + \frac{1}{(1-\Delta_1)\Gamma(\alpha+1)} (L_1 r + L_2 r + N) |\tau_2 - \tau_1| \\
&\quad + \frac{\Gamma(2-\gamma_1)}{1-\Delta_1} \left[\frac{2}{t_1^{1-\gamma_1}} + \frac{1}{z^{\delta_1}\Gamma(1-\delta_1)} + 1 \right] \sum_{i=1}^{n} M_i \cdot r \cdot |\tau_2 - \tau_1| \\
&\le \Biggl[(L_1 r + L_2 r + N) \left(\frac{1}{\Gamma(\alpha)} + \frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1+1)} + \frac{1}{(1-\Delta_1)\Gamma(\alpha+1)} \right) \\
&\quad + \frac{r\Gamma(2-\gamma_1)}{1-\Delta_1} \left(\frac{2}{t_1^{1-\gamma_1}} + \frac{1}{z^{\delta_1}\Gamma(1-\delta_1)} + 1 \right) \sum_{i=1}^{n} M_i \Biggr] |\tau_2 - \tau_1| \\
&= \rho_1 |\tau_2 - \tau_1|, \qquad (3.8)
\end{aligned}
$$

where $\tau_1 < \eta < \tau_2$, $\rho_1 = (L_1 r + L_2 r + N)(\frac{1}{\Gamma(\alpha)} + \frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1+1)} + \frac{1}{(1-\Delta_1)\Gamma(\alpha+1)}) + \frac{r\Gamma(2-\gamma_1)}{1-\Delta_1}(\frac{2}{t_1^{1-\gamma_1}} + \frac{1}{z^{\delta_1}\Gamma(1-\delta_1)} + 1)\sum_{i=1}^{n} M_i$. Similar to (3.8), one has

$$\left|T_2(x,y)(\tau_2) - T_1(x,y)(\tau_1)\right| \le \rho_2|\tau_2 - \tau_1|, \tag{3.9}$$

where $\rho_2 = (\hat{L}_1 r + \hat{L}_2 r + \hat{N})(\frac{1}{\Gamma(\beta)} + \frac{1}{(1-\Delta_2)\Gamma(\beta-\delta_2+1)} + \frac{1}{(1-\Delta_2)\Gamma(\beta+1)}) + \frac{r\Gamma(2-\gamma_2)}{1-\Delta_2}(\frac{2}{t_1^{1-\gamma_2}} + \frac{1}{w^{\delta_2}\Gamma(1-\delta_2)} + 1)\sum_{i=1}^{n} \hat{M}_i$.

Take $\delta = \min\{\xi, \frac{\epsilon}{\rho_1}, \frac{\epsilon}{\rho_1}\}$. According to (3.8) and (3.9), we conclude that, for any $\epsilon > 0$, $\tau_2, \tau_1 \in J = [0,1]$, $(x,y) \in \overline{\Omega}$, there exists $\delta > 0$ such that $\|T(x,y)(\tau_2) - T(x,y)(\tau_1)\| < \epsilon$ if $|\tau_2 - \tau_1| < \delta$, namely, operator T is equicontinuous. Hence, by the Arzela–Ascoli theorem, we know that $T : \overline{\Omega} \to \overline{\Omega}$ is completely continuous.

Finally, we prove that condition (ii) of Lemma 2.4 is not true. In fact, for all $(\overline{x},\overline{y}) \in \partial\Omega$, $0 < \lambda < 1$ and $t \in [0,1]$, analogous to (3.6) and (3.7), we have

$$\left|\lambda T_1(\overline{x},\overline{y})(t)\right| \le \lambda\left(\kappa_1\|\overline{x}\|_{\mathrm{PC}} + N\mathcal{M}_1\right) < \left\|(\overline{x},\overline{y})\right\| = r \tag{3.10}$$

and

$$\left|\lambda T_2(\overline{x},\overline{y})(t)\right| \le \lambda\left(\kappa_2\|\overline{x}\|_{\mathrm{PC}} + \hat{N}\mathcal{M}_2\right) < \left\|(\overline{x},\overline{y})\right\| = r. \tag{3.11}$$

Estimates (3.10) and (3.11) imply that $\|\lambda T(\overline{x},\overline{y})\| < \|(\overline{x},\overline{y})\| = r$, that is, $(\overline{x},\overline{y}) \neq \lambda T(\overline{x},\overline{y})$, for all $(\overline{x},\overline{y}) \in \partial\Omega$. According to Lemma 2.4, we know that the boundary value problem (1.1) has a pair of solutions $(x^*,y^*) \in \overline{\Omega}$. The proof is completed. □

4 Illustrative examples

Consider the following four-point boundary value problem for nonlinear fractional differential coupling system with fractional order impulses:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = f(t, x(t), {}^{C}D_{0^+}^{p}y(t)), & t \in J = [0,1], t \neq t_k, \\ {}^{C}D_{0^+}^{\beta}y(t) = g(t, {}^{C}D_{0^+}^{q}x(t), y(t)), & t \in J = [0,1], t \neq t_k, \\ {}^{C}D_{0^+}^{\gamma_1}x(t_k^+) - {}^{C}D_{0^+}^{\gamma_1}x(t_k^-) = J_{1k}(x(t_k)), & k = 1,\dots,n, \\ {}^{C}D_{0^+}^{\gamma_2}y(t_k^+) - {}^{C}D_{0^+}^{\gamma_2}y(t_k^-) = J_{2k}(y(t_k)), & k = 1,\dots,n, \\ x(0) = y(0) = 0, \quad {}^{\mathrm{LR}}D_{0^+}^{\delta_1}x(z) = x(1), & {}^{\mathrm{LR}}D_{0^+}^{\delta_2}y(w) = y(1). \end{cases} \tag{4.1}$$

Take $\alpha = \frac{5}{4}$, $\beta = \frac{7}{4}$, $p = \frac{1}{2}$, $q = \frac{3}{4}$, $\gamma_1 = \frac{1}{3}$, $\gamma_2 = \frac{2}{3}$, $\delta_1 = \frac{1}{5}$, $\delta_2 = \frac{3}{5}$, $n = 2$, $t_1 = \frac{1}{6}$, $t_2 = \frac{5}{6}$, $z = \frac{1}{7}$, $w = \frac{4}{7}$, $f(t,u,v) = \frac{\sin(\pi t)+u+v}{100}$, $g(t,u,v) = \frac{e^t+\arctan(u^2+v^2)}{100}$, $J_{11}(u) = J_{22}(u) = \frac{u^2}{200}$, $J_{12}(u) = J_{21}(u) = \frac{\sqrt[3]{u}}{100}$. Obviously, $f, g \in C(J \times R^2, R)$, $J_{11}, J_{12}, J_{21}, J_{22} \in C(R,R)$. By a simple calculation, we have

$$\left|f(t,u_1,v_1) - f(t,u_2,v_2)\right| \le \frac{1}{100}|u_1 - u_2| + \frac{1}{100}|v_1 - v_2|,$$

$$\left|g(t,u_1,v_1) - g(t,u_2,v_2)\right| \le \frac{1}{100}|u_1 - u_2| + \frac{1}{100}|v_1 - v_2|,$$

$$\left|J_{11}(u)\right| = \left|J_{22}(u)\right| \le \frac{1}{50}|u|, \qquad \left|J_{12}(u)\right| = \left|J_{21}(u)\right| \le \frac{1}{300}|u|,$$

$$\sup_{t\in[0,1]}\left|f(t,0,0)\right| = \frac{1}{100}, \qquad \sup_{t\in[0,1]}\left|g(t,0,0)\right| = \frac{e}{100},$$

that is, $L_1 = L_2 = \hat{L}_1 = \hat{L}_2 = \frac{1}{100}$, $M_1 = \hat{M}_2 = \frac{1}{100}$, $M_2 = \hat{M}_1 = \frac{1}{300}$, $N = \frac{1}{100}$, $\hat{N} = \frac{e}{100}$. Therefore, we obtain

$$
\begin{aligned}
&0 < \Delta_1 = \frac{z^{1-\delta_1}}{\Gamma(2-\delta_1)} \approx 0.2264 < 1, \qquad 0 < \Delta_2 = \frac{w^{1-\delta_2}}{\Gamma(2-\delta_2)} \approx 0.9010 < 1,\\
&\mathcal{M}_1 = (L_1 + L_2)\left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{(1-\Delta_1)\Gamma(\alpha-\delta_1+1)} + \frac{1}{(1-\Delta_1)\Gamma(\alpha+1)}\right) \approx 0.0658,\\
&\mathcal{N}_1 = \frac{\Gamma(2-\gamma_1)}{1-\Delta_1}\left(\frac{2}{t_1^{1-\gamma_1}} + \frac{1}{z^{\delta_1}\Gamma(1-\delta_1)} + 2 - \Delta_1\right)\sum_{i=1}^{n} M_i \approx 0.1501,\\
&\mathcal{M}_2 = (\hat{L}_1 + \hat{L}_2)\left(\frac{1}{\Gamma(\beta+1)} + \frac{1}{(1-\Delta_2)\Gamma(\beta-\delta_2+1)} + \frac{1}{(1-\Delta_2)\Gamma(\beta+1)}\right) \approx 0.3263,\\
&\mathcal{N}_2 = \frac{\Gamma(2-\gamma_2)}{1-\Delta_2}\left(\frac{2}{t_1^{1-\gamma_2}} + \frac{1}{w^{\delta_2}\Gamma(1-\delta_2)} + 2 - \Delta_2\right)\sum_{i=1}^{n} \hat{M}_i \approx 0.6451,\\
&\kappa_1 = \mathcal{M}_1 + \mathcal{N}_1 \approx 0.2159 < 1, \qquad \kappa_2 = \mathcal{M}_2 + \mathcal{N}_2 \approx 0.9714 < 1.
\end{aligned}
$$

Thus, conditions (H_1)–(H_6) of Theorem 3.1 hold. Then (4.1) has at least a pair of solutions.

5 Conclusions

In describing some phenomena and processes of many fields such as physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, capacitor theory, electrical circuits, biology, control theory, fitting of experimental data, and so on, the fractional differential equation is better and more accurate than the integral-order differential equations. So the study of fractional differential equations has attracted the eyes of many scholars. Especially, the nonlocal boundary value problems have been widely studied by many researchers because of their extensive applications in, e.g., blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so forth. In this paper, we consider the nonlocal boundary value problem for a nonlinear fractional differential coupled system with fractional order impulses. We obtain some new sufficient criteria for the existence of solutions by use of the Leray–Schauder alternative theorem.

Acknowledgements
The authors thank the referees for a number of suggestions which have improved many aspects of this article.

Funding
This work was supported by the National Natural Sciences Foundation of Peoples Republic of China under Grant (Nos. 11161025, 11661047).

Availability of data and materials
Not applicable.

Competing interests
The authors declare to have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

Publisher's Note

Received: 11 November 2018 Accepted: 21 January 2019 Published online: 29 January 2019

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