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Extended central factorial polynomials of the second kind

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Abstract

In this paper, we consider the extended central factorial polynomials and numbers of the second kind, and investigate some properties and identities for these polynomials and numbers. In addition, we give some relations between those polynomials and the extended central Bell polynomials. Finally, we present some applications of our results to moments of Poisson distributions.

MSC: 11B75; 11B83

Keywords: Extended central factorial polynomials of the second kind; Extended central Bell polynomials

1 Introduction

For $n \geq 0$, the central factorial numbers of the second kind are defined by

$$x^n = \sum_{k=0}^n T(n, k)x^{[k]}, \quad n \geq 0, \text{ (see [1, 2, 20–22])}, \quad (1.1)$$

where $x^{[k]} = x(x + \frac{k}{2} - 1)(x + \frac{k}{2} - 2) \cdots (x - \frac{k}{2} + 1)$, $k \geq 1$, $x^{[0]} = 1$.

By (1.1), we see that the generating function of the central factorial numbers of the second kind is given by

$$\frac{1}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k = \sum_{n=k}^{\infty} T(n, k) \frac{t^n}{n!} \quad \text{(see [1–4, 21])}. \quad (1.2)$$

Here the definition of $T(n, k)$ is extended so that $T(n, k) = 0$ for $n < k$. This agreement will be applied to all similar situations without further mention.

Then, by (1.2), we have

$$T(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \left(j - \frac{k}{2}\right)^n, \quad n, k \geq 0. \quad (1.3)$$

Let us recall that the Stirling polynomials of the second kind are defined by

$$\frac{1}{k!} e^{xt} (e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k|x) \frac{t^n}{n!} \quad (\text{see [10, 16, 17]}), \tag{1.4}$$

where k is a nonnegative integer.

When $x = 0$, $S_2(n, k) = S_2(n, k|0)$, $n, k \geq 0$, are the Stirling numbers of the second kind given by

$$x^n = \sum_{k=0}^n S_2(n, k)(x)_k, \quad n \geq 0 \text{ (see [12, 14, 15, 21])},$$

where $(x)_0 = 1$, $(x)_k = x(x - 1) \cdots (x - k + 1)$, $k \geq 1$.

From (1.4), we note that

$$\begin{aligned} S_2(n, k|x) &= \sum_{l=k}^n \binom{n}{l} S_2(l, k)x^{n-l} \\ &= \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} (j+x)^n, \quad n, k \geq 0. \end{aligned} \tag{1.5}$$

The Bell polynomials are given by the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \text{Bel}_n(x) \frac{t^n}{n!} \quad (\text{see [5-11]}). \tag{1.6}$$

Then, from (1.4) and (1.6), we get

$$\text{Bel}_n(x) = \sum_{m=0}^n S_2(n, m)x^m \quad (\text{see [10]}). \tag{1.7}$$

In [17], the extended Stirling polynomials of the second kind are defined by

$$\frac{1}{k!} e^{xt} (e^t - 1 + rt)^k = \sum_{n=k}^{\infty} S_{2,r}(n, k|x) \frac{t^n}{n!}, \tag{1.8}$$

where $n, k \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{R}$.

When $x = 0$, $S_{2,r}(n, k) = S_{2,r}(n, k|0)$, $n, k \geq 0$, are called the extended Stirling numbers of the second kind. Note that $S_{2,0}(n, k) = S_2(n, k)$ and $S_{2,0}(n, k|x) = S_2(n, k|x)$.

From (1.4) and (1.8), we note that

$$S_{2,r}(n, k) = \sum_{l=0}^k \binom{n}{l} r^l S_2(n-l, k-l), \quad n, k \geq 0 \text{ (see [17])}. \tag{1.9}$$

It is known that the extended Bell polynomials are defined by

$$e^{x(e^t-1+rt)} = \sum_{n=0}^{\infty} \text{Bel}_{n,r}(x) \frac{t^n}{n!} \quad (\text{see [13, 18, 19]}). \tag{1.10}$$

For $x = 1$, $\text{Bel}_{n,r} = \text{Bel}_{n,r}(1)$ are called the extended Bell numbers.

Then, from (1.10), we get

$$\text{Bel}_{n,r}(x) = \sum_{m=0}^n x^m S_{2,r}(n, m) \quad (\text{see [17]}). \tag{1.11}$$

Recently, the central Bell polynomials were defined by Kim as

$$\sum_{n=k}^{\infty} \text{Bel}_n^{(c)}(x) \frac{t^n}{n!} = e^{x(e^{\frac{t}{2}} - e^{-\frac{t}{2}})} \quad (\text{see [1-3]}). \tag{1.12}$$

For $x = 1$, $\text{Bel}_n^{(c)} = \text{Bel}_n^{(c)}(1)$ are called the central Bell numbers.

Thus, by (1.12), we get

$$\text{Bel}_n^{(c)}(x) = \sum_{k=0}^n x^k T(n, k), \quad n \geq 0 \quad (\text{see [17]}). \tag{1.13}$$

The purpose of this paper is to consider the extended central factorial polynomials and numbers of the second kind, and investigate some properties and identities for these polynomials and numbers. In addition, we give some relations between those polynomials and the extended central Bell polynomials. Finally, we present some applications of our results to moments of Poisson distributions.

2 Extended central factorial polynomials of the second kind

Motivated by (1.8), we define the extended central factorial polynomials of the second kind by

$$\frac{1}{k!} e^{xt} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^k = \sum_{n=k}^{\infty} T^{(r)}(n, k|x) \frac{t^n}{n!}, \tag{2.1}$$

where $k \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{R}$. When $x = 0$, $T^{(r)}(n, k) = T^{(r)}(n, k|0)$, $n, k \geq 0$, are called the extended central factorial numbers of the second kind. Note here that, when $r = 0$, $T(n, k|x) = T^{(0)}(n, k|x)$ and $T(n, k) = T^{(0)}(n, k)$ are respectively the central factorial polynomials of the second kind and the central factorial numbers of the second kind.

From (2.1), we note that

$$\begin{aligned} \sum_{n=k}^{\infty} T^{(r)}(n, k|x) \frac{t^n}{n!} &= \sum_{m=0}^{\infty} \frac{1}{m!} x^m t^m \sum_{l=k}^{\infty} T^{(r)}(l, k) \frac{t^l}{l!} \\ &= \sum_{n=k}^{\infty} \left(\sum_{l=k}^n \binom{n}{l} T^{(r)}(l, k) x^{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

Therefore, by comparing the coefficients on both sides of (2.2), we obtain the following theorem.

Theorem 2.1 For $n, k \geq 0$, we have

$$T^{(r)}(n, k|x) = \sum_{l=k}^n \binom{n}{l} T^{(r)}(l, k) x^{n-l}. \tag{2.3}$$

From (2.1), we note that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{x^k}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^k &= \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} T^{(r)}(n, k) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n x^k T^{(r)}(n, k) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

On the other hand,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{x^k}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^k &= e^{x(e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)} = e^{x(e^{\frac{t}{2}} - e^{-\frac{t}{2}})} e^{xrt} \\ &= \sum_{k=0}^{\infty} \text{Bel}_k^{(c)}(x) \frac{t^k}{k!} \sum_{m=0}^{\infty} r^m x^m \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} \text{Bel}_k^{(c)}(x) r^{n-k} x^{n-k} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

Now, we define the extended central Bell polynomials by

$$e^{x(e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)} = \sum_{n=0}^{\infty} \text{Bel}_n^{(c,r)}(x) \frac{t^n}{n!}. \tag{2.6}$$

For $x = 1$, $\text{Bel}_n^{(c,r)} = \text{Bel}_n^{(c,r)}(1)$ are called the extended central Bell numbers.

Therefore, by combining (2.4)–(2.6), we obtain the following theorem.

Theorem 2.2 For $n \geq 0$, we have

$$\begin{aligned} \text{Bel}_n^{(c,r)}(x) &= \sum_{k=0}^n x^k T^{(r)}(n, k) \\ &= \sum_{k=0}^n \binom{n}{k} \text{Bel}_k^{(c)}(x) r^{n-k} x^{n-k}. \end{aligned} \tag{2.7}$$

In particular,

$$\text{Bel}_n^{(c,r)} = \sum_{k=0}^n \binom{n}{k} \text{Bel}_k^{(c)} r^{n-k} = \sum_{k=0}^n T^{(r)}(n, k). \tag{2.8}$$

Remark By (2.6), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \text{Bel}_n^{(c,r)}(x) \frac{t^n}{n!} &= e^{xt} e^{x(e^{\frac{t}{2}} - e^{-\frac{t}{2}} + (r-1)t)} \\ &= \sum_{k=0}^{\infty} x^k \frac{1}{k!} e^{xt} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + (r-1)t)^k \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} T^{(r-1)}(n, k|x) \frac{t^n}{n!} \right) x^k \\
 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n x^k T^{(r-1)}(n, k|x) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.9}$$

Comparing the coefficients on both sides of (2.9), we get

$$\text{Bel}_n^{(c,r)}(x) = \sum_{k=0}^n x^k T^{(r-1)}(n, k|x), \quad n \geq 0. \tag{2.10}$$

In particular,

$$\text{Bel}_n^{(c,r)} = \sum_{k=0}^n T^{(r-1)}(n, k|1), \tag{2.11}$$

and, invoking (2.3),

$$\text{Bel}_n^{(c,1)}(x) = \sum_{k=0}^n T(n, k|x)x^k = \sum_{k=0}^n \sum_{l=k}^n \binom{n}{l} T(l, k)x^{n-l+k}. \tag{2.12}$$

Therefore, by (2.10)–(2.12), we obtain the following corollary.

Corollary 2.3 *For $n \geq 0$, we have*

$$\text{Bel}_n^{(c,r)}(x) = \sum_{k=0}^n x^k T^{(r-1)}(n, k|x).$$

In particular,

$$\text{Bel}_n^{(c,r)} = \sum_{k=0}^n T^{(r-1)}(n, k|1)$$

and

$$\text{Bel}_n^{(c,1)}(x) = \sum_{k=0}^n \sum_{l=k}^n \binom{n}{l} T(l, k)x^{n-l+k}.$$

From (2.1), we note that

$$\begin{aligned}
 \sum_{n=0}^{\infty} T^{(r)}(n, k) \frac{t^n}{n!} &= \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} r^l t^l (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^{k-l} \\
 &= \sum_{l=0}^k \frac{r^l}{l!} t^l \frac{1}{(k-l)!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^{k-l}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{l=0}^k \frac{r^l}{l!} t^l \sum_{n=k}^{\infty} T(n-l, k-l) \frac{t^{n-l}}{(n-l)!} \\
 &= \sum_{n=k}^{\infty} \left(\sum_{l=0}^k \binom{n}{l} r^l T(n-l, k-l) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.13}$$

By comparing the coefficients on both sides of (2.13), we obtain the following theorem.

Theorem 2.4 For $n, k \geq 0$, we have

$$T^{(r)}(n, k) = \sum_{l=0}^k \binom{n}{l} r^l T(n-l, k-l). \tag{2.14}$$

From (2.6), we note that

$$\begin{aligned}
 e^{(e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)} &= e^{(e^{\frac{t}{2}} - e^{-\frac{t}{2}})} e^{rt} = \sum_{k=0}^{\infty} \text{Bel}_k^{(c)} \frac{t^k}{k!} \sum_{l=0}^{\infty} r^l \frac{t^l}{l!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} r^{n-k} \text{Bel}_k^{(c)} \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.15}$$

Therefore, by (2.14) and (2.8), we obtain the following theorem.

Theorem 2.5 For $n \geq 0$, we have

$$\text{Bel}_n^{(c,r)} = \sum_{k=0}^n \sum_{l=0}^k \binom{n}{l} r^l T(n-l, k-l). \tag{2.16}$$

From (2.1), we have

$$\begin{aligned}
 \sum_{n=k}^{\infty} T(n, k|x) \frac{t^n}{n!} &= \frac{1}{k!} e^{(x-\frac{k}{2})t} (e^t - 1)^k \\
 &= \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} e^{(j+x-\frac{k}{2})t} \\
 &= \sum_{n=0}^{\infty} \left(\frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \left(j + x - \frac{k}{2} \right)^n \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.17}$$

Therefore, by (2.17), we obtain the following theorem.

Theorem 2.6 For $n \geq 0$, we have

$$\frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \left(j + x - \frac{k}{2} \right)^n = \begin{cases} T(n, k|x), & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases} \tag{2.18}$$

Now, we observe that

$$\begin{aligned}
 e^{(e^t-1+rt)} &= e^{rt} e^{e^{\frac{t}{2}}(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} \\
 &= \sum_{k=0}^{\infty} e^{(\frac{k}{2}+r)t} \frac{1}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^k \\
 &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{k}{2}^j \frac{t^j}{j!} \sum_{m=k}^{\infty} T(m, k) \frac{t^m}{m!} \\
 &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \sum_{m=k}^n \binom{n}{m} T(m, k) \left(\frac{k}{2} + r\right)^{n-m} \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{m=k}^n \binom{n}{m} T(m, k) \left(\frac{k}{2} + r\right)^{n-m} \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.19}$$

Therefore, by (1.10) and (2.19), we obtain the following theorem.

Theorem 2.7 For $n \geq 0$, we have

$$\text{Bel}_{n,r} = \sum_{k=0}^n \sum_{m=k}^n \binom{n}{m} T(m, k) \left(\frac{k}{2} + r\right)^{n-m}. \tag{2.20}$$

From (2.18), we note that

$$\begin{aligned}
 \sum_{n=0}^{\infty} \text{Bel}_n^{(c,r)} \frac{t^n}{n!} &= e^{rt} e^{(e^{\frac{t}{2}}-e^{-\frac{t}{2}})} = \sum_{k=0}^{\infty} \frac{1}{k!} e^{(r-\frac{k}{2})t} (e^t - 1)^k \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} e^{(j+r-\frac{k}{2})t} \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \left(j + r - \frac{k}{2}\right)^n \frac{t^n}{n!} \\
 &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} T(n, k|r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^n T(n, k|r) \frac{t^n}{n!}.
 \end{aligned} \tag{2.21}$$

Therefore, by comparing the coefficients on both sides of (2.21), we obtain the following theorem.

Theorem 2.8 For $n \geq 0$, we have

$$\text{Bel}_n^{(c,r)} = \sum_{k=0}^n T(n, k|r). \tag{2.22}$$

From (1.2), we have

$$\sum_{n=0}^{\infty} T(n, k) \frac{t^n}{n!} = \frac{1}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt - rt)^k$$

$$\begin{aligned}
 &= \sum_{l=0}^k \frac{(-1)^l r^l}{l!} t^l \frac{1}{(k-l)!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^{k-l} \\
 &= \sum_{l=0}^k \frac{(-1)^l r^l}{l!} t^l \sum_{n=k}^{\infty} T^{(r)}(n-l, k-l) \frac{t^{n-l}}{(n-l)!} \\
 &= \sum_{n=k}^{\infty} \left(\sum_{l=0}^k \binom{n}{l} (-1)^l r^l T^{(r)}(n-l, k-l) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.23}$$

Therefore, by comparing the coefficients on both sides of (2.23), we obtain the following theorem.

Theorem 2.9 *For $n, k \geq 0$, we have*

$$T(n, k) = \sum_{l=0}^k \binom{n}{l} (-1)^l r^l T^{(r)}(n-l, k-l). \tag{2.24}$$

Now, we observe that

$$\begin{aligned}
 \frac{1}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^k &= \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} r^l t^l (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^{k-l} \\
 &= \sum_{l=0}^k \frac{r^l}{l!} t^l \sum_{n=k}^{\infty} T(n-l, k-l) \frac{t^{n-l}}{(n-l)!} \\
 &= \sum_{n=k}^{\infty} \left(\sum_{l=0}^k \binom{n}{l} r^l T(n-l, k-l) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.25}$$

Therefore, by (2.1) and (2.25), we obtain the following theorem.

Theorem 2.10 *For $n, k \geq 0$, we have*

$$T^{(r)}(n, k) = \sum_{l=0}^k \binom{n}{l} r^l T(n-l, k-l). \tag{2.26}$$

Let $m, k \in \mathbb{N} \cup \{0\}$. Then we have

$$\begin{aligned}
 &\frac{1}{m!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^m \frac{1}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^k \\
 &= \binom{m+k}{m} \frac{1}{(m+k)!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^{m+k} \\
 &= \binom{m+k}{m} \sum_{n=m+k}^{\infty} T^{(r)}(n, m+k) \frac{t^n}{n!}.
 \end{aligned} \tag{2.27}$$

On the other hand,

$$\begin{aligned} & \frac{1}{m!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^m \frac{1}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^k \\ &= \sum_{l=m}^{\infty} T^{(r)}(l, m) \frac{t^l}{l!} \sum_{j=k}^{\infty} T^{(r)}(j, k) \frac{t^j}{j!} \\ &= \sum_{n=m+k}^{\infty} \left(\sum_{l=m}^{n-k} \binom{n}{l} T^{(r)}(l, m) T^{(r)}(n-l, k) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.28}$$

Therefore, by (2.27) and (2.28), we obtain the following theorem.

Theorem 2.11 For $n \geq m + k$, with $m, k \geq 0$, we have

$$\binom{m+k}{m} T^{(r)}(n, m+k) = \sum_{l=m}^{n-k} \binom{n}{l} T^{(r)}(l, m) T^{(r)}(n-l, k). \tag{2.29}$$

For $m, k \geq 0$ with $m \geq k$, by (2.1), we get

$$\begin{aligned} \sum_{n=m}^{\infty} T^{(r)}(n, m) \frac{t^n}{n!} &= \frac{1}{m!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^{m-k} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^k \\ &= \frac{k!(m-k)!}{m!} \frac{1}{(m-k)!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^{m-k} \frac{1}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^k \\ &= \frac{1}{\binom{m}{k}} \sum_{l=m-k}^{\infty} T^{(r)}(l, m-k) \frac{t^l}{l!} \sum_{j=k}^{\infty} T^{(r)}(j, k) \frac{t^j}{j!} \\ &= \frac{1}{\binom{m}{k}} \sum_{n=m}^{\infty} \left(\sum_{l=m-k}^{n-k} \binom{n}{l} T^{(r)}(l, m-k) T^{(r)}(n-l, k) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.30}$$

By comparing the coefficients on both sides of (2.30), we obtain the following theorem.

Theorem 2.12 For $n, m, k \geq 0$, with $n \geq m \geq k$, we have

$$\binom{m}{k} T^{(r)}(n, m) = \sum_{l=m-k}^{n-k} \binom{n}{l} T^{(r)}(l, m-k) T^{(r)}(n-l, k). \tag{2.31}$$

Next, we observe that

$$\begin{aligned} & \frac{1}{m!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^m \frac{1}{k!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}} + rt)^k \\ &= \sum_{l=0}^m \frac{r^l}{l!} t^l \frac{1}{(m-l)!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^{m-l} \sum_{j=0}^k \frac{r^j}{j!} t^j \frac{1}{(k-j)!} (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^{k-j} \\ &= \sum_{n=m+k}^{\infty} \sum_{n_1=m}^{n-k} \sum_{l=0}^m \sum_{j=0}^k \binom{n_1}{l} \binom{n}{n_1} \binom{n-n_1}{j} r^{l+j} \\ &\quad \times T(n_1-l, m-l) T(n-n_1-j, k-j) \frac{t^n}{n!}. \end{aligned} \tag{2.32}$$

Therefore, by (2.27) and (2.32), we obtain the following theorem.

Theorem 2.13 *For $n, m, k \geq 0$, with $n \geq m + k$, we have*

$$\begin{aligned} & \binom{m+k}{m} T^{(r)}(n, m+k) \\ &= \sum_{n_1=m}^{n-k} \sum_{l=0}^m \sum_{j=0}^k \binom{n_1}{l} \binom{n}{n_1} \binom{n-n_1}{j} r^{l+j} T(n_1-l, m-l) T(n-n_1-j, k-j). \end{aligned} \tag{2.33}$$

Remark From (2.33) with $r = 0$, we can derive the following equation:

$$\binom{m+k}{m} T(n, m+k) = \sum_{l=m}^{n-k} \binom{n}{l} T(l, m) T(n-l, k), \tag{2.34}$$

where $n, m, k \geq 0$ with $n \geq m + k$.

3 Application

A random variable X , taking values $0, 1, 2, \dots$ is said to be a Poisson random variable with parameter $\lambda > 0$ if $P(i) = P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$, $i = 0, 1, 2, \dots$. Then we have $\sum_{i=0}^{\infty} P(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$. It is easy to show that the Bell polynomials $Bel_n(x)$, $n \geq 0$, are connected with the moments of Poisson distribution as follows:

$$E[x^n] = Bel_n(\lambda), \quad n \in \mathbb{N} \cup \{0\}, \lambda > 0.$$

Let X be a Poisson random variable with parameter $\lambda > 0$. Then we note that

$$\begin{aligned} \sum_{n=0}^{\infty} E[(X+r\lambda)^n] \frac{t^n}{n!} &= e^{\lambda e^{\frac{t}{2}} (e^{\frac{t}{2}} - e^{-\frac{t}{2}})} e^{\lambda r t} \\ &= \sum_{m=0}^{\infty} \lambda^m e^{\frac{m t}{2}} \frac{1}{m!} e^{\lambda r t} (e^{\frac{t}{2}} - e^{-\frac{t}{2}})^m \\ &= \sum_{m=0}^{\infty} \lambda^m \sum_{n=m}^{\infty} \sum_{j=m}^n \binom{n}{j} T(j, m|\lambda r) \left(\frac{m}{2}\right)^{n-j} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \sum_{j=m}^n \lambda^m \binom{n}{j} T(j, m|\lambda r) \left(\frac{m}{2}\right)^{n-j} \right) \frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

Thus, by (3.1), we get

$$E[(X+r\lambda)^n] = \sum_{m=0}^n \sum_{j=m}^n \lambda^m \binom{n}{j} T(j, m|\lambda r) \left(\frac{m}{2}\right)^{n-j}.$$

From (2.6), we can derive the following equation:

$$\begin{aligned} & \sum_{n=0}^{\infty} Bel_n^{(c,r)}(\lambda) \frac{t^n}{n!} \\ &= e^{\lambda(e^{-\frac{t}{2}}-1)(e^t-1)} e^{\lambda(e^t-1+rt)} \end{aligned}$$

$$\begin{aligned}
 &= \left(\sum_{k=0}^{\infty} \left(\sum_{l=0}^k \sum_{j=0}^l \lambda^l \binom{l}{j} (-1)^{l-j} S_2 \left(k, l - \frac{j}{2} \right) \right) \frac{t^k}{k!} \right) \left(\sum_{m=0}^{\infty} E[(X + r\lambda)^m] \frac{t^m}{m!} \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{l=0}^k \sum_{j=0}^l \lambda^l \binom{l}{j} \binom{n}{k} (-1)^{l-j} S_2 \left(k, l - \frac{j}{2} \right) E[(X + r\lambda)^{n-k}] \right) \frac{t^n}{n!}. \tag{3.2}
 \end{aligned}$$

Thus, we have

$$\text{Bel}_n^{(c,r)}(\lambda) = \sum_{k=0}^n \sum_{l=0}^k \sum_{j=0}^l \lambda^l \binom{l}{j} \binom{n}{k} (-1)^{l-j} S_2 \left(k, l - \frac{j}{2} \right) E[(X + r\lambda)^{n-k}],$$

where X is the Poisson random variable with parameter $\lambda > 0$, and $n \geq 0$.

4 Conclusions

T. Kim et al. have studied the central factorial polynomials and numbers of the second kind which are represented by some p -adic integrals on \mathbb{Z}_p and investigated some properties of these numbers and polynomials. In this paper, we introduced the extended central factorial numbers and polynomials by means of generating functions, which are useful, for example, in obtaining the moments of Poisson random variables. In addition, we gave some identities for the extended central Bell polynomials in terms of those numbers and polynomials. In more detail, in Sect. 2, we investigated some properties of the extended central factorial numbers and polynomials in connection with the extended central Bell numbers and polynomials, central factorial numbers and polynomials, and central factorial numbers and polynomials of the second kind in Theorems 2.1–2.13. Furthermore, in Sect. 3, we have applied our results to the moments of Poisson distribution.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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