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L^p ($p > 2$)-strong convergence of multiscale integration scheme for jump-diffusion systems

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Abstract

In this paper we shall prove the L^p ($p > 2$)-strong convergence of multiscale integration scheme for stochastic jump-diffusion systems with two-time-scale, which gives a numerical method for effective dynamical systems.

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1 Introduction

This paper focuses on the following two-time-scale jump-diffusion SDEs:

$$\begin{cases} dx_t^\epsilon = a(x_t^\epsilon, y_t^\epsilon) dt + b(x_t^\epsilon) dB_t + c(x_t^\epsilon) dP_t, & x_0^\epsilon = x_0, \\ dy_t^\epsilon = \frac{1}{\epsilon} f(x_t^\epsilon, y_t^\epsilon) dt + \frac{1}{\sqrt{\epsilon}} g(x_t^\epsilon, y_t^\epsilon) dW_t + h(x_t^\epsilon, y_t^\epsilon) dN_t^\epsilon, & y_0^\epsilon = y_0, \end{cases} \quad (1)$$

where $x_t^\epsilon \in \mathbb{R}^n$ and $y_t^\epsilon \in \mathbb{R}^m$ are jump-diffusion processes. The functions $a(x, y) \in \mathbb{R}^n$ and $f(x, y) \in \mathbb{R}^m$ are the drift coefficients, the functions $b(x) \in \mathbb{R}^{n \times d_1}$ and $g(x, y) \in \mathbb{R}^{m \times d_2}$ are the diffusion coefficients, and the functions $c(x) \in \mathbb{R}^n$ and $h(x, y) \in \mathbb{R}^m$ are the jump coefficients; B_t and W_t are d_1 , d_2 -dimensional independent Wiener processes, P_t is a scalar simple Poisson process with intensity λ_1 , and N_t^ϵ is a scalar simple Poisson process with intensity $\frac{\lambda_2}{\epsilon}$. ϵ is a small parameter, which represents the ratio of time scale between the processes x_t^ϵ and y_t^ϵ . With this time scale, the vector x_t^ϵ is referred to as the “slow component” and y_t^ϵ as the “fast component”. Under suitable assumptions the authors [1, 2] proved that when $\epsilon \rightarrow 0$, the slow component x_t^ϵ mean square converges to the solution of SDEs in the following form:

$$\begin{cases} d\bar{x}_t = \bar{a}(\bar{x}_t) dt + b(\bar{x}_t) dB_t + c(\bar{x}_t) dP_t, \\ \bar{x}_0 = x_0, \end{cases} \quad (2)$$

with

$$\bar{a}(x) = \int_{\mathbb{R}^m} a(x, y) \mu^x(dy), \quad x \in \mathbb{R}^n,$$

and μ^x is the invariant, ergodic measure generated by the following equation with frozen x :

$$dy_t = f(x, y_t) dt + g(x, y_t) dW_t + h(x, y_t) dN_t, \quad y_0 = y_0.$$

Multiscale jump-diffusion stochastic differential equations arise in many applications and have already been studied widely. What is usually of interest for this kind of system (1) is the time evolution of the slow variable x_t^ϵ . Thus a simplified equation, which is independent of the fast variable and possesses the essential features of the system, is highly desirable. On the one hand, while averaging principle [1–6] plays an important role in the research of slow component by getting a reduced equation (2), the difficulty of obtaining the effective equation (2) lies in the fact that the coefficient $\bar{a}(\cdot)$ is given via expectation with respect to measure $\mu^x(dy)$, which is usually difficult or impossible to obtain analytically, especially when the dimension m is large. On the other hand, even if we get the reduced equation, the equation cannot be solved explicitly. Therefore, the construction of the efficient computational methods is of great importance. Furthermore, the idea of multiscale integration schemes (cf. [7]) overcomes these difficulties exactly, which solves $\bar{x}_t$ with $\bar{a}(\cdot)$ being estimated on the fly using an empirical average of the original slow coefficients $a(\cdot)$ with respect to numerical solutions of the fast processes. This is one of our motivations.

For another significant motivation, a substantial body of work has been done concerning multiscale integration scheme for fast-slow SDEs. Most of the existing research theories discuss the convergence in L^p ($0 < p \leq 2$), even in a weaker sense [4, 5, 8–10]. Nevertheless, convergence in a stronger sense is what we want. In 2007, the L^2 averaging principle was proposed for a system, in which slow and fast dynamics were driven by Brownian noises and Poisson noises in [1]. Subsequently, the authors gave a multiscale integration scheme for the result in [9]. In 2015, Xu and Miao extended the result of [1] to the L^p ($p > 2$) case under assumptions (H1)–(H5) in [2]. A natural question is as follows: Can we also establish the L^p ($p > 2$) averaging principle by the multiscale integration scheme? It is well known that L^1 convergence and L^2 convergence cannot conclude L^p ($p > 2$) convergence. However, L^1 convergence and L^2 convergence can be deduced by L^p ($p > 2$) convergence. Once the L^p ($p > 2$) convergence has been established, then a much bigger degree of freedom for parameter q in research of L^q ($0 < q < p$) convergence would be obtained.

Based on the above discussion, the aim of this paper is to prove the L^p ($p > 2$) convergence of the multiscale integration scheme under the following assumptions:

- (H1) The measurable functions a, b, c, f, g , and h satisfy the global Lipschitz conditions, i.e., there is a positive constant L such that

$$\begin{aligned} & |a(u_1, v_1) - a(u_2, v_2)|^2 + |b(u_1) - b(u_2)|^2 + |c(u_1) - c(u_2)|^2 \\ & + |f(u_1, v_1) - f(u_2, v_2)|^2 + |g(u_1, v_1) - g(u_2, v_2)|^2 + |h(u_1, v_1) - h(u_2, v_2)|^2 \\ & \leq L(|u_1 - u_2|^2 + |v_1 - v_2|^2) \end{aligned}$$

for all $u_i \in \mathbb{R}^n$, $v_i \in \mathbb{R}^m$, $i = 1, 2$. Here and below we use $|\cdot|$ to denote both the Euclidean vector norm and the Frobenius matrix norm.

Remark 1.1 With the help of (H1), it immediately follows that there is a positive constant K such that

$$\begin{aligned} & |a(u, v)|^2 + |b(u)|^2 + |c(u)|^2 + |f(u, v)|^2 + |g(u, v)|^2 + |h(u, v)|^2 \\ & \leq K(1 + |u|^2 + |v|^2) \end{aligned}$$

for $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$. Thus a, b, c, f, g , and h satisfy the sublinear growth condition.

(H2) a, g , and h are globally bounded.

Remark 1.2 By (H1) and (H2), it is easy to derive that $\bar{a}$ in (2) is bounded and satisfies the Lipschitz condition [11].

(H3) There exist constants $\beta_1 > 0$ and $\beta_j \in \mathbb{R}$, $j = 2, 3, 4$, which are all independent of (u_1, v_1, v_2) , such that

$$\begin{aligned} v_1 \cdot f(u_1, v_1) & \leq -\beta_1 |v_1|^2 + \beta_2, \\ (f(u_1, v_1) - f(u_1, v_2))(v_1 - v_2) & \leq \beta_3 |v_1 - v_2|^2, \end{aligned} \quad (3)$$

and

$$(h(u_1, v_1) - h(u_1, v_2))(v_1 - v_2) \leq \beta_4 |v_1 - v_2|^2 \quad (4)$$

for all $u_1 \in \mathbb{R}^n$ and $v_1, v_2 \in \mathbb{R}^m$.

(H4) $\eta := -(2\beta_3 + 2\lambda_2\beta_4 + C_g + \lambda_2 C_h) > 0$, here β_3 and β_4 are taken from (3) and (4), λ_2 is from N_t^ϵ with intensity λ_2/ϵ , C_g , and C_h are the Lipschitz coefficients for g and h , respectively, i.e.,

$$|g(u_1, v_1) - g(u_2, v_2)|^2 \leq C_g(|u_1 - u_2|^2 + |v_1 - v_2|^2)$$

and

$$|h(u_1, v_1) - h(u_2, v_2)|^2 \leq C_h(|u_1 - u_2|^2 + |v_1 - v_2|^2)$$

for all $u_1, u_2 \in \mathbb{R}^n$, $v_1, v_2 \in \mathbb{R}^m$.

(H5) There exists a constant $\gamma > 0$, which is independent of (u, v) , such that

$$v^T g(u, v) g^T(u, v) v \geq \gamma |v|^2$$

for all $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$.

An example that satisfies (H1)–(H5) is $a(u, v) = \frac{1}{1+(u+v)^2}$, $b(u) = e^{-u^2}$, $c(u) = \sin u$, $f(u, v) = -1.5(\lambda_2 + 1)v$, $g(u, v) = \frac{3+\sin u+\sin v}{\sqrt{2}}$, and $h(u, v) = \frac{\sin u+\sin v}{\sqrt{2}}$.

It is worth pointing out that the L^p ($p > 2$) averaging principle under assumptions (H1)–(H5) had been established in [2].

Now, we will introduce the multiscale integration scheme. The scheme is made up of a macro solver to evolve (2) and a micro solver to simulate the fast dynamics in (1):

1. *Macro solver.* Let Δt be a fixed step, and let X_n be a numerical approximation to the coarse variable $\bar{x}$ at time $t_n = n\Delta t$. The simplest choice is the Euler–Maruyama scheme

$$X_{n+1} = X_n + A(X_n)\Delta t + b(X_n)\Delta B_n + c(X_n)\Delta P_n, \quad X_0 = x_0, \quad (5)$$

where $A(X_n)$ is estimated by an empirical average

$$A(X_n) = \frac{1}{M} \sum_{m=1}^M a(X_n, Y_m^n). \quad (6)$$

2. *Micro solver.* To get $A(X_n)$ used in the macro solver, we adopt the Euler–Maruyama scheme to generate Y_m^n :

$$Y_{m+1}^n = Y_m^n + \frac{1}{\epsilon} f(X_n, Y_m^n) \delta t + \frac{1}{\sqrt{\epsilon}} g(X_n, Y_m^n) \Delta W_m^n + h(X_n, Y_m^n) \Delta N_m^n, \quad (7)$$

with fixed X_n , and we denote the solution by Y_m^n , $m = 0, 1, \dots, M$, where ΔW_m^n are Brownian increments over a time interval δt , and ΔN_m^n are Poisson increments with intensity $\frac{\lambda_2}{\epsilon}$. Due to ergodicity [2] of the fast dynamics, we can select, among other selections, $Y_0^n = y_0$.

Note that the effective dynamics do not rely on ϵ . Meanwhile, since the discrete solution Y_m^n obtained by the micro-solver is for X_n fixed, it only depends on the ratio $\frac{\delta t}{\epsilon}$. Thus, without loss of generality, we may take $\epsilon = 1$. Then we have

$$Y_{m+1}^n = Y_m^n + f(X_n, Y_m^n) \delta t + g(X_n, Y_m^n) \Delta W_m^n + h(X_n, Y_m^n) \Delta N_m^n, \quad (8)$$

where $\Delta W_m^n = W_{(m+1)\delta t}^n - W_{m\delta t}^n$ are the Brownian increments, and $\Delta N_m^n = N_{(m+1)\delta t}^n - N_{m\delta t}^n$ are the Poisson increments with intensity λ_2 .

Simultaneously, Y_m^n are numerically generated discrete solutions of the family of SDEs as well:

$$dz_t^n = f(X_n, z_t^n) dt + g(X_n, z_t^n) dW_t^n + h(X_n, z_t^n) dN_t^n, \quad (9)$$

with initial conditions $z_0^n = Y_0^n = y_0$ and a time step δt (the choice of a fixed Y_0^n for all n simplifies our estimates; in practice, we could take $Y_0^n = Y_M^{n-1}$ for all $n > 0$).

We also present a discrete auxiliary process $\bar{X}_n$, the Euler solution to the effective dynamics (2):

$$\bar{X}_{n+1} = \bar{X}_n + \bar{a}(\bar{X}_n)\Delta t + b(\bar{X}_n)\Delta B_n + c(\bar{X}_n)\Delta P_n. \quad (10)$$

Concretely speaking, we are concentrating on estimating the L^p -strong error between the solution $\bar{x}_t$ of the effective dynamics (2) and the solution X_n of the multiscale integration scheme (5), (6), and (8) in this paper. Furthermore, we may easily obtain that the solution X_n of the multiscale integration scheme can approximate the solution $\bar{x}_t$ of the effective dynamics in both the sense of L^q ($0 < q < p$) and the probability by Hölder's inequality and Chebyshev's inequality. Then the process of proving the main result can be divided into two parts: (I') the difference between the process $\bar{x}_{t_n}$ and the auxiliary process

$\bar{X}_n$ (see Lemma 2.4 below); (II') the difference between the process X_n and the auxiliary process $\bar{X}_n$ (see Lemma 3.8 below).

We now describe the structure of the present paper. In Sect. 2, we introduce some a priori estimates to testify the error between the process $\bar{x}_{t_n}$ and the auxiliary process $\bar{X}_n$. In Sect. 3, we devote ourselves to proving the error between the process X_n and the auxiliary process $\bar{X}_n$. In Sect. 4, based on the above two estimates, we can derive our main result (see Theorem 4.1).

Throughout this paper, we will denote by C or K a generic positive constant which may change its value from line to line. In chains of inequalities, we will adopt $C, C', C'', \dots$ or $C_1, C_2, K_1, K_2, \dots$ to avoid confusion.

2 Some a priori estimates

In this section, we shall give some a priori estimates in the first three lemmas. Then we can apply the obtained results to estimate the difference between the process $\bar{x}_{t_n}$ and the auxiliary process $\bar{X}_n$.

For convenience, we will extend the discrete numerical solution $\bar{X}_n$ of (10) to continuous time. We first define the 'step functions'

$$Z(t) = \sum_k \bar{X}_k 1_{[k\Delta t, (k+1)\Delta t)}(t), \quad (11)$$

where 1_G is the indicator function for the set G . Then we define

$$\bar{X}(t) = x_0 + \int_0^t \bar{a}(Z(s)) ds + \int_0^t b(Z(s)) dB_s + \int_0^t c(Z(s)) dP_s. \quad (12)$$

(Note that by construction $Z(t-) = Z(t)$ for $t \neq k\Delta t$.) It is not difficult to verify $Z(t_k) = \bar{X}(t_k) = \bar{X}_k$. The aim of this section is to prove a convergence result for $\bar{X}(t)$ because the discrete numerical solution is interpolated to $\bar{X}(t)$. Then, we can obtain the convergence result for $\bar{X}_k$ straightly.

Firstly, we show that the discrete numerical solution $\bar{X}_k$ and the continuous approximation $\bar{X}(t)$ have $2p$ bounded moments in the first two lemmas.

Lemma 2.1 *For any $p > 1$ and $T > 0$, there exist positive constants Δt^* and C_1 such that, for all $0 < \Delta t \leq \Delta t^*$,*

$$\mathbb{E}|\bar{X}_k|^{2p} \leq C_1(1 + |x_0|^{2p}) \quad (13)$$

for $k\Delta t \leq T$, where C_1 is independent of $(k, \Delta t)$.

Proof By construction (12), we have

$$\bar{X}_{k+1} = x_0 + \int_0^{(k+1)\Delta t} \bar{a}(Z(s)) ds + \int_0^{(k+1)\Delta t} b(Z(s)) dB_s + \int_0^{(k+1)\Delta t} c(Z(s)) dP_s.$$

Then we obtain

$$\begin{aligned}\mathbb{E}|\bar{X}_{k+1}|^{2p} &\leq C|x_0|^{2p} + C\mathbb{E}\left|\int_0^{(k+1)\Delta t} \bar{a}(Z(s)) ds\right|^{2p} + C\mathbb{E}\left|\int_0^{(k+1)\Delta t} b(Z(s)) dB_s\right|^{2p} \\ &\quad + C\mathbb{E}\left|\int_0^{(k+1)\Delta t} c(Z(s)) dP_s\right|^{2p} \\ &:= C(|x_0|^{2p} + I_1 + I_2 + I_3)\end{aligned}\quad (14)$$

for $(k+1)\Delta t \leq T$. For I_2 and I_3 , using $\tilde{P}_t := P_t - \lambda_1 t$, Burkholder's inequality [12], Hölder's inequality, Remark 1.1, and (11), we have

$$\begin{aligned}I_2 &\leq C\mathbb{E}\left[\int_0^{(k+1)\Delta t} |b(Z(s))|^2 ds\right]^p \\ &\leq C\int_0^{(k+1)\Delta t} \mathbb{E}|b(Z(s))|^{2p} ds \\ &\leq C\int_0^{(k+1)\Delta t} \mathbb{E}(1 + |Z(s)|^{2p}) ds \\ &\leq C + C\Delta t \sum_{i=0}^k \mathbb{E}|\bar{X}_i|^{2p}\end{aligned}\quad (15)$$

and

$$\begin{aligned}I_3 &= \mathbb{E}\left|\int_0^{(k+1)\Delta t} c(Z(s)) d\tilde{P}_s + \lambda_1 \int_0^{(k+1)\Delta t} c(Z(s)) ds\right|^{2p} \\ &\leq C\mathbb{E}\left|\int_0^{(k+1)\Delta t} c(Z(s)) d\tilde{P}_s\right|^{2p} + C\mathbb{E}\left|\lambda_1 \int_0^{(k+1)\Delta t} c(Z(s)) ds\right|^{2p} \\ &\leq C\mathbb{E}\left[\int_0^{(k+1)\Delta t} |c(Z(s))|^2 ds\right]^p + C\mathbb{E}\left|\lambda_1 \int_0^{(k+1)\Delta t} c(Z(s)) ds\right|^{2p} \\ &\leq C\int_0^{(k+1)\Delta t} \mathbb{E}|c(Z(s))|^{2p} ds + C\int_0^{(k+1)\Delta t} \mathbb{E}|c(Z(s))|^{2p} ds \\ &\leq C\int_0^{(k+1)\Delta t} \mathbb{E}(1 + |Z(s)|^{2p}) ds \\ &\leq C + C\Delta t \sum_{i=0}^k \mathbb{E}|\bar{X}_i|^{2p}.\end{aligned}\quad (16)$$

Similarly, we may deal with I_1 and have

$$I_1 \leq C + C\Delta t \sum_{i=0}^k \mathbb{E}|\bar{X}_i|^{2p}.\quad (17)$$

Choosing Δt sufficiently small and by (14)–(17), we have

$$\mathbb{E}|\bar{X}_{k+1}|^{2p} \leq C(1 + |x_0|^{2p}) + C\Delta t \sum_{i=0}^k \mathbb{E}|\bar{X}_i|^{2p},$$

which, with the aid of discrete Gronwall's inequality, gives the result. $\square$

Lemma 2.2 For any $p > 1$ and $T > 0$, there exist positive constants Δt^* and C_2 such that, for all $0 < \Delta t \leq \Delta t^*$,

$$\mathbb{E} \sup_{t \in [0, T]} |\bar{X}(t)|^{2p} \leq C_2 (1 + |x_0|^{2p}), \quad (18)$$

where C_2 is independent of Δt .

Proof From (12), we have

$$|\bar{X}(t)|^{2p} \leq C|x_0|^{2p} + C \left| \int_0^t \bar{a}(Z(s)) ds \right|^{2p} + C \left| \int_0^t b(Z(s)) dB_s \right|^{2p} + C \left| \int_0^t c(Z(s)) dP_s \right|^{2p}.$$

Thus, by the definition of $\tilde{P}_t$, we have

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |\bar{X}(t)|^{2p} &\leq C|x_0|^{2p} + C \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t \bar{a}(Z(s)) ds \right|^{2p} + C \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t b(Z(s)) dB_s \right|^{2p} \\ &\quad + C \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t c(Z(s)) d\tilde{P}_s \right|^{2p} + C \mathbb{E} \sup_{t \in [0, T]} \left| \lambda_1 \int_0^t c(Z(s)) ds \right|^{2p}. \end{aligned} \quad (19)$$

By the same method as in the previous lemma, we obtain

$$\mathbb{E} \sup_{t \in [0, T]} |\bar{X}(t)|^{2p} \leq C(1 + |x_0|^{2p}) + C \int_0^T \mathbb{E} |Z(s)|^{2p} ds.$$

Applying (11) and Lemma 2.1 over the interval $[0, T]$, we obtain result (18). $\square$

Secondly, we show that the continuous-time approximation remains close to the step functions $Z(s)$ in a strong sense.

Lemma 2.3 For any $p > 1$ and $T > 0$, there exist positive constants Δt^* and C_3 such that, for all $0 < \Delta t \leq \Delta t^*$,

$$\mathbb{E} \sup_{t \in [0, T]} |\bar{X}(t) - Z(t)|^{2p} \leq C_3 \Delta t^p (1 + |x_0|^{2p}), \quad (20)$$

where C_3 is independent of Δt .

Proof Consider $t \in [k\Delta t, (k+1)\Delta t] \subseteq [0, T]$, we have

$$\bar{X}(t) - Z(t) = \bar{X}(t) - \bar{X}_k = \int_{k\Delta t}^t \bar{a}(Z(s)) ds + \int_{k\Delta t}^t b(Z(s)) dB_s + \int_{k\Delta t}^t c(Z(s)) dP_s.$$

Thus

$$|\bar{X}(t) - Z(t)|^{2p} \leq C \left| \int_{k\Delta t}^t \bar{a}(Z(s)) ds \right|^{2p} + C \left| \int_{k\Delta t}^t b(Z(s)) dB_s \right|^{2p} + C \left| \int_{k\Delta t}^t c(Z(s)) dP_s \right|^{2p}$$

for each $t \in [k\Delta t, (k+1)\Delta t]$. Then

$$\begin{aligned} \sup_{t \in [0, T]} |\bar{X}(t) - Z(t)|^{2p} &\leq \max_{k=0,1,\dots,T/\Delta t-1} \sup_{\tau \in [k\Delta t, (k+1)\Delta t]} \left\{ C \left| \int_{k\Delta t}^{\tau} \bar{a}(Z(s)) ds \right|^{2p} \right. \\ &\quad + C \left| \int_{k\Delta t}^{\tau} b(Z(s)) dB_s \right|^{2p} + C \left| \int_{k\Delta t}^{\tau} c(Z(s)) d\tilde{P}_s \right|^{2p} \\ &\quad \left. + C \left| \lambda_1 \int_{k\Delta t}^{\tau} c(Z(s)) ds \right|^{2p} \right\}. \end{aligned} \quad (21)$$

Now, taking expectations on both sides of (21), then using Burkholder's inequality on the martingale integrals and by Hölder's inequality, we have

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |\bar{X}(t) - Z(t)|^{2p} &\leq \max_{k=0,1,\dots,T/\Delta t-1} \left\{ C \Delta t^{2p-1} \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E} |\bar{a}(Z(s))|^{2p} ds \right. \\ &\quad + C \Delta t^{p-1} \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E} |b(Z(s))|^{2p} ds \\ &\quad + C \Delta t^{p-1} \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E} |c(Z(s))|^{2p} ds \\ &\quad \left. + C \Delta t^{2p-1} \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E} |c(Z(s))|^{2p} ds \right\}. \end{aligned}$$

Applying Remarks 1.1 and 1.2, we have

$$\mathbb{E} \sup_{t \in [0, T]} |\bar{X}(t) - Z(t)|^{2p} \leq \max_{k=0,1,\dots,T/\Delta t-1} \left\{ C (\Delta t^{p-1} + \Delta t^{2p-1}) \int_{k\Delta t}^{(k+1)\Delta t} (1 + \mathbb{E} |Z(s)|^{2p}) ds \right\}.$$

But $Z(s) \equiv \bar{X}_k$ on $[k\Delta t, (k+1)\Delta t]$, hence, it follows from Lemma 2.1 that

$$\mathbb{E} \sup_{t \in [0, T]} |\bar{X}(t) - Z(t)|^{2p} \leq C (\Delta t^p + \Delta t^{2p}) [1 + C_1 (1 + |x_0|^{2p})],$$

which yields result (20). $\square$

Lastly, we prove a strong convergence result for $\bar{X}(t)$.

Lemma 2.4 *For any $p > 1$ and $T > 0$, there exist positive constants Δt^* and C_4 such that, for all $0 < \Delta t \leq \Delta t^*$,*

$$\mathbb{E} \sup_{t \in [0, T]} |\bar{X}(t) - \bar{x}(t)|^{2p} \leq C_4 \Delta t^p (1 + |x_0|^{2p}), \quad (22)$$

where C_4 is independent of Δt .

Proof By construction (12), we get

$$\begin{aligned} \bar{X}(t) - \bar{x}(t) &= \int_0^t [\bar{a}(Z(s)) - \bar{a}(\bar{x}(s))] ds + \int_0^t [b(Z(s)) - b(\bar{x}(s))] dB_s \\ &\quad + \int_0^t [c(Z(s)) - c(\bar{x}(s))] dP_s. \end{aligned}$$

Hence, we have

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, t_1]} |\bar{X}(t) - \bar{x}(t)|^{2p} &\leq C \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t [\bar{a}(Z(s)) - \bar{a}(\bar{x}(s))] ds \right|^{2p} \\ &\quad + C \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t [b(Z(s)) - b(\bar{x}(s))] dB_s \right|^{2p} \\ &\quad + C \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t [c(Z(s)) - c(\bar{x}(s))] dP_s \right|^{2p} \\ &:= C(I_1 + I_2 + I_3) \end{aligned} \quad (23)$$

for any $0 \leq t_1 \leq T$. By the definition of $\tilde{P}_t$, Burkholder's inequality, Hölder's inequality, and (H1), we obtain

$$\begin{aligned} I_2 &\leq C \mathbb{E} \left[\int_0^{t_1} |b(Z(s)) - b(\bar{x}(s))|^2 ds \right]^p \\ &\leq C \int_0^{t_1} \mathbb{E} |b(Z(s)) - b(\bar{x}(s))|^{2p} ds \\ &\leq C \int_0^{t_1} \mathbb{E} |Z(s) - \bar{x}(s)|^{2p} ds, \end{aligned} \quad (24)$$

$$\begin{aligned} I_3 &\leq C \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t [c(Z(s)) - c(\bar{x}(s))] d\tilde{P}_s \right|^{2p} \\ &\quad + C \mathbb{E} \sup_{t \in [0, t_1]} \left| \lambda_1 \int_0^t [c(Z(s)) - c(\bar{x}(s))] ds \right|^{2p} \\ &\leq C \mathbb{E} \left[\int_0^{t_1} |c(Z(s)) - c(\bar{x}(s))|^2 ds \right]^p \\ &\quad + C \mathbb{E} \sup_{t \in [0, t_1]} \left| \lambda_1 \int_0^t [c(Z(s)) - c(\bar{x}(s))] ds \right|^{2p} \\ &\leq C \int_0^{t_1} \mathbb{E} |c(Z(s)) - c(\bar{x}(s))|^{2p} ds \\ &\leq C \int_0^{t_1} \mathbb{E} |Z(s) - \bar{x}(s)|^{2p} ds. \end{aligned} \quad (25)$$

Dealing with I_1 similarly and combining (23)–(25), it follows that

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |\bar{X}(t) - \bar{x}(t)|^{2p} &\leq C \int_0^{t_1} \mathbb{E} |Z(s) - \bar{x}(s)|^{2p} ds \\ &\leq C \int_0^{t_1} \mathbb{E} |\bar{X}(s) - \bar{x}(s)|^{2p} ds + C \int_0^{t_1} \mathbb{E} |\bar{X}(s) - Z(s)|^{2p} ds. \end{aligned}$$

Applying Lemma 2.3, we obtain

$$\mathbb{E} \sup_{t \in [0, T]} |\bar{X}(t) - \bar{x}(t)|^{2p} \leq C_5 \Delta t^p (1 + |x_0|^{2p}) + C_6 \int_0^{t_1} \mathbb{E} \sup_{t \in [0, s]} |\bar{X}(t) - \bar{x}(t)|^{2p} ds.$$

By continuous Gronwall's inequality, the desired estimate (22) is obtained. $\square$

3 Strong convergence of the scheme

In this section, some a priori estimates would be provided in the first seven lemmas. Then we can use the established estimates to get the error between the process X_n and the auxiliary process $\tilde{X}_n$.

Now, we firstly show the $2p$ th moment estimates for the processes z_t^n , X_n , and Y_m^n .

Lemma 3.1 *For any $p > 1$ and $T > 0$, there exists a positive constant K_1 such that*

$$\sup_{0 \leq t \leq T} \mathbb{E} |z_t^n|^{2p} \leq K_1. \quad (26)$$

Proof For $|z_t^n|^{2p}$, direct computation with Itô's formula gives that

$$\begin{aligned} |z_t^n|^{2p} &= |y_0|^{2p} + 2p \int_0^t |z_s^n|^{2p-2} (f(X_n, z_s^n), z_s^n) ds + 2p \int_0^t |z_s^n|^{2p-2} (g(X_n, z_s^n), z_s^n) dW_s^n \\ &\quad + 2p \int_0^t |z_s^n|^{2p-2} (h(X_n, z_s^n), z_s^n) dN_s^n + 2p(p-1) \int_0^t |z_s^n|^{2(p-2)} (g(X_n, z_s^n), z_s^n)^2 ds \\ &\quad + 2p(p-1)\lambda_2 \int_0^t |z_s^n|^{2(p-2)} (h(X_n, z_s^n), z_s^n)^2 ds + p \int_0^t |z_s^n|^{2p-2} |g(X_n, z_s^n)|^2 ds \\ &\quad + p\lambda_2 \int_0^t |z_s^n|^{2p-2} |h(X_n, z_s^n)|^2 ds. \end{aligned} \quad (27)$$

We have by (H3)

$$(f(X_n, z_s^n), z_s^n) \leq -\beta_1 |z_s^n|^2 + \beta_2. \quad (28)$$

By Young's inequality and (H2), we have

$$2\lambda_2 (h(X_n, z_s^n), z_s^n) \leq \frac{\lambda_2^2}{\beta_1} |h(X_n, z_s^n)|^2 + \beta_1 |z_s^n|^2 \leq C + \beta_1 |z_s^n|^2, \quad (29)$$

$$(g(X_n, z_s^n), z_s^n)^2 \leq C |z_s^n|^2, \quad (30)$$

and

$$(h(X_n, z_s^n), z_s^n)^2 \leq C |z_s^n|^2. \quad (31)$$

Taking expectations on both sides of (27) and combining (28)–(31), we have

$$\mathbb{E} |z_t^n|^{2p} \leq |y_0|^{2p} - p\beta_1 \mathbb{E} \int_0^t |z_s^n|^{2p} ds + C_{p,\lambda_2,\beta_1,\beta_2} \mathbb{E} \int_0^t |z_s^n|^{2p-2} ds.$$

Moreover, taking $k > 0$ small enough for Young's inequality in the form $ab \leq k|b|^m + C_{k,m}|a|^{m/m-1}$, we have

$$\mathbb{E} |z_s^n|^{2p} \leq |y_0|^{2p} - C_{p,\lambda_2,\beta_1,\beta_2} \mathbb{E} \int_0^t |z_s^n|^{2p} ds + C'_{p,\lambda_2,\beta_1,\beta_2} t,$$

which, with the help of continuous Gronwall's inequality, yields the result. $\square$

The proof of the following lemma is similar to Sect. 2. We omit the details.

Lemma 3.2 *For any $p > 1$ and small enough Δt , there exists a positive constant K_2 such that*

$$\sup_{0 \leq n \leq T/\Delta t} \mathbb{E}|X_n|^{2p} \leq K_2, \quad (32)$$

where K_2 is independent of Δt .

Lemma 3.3 *For small enough δt and $p > 1$, there exists a positive constant K_3 such that*

$$\sup_{\substack{0 \leq n \leq \frac{T}{\Delta t} \\ 0 < m < M}} E|Y_m^n|^{2p} \leq K_3, \quad (33)$$

where K_3 is independent of $(M, \delta t)$.

Proof Now we give a definition of Y_t^n by

$$Y_t^n := y_0 + \int_0^t f(X_n, \hat{Y}_s^n) ds + \int_0^t g(X_n, \hat{Y}_s^n) dW_s^n + \int_0^t h(X_n, \hat{Y}_s^n) dN_s^n,$$

where $\hat{Y}_t^n := Y_k^n$ for $t \in [k\delta t, (k+1)\delta t)$, $k = 1, 2, \dots, M$, and $\hat{Y}_{t_k}^n = Y_k^n = Y_k^n(t_k = k\delta t)$.

Thus we have

$$\begin{aligned} Y_{k+1}^n &= y_0 + \int_0^{(k+1)\delta t} f(X_n, \hat{Y}_s^n) ds + \int_0^{(k+1)\delta t} g(X_n, \hat{Y}_s^n) dW_s^n \\ &\quad + \int_0^{(k+1)\delta t} h(X_n, \hat{Y}_s^n) dN_s^n. \end{aligned} \quad (34)$$

Taking the 2pth moment and expectations on both sides of (34), we get

$$\begin{aligned} \mathbb{E}|Y_{k+1}^n|^{2p} &= \mathbb{E} \left| y_0 + \int_0^{(k+1)\delta t} f(X_n, \hat{Y}_s^n) ds + \int_0^{(k+1)\delta t} g(X_n, \hat{Y}_s^n) dW_s^n \right. \\ &\quad \left. + \int_0^{(k+1)\delta t} h(X_n, \hat{Y}_s^n) dN_s^n \right|^{2p} \\ &\leq C|y_0|^{2p} + C\mathbb{E} \left| \int_0^{(k+1)\delta t} f(X_n, \hat{Y}_s^n) ds \right|^{2p} + C\mathbb{E} \left| \int_0^{(k+1)\delta t} g(X_n, \hat{Y}_s^n) dW_s^n \right|^{2p} \\ &\quad + C\mathbb{E} \left| \int_0^{(k+1)\delta t} h(X_n, \hat{Y}_s^n) dN_s^n \right|^{2p}. \end{aligned} \quad (35)$$

Using Hölder's inequality, Remark 1.1, and the definition of $\hat{Y}_t^n$, we have

$$\mathbb{E} \left| \int_0^{(k+1)\delta t} f(X_n, \hat{Y}_s^n) ds \right|^{2p} \leq C \int_0^{(k+1)\delta t} \mathbb{E}|f(X_n, \hat{Y}_s^n)|^{2p} ds$$

$$\begin{aligned}
&\leq C \int_0^{(k+1)\delta t} \mathbb{E}(1 + |X_n|^{2p} + |\hat{Y}_s^n|^{2p}) ds \\
&\leq C + C\mathbb{E}|X_n|^{2p} + C\delta t \sum_{i=0}^k \mathbb{E}|Y_i^n|^{2p}.
\end{aligned} \tag{36}$$

By the definition of $\tilde{N}_t$, Burkholder's inequality, Hölder's inequality, and (H2), we obtain

$$\begin{aligned}
\mathbb{E} \left| \int_0^{(k+1)\delta t} g(X_n, \hat{Y}_s^n) dW_s^n \right|^{2p} &\leq C \mathbb{E} \left[\int_0^{(k+1)\delta t} |g(X_n, \hat{Y}_s^n)|^2 ds \right]^p \\
&\leq C \int_0^{(k+1)\delta t} \mathbb{E} |g(X_n, \hat{Y}_s^n)|^{2p} ds \\
&\leq C_{p,T}
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&\mathbb{E} \left| \int_0^{(k+1)\delta t} h(X_n, \hat{Y}_s^n) dN_s^n \right|^{2p} \\
&= \mathbb{E} \left| \int_0^{(k+1)\delta t} h(X_n, \hat{Y}_s^n) d\tilde{N}_s^n + \lambda_2 \int_0^{(k+1)\delta t} h(X_n, \hat{Y}_s^n) ds \right|^{2p} \\
&\leq C \mathbb{E} \left[\int_0^{(k+1)\delta t} |h(X_n, \hat{Y}_s^n)|^2 ds \right]^p \\
&\quad + C \mathbb{E} \left| \lambda_2 \int_0^{(k+1)\delta t} h(X_n, \hat{Y}_s^n) ds \right|^{2p} \\
&\leq C \int_0^{(k+1)\delta t} \mathbb{E} |h(X_n, \hat{Y}_s^n)|^{2p} ds \\
&\leq C_{p,T,\lambda_2}.
\end{aligned} \tag{38}$$

Substituting (36)–(38) into (35) gives that

$$\mathbb{E} |Y_{k+1}^n|^{2p} \leq C(1 + |y_0|^{2p}) + C\mathbb{E}|X_n|^{2p} + C\delta t \sum_{i=0}^k \mathbb{E}|Y_i^n|^{2p}.$$

Using Lemma 3.2 and discrete Gronwall's inequality, we get the result. $\square$

Next, we give the 2pth moment deviation between two successive iterations of the micro-solver.

Lemma 3.4 *For small enough δt and $p > 1$, there exists a positive constant K_4 such that*

$$\sup_{\substack{0 \leq n \leq \frac{T}{\delta t} \\ 0 < m < M}} \mathbb{E} |Y_{m+1}^n - Y_m^n|^{2p} \leq K_4(\delta t)^p, \tag{39}$$

where K_4 is independent of $(M, \delta t)$.

Proof It is clear that

$$Y_{m+1}^n - Y_m^n = f(X_n, Y_m^n)\delta t + g(X_n, Y_m^n)\Delta W_m^n + h(X_n, Y_m^n)\Delta \tilde{N}_m^n + \lambda_2 h(X_n, Y_m^n)\delta t. \quad (40)$$

Taking the $2p$ th moment and expectations on both sides of (40), we get

$$\begin{aligned} & \mathbb{E}|Y_{m+1}^n - Y_m^n|^{2p} \\ &= \mathbb{E}|f(X_n, Y_m^n)\delta t + g(X_n, Y_m^n)\Delta W_m^n + h(X_n, Y_m^n)\Delta \tilde{N}_m^n + \lambda_2 h(X_n, Y_m^n)\delta t|^{2p} \\ &\leq C\delta t^{2p}\mathbb{E}|f(X_n, Y_m^n)|^{2p} + C(\delta t)^p\mathbb{E}|g(X_n, Y_m^n)|^{2p} \\ &\quad + C\delta t^{2p}\mathbb{E}|h(X_n, Y_m^n)|^{2p} + C\delta t^p\mathbb{E}|h(X_n, Y_m^n)|^{2p}. \end{aligned}$$

By Remark 1.1 and (H2), we have

$$\mathbb{E}|Y_{m+1}^n - Y_m^n|^{2p} \leq C\delta t^{2p}(1 + \mathbb{E}|X_n|^{2p} + \mathbb{E}|Y_m^n|^{2p}) + C\delta t^p.$$

Using Lemmas 3.2 and 3.3, for small enough δt , we get

$$\mathbb{E}|Y_{m+1}^n - Y_m^n|^{2p} \leq K_4\delta t^p. \quad \square$$

Lemma 2.1 in [9] shows that z_t^n is statistically equivalent to a shifted and rescaled version of y_t^ϵ , with x being a parameter, that is, $z_t^k \sim y_{t-t_k/\epsilon}^\epsilon$.

It is proved in [2] that dynamic (9) is ergodic with a unique invariant measure μ^{X_n} (Assumptions H3–H5), which possesses exponentially mixing property in the following sense. Let $P^{X_n}(t, z, E)$ denote the transition probability of (9). Then there exist positive constants $\eta, \alpha < 1$ such that

$$|P^{X_n}(t, z, E) - \mu^{X_n}(E)| < \eta\alpha^t$$

for every $E \in \mathcal{B}(R^m)$.

Then we establish the mixing properties of the auxiliary processes z_t^n . Note that $\bar{a}(X_n)$ is the average of $a(X_n, y)$ with respect to μ^{X_n} , which is the invariant measure induced by z_t^n . We denote $z_m^n = z_{m\delta t}^n$.

Lemma 3.5 *For small enough δt and $p > 1$, there exists a positive constant K_5 such that*

$$\mathbb{E}\left|\frac{1}{M}\sum_{m=1}^M a(X_n, z_m^n) - \bar{a}(X_n)\right|^{2p} \leq K_5\left[\frac{-\log_\alpha M\delta t + 1}{M\delta t} + \frac{1}{M}\right], \quad (41)$$

where K_5 is independent of $(M, \delta t)$.

Proof By (H2) and Remark (1.2), we have

$$\begin{aligned} & \mathbb{E}\left|\frac{1}{M}\sum_{m=1}^M a(X_n, z_m^n) - \bar{a}(X_n)\right|^{2p} \\ &= \mathbb{E}\left[\left|\frac{1}{M}\sum_{m=1}^M a(X_n, z_m^n) - \bar{a}(X_n)\right|^{2p-2} \times \left|\frac{1}{M}\sum_{m=1}^M a(X_n, z_m^n) - \bar{a}(X_n)\right|^2\right] \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E} \left[\left(\frac{1}{M} \sum_{m=1}^M |a(X_n, z_m^n) - \bar{a}(X_n)|^{2p-2} \right) \times \left| \frac{1}{M} \sum_{m=1}^M a(X_n, z_m^n) - \bar{a}(X_n) \right|^2 \right] \\
&\leq C_{a, \bar{a}} \mathbb{E} \left| \frac{1}{M} \sum_{m=1}^M a(X_n, z_m^n) - \bar{a}(X_n) \right|^2.
\end{aligned} \quad (42)$$

It remains to estimate the mean-square term, and the proof for the term is similar to the method in [9] (Lemma 2.6). We omit the details. Thus we obtain the desired result (41). $\square$

Afterwards, we establish the $2p$ th moments deviation between (9) and its numerical approximation (8).

Lemma 3.6 *Let z_t^n be the family of processes defined by (9). For small enough δt and $p > 1$, there exists a positive constant K_8 such that*

$$\max_{\substack{0 \leq n \leq \lfloor \frac{T}{\delta t} \rfloor \\ 0 \leq m \leq M}} \mathbb{E} |Y_m^n - z_m^n|^{2p} \leq K_8 \delta t^p, \quad (43)$$

where K_8 is independent of $(M, \delta t)$.

Proof Define $t_{\delta t} = \lfloor t/\delta t \rfloor \delta t$. Let Y_t^n be the Euler approximation $Y_{t_{\delta t}}^n$, interpolated continuously by

$$Y_t^n = \int_0^t f(X_n, Y_{s_{\delta t}}^n) ds + \int_0^t g(X_n, Y_{s_{\delta t}}^n) dW_s^n + \int_0^t h(X_n, Y_{s_{\delta t}}^n) dN_s^n. \quad (44)$$

Hence, we have

$$\begin{aligned}
\mathbb{E} \sup_{t \in [0, t_1]} |Y_t^n - z_t^n|^{2p} &\leq C \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t [f(X_n, Y_{s_{\delta t}}^n) - f(X_n, z_s^n)] ds \right|^{2p} \\
&\quad + C \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t [g(X_n, Y_{s_{\delta t}}^n) - g(X_n, z_s^n)] dW_s^n \right|^{2p} \\
&\quad + C \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t [h(X_n, Y_{s_{\delta t}}^n) - h(X_n, z_s^n)] d\tilde{N}_s^n \right|^{2p} \\
&\quad + C \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t \lambda_2 [h(X_n, Y_{s_{\delta t}}^n) - h(X_n, z_s^n)] ds \right|^{2p}
\end{aligned} \quad (45)$$

for any $0 \leq t_1 \leq T$, where we have used the definition of $\tilde{N}$. Now, we use Burkholder's inequality and Hölder's inequality on the two martingale terms to get

$$\begin{aligned}
&\mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t [g(X_n, Y_{s_{\delta t}}^n) - g(X_n, z_s^n)] dW_s^n \right|^{2p} \\
&\leq C \mathbb{E} \left[\int_0^{t_1} |g(X_n, Y_{s_{\delta t}}^n) - g(X_n, z_s^n)|^2 ds \right]^p \\
&\leq C \int_0^{t_1} \mathbb{E} |g(X_n, Y_{s_{\delta t}}^n) - g(X_n, z_s^n)|^{2p} ds
\end{aligned} \quad (46)$$

and

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t [h(X_n, Y_{s\delta t}^n) - h(X_n, z_s^n)] d\tilde{N}_s^n \right|^{2p} \\ & \leq C \mathbb{E} \left[\int_0^{t_1} \lambda_2 |h(X_n, Y_{s\delta t}^n) - h(X_n, z_s^n)|^2 ds \right]^p \\ & \leq C \int_0^{t_1} \mathbb{E} |h(X_n, Y_{s\delta t}^n) - h(X_n, z_s^n)|^{2p} ds. \end{aligned} \quad (47)$$

By Hölder's inequality, we have

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t [f(X_n, Y_{s\delta t}^n) - f(X_n, z_s^n)] ds \right|^{2p} \\ & \leq C \int_0^{t_1} \mathbb{E} |f(X_n, Y_{s\delta t}^n) - f(X_n, z_s^n)|^{2p} ds, \end{aligned} \quad (48)$$

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, t_1]} \left| \int_0^t \lambda_2 [h(X_n, Y_{s\delta t}^n) - h(X_n, z_s^n)] ds \right|^{2p} \\ & \leq C \int_0^{t_1} \mathbb{E} |h(X_n, Y_{s\delta t}^n) - h(X_n, z_s^n)|^{2p} ds. \end{aligned} \quad (49)$$

Combining (44)–(49) and applying the Lipschitz condition in (H1), we have

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, t_1]} |Y_t^n - z_t^n|^{2p} & \leq C \int_0^{t_1} \mathbb{E} |Y_{s\delta t}^n - z_s^n|^{2p} ds \\ & \leq C \int_0^{t_1} \mathbb{E} |Y_{s\delta t}^n - Y_s^n|^{2p} ds + C \int_0^{t_1} \mathbb{E} \sup_{t \in [0, s]} |Y_t^n - z_t^n|^{2p} ds \\ & \leq C' \delta t^p + C'' \mathbb{E} \int_0^{t_1} \sup_{t \in [0, s]} |Y_t^n - z_t^n|^{2p} ds, \end{aligned}$$

which, with the help of continuous Gronwall's inequality, yields the result. $\square$

Lemma 3.7 *There exists a positive constant K_6 such that, for all $p > 1$ and $0 \leq n \leq \lfloor \frac{T}{\Delta t} \rfloor$,*

$$\mathbb{E} |\tilde{a}(X_n) - A(X_n)|^{2p} \leq K_6 \left(\frac{-\log_\alpha M \delta t + 1}{M \delta t} + \frac{1}{M} + \delta t^p \right), \quad (50)$$

where K_6 is independent of $(M, \delta t)$.

Proof By definition, we have

$$\begin{aligned} \mathbb{E} |\tilde{a}(X_n) - A(X_n)|^{2p} & = \mathbb{E} \left| \int_{\mathbb{R}^m} a(X_n, y) \mu^{X_n}(dy) - \frac{1}{M} \sum_{m=1}^M a(X_n, Y_m^n) \right|^{2p} \\ & \leq CI_1^n + CI_2^n, \end{aligned} \quad (51)$$

where

$$I_1^n := \mathbb{E} \left| \int_{\mathbb{R}^m} a(X_n, y) \mu^{X_n}(dy) - \frac{1}{M} \sum_{m=1}^M a(X_n, z_m^n) \right|^{2p},$$

$$I_2^n := \mathbb{E} \left| \frac{1}{M} \sum_{m=1}^M a(X_n, z_m^n) - \frac{1}{M} \sum_{m=1}^M a(X_n, Y_m^n) \right|^{2p},$$

where z_t^n is the family of processes defined by (9). I_1^n is the difference between the ensemble average of $a(X_n, \cdot)$ with respect to the (exact) invariant measure of z_t^n and its empirical average over M equi-distanced sample points. I_2^n is the difference between empirical averages of $a(X_n, \cdot)$ over M equi-distanced sample points, once for the process z_t^n and once for its Euler approximation Y_m^n .

The estimation of I_1^n is given in Lemma 3.5,

$$I_1^n = \mathbb{E} \left| \int_{\mathbb{R}^m} a(X_n, y) \mu^{X_n}(dy) - \frac{1}{M} \sum_{m=1}^M a(X_n, z_m^n) \right|^{2p}$$

$$\leq K_5 \left[\frac{-\log_\alpha M\delta t + 1}{M\delta t} + \frac{1}{M} \right]. \quad (52)$$

Then we estimate I_2^n by (H2)

$$I_2^n = \mathbb{E} \left| \frac{1}{M} \sum_{m=1}^M a(X_n, z_m^n) - \frac{1}{M} \sum_{m=1}^M a(X_n, Y_m^n) \right|^{2p}$$

$$\leq \sum_{m=1}^M \mathbb{E} |a(X_n, z_m^n) - a(X_n, Y_m^n)|^{2p}$$

$$\leq C \max_{m \leq M} \mathbb{E} |Y_m^n - z_m^n|^{2p}.$$

Using Lemma 3.6, we obtain

$$I_2^n \leq CK_8 \delta t^p. \quad (53)$$

Combining (51)–(53), we get

$$\mathbb{E} |\bar{a}(X_n) - A(X_n)|^{2p} \leq K_6 \left[\frac{-\log_\alpha M\delta t + 1}{M\delta t} + \frac{1}{M} + \delta t^p \right],$$

which is uniform in $n \leq T/\Delta t$. $\square$

Finally, we estimate the difference between the process X_n and the auxiliary process $\bar{X}_n$.

Lemma 3.8 *There exist positive constants Δt^* and K_7 such that, for $p > 1$ and $0 < \Delta t \leq \Delta t^*$,*

$$\mathbb{E} \sup_{0 \leq n \leq \lfloor T/\Delta t \rfloor} |X_n - \bar{X}_n|^{2p} \leq K_7 \left(\frac{-\log_\alpha M\delta t + 1}{M\delta t} + \frac{1}{M} + \delta t^p \right), \quad (54)$$

where K_7 is independent of $(M, \delta t, \Delta t)$.

Proof Set $E_n = \mathbb{E} \sup_{l \leq n} |\bar{X}_l - X_l|^{2p}$, then

$$\begin{aligned} E_n &= \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [\bar{a}(\bar{X}_i) - A(X_i)] \Delta t + \sum_{i=0}^{l-1} [b(\bar{X}_i) - b(X_i)] \Delta W_i + \sum_{i=0}^{l-1} [c(\bar{X}_i) - c(X_i)] \Delta P_i \right|^{2p} \\ &= \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [\bar{a}(\bar{X}_i) - A(X_i)] \Delta t + \sum_{i=0}^{l-1} [b(\bar{X}_i) - b(X_i)] \Delta W_i + \sum_{i=0}^{l-1} [c(\bar{X}_i) - c(X_i)] \Delta \tilde{P}_i \right. \\ &\quad \left. + \lambda_1 \sum_{i=0}^{l-1} [c(\bar{X}_i) - c(X_i)] \Delta t \right|^{2p} \\ &\leq C \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [\bar{a}(\bar{X}_i) - A(X_i)] \Delta t \right|^{2p} + C \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [b(\bar{X}_i) - b(X_i)] \Delta W_i \right|^{2p} \\ &\quad + C \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [c(\bar{X}_i) - c(X_i)] \Delta \tilde{P}_i \right|^{2p} + C \mathbb{E} \sup_{l \leq n} \left| \lambda_1 \sum_{i=0}^{l-1} [c(\bar{X}_i) - c(X_i)] \Delta t \right|^{2p}. \end{aligned}$$

We split the first term on the right-hand side:

$$\begin{aligned} E_n &\leq C \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [\bar{a}(\bar{X}_i) - \bar{a}(X_i)] \Delta t \right|^{2p} + C \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [\bar{a}(X_i) - A(X_i)] \Delta t \right|^{2p} \\ &\quad + C \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [b(\bar{X}_i) - b(X_i)] \Delta W_i \right|^{2p} + C \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [c(\bar{X}_i) - c(X_i)] \Delta \tilde{P}_i \right|^{2p} \\ &\quad + C \mathbb{E} \sup_{l \leq n} \left| \lambda_1 \sum_{i=0}^{l-1} [c(\bar{X}_i) - c(X_i)] \Delta t \right|^{2p}. \end{aligned} \quad (55)$$

The first and fifth terms on the right-hand side are estimated using the Lipschitz continuity of $\bar{a}$ and c :

$$\begin{aligned} \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [\bar{a}(\bar{X}_i) - \bar{a}(X_i)] \Delta t \right|^{2p} &\leq C \mathbb{E} \sup_{l \leq n} \sum_{i=0}^{l-1} |\bar{a}(\bar{X}_i) - \bar{a}(X_i)|^{2p} \Delta t^{2p} \\ &\leq C \sum_{i=0}^{n-1} \mathbb{E} |\bar{X}_i - X_i|^{2p} \Delta t \leq C \sum_{i=0}^{n-1} E_i \Delta t \end{aligned} \quad (56)$$

and

$$\begin{aligned} \mathbb{E} \sup_{l \leq n} \left| \lambda_1 \sum_{i=0}^{l-1} [c(\bar{X}_i) - c(X_i)] \Delta t \right|^{2p} &\leq C \mathbb{E} \sup_{l \leq n} \sum_{i=0}^{l-1} |c(\bar{X}_i) - c(X_i)|^{2p} \Delta t^{2p} \\ &\leq C \sum_{i=0}^{n-1} \mathbb{E} |\bar{X}_i - X_i|^{2p} \Delta t \leq C \sum_{i=0}^{n-1} E_i \Delta t. \end{aligned} \quad (57)$$

Now using Burkholder's inequality and (H2) on the two martingale terms, we get

$$\begin{aligned} \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [c(\bar{X}_i) - c(X_i)] \Delta \tilde{P}_i \right|^{2p} &\leq C \mathbb{E} \left| \sum_{i=0}^{n-1} \lambda_1 \Delta t [c(\bar{X}_i) - c(X_i)]^2 \right|^p \\ &\leq C \sum_{i=0}^{n-1} \mathbb{E} |\bar{X}_i - X_i|^{2p} \Delta t \leq C \sum_{i=0}^{n-1} E_i \Delta t \end{aligned} \quad (58)$$

and

$$\begin{aligned} \mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [b(\bar{X}_i) - b(X_i)] \Delta W_i \right|^{2p} &\leq C \mathbb{E} \left| \sum_{i=0}^{n-1} \Delta t [b(\bar{X}_i) - b(X_i)]^2 \right|^p \\ &\leq C \sum_{i=0}^{n-1} \mathbb{E} |\bar{X}_i - X_i|^{2p} \Delta t \leq C \sum_{i=0}^{n-1} E_i \Delta t. \end{aligned} \quad (59)$$

The second term on the right-hand side can be bounded as follows:

$$\mathbb{E} \sup_{l \leq n} \left| \sum_{i=0}^{l-1} [\bar{a}(X_i) - A(X_i)] \Delta t \right|^{2p} \leq C \max_{i < n} \mathbb{E} |\bar{a}(X_i) - A(X_i)|^{2p}. \quad (60)$$

Combining (55)–(60) with Lemma 3.7, we obtain a discrete linear integral inequality

$$E_n \leq C \sum_{i=0}^{n-1} E_i \Delta t + CK_6 \left(\frac{-\log_\alpha M \delta t + 1}{M \delta t} + \frac{1}{M} + \delta t^p \right),$$

with the initial condition $E_0 = 0$. It follows that, for sufficiently small Δt ,

$$\begin{aligned} E_n &\leq CK_6 \left(\frac{-\log_\alpha M \delta t + 1}{M \delta t} + \frac{1}{M} + \delta t^p \right) \{1 + C \Delta t\}^n \\ &\leq CK_6 \left(\frac{-\log_\alpha M \delta t + 1}{M \delta t} + \frac{1}{M} + \delta t^p \right) e^{CT}. \end{aligned}$$

This estimate proves the lemma with $K_7 = CK_6 e^{CT}$. $\square$

4 Main result

Now we can state and prove our main theorem readily.

Theorem 4.1 *Suppose that conditions (H1)–(H5) hold, then there exist positive constants K' and K'' such that*

$$\mathbb{E} \sup_{0 \leq n \leq \lfloor T/\Delta t \rfloor} |X_n - \bar{x}(t_n)|^{2p} \leq K' \Delta t^p + K'' \left(\frac{-\log_\alpha M \delta t + 1}{M \delta t} + \frac{1}{M} + \delta t^p \right),$$

where K', K'' are independent of $(\Delta t, \delta t, M)$.

Proof We begin our proof with subtracting and adding the term $\bar{X}_n$:

$$\begin{aligned} \mathbb{E} \sup_{0 \leq n \leq \lfloor T/\Delta t \rfloor} |X_n - \bar{x}(t_n)|^{2p} &\leq C \mathbb{E} \sup_{0 \leq n \leq \lfloor T/\Delta t \rfloor} |X_n - \bar{X}_n|^{2p} + C \mathbb{E} \sup_{0 \leq n \leq \lfloor T/\Delta t \rfloor} |\bar{X}_n - \bar{x}(t_n)|^{2p} \\ &\leq K' \Delta t^p + K'' \left(\frac{-\log_\alpha M \delta t + 1}{M \delta t} + \frac{1}{M} + \delta t^p \right), \end{aligned}$$

where we have used the result of Lemmas 2.4 and 3.8. $\square$

5 Conclusions

In this paper, the L^p ($p > 2$)-strong convergence of the multiscale integration scheme has been studied for the two-time-scale jump-diffusion systems. By Lemmas 2.4 and 3.8, we obtained our desired main result. The results in [2, 9] are extended in this paper. First, we provide a numerical method for the L^p ($p > 2$) averaging principle in [2]; second, in [9], the authors only studied L^2 convergence of the multiscale integration scheme, and we extended the result into the L^p ($p > 2$) case.

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