Jianzhi and Hongyan *Advances in Difference Equations* (2019) 2019:107
https://doi.org/10.1186/s13662-019-1948-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Bifurcation analysis for the Kaldor–Kalecki model with two delays

Cao Jianzhi[1*] and Sun Hongyan[1]

*Correspondence:
cjz2004987@163.com
[1]College of Mathematics and Information Sciences, Key Laboratory of Machine Learning and Computational Intelligence, Hebei University, Baoding, China

Abstract

In this paper, a Kaldor–Kalecki model of business cycle with two discrete time delays is considered. Firstly, by analyzing the corresponding characteristic equations, the local stability of the positive equilibrium is discussed. Choosing delay (or the adjustment coefficient in the goods market α) as bifurcation parameter, the existence of Hopf bifurcation is investigated in detail. Secondly, by combining the normal form method with the center manifold theorem, we are able to determine the direction of the bifurcation and the stability of the bifurcated periodic solutions. Finally, some numerical simulations are carried out to illustrate the theoretical results.

Keywords: Kaldor–Kalecki model; Hopf bifurcation; Stability switch; Periodic solution; Two delays

1 Introduction

Business cycle (or named economic cycle) is a hot topic in the study of the macroeconomic theory. The definition of the business cycle refers to the overall economic performance in the period of economic expansion appears alternated with economic contraction, a phenomenon of the cycle, expressed as gross domestic product, changes in industrial production, prices, employment and unemployment, and other economic variables. Thus, the study of factors that cause fluctuations in the economic cycle and the duration of the economic cycle have important theoretical and practical significance and will help us to better understand the law of economic operation and to gain a reasonable understanding of the leading role of investment in economic development.

As we all know, the model proposed by Kaldor (1940, [1]) is one of the first and best known endogenous business cycle models. According to Kaldor's idea, the main economic proxy toward business fluctuations is a nonlinearity in the investment-saving mechanism. This idea was formalized in a model and studied by means of the mathematical theory of dynamical systems in Chang and Smyth (1971, [2]),

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y,K) - S(Y,K)], \\ \frac{dK}{dt} = I(Y,K) - \delta K. \end{cases}$$

On the other hand, in 1935 Kalecki released a business cycle model where he pointed out the existence of a time lag between a decision of investment and its effect on the capital stock. He assumed that the saved part of profit is invested and the capital growth is due to past investment decisions. There is a gestation period or a time lag, after which capital equipment is available for production. The change in the capital stock is due to the past investment orders (see [3])

$$\frac{dK}{dt} = D(t) - U = I(t - T) - U,$$

where D denotes the third investment stage, i.e., deliveries of finished capital goods; U is the capital depreciation.

Based on Kaldor's idea of introducing nonlinear functional forms and Kalecki's idea of introducing time lags, a Kaldor–Kalecki type model was proposed in [4]:

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y,K) - S(Y,K)], \\ \frac{dK}{dt} = I(Y(t-T),K) - \delta K. \end{cases} \tag{1.1}$$

Since delay could bring a switch in the stability of equilibrium and induce various oscillations and periodic solutions, researches showed that a system with time delay exhibits more complicated dynamics than that without time delay. In [5], Szydlowski et al. showed that System (1.1) can undergo a Hopf bifurcation when the parameter τ spans the critical values. Also Zhang and Wei [6] investigated local and global existence of Hopf bifurcation for System (1.1).

Taking into account the impact of capital stock in the past also, in 2008, Kaddar proposed a new Kaldor–Kalecki model of business cycle with time delay in the following form [7]:

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t),K(t)) - S(Y(t),K(t))], \\ \frac{dK(t)}{dt} = I(Y(t-\tau),K(t-\tau)) - \delta K(t). \end{cases} \tag{1.2}$$

Recently, model (1.2) has aroused enthusiasm among many scholars (see [7–11] and the references cited therein). For instance, taking the delay τ as a bifurcation parameter, Kaddar [7, 8] showed that local and global Hopf bifurcations can occur as the delay crosses some critical values. In [9, 10], Wu XP investigated the simple-zero, double-zero, and zero-Hopf singularity of System (1.2), got bifurcation diagrams, and hence obtained double limit cycle and homoclinic bifurcations. In [11], Wu XP studied triple zero singularity of System (1.2) and for this singularity derived the normal form on the center manifold.

All results mentioned above pay attention to the study of Kaldor–Kalecki model with discrete time delay. Considering the essential idea of endogenous business cycle theory, in 2016, Yu and Peng [12] introduced a distributed delay and modified the Kaldor–Kalecki model in the following form:

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t),K(t)) - S(Y(t),K(t))], \\ \frac{dK(t)}{dt} = I(Y(t-\tau),\int_{-\infty}^{t} F(t-s)K(s)\,ds) - qK(t), \end{cases}$$

where $F(s)$ is the weak kernel function. With the corresponding characteristic equation analyzed, the local stability of the positive equilibrium was investigated. Furthermore, it was found that there exists a Hopf bifurcation when the discrete time delay passes a sequence of critical values.

On the other hand, it is well known that the investment delays caused by gross product in the past and capital stock in the past are not always unified (see [13]). For a variety of different models with two different delays, the dynamic behaviors of the system are fruitful (see [14–18]). The idea of introducing two discrete delays into the capital stock accumulation equation was introduced for the first time in 2009 by Zhou and Li [19]. To the best of our knowledge, there is no mathematical investigation on the Kaldor–Kalecki model with two different delays. Motivated by the aforementioned discussion, in this paper, we consider the following system:

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t),K(t)) - S(Y(t),K(t))], \\ \frac{dK(t)}{dt} = I(Y(t-\tau_1),K(t-\tau_2)) - \delta K(t), \end{cases} \tag{1.3}$$

where Y is the gross product, K is the capital stock, α is the adjustment coefficient in the goods market, δ is the depreciation rate of capital stock, $I(Y,K)$ is the investment function, $S(Y,K)$ is the saving function, $\tau_1 \geq 0$ is the time delay for the investment due to the past gross product, $\tau_2 \geq 0$ is the time delay for capital stock in the past.

The remaining part is organized as follows: in the next section, employing the characteristic equation, the stability of the positive equilibrium, and the occurrence of local Hopf bifurcation are investigated. In Sect. 3, by using the normal form theory and the center manifold theorem, we derive some formulas that can determine the direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions. In Sect. 4, some numerical simulations are carried out to illustrate the main results.

2 Local stability and Hopf bifurcation

In this section, the stability and Hopf bifurcation of the positive equilibrium point will be investigated.

As usual in a Keynesian framework, savings are assumed to be proportional to the current level of income, $S(Y,K) = \gamma Y$, where the coefficient γ, $0 < \gamma < 1$, represents the propensity to save. While in many versions of the Kaldor model the saving function is assumed to be nonlinear, we prefer a linear specification, both for its analytical simplicity and for its sounder microfoundation. Moreover, in our case this assumption does not affect the nonlinearity of the model, which is ensured by the nonlinearity of the investment function.

As usual, the investment demand is assumed to be an increasing and sigmoid-shaped function of income. For example, Bischi et al. consider the form proposed in [20]

$$I(Y,K) = \sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K + \arctan(Y-\mu)\right),$$

where $\frac{\sigma\mu}{\delta}$ is the "normal" level of capital stock.

In [21], using the French quarterly data for 1960–1974, Dana and Malgrange obtain the following form:

$$I(Y,K) = K\Phi\left(\frac{Y}{K}\right) = K\Phi(x),$$

where

$$\Phi(x) = c + \frac{d}{1 + \exp[-a(\nu x - 1)]}$$

or equivalently

$$\Phi(x) = c + \frac{d}{2}\exp^{\frac{a}{2}(\nu x-1)}\cosh^{-1}\left[\frac{a}{2}(\nu x - 1)\right].$$

As far as $I(Y,K)$ is concerned, in [22], authors assume the following increasing S-shaped function:

$$I(Y,K) = \frac{1}{1 + \exp(-b(Y-c)) + d} - \beta K.$$

The function $I(Y,K)$ takes a variety of ways, because of the limitation of length, no more tautology here.

In order to provide some specific cases to analyze how delays influence the dynamics, rather than paying attention to the concrete form of functions I and S, we assume two specific functional forms for the investment function and the saving function (see [8, 12, 23]). The investment function is chosen to be additive in Y and K, and it takes the form

$$I(Y,K) = I(Y) - \beta K,$$

then (1.3) becomes

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t)) - \beta K(t) - \gamma Y(t)], \\ \frac{dK(t)}{dt} = I(Y(t-\tau_1)) - \beta K(t-\tau_2) - \delta K(t). \end{cases} \tag{2.1}$$

In the existing literature, Hopf bifurcation occurs due to the nonlinearity of the investment function or time delay in output; however, almost no attention is given to two different delays effect in the Kaldor–Kalecki model. Overall, the dynamical behaviors of (2.1) is one of the special cases of System (1.3), the study in Hopf bifurcation of (2.1) is just a case where the Kaldor–Kalecki model produces periodic solutions.

2.1 Existence and uniqueness of the positive equilibrium

It is easy to verify that System (2.1) has a unique positive equilibrium point $E(Y^*,K^*)$ if the conditions of the following lemma hold.

Lemma 2.1 *Suppose that*

(A1) $I(0) > 0$;

(A2) $I'(Y) < (1 + \frac{\beta}{\delta})\gamma$;

(A3) *there exists a constant* $L > 0$ *such that* $|I(Y)| \leq L$ *for all* $Y \in \mathbf{R}$.

Then there exists a unique equilibrium $E(Y^*,K^*)$ *of System* (2.1).

Proof (Y,K) is a steady state of System (2.1) if

$$\frac{dY}{dt} = \frac{dK}{dt} = 0,$$

that is,

$$\begin{cases} I(Y) - \beta K - \gamma Y = 0, \\ I(Y) - (\beta + \delta)K = 0. \end{cases} \tag{2.2}$$

From (2.2), we can obtain $K = \frac{\gamma}{\delta}Y$, substituting it into the first equation of (2.2) and the following equation is yielded:

$$I(Y) = \left(\beta\frac{\gamma}{\delta} + \gamma\right)Y. \tag{2.3}$$

Let

$$u(Y) \doteq \left(\beta\frac{\gamma}{\delta} + \gamma\right)Y,$$

then the existence of positive steady state of (2.1) is transformed into whether $I(Y)$ and $u(Y)$ intersect in the first quadrant.

As we all know, $u(Y)$ is a straight line passing through the origin with the slope of $\beta\frac{\gamma}{\delta} + \gamma > 0$, from (A1) and (A3), $I(Y)$ is a bounded function on its existence interval, then by intermediate value theorem, the curve $I(Y)$ and the line $u(Y)$ must intersect in the first quadrant.

Next we will prove that the intersection in the first quadrant is unique. Otherwise, let

$$\bigl(Y_1, u(Y_1)\bigr) \quad \text{and} \quad \bigl(Y_2, u(Y_2)\bigr)$$

be two adjacent intersections in the first quadrant, where $Y_1 < Y_2$, $u(Y_1) < u(Y_2)$. From (A1), we claim that the curve $I(Y)$ is below (or above) the line $u(Y)$ for $Y \in [Y_1, Y_2]$. By Lagrange's mean value theorem, there must be a point $\xi \in (Y_1, Y_2)$ such that

$$I'(\xi) = \beta\frac{\gamma}{\delta} + \gamma,$$

which is a contradiction with (A2). Therefore, the uniqueness is proved.

Let $Y = Y^*$ be the unique solution of (2.3), then $K = K^*$ can be given by the formula $K = \frac{\gamma}{\delta}Y$, one can claim that under hypotheses (A1)–(A3), System (2.1) has a unique equilibrium E. This concludes the proof. □

2.2 Local stability and Hopf bifurcation

Let $y = Y - Y^*$, $k = K - K^*$, then by linearizing System (2.1) around $(0, 0)$ we have

$$\begin{cases} \frac{dy(t)}{dt} = \alpha(I'(Y^*) - \gamma)y(t) - \alpha\beta k(t), \\ \frac{dk(t)}{dt} = I'(Y^*)y(t - \tau_1) - \beta k(t - \tau_2) - \delta k(t). \end{cases} \tag{2.4}$$

The associated characteristic equation of System (2.4) is

$$\lambda^2 + (\delta - a)\lambda + be^{-\lambda\tau_1} + \beta(\lambda - a)e^{-\lambda\tau_2} - \delta a = 0, \tag{2.5}$$

where $a = \alpha(I'(Y^*) - \gamma)$, $b = \alpha\beta I'(Y^*)$.

Since the system contains two time delays, that is, τ_1 and τ_2, therefore the following six cases are considered.

Case I $\tau_1 = \tau_2 = 0$.

When there is no time delay, System (2.1) becomes (1.1) and the characteristic equation (2.5) reduces to

$$\lambda^2 + e\lambda + f = 0, \tag{2.6}$$

where $e = \delta + \beta - \alpha(I'(Y^*) - \gamma), f = \alpha((\beta + \delta)\gamma - \delta I'(Y^*))$. Then we have

$$\lambda_{1,2} = \frac{-e \pm \sqrt{e^2 - 4f}}{2}.$$

To establish our main results, it is necessary to make the following assumptions:

(H) $I'(Y^*) > \gamma$.

By (A2) of Lemma 2.1, it follows that $f > 0$. Then, if $e > 0$, i.e., $\alpha < \frac{\delta+\beta}{I'(Y^*)-\gamma} \doteq \alpha^*$, all the roots of Eq. (2.6) have negative real parts; if $e < 0$, i.e., $\alpha > \alpha^*$, all the roots of Eq. (2.6) have positive real parts; if $e = 0$, i.e., $\alpha = \alpha^*$, Eq. (2.6) has a pair of conjugate purely imaginary roots $\pm i\sqrt{f}$. According to the Hopf bifurcation theorem, it is necessary to verify the transversality condition. Differentiating both sides of Eq. (2.6) with α, we have

$$\frac{d\lambda}{d\alpha} = \frac{(I'(Y^*) - \gamma)\lambda - [(\delta + \beta)\gamma - \delta I'(Y^*)]}{2\lambda + [\delta - \alpha(I'(Y^*) - \gamma) + \beta]},$$

hence,

$$\begin{aligned} \operatorname{Re}\left(\frac{d\lambda}{d\alpha}\right)\bigg|_{\alpha=\alpha^*} &= \operatorname{Re}\left\{\frac{(I'(Y^*) - \gamma)\lambda - [(\delta + \beta)\gamma - \delta I'(Y^*)]}{2\lambda + [\delta - \alpha(I'(Y^*) - \gamma) + \beta]}\right\}\bigg|_{\alpha=\alpha^*} \\ &= \operatorname{Re}\left\{\frac{(I'(Y^*) - \gamma)i\sqrt{f} - [(\delta + \beta)\gamma - \delta I'(Y^*)]}{2i\sqrt{f}}\right\} \\ &= \frac{1}{2}\big(I'\big(Y^*\big) - \gamma\big) > 0. \end{aligned}$$

From what has been discussed above, taking the adjustment coefficient in the goods market α as the bifurcation parameter, we have the following result.

Theorem 2.1 *For System* (2.1), $\tau_1 = \tau_2 = 0$, *if the hypotheses* (A1)–(A3) *of Lemma* 2.1 *and* (H) *are established, then there exists* $\alpha^* \in (0, \infty)$ *such that the unique positive equilibrium* $E(Y^*, K^*)$ *of System* (2.1) *is locally asymptotically stable when* $0 < \alpha < \alpha^*$; $E(Y^*, K^*)$ *is unstable when* $\alpha > \alpha^*$; *and when* $\alpha = \alpha^*$, *the associated characteristic equation has a pair of purely imaginary roots* $\pm i\sqrt{f}$, *System* (2.1) *undergoes a Hopf bifurcation at* $E(Y^*, K^*)$.

Remark 2.1 From the economic point of view, when the speed of adjustment of the goods market α is low enough, regardless of any economic system in the initial position, it will eventually converge to a stable equilibrium point; in this equilibrium, the level of output and capital stock is constant. The principle of economics is that when the aggregate demand and aggregate supply gap appears, lower commodity market correction will lead to a more moderate rate of change in output, thus reducing the economic volatility. When the adjustment speed of the commodity market gradually increases and exceeds a certain critical value, the economic system also begins to change from stable to cyclical fluctuations.

Remark 2.2 In [24], taking the savings rate γ as the bifurcation parameter, authors study the stability and Hopf bifurcation of System (1.3) with $\tau_1 = \tau_2 = 0$. Comparing with Theorem 2.1, we conclude that the Kaldor–Kalecki model may exhibit various nonlinear dynamic behaviors depending on the choice of parameters.

In order to investigate the distribution of roots of the transcendental equation (2.5), we introduce the following results, the details can be found in [25]. Firstly, consider the second degree transcendental polynomial equation

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau} = 0. \tag{2.7}$$

Suppose the following assumptions hold:
(H1) $p + s > 0$;
(H2) $q + r > 0$;
(H3) either $s^2 - p^2 + 2r < 0$ and $r^2 - q^2 > 0$ or $(s^2 - p^2 + 2r)^2 < 4(r^2 - q^2)$;
(H4) either $r^2 - q^2 < 0$ or $s^2 - p^2 + 2r > 0$ and $(s^2 - p^2 + 2r)^2 = 4(r^2 - q^2)$;
(H5) $r^2 - q^2 > 0$, $s^2 - p^2 + 2r > 0$ and $(s^2 - p^2 + 2r)^2 > 4(r^2 - q^2)$;
then the result about the distribution of the roots of Eq. (2.7) can be obtained.

Lemma 2.2 ([25]) *For Eq.* (2.7), *we have*
(i) *if* (H1)–(H3) *hold, then all roots of Eq.* (2.7) *have negative real parts for all* $\tau \geq 0$;
(ii) *if* (H1), (H2), *and* (H4) *hold, then when* $\tau \in [0, \tau_0^+)$ *all roots of Eq.* (2.7) *have negative real parts, when* $\tau = \tau_0^+$, *Eq.* (2.7) *has a pair of purely imaginary roots* $\pm i\omega_+$, *and when* $\tau > \tau_0^+$, *Eq.* (2.7) *has at least one root with positive real part*;
(iii) *if* (H1), (H2), *and* (H5) *hold, then there is a positive integer k such that there are k switches from stability to instability to stability*; *that is, when*

$$\tau \in \left[0, \tau_0^+\right], \left(\tau_0^-, \tau_1^+\right), \ldots, \left(\tau_{k-1}^-, \tau_k^+\right),$$

all roots of Eq. (2.7) *have negative real parts, when*

$$\tau \in [\tau_0^+, \tau_0^-), [\tau_1^+, \tau_1^-), \ldots, [\tau_{k-1}^+, \tau_{k-1}^-) \quad \text{and} \quad \tau > \tau_k^+,$$

Eq. (2.7) *has at least one root with positive real part.*

Here

$$\omega_\pm^2 = \frac{1}{2}\left(s^2 - p^2 + 2r\right) \pm \frac{1}{2}\left[\left(s^2 - p^2 + 2r\right)^2 - 4\left(r^2 - q^2\right)\right]^{\frac{1}{2}}$$

and

$$\tau_j^\pm = \frac{1}{\omega_\pm} \arccos\left\{\frac{q(\omega_\pm^2 - r) - ps\omega_\pm^2}{s^2\omega_\pm^2 + q^2}\right\} + \frac{2j\pi}{\omega_\pm}, \quad j = 0, 1, 2, \ldots.$$

Case II $\tau_1 = 0$, $\tau_2 > 0$.

For $\tau_1 = 0$, $\tau_2 > 0$, Eq. (2.5) becomes

$$\lambda^2 + (\delta - a)\lambda + \beta(\lambda - a)e^{-\lambda\tau_2} + b - \delta a = 0. \tag{2.8}$$

Compared with Eq. (2.7), we get

$$p = \delta - a, \qquad r = b - \delta a, \qquad s = \beta, \qquad q = -a\beta.$$

If $\delta > \frac{-\beta+\sqrt{\beta^2+4\alpha\beta\gamma}}{2} \doteq \delta^*$, condition (H1)

$$p + s = \delta + \beta - \alpha\big(I'\big(Y^*\big) - \gamma\big) > \delta + \beta - \frac{\alpha\beta}{\delta}\gamma = \frac{1}{\delta}\big(\delta^2 + \beta\delta - \alpha\beta\gamma\big) > 0$$

is satisfied. Condition (H2) $q + r = b - a\beta - \delta a = \alpha((\beta + \delta)\gamma - \delta I'(Y^*)) > 0$ can be yielded by (A2) of Lemma 2.1.

Let $\lambda = i\omega$ ($\omega > 0$) be the root of (2.8), then

$$-\omega^2 + (\delta - a)i\omega + \beta(i\omega - a)e^{-i\omega\tau_2} + b - \delta a = 0.$$

Separating the real and the imaginary parts, we have

$$\begin{cases} -\omega^2 - a\beta\cos\omega\tau_2 + \beta\omega\sin\omega\tau_2 + b - \delta a = 0, \\ (\delta - a)\omega + \beta\omega\cos\omega\tau_2 + a\beta\sin\omega\tau_2 = 0, \end{cases} \tag{2.9}$$

which leads to the following equation:

$$\omega^4 + \big(a^2 + \delta^2 - \beta^2 - 2b\big)\omega^2 + (\delta a - b)^2 - a^2\beta^2 = 0. \tag{2.10}$$

For condition (H3),

$$r^2 - q^2 = (r + q)(r - q) = \big[\alpha\big((\beta + \delta)\gamma - \delta I'\big(Y^*\big)\big)\big]\big[\alpha\big((\delta - \beta)\gamma - (2\beta - \delta)I'\big(Y^*\big)\big)\big],$$

if $\gamma < I'(Y^*) < (1 + \frac{\beta}{\delta})\gamma$, one has

$$\begin{aligned} (r - q) &= \alpha\big((\delta - \beta)\gamma - (2\beta - \delta)I'\big(Y^*\big)\big) \\ &> 2\alpha\beta\gamma - \alpha\delta\left(1 + \frac{\beta}{\delta}\right)\gamma + \alpha\delta\gamma - \alpha\beta\gamma \\ &= 2\alpha\beta\gamma - \alpha\delta\gamma - \alpha\beta\gamma + \alpha\delta\gamma - \alpha\beta\gamma \\ &= 0, \end{aligned}$$

combined with (H1), $r^2 - q^2 > 0$ can be established,

$$\begin{aligned} s^2 - p^2 + 2r &= \beta^2 - \delta^2 - \alpha^2\big(I'\big(Y^*\big) - \gamma\big)^2 + 2\alpha\beta I'\big(Y^*\big) \\ &< \beta^2 - \delta^2 - \alpha^2\gamma^2 - \alpha^2\gamma^2 + 2\alpha^2\gamma^2\left(1 + \frac{\beta}{\delta}\right) + 2\alpha\beta\gamma\left(1 + \frac{\beta}{\delta}\right) \\ &= 2\alpha\beta\gamma\left(\frac{\alpha\gamma + \beta}{\delta} + 1\right) + \beta^2 - \delta^2. \end{aligned}$$

Let

$$h(\delta) = 2\alpha\beta\gamma\left(\frac{\alpha\gamma+\beta}{\delta}+1\right)+\beta^2-\delta^2,$$

obviously,

$$h'(\delta) = -2\alpha\beta\gamma\frac{\alpha\gamma+\beta}{\delta^2}-2\delta<0.$$

By direct calculation, one can obtain $\lim_{\delta\to+\infty} h(\delta) = -\infty$ and $\lim_{\delta\to 0^+} h(\delta) = +\infty$. From the intermediate value theorem, there exists unique $\delta^{**}>0$ such that $h(\delta^{**})=0$, then the first inequality of assumption (H3) $s^2-p^2+2r<0$ holds when $\delta \geq \delta^{**}$.

On the other hand,

$$\begin{aligned} s^2-p^2+2r &= \beta^2-\delta^2-\alpha^2\bigl(I'(Y^*)-\gamma\bigr)^2+2\alpha\beta I'(Y^*) \\ &> \beta^2-\delta^2-\frac{\alpha^2\beta^2\gamma^2}{\delta^2}+2\alpha\beta\gamma. \end{aligned}$$

Let

$$g(\delta) = -\delta^2-\frac{\alpha^2\beta^2\gamma^2}{\delta^2}+\beta^2+2\alpha\beta\gamma,$$

then

$$g'(\delta) = -2\delta+\frac{2\alpha^2\beta^2\gamma^2}{\delta^3}.$$

By direct calculation, one can obtain $\lim_{\delta\to+\infty} h(\delta) = -\infty$, $\lim_{\delta\to 0^+} h(\delta) = -\infty$, and $g'(\sqrt{\alpha\beta\gamma}) = 0$, then there exist two positive numbers δ_1^* and δ_1^{**} with $\delta_1^* < \sqrt{\alpha\beta\gamma} < \delta_1^{**}$ such that $s^2-p^2+2r>0$ when $\delta \in [\delta_1^*, \delta_1^{**}]$.

Differentiating both sides of Eq. (2.8) with τ_2

$$\begin{aligned} \operatorname{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\bigg|_{\tau_2=\tau_{20}^+} &= \operatorname{Re}\left\{\frac{2\lambda+\delta-a-\beta e^{-\lambda\tau_2}}{\lambda\beta(\lambda-a)e^{-\lambda\tau_2}}-\frac{\tau_2}{\lambda}\right\}\bigg|_{\tau_2=\tau_{20}^+} \\ &= \operatorname{Re}\left\{\frac{-\omega_+^2+a^2-b-2ia\omega_+}{-\omega_+^2[\omega_+^2-(a^2-2a\delta+b)]+i\omega_+[(\delta-2a)\omega_+^2+(ab-a^2\delta)]}\right\} \\ &= \frac{\omega_+^2[\omega_+^4+2a^2\omega_+^2+a^4+2ab\delta-b^2-2a^2b]}{E^2+F^2}, \end{aligned}$$

where

$$E = -\omega_+^4+\bigl(a^2-2a\delta+b\bigr)\omega_+^2, \qquad F = \omega_+\bigl[\delta\omega_+^2+ab-a^2\delta\bigr].$$

If $b+2a(b-\delta)<0$, then

$$4a^4-4\bigl(a^4+2ab\delta-b^2-2a^2b\bigr) = 4b\bigl[b+2a(b-\delta)\bigr]<0,$$

one can obtain $\omega_+^4 + 2a^2\omega_+^2 + a^4 + 2ab\delta - b^2 - 2a^2b > 0$. Moreover, the transversality condition $\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau_2=\tau_{20}^+} > 0$ can be established.

We summarize the above analysis in the following theorem for model (2.1).

Theorem 2.2 *For System* (2.1), $\tau_1 = 0$, $\tau_2 > 0$, *assume that* $\gamma < I'(Y^*) < (1 + \frac{\beta}{\delta})\gamma$ *and* (A1)–(A3) *of Lemma* 2.1 *are satisfied, then*

(i) *if* $\delta > \max\{\delta^*, \delta^{**}\}$, *then conditions* (H1)–(H3) *in Lemma* 2.2 *are satisfied. Furthermore, the unique positive equilibrium* $E(Y^*, K^*)$ *of System* (2.1) *is locally asymptotically stable for all* $\tau_2 > 0$;

(ii) *if* $\delta^* < \delta_1^*$, $\delta \in (\delta_1^*, \delta_1^{**})$ *or* $\delta_1^* < \delta^* < \delta_1^{**}$, $\delta \in (\delta^*, \delta_1^{**})$ *and* $(s^2 - p^2 + 2r)^2 = 4(r^2 - q^2)$ *hold, then* (H1), (H2), *and* (H4) *in Lemma* 2.2 *are satisfied.* $E(Y^*, K^*)$ *is locally asymptotically stable when* $\tau_2 \in [0, \tau_{20}^+)$; $E(Y^*, K^*)$ *is unstable, when* $\tau > \tau_{20}^+$; *Eq.* (2.8) *has a pair of purely imaginary roots* $\pm i\omega_+$, *when* $\tau = \tau_{20}^+$. *Furthermore, if* $b + 2a(b - \delta) < 0$, *then the transversality condition is established, System* (2.1) *undergoes a Hopf bifurcation at* $E(Y^*, K^*)$, *where* ω_+ *and* τ_{20}^+ *can be calculated by the formula in Lemma* 2.2.

Remark 2.3 From Theorem 2.2, we find that conditions for generating oscillation are far more stringent than stable. From the viewpoint of economics, if the time delay for investment τ_1 is ignored and the depreciation rate of capital stock δ is high enough, no matter what the value of $\tau_2 > 0$, the gross product and the capital stock will eventually converge to a stable equilibrium point. On the other hand, if δ is small enough, with finite time τ_{20}^+, when the time delay for capital stock in the past $\tau_2 < \tau_{20}^+$, the economic system is stable, when $\tau_2 > \tau_{20}^+$, the economic operation can appear unstable fluctuation.

Case III $\tau_1 > 0$, $\tau_2 = 0$.

In this case, the characteristic equation (2.5) becomes

$$\lambda^2 + (\delta - a - \beta)\lambda + be^{-\lambda\tau_1} - a(\delta + \beta) = 0. \tag{2.11}$$

Notice that

$$p = \delta - a - \beta, \qquad r = -a(\delta + \beta), \qquad s = 0, \qquad q = b.$$
$$p + s = \delta - \alpha\big(I'(Y^*) - \gamma\big) - \beta > \delta - \beta - \frac{\alpha\beta}{\delta}\gamma = \frac{1}{\delta}\big(\delta^2 - \beta\delta - \alpha\beta\gamma\big),$$

if $\delta > \frac{\beta + \sqrt{\beta^2 + 4\alpha\beta\gamma}}{2} \doteq \delta_2^*$, then (H1) $p + s > 0$ is satisfied. Condition (H2) $q + r > 0$ is satisfied from (A2).

By straightforward computation, if $\gamma < I'(Y^*) < (1 + \frac{\beta}{\delta})\gamma$, then

$$s^2 - p^2 = 2r = -\big(\delta - \alpha\big(I'(Y^*) - \gamma\big) - \beta\big)^2 - 2\alpha(\delta + \beta)\big(I'(Y^*) - \gamma\big) < 0,$$

and

$$\begin{aligned} r^2 - q^2 &= \alpha^2\big[\big(I'(Y^*) - \gamma\big)^2(\delta + \beta)^2 - \beta^2 I'(Y^*)^2\big] \\ &= \alpha^2\big[\big(I'(Y^*) - \gamma\big)(\delta + \beta) - \beta I'(Y^*)\big]\big[\big(I'(Y^*) - \gamma\big)(\delta + \beta) + \beta I'(Y^*)\big]. \end{aligned}$$

Obviously, $(I'(Y^*)-\gamma)(\delta+\beta)+\beta I'(Y^*)>0$,

$$\begin{aligned}\bigl(I'\bigl(Y^*\bigr)-\gamma\bigr)(\delta+\beta)-\beta I'\bigl(Y^*\bigr)&=\delta I'\bigl(Y^*\bigr)-\gamma(\delta+\beta)\\&<\delta\left(1+\frac{\beta}{\delta}\right)\gamma-\gamma(\delta+\beta)=0,\end{aligned}$$

hence, $r^2-q^2<0$, (H4) is satisfied.

Let $\lambda=i\omega$ ($\omega>0$) be the root of (2.11), then

$$-\omega^2+(\delta-a-\beta)i\omega+be^{-i\omega\tau_1}-a(\delta+\beta)=0.$$

Separating the real and the imaginary parts, we have

$$\begin{cases}-\omega^2+b\cos\omega\tau_1-a(\delta+\beta)=0,\\(\delta-a-\beta)\omega-b\sin\omega\tau_1=0,\end{cases}\tag{2.12}$$

which leads to the following equation:

$$\omega^4+\bigl(a^2+\delta^2+\beta^2-4a\beta-2\delta\beta\bigr)\omega^2+a^2(\delta+\beta)^2-b^2=0.\tag{2.13}$$

From (H4) $r^2-q^2<0$, i.e., $a^2(\delta+\beta)^2-b^2<0$, we can prove Eq. (2.13) has a unique solution

$$\begin{aligned}\omega_0=\Biggl(\frac{1}{2}\bigl\{-\bigl(a^2+\delta^2+\beta^2-4a\beta-2\delta\beta\bigr)\\+\sqrt{\bigl(a^2+\delta^2+\beta^2-4a\beta-2\delta\beta\bigr)^2-4\bigl[a^2(\delta+\beta)^2-b^2\bigr]}\bigr\}\Biggr)^{1/2}.\end{aligned}$$

Denote

$$\tau_1^{(k)}=\frac{1}{\omega_0}\arcsin\frac{\delta-a-\beta}{b}\omega_0+\frac{2k\pi}{\omega_0},\quad k=0,1,2,\ldots,$$

then $\pm i\omega_0$ is a pair of purely imaginary roots of (2.11) with $\tau_1=\tau_1^{(k)}$, $k=0,1,2,\ldots$.

We will prove that the transversality condition is satisfied. Differentiating both sides of Eq. (2.11) with τ_1

$$\begin{aligned}\operatorname{Re}\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\bigg|_{\tau_1=\tau_1^{(k)}}&=\operatorname{Re}\left\{\frac{2\lambda+\delta-a-\beta}{b\lambda e^{-\lambda\tau_1}}-\frac{\tau_1}{\lambda}\right\}\bigg|_{\tau_1=\tau_1^{(k)}}\\&=\operatorname{Re}\left\{\frac{2i\omega_0+\delta-a-\beta}{i\omega_0(\omega_0^2-(\delta-a-\beta)i\omega_0+a(\delta+\beta))}\right\}\\&=\frac{\omega_0^2[(\delta-a-\beta)^2+2\omega_0^2+2a(\delta+\beta)]}{(\delta-a-\beta)^2\omega_0^4+[\omega_0^3+a(\delta+\beta)\omega_0]^2}>0.\end{aligned}$$

We conclude the discussions above as follows.

Theorem 2.3 *For System* (2.1), $\tau_1>0$, $\tau_2=0$, *if* $\gamma<I'(Y^*)<(1+\frac{\beta}{\delta})\gamma$ *and* (A1)–(A3) *of Lemma* 2.1 *are satisfied, then*

(i) *when* $0 < \delta < \delta_2^*$, *the unique positive equilibrium* $E(Y^*, K^*)$ *of System* (2.1) *is unstable for all* $\tau_1 > 0$;
(ii) *when* $\delta > \delta_2^*$, *the unique positive equilibrium* $E(Y^*, K^*)$ *of System* (2.1) *is locally asymptotically stable when* $\tau_1 \in [0, \tau_1^{(0)})$; *when* $\tau_1 > \tau_1^{(0)}$, $E(Y^*, K^*)$ *is unstable*; *when* $\tau = \tau_1^{(k)}$, $k = 0, 1, 2, \ldots$, *the characteristic equation* (2.11) *has a pair of purely imaginary roots* $\pm i\omega_0$, *System* (2.1) *undergoes a Hopf bifurcation at* $E(Y^*, K^*)$.

Remark 2.4 In [5], different from the method in this paper, model (2.1) for $\tau_1 > 0$, $\tau_2 = 0$ is formulated in terms of a second-order nonlinear delay differential equation, the Hopf bifurcation theorem is obtained by computing the normal form on the center manifold, which requires tedious calculation.

Remark 2.5 The theorem of stability and Hopf bifurcation of System (2.1) for $\tau_1 > 0$, $\tau_2 = 0$ in the present paper is obtained by the existence and uniqueness of the equilibrium point. However, in Ref. [6] the existence of equilibrium point is only a hypothesis, thus the conclusion we obtained is more clear.

Case IV $\tau_1 = \tau_2 > 0$.

For $\tau_1 = \tau_2 > 0$, System (2.1) becomes (1.3) and the characteristic equation (2.5) becomes

$$\lambda^2 + (\delta - a)\lambda + (\beta\lambda + b - a\beta)e^{-\lambda\tau_1} - \delta a = 0. \tag{2.14}$$

Compared with Eq. (2.7), we get

$$p = \delta - a, \qquad r = -\delta a, \qquad s = \beta, \qquad q = \alpha\beta\gamma.$$

If $\delta > \delta^*$ (see Case II), condition (H1)

$$p + s = \delta + \beta - \alpha\bigl(I'\bigl(Y^*\bigr) - \gamma\bigr) > 0$$

is satisfied. Condition (H2) $q + r = \alpha(\beta\gamma - \delta(I'(Y^*) - \gamma)) > 0$ can be yielded by (A2) of Lemma 2.1.

Let $\lambda = i\omega$ ($\omega > 0$) be the root of (2.14), then

$$-\omega^2 + (\delta - a)i\omega + (\beta i\omega + b - a\beta)e^{-i\omega\tau_1} + b - \delta a = 0.$$

Separating the real and the imaginary parts, we have

$$\begin{cases} -\omega^2 + (b - a\beta)\cos\omega\tau_1 + \beta\omega\sin\omega\tau_1 - \delta a = 0, \\ (\delta - a)\omega + \beta\omega\cos\omega\tau_1 - (b - a\beta)\sin\omega\tau_1 = 0, \end{cases} \tag{2.15}$$

which leads to the following equation:

$$\omega^4 + \bigl(a^2 + \delta^2 - \beta^2\bigr)\omega^2 + \delta^2 a^2 - (b - a\beta)^2 = 0. \tag{2.16}$$

Compared with (H3)–(H5), we have

$$r^2 - q^2 = \delta^2 a^2 - \alpha^2\beta^2\gamma^2 = \alpha^2\bigl[\delta\bigl(I'\bigl(Y^*\bigr) - \gamma\bigr) - \beta\gamma\bigr]\bigl[\delta\bigl(I'\bigl(Y^*\bigr) - \gamma\bigr) + \beta\gamma\bigr]$$

and

$$s^2 - p^2 + 2r = \beta^2 - (\delta - a)^2 - 2a\delta = \beta^2 - \delta^2 - \alpha^2\bigl(I'\bigl(Y^*\bigr) - \gamma\bigr)^2.$$

If $r^2 - q^2 < 0$, Eq. (2.16) has a unique positive root

$$\omega_+ = \sqrt{\frac{-(a^2+\delta^2-\beta^2) + \sqrt{(a^2+\delta^2-\beta^2)^2 - 4(a^2\delta^2 - (b-\alpha\beta)^2)}}{2}},$$

then there exists a sequence of positive numbers $\{\tau_{1j}^+\}_{j=0}^{\infty}$ such that Eq. (2.16) has a pair of purely imaginary roots $\pm i\omega_+$.

If $r^2 - q^2 > 0$, $s^2 - p^2 + 2r > 0$ and $(s^2 - p^2 + 2r)^2 > 4(r^2 - q^2)$, Eq. (2.16) has two positive roots

$$\omega_\pm = \sqrt{\frac{-(a^2+\delta^2-\beta^2) \pm \sqrt{(a^2+\delta^2-\beta^2)^2 - 4(a^2\delta^2 - (b-\alpha\beta)^2)}}{2}},$$

then there exist two sequences of positive numbers $\{\tau_{1j}^+\}_{j=0}^{\infty}$ and $\{\tau_{1j}^-\}_{j=0}^{\infty}$ such that Eq. (2.16) has two pairs of purely imaginary roots $\pm i\omega_\pm$.

We will list the transversality condition and the proof can be found in [26],

$$\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau_1}\right\}_{\tau_1=\tau_{1j}^-} < 0 \quad \text{and} \quad \operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau_1}\right\}_{\tau_1=\tau_{1j}^+} > 0.$$

By similar discussion to [26, 27], we arrive at the following theorem.

Theorem 2.4 *For System* (2.1), $\tau_1 = \tau_2 > 0$,

- *assume* $\delta^* < \delta < \beta$, *then we have*
  - (i) *if* $I'(Y^*) < \min\{\gamma(1 - \frac{\beta}{\delta}), \gamma - \frac{\sqrt{\beta^2-\delta^2}}{\alpha}\}$, *then conditions* (H1)–(H3) *hold, the unique positive equilibrium* $E(Y^*, K^*)$ *of System* (2.1) *is locally asymptotically stable for all* $\tau_1 \geq 0$;
  - (ii) *if* $|I'(Y^*) - \gamma| < \frac{\beta}{\delta}\gamma$, *then conditions* (H1), (H2), *and* (H4) *hold,* $E(Y^*, K^*)$ *is locally asymptotically stable for* $\tau_1 \in [0, \tau_{10}^+)$. *System* (2.1) *undergoes a Hopf bifurcation at* $E(Y^*, K^*)$ *when* $\tau_1 = \tau_{1j}^+$, $j = 0, 1, 2, \dots$.
  - (iii) *if* $(s^2 - p^2 + 2r)^2 > 4(r^2 - q^2)$ *and* $\frac{\beta\gamma}{\delta} < |I'(Y^*) - \gamma| < \frac{\sqrt{\beta^2-\delta^2}}{\alpha}$, *then conditions* (H1), (H2), *and* (H5) *hold. System* (2.1) *undergoes* k (*a finite number*) *switches from stability to instability to stability when the parameters are such that*

$$\tau_{10}^- < \tau_{10}^+ < \tau_{11}^- < \cdots < \tau_{1,k-1}^- < \tau_{1,k-1}^+ < \tau_{1k}^- < \tau_{1,k+1}^- < \tau_{1k}^+ \cdots,$$

    and eventually it becomes unstable.
- *assume* $\delta > \max\{\beta, \delta^*\}$, *then we have*
  - (i) *if* $I'(Y^*) < \gamma(1 - \frac{\beta}{\delta})$, *then conditions* (H1)–(H3) *hold, the unique positive equilibrium* $E(Y^*, K^*)$ *of System* (2.1) *is locally asymptotically stable for all* $\tau_1 \geq 0$;

(ii) *if* $|I'(Y^*) - \gamma| < \frac{\beta}{\delta}\gamma$, *then conditions* (H1), (H2), *and* (H4) *hold*, $E(Y^*, K^*)$ *is locally asymptotically stable for* $\tau_1 \in [0, \tau_{10}^+)$. *System* (2.1) *undergoes a Hopf bifurcation at* $E(Y^*, K^*)$ *when* $\tau_1 = \tau_{1j}^+$, $j = 0, 1, 2, \ldots$.

Case V $\tau_1 = 2\tau_2 > 0$.

For $\tau_1 = 2\tau_2 > 0$, the characteristic equation (2.5) becomes

$$\lambda^2 + (\delta - a)\lambda + be^{-2\lambda\tau_2} + \beta(\lambda - a)e^{-\lambda\tau_2} - \delta a = 0. \tag{2.17}$$

We will employ the method proposed in [28, 29] to analyze the distribution of characteristic roots of (2.17). Obviously, $\pm i\omega$ ($\omega > 0$) is a pair of roots of (2.17) if and only if ω satisfies

$$-\omega^2 + (\delta - a)i\omega + be^{-2i\omega\tau_2} + \beta(i\omega - a)e^{-i\omega\tau_2} - \delta a = 0. \tag{2.18}$$

If $\frac{\omega\tau_2}{2} \neq \frac{\pi}{2} + j\pi$, $j \in \mathbf{Z}$, then let $\theta = \tan\frac{\omega\tau_2}{2}$, we have $e^{-i\omega\tau_2} = \frac{1-i\theta}{1+i\theta}$. Separating the real and the imaginary parts, we find that θ satisfies

$$\begin{cases} (\omega^2 - b + a\delta - a\beta)\theta^2 + 2\omega(a - \delta)\theta = \omega^2 - b + a\delta + a\beta, \\ \omega(a - \delta + \beta)\theta^2 - 2(\omega^2 + b + a\delta)\theta = \omega(a - \delta - \beta). \end{cases} \tag{2.19}$$

Denote

$$M = \begin{bmatrix} \omega^2 - b + a\delta - a\beta & 2\omega(a - \delta) & \omega^2 - b + a\delta + a\beta \\ \omega(a - \delta + \beta) & -2(\omega^2 + b + a\delta) & \omega(a - \delta - \beta) \end{bmatrix},$$

$$M_1 = \begin{bmatrix} \omega^2 - b + a\delta - a\beta & 2\omega(a - \delta) \\ \omega(a - \delta + \beta) & -2(\omega^2 + b + a\delta) \end{bmatrix},$$

$$M_2 = \begin{bmatrix} \omega^2 - b + a\delta - a\beta & \omega^2 - b + a\delta + a\beta \\ \omega(a - \delta + \beta) & \omega(a - \delta - \beta) \end{bmatrix},$$

$$M_3 = \begin{bmatrix} \omega^2 - b + a\delta + a\beta & 2\omega(a - \delta) \\ \omega(a - \delta - \beta) & -2(\omega^2 + b + a\delta) \end{bmatrix}.$$

We define

$$D(\omega) = \det(M_1), \qquad E(\omega) = \det(M_2) \quad \text{and} \quad F(\omega) = \det(M_3).$$

According to Cramer's rule, if $D(\omega) \neq 0$, one can solve from Eq. (2.19) that

$$\theta^2 = \frac{E(\omega)}{D(\omega)} \quad \text{and} \quad \theta = \frac{F(\omega)}{D(\omega)},$$

where ω satisfies

$$D(\omega)E(\omega) = F^2(\omega). \tag{2.20}$$

If $D(\omega) = 0$, in order to make sure the solvability of (2.19) for θ, we have

$$E(\omega) = F(\omega) = 0,$$

and hence ω satisfies (2.20) in this case as well. Simplifying (2.20), we conclude that ω satisfies a polynomial equation with degree 8:

$$\omega^8 + s_1\omega^6 + s_2\omega^4 + s_3\omega^2 + s_4 = 0, \tag{2.21}$$

where

$$\begin{aligned}
s_1 &= 2\delta^2 + 2a^2 - \beta^2,\\
s_2 &= \delta^4 + 4a^2\delta^2 - 2b^2 - 2a^2\beta^2 + a^4 + 2b\beta^2,\\
s_3 &= 2a^2\delta^4 + 2\bigl(a^4 - a^2\beta^2 - b^2\bigr)\delta^2 - 2ab\beta^2\delta + a^4\beta^2 + 2a^2\beta^2 b - 2a^2b^2 - b^2\beta^2,\\
s_4 &= (b + a\delta)^2\bigl[(b - a\delta + a\beta)(b - a\delta - a\beta)\bigr],
\end{aligned}$$

and ω^2 is a positive root of

$$z^4 + s_1z^3 + s_2z^2 + s_3z + s_4 = 0. \tag{2.22}$$

From (A2) of Lemma 2.1, $b - a\delta - a\beta = \alpha(\beta\gamma + \delta\gamma - \delta I'(Y^*)) > 0$; on the other hand,

$$b - a\delta + a\beta = \alpha I'\bigl(Y^*\bigr)(2\beta - \delta) + \alpha\gamma(\delta - \beta). \tag{2.23}$$

Analyzing (2.23), we have the following results:

- if $\beta - \delta > 0$, then $b - a\delta + a\beta < 0$ when $I'(Y^*) < \gamma(1 - \frac{\beta}{2\beta-\delta})$;
- if $\beta - \delta < 0$ and $2\beta - \delta > 0$, then $b - a\delta + a\beta > 0$ all the time;
- if $2\beta - \delta < 0$, then $b - a\delta + a\beta < 0$ when $I'(Y^*) > \gamma(1 + \frac{\beta}{\delta-2\beta})$.

If $\frac{\omega\tau_2}{2} = \frac{\pi}{2} + j\pi, j \in \mathbf{Z}$, from (2.18) we obtain

$$-\omega^2 + (\delta - a)i\omega + b - \beta(i\omega - a) - \delta a = 0.$$

Separating the real and the imaginary parts, we have

$$\delta - \beta = \alpha\bigl(I'\bigl(Y^*\bigr) - \gamma\bigr) \quad \text{and} \quad \omega^2 = b - a\delta + a\beta.$$

From the above analysis we have the following lemma.

Lemma 2.3 *Either $\beta - \delta > 0$ and $I'(Y^*) < \gamma(1 - \frac{\beta}{2\beta-\delta})$ or $2\beta - \delta < 0$ and $I'(Y^*) > \gamma(1 + \frac{\beta}{\delta-2\beta})$ are satisfied, Eq. (2.22) has a positive root ω_*^2 ($\omega_* > 0$). Furthermore, if $D(\omega_*) \neq 0$, then Eq. (2.19) has a unique real root $\theta^* = \frac{F(\omega_*)}{D(\omega_*)}$. Hence Eq. (2.17) has a pair of purely imaginary roots $\pm i\omega_*$, when*

$$\tau_2 = \tau_{2j} = \frac{2\arctan\theta^* + 2j\pi}{\omega_*}, \quad j \in \mathbf{Z}.$$

Remark 2.6 The conditions in Lemma 2.3 guarantee that $b - a\delta + a\beta < 0$. However, $\frac{\omega\tau_2}{2} = \frac{\pi}{2} + j\pi, j \in \mathbf{Z}$ is a root of (2.18) if and only if $\omega^2 = b - a\delta + a\beta > 0$, this is a contradiction, in other words, $\omega\tau_2 = \pi + 2j\pi, j \in \mathbf{Z}$ is not a root of (2.18).

To make sure the Hopf bifurcation occurs, the transversality condition should be checked. Differentiating both sides of Eq. (2.17) with τ_2, one reaches

$$\big[2\lambda + (\delta - a) - 2\tau_2 be^{-2\lambda\tau_2} + \beta e^{-\lambda\tau_2} - \tau_2\beta(\lambda - a)e^{-\lambda\tau_2}\big]\frac{d\lambda}{d\tau_2} = 2\lambda be^{-2\lambda\tau_2} + \lambda\beta(\lambda - a)e^{-\lambda\tau_2},$$

then

$$\begin{aligned}
&\mathrm{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\bigg|_{\tau_2=\tau_{2j}} \\
&\quad = \mathrm{Re}\left\{\frac{(2\lambda + \delta - a)e^{\lambda\tau_2} - 2\tau_2 be^{-\lambda\tau_2} + \beta(1 - \lambda\tau_2 + a\tau_2)}{2\lambda be^{-\lambda\tau_2} + \lambda\beta(\lambda - a)}\right\}_{\tau_2=\tau_{2j}} \\
&\quad = \mathrm{Re}\left\{\frac{(2i\omega_* + \delta - a)(\cos\xi + i\sin\xi) - 2b\tau_{2j}(\cos\xi - i\sin\xi) + \beta - i\beta\omega_*\tau_{2j} + a\beta\tau_{2j}}{2i\omega_* b(\cos\xi - i\sin\xi) + i\omega_*\beta(i\omega_* - a)}\right\} \\
&\quad = \big(\omega_*\big[4b(\delta - a)\sin\xi\cos\xi - \beta\omega(a + \delta)\cos\xi + \beta\big(2\omega_*^2 + 2b - a\delta + a^2\big)\sin\xi \\
&\qquad + 4b\omega\big(\cos^2\xi\sin^2\xi - \omega\beta^2\big)\big]\big)/\big(A^2 + B^2\big),
\end{aligned} \tag{2.24}$$

where

$$\xi = \omega_*\tau_{2j}, \qquad A^2 = \omega_*^2(2b\sin\omega_*\tau_{2j} - \beta\omega_*)^2, \quad \text{and} \quad B^2 = \omega_*^2(2b\cos\omega_*\tau_{2j} - a\beta)^2.$$

Obviously, it is difficult to distinguish the sign of (2.24), we will demonstrate it by calculating examples in Sect. 4.

One has the following theorem by the Hopf bifurcation.

Theorem 2.5 *For System* (2.1), $\tau_1 = 2\tau_2 > 0$, *if* $\delta^* < \delta < \beta$, $I'(Y^*) < (1 - \frac{\beta}{2\beta-\delta})\gamma$, (A1)–(A3) *of Lemma* 2.1 *and the transversality condition* (2.24) *are satisfied, then the unique positive equilibrium* $E(Y^*, K^*)$ *of System* (2.1) *is locally asymptotically stable when* $\tau_2 \in [0, \tau_{20})$; *when* $\tau_2 > \tau_{20}$, $E(Y^*, K^*)$ *is unstable*; *when* $\tau = \tau_{2j}, j \in \mathbf{Z}$, *the characteristic equation* (2.17) *has a pair of purely imaginary roots* $\pm i\omega_*$, *System* (2.1) *undergoes a Hopf bifurcation at* $E(Y^*, K^*)$.

Case VI $\tau_1 \neq \tau_2$.

For $\tau_1 > 0$, $\tau_2 > 0$, and $\tau_1 \neq \tau_2$, the characteristic equation is in the form of (2.5)

$$\lambda^2 + (\delta - a)\lambda + be^{-\lambda\tau_1} + \beta(\lambda - a)e^{-\lambda\tau_2} - \delta a = 0. \tag{2.5}$$

Let $\lambda = i\omega$ ($\omega > 0$) be the root of (2.5), then

$$-\omega^2 + (\delta - a)i\omega + be^{-i\omega\tau_1} + \beta(i\omega - a)e^{-i\omega\tau_2} - \delta a = 0.$$

Separating the real and the imaginary parts, we have

$$\begin{cases} -\omega^2 + b\cos\omega\tau_1 - a\beta\cos\omega\tau_2 + \beta\omega\sin\omega\tau_2 - \delta a = 0, \\ (\delta - a)\omega - b\sin\omega\tau_1 + \beta\omega\cos\omega\tau_2 + a\beta\sin\omega\tau_2 = 0. \end{cases} \tag{2.25}$$

Eliminating $\sin\omega\tau_2$ and $\cos\omega\tau_2$, we obtain the following transcendental equation:

$$\begin{aligned} &\omega^4 + \left(a^2 + \delta^2 - \beta^2\right)\omega^2 + a^2\left(\delta^2 - \beta^2\right) + b^2 \\ &\quad - 2b\left(a\delta + \omega^2\right)\cos\omega\tau_1 - 2(a - \delta)b\omega\sin\omega\tau_1 = 0. \end{aligned} \tag{2.26}$$

Let

$$\begin{aligned} h(\omega) = {}& \omega^4 + \left(a^2 + \delta^2 - \beta^2\right)\omega^2 + a^2\left(\delta^2 - \beta^2\right) + b^2 \\ & - 2b\left(a\delta + \omega^2\right)\cos\omega\tau_1 - 2(a - \delta)b\omega\sin\omega\tau_1, \end{aligned}$$

since $h(\omega) \to +\infty$ when $\omega \to +\infty$, Eq. (2.26) has at most finite positive roots. If there exists a positive root, without loss of generality, we suppose that (2.26) has N_1 ($0 < N_1 < \infty$, $N_1 \in \mathbf{Z}$) positive roots as $\omega_1, \omega_2, \ldots, \omega_{N_1}$.

Denote

$$\tau_{2i}^{(j)} = \frac{1}{\omega_i}\arccos\left[-\frac{\delta}{\beta} + \frac{b\omega_i\sin\omega_i\tau_1 + ab\cos\omega_i\tau_1}{\beta(a^2 + \beta^2)}\right] + \frac{2k\pi}{\omega_i}, \quad j = 0, 1, \ldots, i = 1, 2, \ldots, N_1,$$

then $\pm i\omega_i$, $i = 1, 2, \ldots, N_1$, is a pair of purely imaginary roots of Eq. (2.5) with $\tau_2 = \tau_{2i}^{(j)}$, $j = 0, 1, \ldots$, $i = 1, 2, \ldots, N_1$.

In this case, τ_2 is regarded as the varying parameter, the necessary condition is that the critical eigenvalue passes through the imaginary axis having the nonzero velocity. Differentiating λ with respect to τ_2 in (2.5), one reaches

$$\begin{aligned} &\operatorname{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\bigg|_{\tau_2 = \tau_{2i}^{(j)}} \\ &\quad = \operatorname{Re}\left\{\frac{2\lambda + \delta - a - \tau_1 b e^{-\lambda\tau_1} + \beta e^{-\lambda\tau_2} - \beta\tau_2(\lambda - a)e^{-\lambda\tau_2}}{\beta\lambda(\lambda - a)e^{-\lambda\tau_2}}\right\}_{\tau_2 = \tau_{2i}^{(j)}} \\ &\quad = \operatorname{Re}\left\{-\frac{\tau_{2i}^{(j)}}{\lambda} + \frac{1}{\lambda(\lambda - a)} + \frac{2\lambda + \delta - a - \tau_1 b e^{-\lambda\tau_1}}{-\beta\lambda(\lambda - a)(\lambda^2 + (\delta - a)\lambda + be^{-\lambda\tau_1} - \delta a)}\right\} \\ &\quad = -\frac{1}{a^2 + \omega_i^2} + \frac{(\delta - a - \tau_1 b\cos\omega_i\tau_1)P - (2\omega + \tau_1 b\sin\omega_i\tau_1)Q}{P^2 + Q^2}, \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} P &= -\beta\omega_i^4 - a\beta(a - 2\delta)\omega_i^2 + \beta b\omega_i^2\cos\omega_i\tau_1 + \alpha\beta b\sin\omega_i\tau_1, \\ Q &= \beta(\delta - 2a)\omega_i^3 - a^2\beta\delta\omega_i + a\beta\beta b\omega_i\cos\omega_i\tau_1 - b\beta\omega_i^2\sin\omega_i\tau_1. \end{aligned}$$

We conclude the discussions above as follows.

Theorem 2.6 *For System* (2.1), $\tau_1 > 0$, $\tau_2 > 0$, $\tau_1 \neq \tau_2$, *let* $\delta > \delta^*$, $I'(Y^*) < (1 + \frac{\beta}{\delta})\gamma$ *and* (2.27) *is not equal to zero, then*

(i) *if* $h(\omega) = 0$ *exhibits no positive root, the unique positive equilibrium* $E(Y^*, K^*)$ *of System* (2.1) *is locally asymptotically stable for all* $\tau_2 > 0$;

(ii) *if* $h(\omega) = 0$ *has* N_1 *positive roots, then there exists a positive number* $\tau_{20}^* = \min\{\tau_{2i}^{(0)}, i = 1, 2, \ldots, N_1\}$ *such that* $E(Y^*, K^*)$ *is locally asymptotically stable for* $\tau_2 \in [0, \tau_{20}^*)$ *and unstable for* $\tau_2 > \tau_{20}^*$. *Furthermore, System* (2.1) *undergoes a Hopf bifurcation at* $E(Y^*, K^*)$ *when* $\tau_2 = \tau_{20}^*$.

Remark 2.7 Since the set $\{(\tau_1, \tau_2) | \tau_1 = 2\tau_2\}$ belongs to $\{(\tau_1, \tau_2) | \tau_1 \neq \tau_2, \tau_1 > 0, \tau_2 > 0\}$, it can be seen from the above discussion that Case V just is a special case of Case VI.

On the other hand, one can take $b = \alpha\beta I(Y^*)$ as the bifurcation parameter also. From (2.25), the following equation is established:

$$\tan\omega\tau_1 = \frac{(\delta - a)\omega + \beta\omega\cos\omega\tau_2 + a\beta\sin\omega\tau_2}{\omega^2 + a\beta\cos\omega\tau_2 - \beta\omega\sin\omega\tau_2 + \delta a}. \tag{2.28}$$

Let

$$m(\omega) = \frac{(\delta - a)\omega + \beta\omega\cos\omega\tau_2 + a\beta\sin\omega\tau_2}{\omega^2 + a\beta\cos\omega\tau_2 - \beta\omega\sin\omega\tau_2 + \delta a},$$

then

$$\lim_{\omega\to+\infty} m(\omega) = 0.$$

Obviously, there are an infinite number of intersecting points for the two curves $\tan\omega\tau_1$ and $m(\omega)$, i.e., Eq. (2.28) has a sequence of roots $\{\omega_j\}_{j\geq 1}$.

Define

$$b_j = \frac{\omega_j^2 + a\delta + a\beta\cos\omega_j\tau_2 - \beta\omega_j\sin\omega_j\tau_2}{\cos\omega_j\tau_1}, \quad j = 1, 2, \ldots,$$

then we claim that the characteristic equation (2.5) has purely imaginary roots if and only if $b = b_j$ and the purely imaginary roots are $\pm i\omega_j$.

Differentiating λ with respect to b in (2.5), we have

$$\left[2\lambda + \delta - a - \tau_1 be^{-\lambda\tau_1} + \beta e^{-\lambda\tau_2} - \beta\tau_2(\lambda - a)e^{-\lambda\tau_2}\right]\frac{d\lambda}{db} = -e^{-\lambda\tau_1}, \tag{2.29}$$

hence,

$$\frac{d\lambda}{db} = \frac{-e^{-\lambda\tau_1}}{2\lambda + \delta - a - \tau_1 be^{-\lambda\tau_1} + \beta e^{-\lambda\tau_2} - \beta\tau_2(\lambda - a)e^{-\lambda\tau_2}}.$$

Substituting b_j into the above equation, we obtain

$$\begin{aligned}\mathrm{Re}\left(\frac{d\lambda}{db}\right)\bigg|_{b=b_j} &= \mathrm{Re}\left\{\frac{-e^{-\lambda\tau_1}}{2\lambda + \delta - a - \tau_1 be^{-\lambda\tau_1} + \beta e^{-\lambda\tau_2} - \beta\tau_2(\lambda - a)e^{-\lambda\tau_2}}\right\}_{b=b_j} \\ &= \frac{-M\cos\omega\tau_1 + N\sin\omega\tau_1}{M^2 + N^2},\end{aligned} \tag{2.30}$$

where

$$M = \delta - a - \tau_1 b_j \cos\omega\tau_1 + \beta(1 + a\tau_2)\cos\omega\tau_2 - \beta\tau_2\omega\sin\omega\tau_2,$$
$$N = 2\omega + \tau_1 b_j \sin\omega\tau_1 - \beta(1 + a\tau_2)\sin\omega\tau_2 - \beta\tau_2\omega\cos\omega\tau_2.$$

When $b = 0$, $\tau_2 = 0$, Eq. (2.5) becomes

$$\lambda^2 + (\delta - a + \beta)\lambda - a(\delta + \beta) = 0. \tag{2.31}$$

Obviously, if $I'(Y^*) < \gamma$ and $\delta > \delta^*$, all the roots of Eq. (2.31) have negative real parts. Let $\lambda = i\omega$ ($\omega > 0$) be the root of (2.5) with $b = 0$, $\tau_2 > 0$, then

$$-\omega^2 + (\delta - a)i\omega + \beta(i\omega - a)(\cos\omega\tau_2 - i\sin\omega\tau_2) - \delta\alpha = 0,$$

separating the real and the imaginary parts, we have

$$\begin{cases} -\omega^2 + \beta\omega\sin\omega\tau_2 - a\beta\cos\omega\tau_2 - \delta a = 0, \\ (\delta - a)\omega + \beta\omega\cos\omega\tau_2 + a\beta\sin\omega\tau_2 = 0, \end{cases} \tag{2.32}$$

which leads to the following equation:

$$\omega^4 + \left(a^2 + \delta^2 - \beta^2\right)\omega^2 + a^2\left(\delta^2 - \beta^2\right) = 0. \tag{2.33}$$

As is known to all, if $\delta > \beta$, Eq. (2.33) has no real roots, i.e., all the roots of Eq. (2.5) have negative real parts when $b = 0$, $\tau_2 > 0$.

Now we can state the following result.

Theorem 2.7 *For System (2.1), $\tau_1 > 0$, $\tau_2 > 0$, $\tau_1 \neq \tau_2$, b is regarded as the bifurcation parameter, if $\delta > \max\{\delta^*, \beta\}$, $I'(Y^*) < \gamma$, and the transversality condition (2.30) is satisfied, then*

(i) *the unique positive equilibrium $E(Y^*, K^*)$ of System (2.1) is locally asymptotically stable if and only if $b \in (b_0^-, b_0^+)$. If $b \in (-\infty, b_0^-)$ or $b \in (b_0^+, +\infty)$, $E(Y^*, K^*)$ is unstable;*

(ii) *Eq. (2.1) undergoes Hopf bifurcations at the equilibrium $E(Y^*, K^*)$ when $b = b_j$, $j = 1, 2, \dots$,*

where

$$b_0^+ = \min\{b_j : b_j > 0\}, \qquad b_0^- = \max\{b_j : b_j < 0\}.$$

3 Direction and stability of the Hopf bifurcation

In this section, by using the algorithm developed in Hassard et al. [30], we will study the direction of the Hopf bifurcation and the stability of the bifurcated periodic solutions when $\tau_1 \neq \tau_2$ (Case VI). For the other five cases, most of the derivations are nearly the same steps, hence we omit them.

Without loss of generality, assume $\tau_1 < \tau_2$. We fix τ_1 in an appropriate interval such that $h(\omega)$ at least one positive root. Let $\tau_2 = \tau_{20}^* + \mu$, then $\mu = 0$ is the Hopf bifurcation value for System (2.1) in terms of the new bifurcation parameter μ. Let $y(t) = Y(t) - Y^*$, $k(t) = K(t) - K^*$, and normalize the delay with the scaling $t \mapsto (t/\tau_2)$, then System (2.1) can be rewritten as a functional differential equation in the phase space $C = C([-1,0], \mathbf{R}^2)$

$$\dot{u}(t) = L_\mu(u_t) + F(u_t), \tag{3.1}$$

where $L_\mu : C \to \mathbf{R}^2$ and $F : C \to \mathbf{R}^2$. L_μ and F are given by

$$L_\mu\varphi = \tau_2 B_1\varphi(0) + \tau_2 B_2\varphi(-\tau_1/\tau_2) + \tau_2 B_3\varphi(-1), \tag{3.2}$$

where $\varphi(t) = (\varphi_1(t), \varphi_2(t))^T$,

$$B_1 = \begin{pmatrix} \alpha(I'(Y^*) - \gamma) & -\alpha\beta \\ 0 & -\delta \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 \\ I'(Y^*) & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 \\ 0 & -\beta \end{pmatrix},$$
$$F(\varphi) = \left[\begin{pmatrix} \frac{\alpha I''(Y^*)}{2} \\ 0 \end{pmatrix} \varphi^2(0) + \begin{pmatrix} 0 \\ \frac{I''(Y^*)}{2} \end{pmatrix} \varphi^2(-\tau_1/\tau_2) + o(3) \right].$$

By using the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\cdot,\mu) : [-1,0] \to \mathbf{R}^{2^2}$, $\theta \in [-1,0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta) \quad \text{for } \phi \in C. \tag{3.3}$$

Here we can choose

$$\eta(\theta,\mu) = \begin{cases} B_1, & \theta = 0, \\ 0, & \theta \in [-\tau_1/\tau_2, 0), \\ -B_3, & \theta \in (-1, -\tau_1/\tau_2), \\ -B_2 - B_3, & \theta = -1. \end{cases}$$

For $\varphi \in C^1([-1,0], \mathbf{R}^2)$, define

$$A(\mu)\varphi = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(\mu,\xi)\varphi(\xi), & \theta = 0. \end{cases}$$

Furthermore, define the operator R as

$$R(\varphi) = \begin{cases} 0, & \theta \in [-1,0), \\ F(\varphi), & \theta = 0. \end{cases}$$

Then Eq. (3.2) is equivalent to

$$\dot{u}_t = A(\mu)u_t + Ru_t,$$

where $u_t(\theta) = u(t+\theta)$ for $\theta \in [-1,0]$.

For $\psi \in C^1([0,1],(\mathbf{R}^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} \psi(-\xi)\, d\eta^T(\xi,0), & s = 0, \end{cases}$$

and a bilinear inner product

$$\langle \psi(s), \varphi(\theta) \rangle = \overline{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)\, d\eta(\theta)\varphi(\xi)\, d\xi,$$

where $\eta(\theta) = \eta(\theta,0)$, then $A = A(0)$ and A^* are adjoint operators. Let $q(\theta)$ and $q^*(\theta)$ be eigenvectors of A and A^* corresponding to $i\omega_1\tau_{20}$ and $-i\omega_1\tau_{20}$, respectively. By direct computation, we obtain that

$$q(\theta) = (1,\rho)^T e^{i\omega_1\tau_{20}\theta} \quad \text{and} \quad q^*(s) = \frac{1}{\bar{D}}(\sigma,1)e^{i\omega_1\tau_{20}s},$$

where

$$\rho = \frac{\alpha(I'(Y^*)-\gamma) - i\omega_1}{\alpha\beta}, \qquad \sigma = -\frac{I'(Y^*)e^{-i\omega_1\tau_1}}{i\omega_1 + \beta e^{-i\omega_1\tau_{20}}},$$
$$\bar{D} = \big(\rho + \bar{\sigma} + \tau_1 e^{-i\omega_1\tau_1}\beta\rho - \tau_{20}e^{-i\omega_1\tau_{20}}I'\big(Y^*\big)\big)^{-1}.$$

Based on algorithms given in [30], the coefficients for determining the important quantities are obtained:

$$\begin{aligned} g_{20} &= I''\big(Y^*\big)\big(\alpha\bar{\sigma} + \rho^2 e^{-2i\omega_1\tau_1}\big), \\ g_{11} &= I''\big(Y^*\big)(\alpha\bar{\sigma} + \rho\bar{\rho}), \\ g_{02} &= I''\big(Y^*\big)\big(\alpha\bar{\sigma} + \bar{\rho}^2 e^{2i\omega_1\tau_1}\big), \\ g_{21} &= I''\big(Y^*\big)\big[2\alpha\bar{\sigma} W_{11}^{(1)}(0) + \bar{\rho}e^{i\omega\tau_1} W_{20}^{(2)}(-\tau_1) + 2\rho e^{-i\omega\tau_1} W_{11}^{(2)}(-\tau_1)\big] \\ &\quad + I'''\big(Y^*\big)\big(\alpha\bar{\sigma} + \rho^2\bar{\rho}e^{-i\omega\tau_1}\big), \end{aligned}$$

where

$$\begin{aligned} W_{20}(\theta) &= -\frac{g_{20}q(0)}{i\omega_1\tau_{20}}e^{i\omega_1\tau_{20}\theta} - \frac{\overline{g}_{02}\overline{q}(0)}{3i\omega_1\tau_{20}}e^{-i\omega_1\tau_{20}\theta} + E_1 e^{2i\omega_1\tau_{20}\theta}, \\ W_{11}(\theta) &= \frac{g_{11}q(0)}{i\omega_1\tau_{20}}e^{i\omega_1\tau_{20}\theta} - \frac{\overline{g}_{11}\overline{q}(0)}{i\omega_1\tau_{20}}e^{-i\omega_1\tau_{20}\theta} + E_2, \end{aligned}$$

and

$$E_1 = \big(E_1^{(1)}, E_1^{(2)}\big)^T, \qquad E_2 = \big(E_2^{(1)}, E_2^{(2)}\big)^T,$$

where

$$E_1^{(1)} = \frac{1}{D_1}\begin{vmatrix} \alpha I'(Y^*) & \alpha\beta \\ 2\rho^2 e^{-2i\omega_1\tau_{20}} & \beta e^{-2i\omega_1\tau_{20}} \end{vmatrix},$$

$$
\begin{aligned}
E_1^{(2)} &= \frac{1}{D_1}\begin{vmatrix} 2i\omega_1\tau_{20} - \alpha(I'(Y^*) - \gamma) & \alpha I''(Y^*) \\ -I'(Y^*)e^{-2i\omega_1\tau_1} & 2\rho^2 e^{-2i\omega_1\tau_{20}} \end{vmatrix}, \\
E_2^{(1)} &= \frac{1}{D_2}\begin{vmatrix} \alpha I''(Y^*) & \alpha\beta \\ 2\rho\bar{\rho} & \beta \end{vmatrix}, \\
E_2^{(2)} &= \frac{1}{D_2}\begin{vmatrix} -\alpha(I'(Y^*) - \gamma) & \alpha I''(Y^*) \\ -I'(Y^*) & 2\rho\bar{\rho} \end{vmatrix}, \\
D_1 &= \begin{vmatrix} 2i\omega_1\tau_{20} - \alpha(I'(Y^*) - \gamma) & \alpha\beta \\ -I'(Y^*)e^{-2i\omega_1\tau_1} & \beta e^{-2i\omega_1\tau_{20}} \end{vmatrix}, \\
D_2 &= \begin{vmatrix} -\alpha(I'(Y^*) - \gamma) & \alpha\beta \\ -I'(Y^*) & \beta \end{vmatrix}.
\end{aligned}
$$

Therefore, g_{21} can be determined. Furthermore, we can compute the following values:

$$
\begin{aligned}
C_1(0) &= \frac{i}{2\omega_1}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(0)\}}, \\
\beta_2 &= 2\operatorname{Re}\bigl(C_1(0)\bigr), \\
T_2 &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_{20})\}}{\omega_1\tau_{20}}.
\end{aligned}
$$

Theorem 3.1 *The following assertions hold.*

(i) μ_2 *determines the direction of the Hopf bifurcation*: *if* $\mu_2 > 0$ (< 0), *then the Hopf bifurcation is supercritical*(*subcritical*) *and the bifurcating periodic solutions exist for* $\tau_2 > \tau_{20}$ $(< \tau_{20})$;

(ii) β_2 *determines the stability of the bifurcating periodic solutions*: *the bifurcating periodic solutions are orbitally stable* (*unstable*) *if* $\beta_2 < 0$ (> 0);

(iii) T_2 *determines the period of the bifurcating periodic solutions*: *the period increases* (*decreases*) *if* $T_2 > 0$ (< 0).

4 Numerical examples

In this section, we shall give some numerical examples to illustrate the conditions required in our theorems. The investment function $I(Y)$ is taken from the published literature (see [7]):

$$
I(Y) = \frac{e^Y}{1 + e^Y}.
$$

Obviously, $I(0) = \frac{1}{2} > 0$ and $|I(Y)| = |\frac{e^Y}{1+e^Y}| < 1$. It is easy to check that the condition $I'(Y) < (1 + \frac{\beta}{\delta})\gamma$ in the following four examples, then (A1)–(A3) of Lemma 2.1 are all satisfied, System (2.1) has a unique positive equilibrium.

4.1 $\tau_1 = \tau_2 = 0$

In this case, we choose $\beta = 0.2$, $\gamma = 0.2$, $\delta = 0.05$, then System (2.1) becomes

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[\frac{e^Y}{1+e^Y} - 0.2K(t) - 0.2Y(t)], \\ \frac{dK(t)}{dt} = \frac{e^Y}{1+e^Y} - 0.25K(t). \end{cases} \tag{4.1}$$

Since $I'(Y^*) = \frac{e^{0.659}}{(1+e^{0.659})^2} = 0.2247 > 0.2 = \gamma$, then condition (H) is satisfied. Furthermore, we can obtain $\alpha^* = 10.1215$ and $\sqrt{f} = 0.6267$.

From Theorem 2.1, we know that the positive equilibrium E is asymptotically stable when the speed of adjustment of the goods market $\alpha = 8.1 < \alpha^*$ (see Fig. 1(a)), when α passes through the critical value α^*, the positive equilibrium E loses its stability and a Hopf bifurcation occurs. Let $\alpha = 12.4 > \alpha^*$, the periodic oscillations bifurcating from E are depicted in Fig. 1(b).

4.2 $\tau_1 = 0$, $\tau_2 > 0$

In this case, we choose $\alpha = 3$, $\beta = 0.2$, $\gamma = 0.2$. Some calculations indicate that $\delta^* = 0.2606$ and $\delta^{**} = 0.7533$. If we take $\delta = 0.8 > \max\{0.2606, 0.7533\}$, then by (i) of Theorem 2.2, the positive equilibrium is locally asymptotically stable for all $\tau_2 > 0$ (see Fig. 2). Moreover,

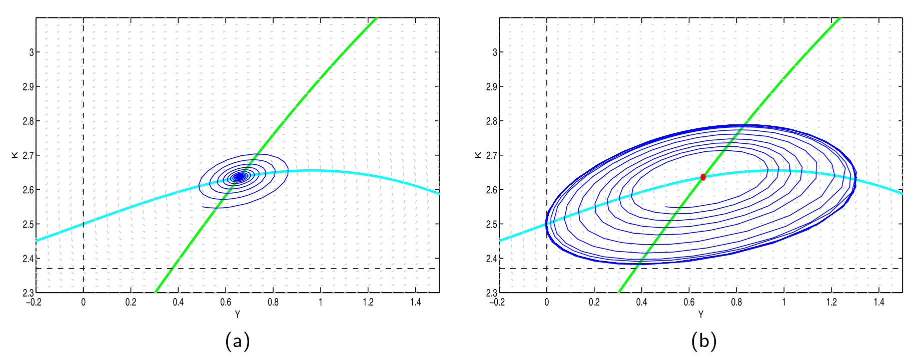


Figure 1 (**a**) $\alpha = 8.1$, the positive equilibrium is asymptotically stable. (**b**) $\alpha = 12.4$, a periodic solution appears near the positive equilibrium

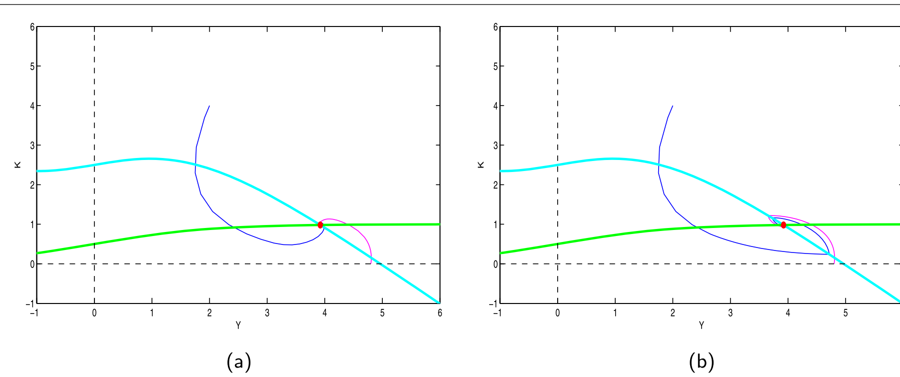


Figure 2 (**a**) $\tau_2 = 2.9$, the positive equilibrium is asymptotically stable. (**b**) $\tau_2 = 21.7$, the positive equilibrium is asymptotically stable also

let $\delta_1^* = 0.2312$ and $\delta_1^{**} = 0.4921$, then one can take $\delta_1^* < \delta = 0.24 < \delta_1^{**}$. From (ii) of Theorem 2.2, we have the positive equilibrium is asymptotically stable when $\tau_2 < \tau_{20}^+ = 3.0719$, unstable when $\tau_2 > \tau_{20}^+$, and System (2.1) undergoes a Hopf bifurcation when $\tau_2 = \tau_{20}^+$ (see Fig. 3).

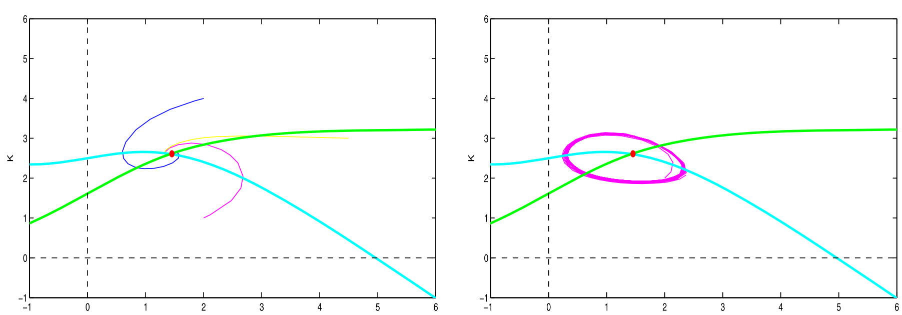
Remark 4.1 The numerical simulations of Case III $\tau_1 > 0$, $\tau_2 = 0$ and Case IV $\tau_1 = \tau_2 > 0$ can be found in [5, 6] and [7, 10]. In the present paper, we omit them.

4.3 $\tau_1 = 2\tau_2 > 0$

Consider System (2.1) and choose the following parameters:

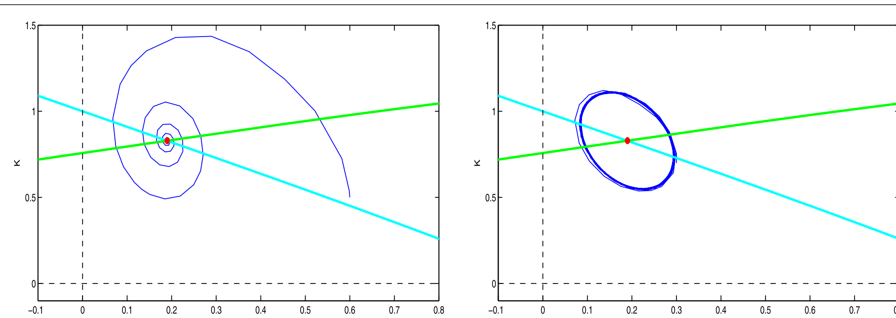
$$\alpha = 0.3, \qquad \beta = 0.5, \qquad \delta = 0.16, \qquad \gamma = 0.7.$$

We can obtain the positive equilibrium is $E(0.1921, 0.8372)$. Furthermore, we have $\delta^* = \frac{-0.5+\sqrt{0.25+4\times 0.3\times 0.7\times 0.5}}{2} = 0.1593 < \delta < \beta$, and $I'(0.1921) = \frac{e^{0.1921}}{(1+e^{0.1921})^2} = 0.2478 < 0.2834 = (1 - \frac{\beta}{2\beta-\delta})\gamma$, $\tau_{20} = 4.2371$, and $\operatorname{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau_2=\tau_{20}} > 0$. Then, by Theorem 2.5, the positive equilibrium is asymptotically stable when $\tau_2 < \tau_{20}$, unstable when $\tau_2 > \tau_{20}$, and System (2.1) undergoes a Hopf bifurcation when $\tau_2 = \tau_{20}$ (see Fig. 4).



(a) (b)

Figure 3 (**a**) $\tau_2 = 1.89$, the positive equilibrium is asymptotically stable. (**b**) $\tau_2 = 4.27$, a periodic solution appears near the positive equilibrium



(a) (b)

Figure 4 (**a**) $\tau_2 = 3.5$ and $\tau_1 = 2\tau_2 = 7.0$, the positive equilibrium is asymptotically stable. (**b**) $\tau_2 = 5.3$ and $\tau_1 = 2\tau_2 = 10.6$, a periodic solution appears near the positive equilibrium

4.4 $\tau_1 \neq \tau_2 > 0$

In this case, we choose τ_1 and τ_2 as the varying parameters for the fixed parameters

$$\alpha = 2, \qquad \beta = 0.3, \qquad \delta = 0.25, \qquad \gamma = 0.2.$$

By numerical calculation, we have $\delta^* = 0.2275 < \delta$ and $I'(2.0024) = 0.1048 < (1 + \frac{\beta}{\delta})\gamma = 0.44$. If we take $\tau_1 = 0.3$, equation $h(\omega) = 0$ has no positive real root(see Fig. 5(a)). From (i) of Theorem 2.6, System (2.1) is locally asymptotically stable for all $\tau_2 > 0$ (see Fig. 6). If we take $\tau_1 = 0.6$, equation $h(\omega) = 0$ has two positive real roots; if we take $\tau_1 = 14.3$, equation $h(\omega) = 0$ has four positive real roots (see Fig. 5(b)). Taking $\tau_1 = 1.9$, we have

$$\tau_{20}^* = 4.7162, \qquad \operatorname{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Bigg|_{\tau_2=\tau_{20}^*} > 0,$$

$$\mu_2 = 0.6448, \qquad \beta_2 = -8.4263, \qquad T_2 = 13.4972.$$

Hence, from (ii) of Theorem 2.6 and Theorem 3.1, we conclude that when $\tau_2 < \tau_{20}^*$ the positive equilibrium is asymptotically stable, the bifurcation occurs when τ_2 increases to pass τ_{20}^*, the bifurcated periodic solution is orbitally asymptotically stable, and the period increases as well as τ_2 increases. These are illustrated in Fig. 7.

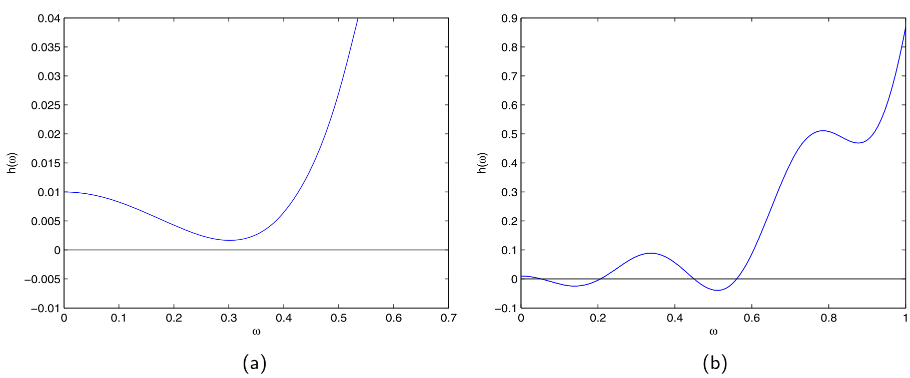


Figure 5 (**a**) When $\tau_1 = 0.3$, equation $h(\omega) = 0$ has no positive real root. (**b**) When $\tau_1 = 14.6$, equation $h(\omega) = 0$ has four positive real roots

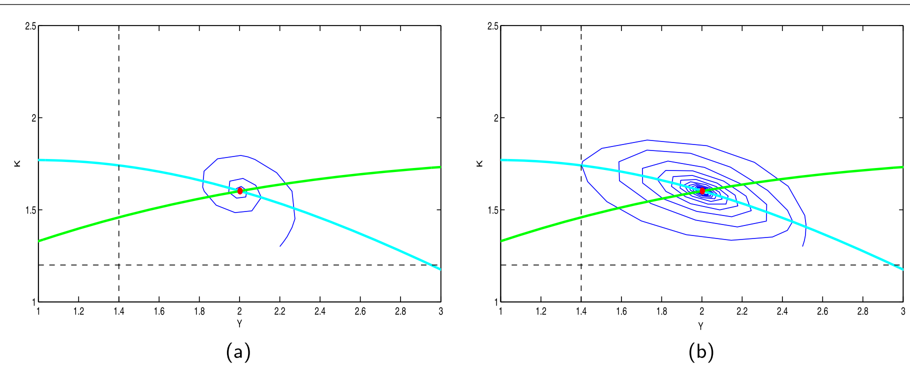
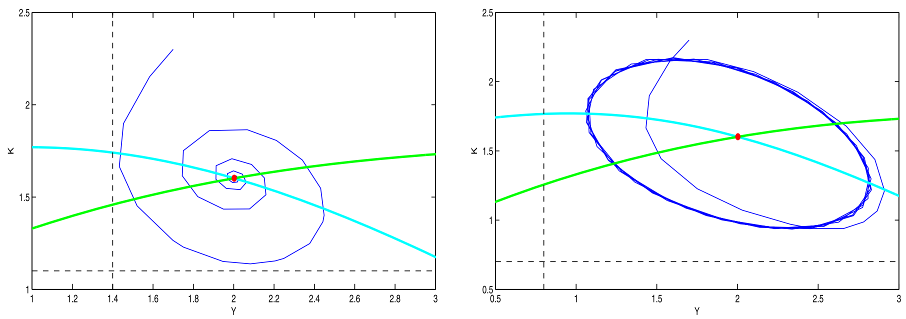


Figure 6 (**a**) $\tau_1 = 0.3$ and $\tau_2 = 4.0$, the positive equilibrium is asymptotically stable. (**b**) $\tau_1 = 0.3$ and $\tau_2 = 12.6$, the positive equilibrium is asymptotically stable also



(a) (b)

Figure 7 (**a**) $\tau_1 = 1.9$ and $\tau_2 = 3.6$, the positive equilibrium is asymptotically stable. (**b**) $\tau_1 = 1.9$ and $\tau_2 = 5.02$, a periodic solution appears near the positive equilibrium

Acknowledgements
The authors are grateful to both reviewers for their helpful suggestions and comments.

Funding
This work was supported by the National Natural Science Foundation of China (No: 11771115), Natural Science Foundation of Hebei Province of China (No: A2016201206), and Hebei Education Department (No.: QN2016030, 2017018).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 13 February 2018 Accepted: 6 January 2019 Published online: 13 March 2019

References

1. Kaldor, N.: A model of the trade cycle. Econ. J. **50**(197), 78–92 (1940)
2. Chang, W., Smyth, D.: The existence and persistence of cycles in a non-linear model: Kaldor's 1940 model re-examined: a comment. Rev. Econ. Stud. **38**(1), 37–44 (1971)
3. Krawiec, A., Szydlowski, M.: The Kaldor–Kalecki business cycle model. Ann. Oper. Res. **89**(89), 89–100 (1999)
4. Kalecki, M.: A macrodynamic theory of business cycles. Econometrica **3**(3), 327–344 (1935)
5. Szydowski, M., Krawiec, A.: The stability problem in the Kaldor–Kalecki business cycle model. Chaos Solitons Fractals **25**(2), 299–305 (2005)
6. Zhang, C., Wei, J.: Stability and bifurcation analysis in a kind of business cycle model with delay. Chaos Solitons Fractals **22**(4), 883–896 (2004)
7. Kaddar, A., Alaoui, H.T.: Hopf bifurcation analysis in a delayed Kaldor–Kalecki model of business cycle. Nonlinear Anal., Model. Control **13**(4), 439–449 (2008)
8. Kaddar, A., Talibi Alaoui, H.: Local Hopf bifurcation and stability of limit cycle in a delayed Kaldor–Kalecki model. Nonlinear Anal., Model. Control **14**(3), 333–343 (2009)
9. Wu, X.P.: Simple-zero and double-zero singularities of a Kaldor–Kalecki model of business cycles with delay. Discrete Dyn. Nat. Soc. **2009**(3), 332–337 (2010)
10. Wu, X.P.: Zero-Hopf bifurcation analysis of a Kaldor–Kalecki model of business cycle with delay. Nonlinear Anal., Real World Appl. **13**(2), 736–754 (2012)
11. Wu, X.P.: Triple-zero singularity of a Kaldor–Kalecki model of business cycles with delay. Nonlinear Anal., Model. Control **18**, 359–376 (2013)
12. Yu, J., Peng, M.: Stability and bifurcation analysis for the Kaldor–Kalecki model with a discrete delay and a distributed delay. Phys. A, Stat. Mech. Appl. **460**, 66–75 (2016)
13. David, R.: Advanced Macroeconomics, 4th edn. Business and Economics, New York (2012)
14. Wang, L., Niu, B., Wei, J.: Dynamical analysis for a model of asset prices with two delays. Phys. A, Stat. Mech. Appl. **447**, 297–313 (2016)
15. Li, A., Song, Y., Xu, D.: Dynamical behavior of a predator–prey system with two delays and stage structure for the prey. Nonlinear Dyn. **85**(3), 2017–2033 (2016)
16. Li, K., Wei, J.: Stability and Hopf bifurcation analysis of a predator–prey system with two delays. Chaos Solitons Fractals **42**(5), 2606–2613 (2009)
17. Li, J., Kuang, Y.: Analysis of a model of the glucose–insulin regulatory system with two delays. SIAM J. Appl. Math. **67**(3), 757–776 (2007)
18. Freedman, H.I.: Stability criteria for a system involving two time delays. SIAM J. Appl. Math. **46**(4), 552–560 (1986)

19. Zhou, L., Li, Y.: A dynamic IS-LM business cycle model with two time delays in capital accumulation equation. J. Comput. Appl. Math. **228**(1), 182–187 (2009)
20. Bischi, G.I., Dieci, R., Rodano, G., Saltari, E.: Multiple attractors and global bifurcations in a Kaldor-type business cycle model. J. Evol. Econ. **11**(5), 527–554 (2001)
21. Dana, R.A., Malgrange, P.: The dynamics of a discrete version of a growth cycle model. In: Analysing the Structure of Econometric Models, pp. 115–142 (1984)
22. Bashkirtseva, I., Radi, D., Ryashko, L., Ryazanova, T.: On the stochastic sensitivity and noise-induced transitions of a Kaldor-type business cycle model. Comput. Econ. **51**(3), 1–20 (2018)
23. Wu, X.P.: Codimension-2 bifurcations of the Kaldor model of business cycle. Chaos Solitons Fractals **44**(1), 28–42 (2011)
24. Grasman, J., Wentzel, J.J.: Co-existence of a limit cycle and an equilibrium in Kaldor's business cycle model and its consequences. J. Econ. Behav. Organ. **24**(3), 369–377 (1994)
25. Ruan, S.: Absolute stability, conditional stability and bifurcation in Kolmogorov-type predator–prey systems with discrete delays. Q. Appl. Math. **59**(1), 159–173 (2001)
26. Wang, L., Wu, X.P.: Bifurcation analysis of a Kaldor–Kalecki model of business cycle with time delay. Electron. J. Qual. Theory Differ. Equ. **2009**, 27 (2009)
27. Kuang, Y.: Delay Differential Equations: With Applications in Population Dynamics. Academic Press, Boston (1993)
28. Chen, S., Wei, J.: Time delay-induced instabilities and Hopf bifurcations in general reaction–diffusion systems. J. Nonlinear Sci. **23**(1), 1–38 (2013)
29. Cao, J., Yuan, R.: Bifurcation analysis in a modified Leslie–Gower model with Holling type II functional response and delay. Nonlinear Dyn. **84**(3), 1341–1352 (2016)
30. Hassard, B.D., Hassard, D., Kazarinoff, N.D., Wan, Y.-H., Wan, Y.W.: Theory and Applications of Hopf Bifurcation. Cambridge University Press, Cambridge (1981)