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# Application of reproducing kernel Hilbert space method for solving second-order fuzzy Volterra integro-differential equations

Ghaleb N. Gumah<sup>1</sup>, Mohammad F.M. Naser<sup>1</sup>, Mohammed Al-Smadi<sup>2</sup> and Shrideh K. Al-Omari<sup>1\*</sup> 

\*Correspondence:

[s.k.alomari@fet.edu.jo](mailto:s.k.alomari@fet.edu.jo)

<sup>1</sup>Faculty of Engineering Technology,  
Al-Balqa Applied University,  
Amman, Jordan

Full list of author information is  
available at the end of the article

## Abstract

In this article, we propose a new method that determines an efficient numerical procedure for solving second-order fuzzy Volterra integro-differential equations in a Hilbert space. This method illustrates the ability of the reproducing kernel concept of the Hilbert space to approximate the solutions of second-order fuzzy Volterra integro-differential equations. Additionally, we discuss and derive the exact and approximate solutions in the form of Fourier series with effortlessly computable terms in the reproducing kernel Hilbert space  $W_2^3[a, b] \oplus W_2^3[a, b]$ . The convergence of the method is proven and its exactness is illustrated by three numerical examples.

**Keywords:** Reproducing kernel Hilbert space; Complete orthonormal system; Fuzzy Volterra integro-differential equation

## 1 Introduction

Fuzzy integro-differential equations are very useful tools for modeling physical systems under the differential sense and dynamical systems under the possibilistic uncertainty [1]. In recent years, fuzzy Volterra integro-differential equations (FVIDEs) in both theoretical and numerical calculations have been growing in many scientific applications [2–4]. They are usually encountered in many applications including fuzzy modeling in population dynamics, experimental quantum optics to quantum gravity, and averaging fuzzy biopolymers (see [5–7] and the references therein). Biswas et al. [8] have recently proposed a numerical fuzzy differential transform (FDT) method for solving second-order FVIDEs under the generalized concept of Seikkala differentiability of fuzzy functions. A further literature review of fuzzy integro-differential equations can be found in [2, 3] and [8].

The main goal of this article is to solve second-order FVIDEs in the Hilbert space  $W_2^3[a, b] \oplus W_2^3[a, b]$  under the assumption of strongly generalized differentiability. More precisely, we provide a numerical approximate solution for fuzzy Volterra integro-differential equation of the general form

$$y''(t) = f(t) + \int_a^t k(t, s)g(y(s))ds, \quad t \geq a, \quad (1)$$

subject to the fuzzy initial conditions

$$y(a) = \gamma_1, \quad y'(a) = \gamma_2, \quad (2)$$

where  $k(s, t)$  is an arbitrary kernel function,  $g$  is a continuous function,  $f$  is a continuous fuzzy-valued function, and  $\gamma_1, \gamma_2$  are fuzzy real numbers. In special cases the analytic solution of FVIDE (1)–(2) can be found using the fuzzy Laplace transform (FLT) method. However, a fuzzy solution of FVIDE (1)–(2) is usually very difficult to be derived analytically. To solve this issue, FVIDE (1)–(2) can be converted into two systems of Volterra integro-differential equations which are solvable numerically using the reproducing kernel Hilbert space (RKHS) method based on the Gram–Schmidt orthogonalization process. Moreover, the authors in [9] have proven the existence and uniqueness of a fuzzy solution for the second-order FVIDE (1)–(2) with the fuzzy kernel under strongly generalized differentiability.

Reproducing kernel theory has important scientific applications in numerical analysis, ordinary and fractional differential equations, probability, statistics, and learning theory. Recently, RKHS method for solving a variety of fuzzy differential equations and fuzzy fractional differential equations has been presented by Ahmadian et al. [10, 11]. The RKHS method possesses many virtues; to begin with, it is conceivable to choose any point in the interval of integration and as well the approximate solution would be appropriate and applicable. Secondly, it is precise and requires less exertion to discover the numerical results. Thirdly, it is a proficient application to solve some scientific models in comparison with the numerical solutions and the exact solutions. Fourthly, the method is easy to understand, and it needs little computational prerequisites. For more points of interest and descriptions about the RKHS method, we allude to [12–17].

Our paper has the following structure. Section 2 presents some concepts and results about the fuzzy numbers and the properties of fuzzy-number-valued functions, including the  $H$ -differentiability or Hukuhara differentiability concept and the fuzzy integral. The fuzzy Volterra integro-differential equations of the second order are introduced in Sect. 3. Section 4 proposes a numerical method for solving fuzzy Volterra integro-differential equations of second order. In Sect. 5, we present three numerical examples to illustrate our method in Sect. 4. Finally, we end the paper with the conclusion in Sect. 6.

## 2 Preliminaries

We recall some basic definitions and theorems needed throughout the paper such as fuzzy number, fuzzy-valued function, and the derivative of the fuzzy-valued functions.

**Definition 2.1** (see [18]) A fuzzy number  $u : \mathbb{R} \rightarrow [0, 1]$  is a fuzzy subset of  $\mathbb{R}$  with normal, convex, and upper semicontinuous membership function of bounded support.

Let  $R_f$  denote the space of fuzzy real numbers. For  $0 < \alpha \leq 1$ , set  $[u]_\alpha = \{s \in \mathbb{R} \mid u(s) \geq \alpha\}$  and  $[u]_0 = \overline{\{s \in \mathbb{R} \mid u(s) > 0\}}$  (the closure of  $\{s \in \mathbb{R} \mid u(s) > 0\}$ ). Then the  $\alpha$ -level set  $[u]_\alpha$  is a nonempty compact interval for all  $0 \leq \alpha \leq 1$  and any  $u \in R_f$ . The notation  $[u]_\alpha = [u_1(\alpha), u_2(\alpha)]$  denotes explicitly the  $\alpha$ -level set of  $u$ . We refer to  $u_1$  and  $u_2$  as the lower and upper branches on  $u$ , respectively.

**Theorem 2.2** (see [19]) *A mapping  $u : \mathbb{R} \rightarrow [0, 1]$  is a fuzzy number with  $\alpha$ -cut representation  $[u_1(\alpha), u_2(\alpha)]$  if and only if the following conditions are satisfied:*

- (i) *the function  $u_1 : [0, 1] \rightarrow \mathbb{R}$  is bounded nondecreasing;*
- (ii) *the function  $u_2 : [0, 1] \rightarrow \mathbb{R}$  is bounded nonincreasing;*
- (iii) *for each  $r \in (0, 1]$ ,  $\lim_{\alpha \rightarrow r^-} u_1(\alpha) = u_1(r)$  and  $\lim_{\alpha \rightarrow r^-} u_2(\alpha) = u_2(r)$ ;*
- (iv) *for each  $r \in (0, 1]$ ,  $\lim_{\alpha \rightarrow r^+} u_1(\alpha) = u_1(r)$  and  $\lim_{\alpha \rightarrow r^+} u_2(\alpha) = u_2(r)$ ;*
- (v)  *$u_1(\alpha) \leq u_2(\alpha)$  for all  $\alpha \in [0, 1]$ .*

For  $u, v \in R_F$  and  $\lambda \in \mathbb{R}$ , the sum  $u + v$  and the product  $\lambda u$  are defined by  $[u + v]_\alpha = [u]_\alpha + [v]_\alpha$ ,  $[\lambda u]_\alpha = \lambda[u]_\alpha$ ,  $\forall \alpha \in [0, 1]$ , where  $[u]_\alpha + [v]_\alpha$  means the usual addition of two intervals (subsets) of  $\mathbb{R}$  and  $\lambda[u]_\alpha$  is the usual product between a scalar and a subset of  $\mathbb{R}$ . The metric structure is given by the Hausdorff distance  $D : R_F \times R_F \rightarrow \mathbb{R}^+ \cup \{0\}$ , by  $D(u, v) = \sup_{\alpha \in [0, 1]} \max\{|u_1 - v_1|, |u_2 - v_2|\}$ .

**Definition 2.3** (see [18]) Let  $y : [a, b] \rightarrow R_F$  and  $t_0 \in (a, b)$ . We say  $y$  is (1)-differentiable at  $t_0$  if there is some element  $y'(t_0) \in R_F$  such that, for all  $h > 0$  sufficiently close to 0, there exist  $y(t_0 + h) \odot y(t_0)$ ,  $y(t_0) \odot y(t_0 - h)$ , and the limits (in the metric  $D$ )

$$\lim_{h \rightarrow 0^+} \frac{y(t_0 + h) \odot y(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{y(t_0) \odot y(t_0 - h)}{h} = y'(t_0).$$

In this case, we denote  $y'(t_0)$  by  $D_1^1 y(t_0)$ . Also,  $y$  is (2)-differentiable if, for all  $h < 0$  sufficiently close to 0, there exist  $y(t_0 + h) \odot y(t_0)$ ,  $y(t_0) \odot y(t_0 - h)$ , and the limits (in the metric  $D$ )

$$\lim_{h \rightarrow 0^-} \frac{y(t_0 + h) \odot y(t_0)}{h} = \lim_{h \rightarrow 0^-} \frac{y(t_0) \odot y(t_0 - h)}{h} = y'(t_0).$$

In this case, this derivative is denoted by  $D_2^1 y(t_0)$ .

**Theorem 2.4** (see [20]) Let  $y : [a, b] \rightarrow R_F$  be a fuzzy-valued function, where  $[y(t)]_\alpha = [y_{1,\alpha}(t), y_{2,\alpha}(t)]$  for each  $\alpha \in [0, 1]$ .

- (i) *If  $y$  is (1)-differentiable, then  $y_{1,\alpha}$  and  $y_{2,\alpha}$  are differentiable functions and  $[D_1^1 y(t)]_\alpha = [y'_{1,\alpha}(t), y'_{2,\alpha}(t)]$ .*
- (ii) *If  $y$  is (2)-differentiable, then  $y_{1,\alpha}$  and  $y_{2,\alpha}$  are differentiable functions and  $[D_2^1 y(t)]_\alpha = [y'_{2,\alpha}(t), y'_{1,\alpha}(t)]$ .*

**Theorem 2.5** (see [21]) Let  $D_1^1 y : [a, b] \rightarrow R_F$  or  $D_2^1 y : [a, b] \rightarrow R_F$  be fuzzy-valued functions, where  $[y(t)]_\alpha = [y_{1,\alpha}(t), y_{2,\alpha}(t)]$  for each  $\alpha \in [0, 1]$ .

- (i) *If  $D_1^1 y$  is (1)-differentiable, then  $y'_{1,\alpha}$  and  $y'_{2,\alpha}$  are differentiable functions and  $[y''(t)]_\alpha = [y''_{1,\alpha}(t), y''_{2,\alpha}(t)]$ .*
- (ii) *If  $D_1^1 y$  is (2)-differentiable, then  $y'_{1,\alpha}$  and  $y'_{2,\alpha}$  are differentiable functions and  $[y''(t)]_\alpha = [y''_{2,\alpha}(t), y''_{1,\alpha}(t)]$ .*
- (iii) *If  $D_2^1 y$  is (1)-differentiable, then  $y'_{1,\alpha}$  and  $y'_{2,\alpha}$  are differentiable functions and  $[y''(t)]_\alpha = [y''_{2,\alpha}(t), y''_{1,\alpha}(t)]$ .*
- (iv) *If  $D_2^1 y$  is (2)-differentiable, then  $y'_{1,\alpha}$  and  $y'_{2,\alpha}$  are differentiable functions and  $[y''(t)]_\alpha = [y''_{1,\alpha}(t), y''_{2,\alpha}(t)]$ .*

**Theorem 2.6** (see [22]) *Let  $y : [a, b] \rightarrow R_F$  be a continuous fuzzy function, where  $[y(t)]_\alpha = [y_{1,\alpha}(t), y_{2,\alpha}(t)]$ . If  $y_{1,\alpha}(t)$  and  $y_{2,\alpha}(t)$  are integrable functions over  $[a, b]$ , then  $\int_a^b y(t) dt \in R_F$  and*

$$\left[ \int_a^b y(t) dt \right]_\alpha = \left[ \int_a^b y_{1,\alpha}(t) dt, \int_a^b y_{2,\alpha}(t) dt \right]. \quad (3)$$

### 3 Formulation of second-order FVIDE

This section studies the second-order FVIDEs under the concept of strongly generalized differentiability in which the FVIDE is converted into an equivalent system. This system consists of the crisp system of Volterra integro-differential equations for each type of differentiability.

In order to design a numerical scheme for solving FVIDE (1)–(2), we first replace it according to Theorem 2.5 by the following two systems:

(1, 1)-system

$$\begin{aligned} y''_{1,\alpha}(t) &= f_{1,\alpha}(t) + \int_a^t \underline{U}(t, \alpha) ds, \\ y''_{2,\alpha}(t) &= f_{2,\alpha}(t) + \int_a^t \overline{U}(t, \alpha) ds; \end{aligned}$$

(1, 2)-system

$$\begin{aligned} y''_{2,\alpha}(t) &= f_{1,\alpha}(t) + \int_a^t \underline{U}(t, \alpha) ds, \\ y''_{1,\alpha}(t) &= f_{2,\alpha}(t) + \int_a^t \overline{U}(t, \alpha) ds. \end{aligned}$$

Depending on the Zadeh extension principle in [23], if  $g$  in Eq. (1) is a strictly increasing function, then

$$\underline{U}(t, \alpha) = \begin{cases} k(t, s)g(y_{1,\alpha}(s)), & k(t, s) \geq 0, \\ k(t, s)g(y_{2,\alpha}(s)), & k(t, s) < 0, \end{cases}$$

and

$$\overline{U}(t, \alpha) = \begin{cases} k(t, s)g(y_{2,\alpha}(s)), & k(t, s) \geq 0, \\ k(t, s)g(y_{1,\alpha}(s)), & k(t, s) < 0. \end{cases}$$

Moreover, if  $g$  in Eq. (1) is a strictly decreasing function, then

$$\underline{U}(t, \alpha) = \begin{cases} k(t, s)g(y_{2,\alpha}(s)), & k(t, s) \geq 0, \\ k(t, s)g(y_{1,\alpha}(s)), & k(t, s) < 0, \end{cases}$$

and

$$\overline{U}(t, \alpha) = \begin{cases} k(t, s)g(y_{1,\alpha}(s)), & k(t, s) \geq 0, \\ k(t, s)g(y_{2,\alpha}(s)), & k(t, s) < 0. \end{cases}$$

Here, we want to mention that the sufficient conditions for the existence of a unique solution to the second-order FVIDE (1)–(2) have been given in [9].

#### 4 Analysis of the method with its convergence

In this section, we present a review of the notations and preliminary definitions of the RKHS theory. Additionally, we demonstrate how to solve FVIDE (1)–(2) using the RKHS method. Accordingly, we build orthonormal function systems of the space  $W_2^3[0, 1] \oplus W_2^3[0, 1]$  based on the Gram–Schmidt orthogonalization process.

**Definition 4.1** (see [24]) Let  $H$  be a Hilbert space. A function  $K : \Omega \times \Omega \rightarrow \mathbb{R}$  is a reproducing kernel of  $H$  if the following conditions are satisfied:

- (i) for each  $x \in \Omega$ ,  $K(\cdot, x) \in H$ ;
- (ii) for each  $x \in \Omega$  and  $\phi \in H$ ,  $\langle \phi, K(\cdot, x) \rangle = \phi(x)$ , which is called the reproducing property.

**Definition 4.2** (see [24]) The Hilbert space  $W_2^m[a, b]$  is defined as  $W_2^m[a, b] = \{y(t) | y(t), y'(t), \dots, y^{(m-1)}(t) \text{ are absolutely continuous, } y^{(m)}(t) \in L^2[a, b] \text{ and } y(a) = y'(a) = \dots = y^{(m-1)}(a) = 0 \text{ whenever } m \neq 1\}$ . Whilst the inner product and the norm in  $W_2^m[a, b]$  are defined by

$$\langle y_1(t), y_2(t) \rangle_{W_2^m} = \sum_{i=0}^{m-1} y_1^{(i)}(a) y_2^{(i)}(a) + \int_a^b y_1^{(m)}(t) y_2^{(m)}(t) dt \quad (4)$$

and  $\|y_1(t)\|_{W_2^m} = \sqrt{\langle y_1(t), y_1(t) \rangle_{W_2^m}}$ , where  $y_1, y_2 \in W_2^m[a, b]$ .

**Definition 4.3** The Hilbert space  $W_2^m[a, b] \oplus W_2^m[a, b]$ ,  $m = 1, 2, 3, \dots, n$ , can be defined as  $W_2^m[a, b] \oplus W_2^m[a, b] = \{y = (y_1, y_2)^T | y_1, y_2 \in W_2^m[a, b]\}$ . Accordingly, the inner product and the norm in  $W_2^m[a, b] \oplus W_2^m[a, b]$  are built, respectively, by  $\langle y(t), z(t) \rangle_{W_2^m \oplus W_2^m} = \sum_{i=1}^2 \langle y_i(t), z_i(t) \rangle_{W_2^m}$  and  $\|y(t)\|_{W_2^m \oplus W_2^m} = \sqrt{\sum_{i=1}^2 \|y_i(t)\|_{W_2^m}^2}$ .

**Theorem 4.4** (see [25]) The space  $W_2^m[a, b]$  is a complete reproducing kernel space. That is, for each fixed  $t \in [a, b]$ , there exists  $R_t(s) \in W_2^m[a, b]$  such that  $\langle y(s), R_t(s) \rangle_{W_2^m} = y(t)$  for any  $y(t) \in W_2^m[a, b]$  and any  $t \in [a, b]$ . The reproducing kernel  $R_t(s)$  can be written as

$$R_t(s) = \begin{cases} \sum_{i=0}^{2m-1} p_i(t) s^i, & s \leq t, \\ \sum_{i=0}^{2m-1} q_i(t) s^i, & s > t. \end{cases} \quad (5)$$

The representation of the reproducing kernel function  $R_t(s)$  in  $W_2^3[0, 1]$ , using Mathematica 7.0 software package, is provided by

$$R_t(s) = \begin{cases} 1 + \frac{1}{12} t^2 s^2 (3+t) + ts(1 - \frac{1}{24} t^4) + \frac{1}{120} t^5, & s \leq t, \\ 1 + \frac{1}{12} t^2 s^2 (3+s) + ts(1 - \frac{1}{24} s^4) + \frac{1}{120} s^5, & s > t. \end{cases} \quad (6)$$

To apply our technique on the Hilbert space  $W_2^3[a, b] \oplus W_2^3[a, b]$ , we define the linear invertible operator  $L : W_2^3[a, b] \oplus W_2^3[a, b] \rightarrow W_2^1[a, b] \oplus W_2^1[a, b]$  as  $Ly(t) = y''(t)$  such

that  $L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$  and  $y(t) = (y_{1,\alpha}(t), y_{2,\alpha}(t))^T$ . Thus, Eqs. (1) and (2) can be converted into the form

$$Ly(t) = G(t, f(t), h(y(t))), \quad (7)$$

subject to the fuzzy initial conditions

$$y(a) = \gamma_1, \quad y'(a) = \gamma_2, \quad (8)$$

where  $h(y(t)) = \int_a^t k(t, s)g(y(t))ds$ ,  $y(t) \in W_2^3[a, b] \oplus W_2^3[a, b]$ , and  $f(t) \in W_2^1[a, b] \oplus W_2^1[a, b]$ . Here,  $G(t, f(t), h(y(t)))$  is  $G(t, f_{1,\alpha}(t), f_{2,\alpha}(t), h(y_{1,\alpha}(t)), h(y_{2,\alpha}(t)))$ .

**Lemma 4.5**  $L : W_2^3[a, b] \oplus W_2^3[a, b] \longrightarrow W_2^1[a, b] \oplus W_2^1[a, b]$  is a bounded linear operator.

*Proof* Firstly, it is easy to prove that  $L$  is a linear operator from  $W_2^3[a, b] \oplus W_2^3[a, b]$  into  $W_2^1[a, b] \oplus W_2^1[a, b]$ . Secondly, we need to prove that  $\|Ly\|_{W_2^1 \oplus W_2^1} \leq \xi \|y\|_{W_2^3 \oplus W_2^3}$ , where  $\xi > 0$ . For each  $y \in W_2^3[a, b] \oplus W_2^3[a, b]$ , we have

$$\begin{aligned} \|Ly(t)\|_{W_2^1 \oplus W_2^1}^2 &= \sum_{i=1}^2 \|L_i y_{i,\alpha}(t)\|_{W_2^1}^2 = \|L_1 y_{1,\alpha}(t)\|_{W_2^1}^2 + \|L_2 y_{2,\alpha}(t)\|_{W_2^1}^2 \\ &= \langle L_1 y_{1,\alpha}(t), L_1 y_{1,\alpha}(t) \rangle_{W_2^1} + \langle L_2 y_{2,\alpha}(t), L_2 y_{2,\alpha}(t) \rangle_{W_2^1} \\ &= [L_1 y_{1,\alpha}(a)]^2 + \int_a^b \left[ \frac{d}{dt} (L_1 y_{1,\alpha}(t)) \right]^2 dt + [L_2 y_{2,\alpha}(a)]^2 \\ &\quad + \int_a^b \left[ \frac{d}{dt} (L_2 y_{2,\alpha}(t)) \right]^2 dt. \end{aligned}$$

By the reproducing property of  $R_t(s)$ , then  $y_{1,\alpha}(t) = \langle y_{1,\alpha}(s), R_t(s) \rangle_{W_2^3}$  and  $y_{2,\alpha}(t) = \langle y_{2,\alpha}(s), R_t(s) \rangle_{W_2^3}$ . Thus,

$$\begin{aligned} L_1 y_{1,\alpha}(t) &= \langle y_{1,\alpha}(s), L_1 R_t(s) \rangle_{W_2^3}, \quad L_2 y_{2,\alpha}(t) = \langle y_{2,\alpha}(s), L_2 R_t(s) \rangle_{W_2^3}, \\ \frac{d}{dt} (L_1 y_{1,\alpha}(t)) &= \left\langle y_{1,\alpha}(s), \frac{d}{dt} (L_1 R_t(s)) \right\rangle_{W_2^3}, \\ \frac{d}{dt} (L_2 y_{2,\alpha}(t)) &= \left\langle y_{2,\alpha}(s), \frac{d}{dt} (L_2 R_t(s)) \right\rangle_{W_2^3}. \end{aligned}$$

From the continuity of  $R_t(s)$  on  $[a, b]$ , we have

$$\begin{aligned} |L_1 y_{1,\alpha}(t)| &= |\langle y_{1,\alpha}(s), L_1 R_t(s) \rangle_{W_2^3}| \leq \|y_{1,\alpha}\|_{W_2^3} \|L_1 R_t(s)\|_{W_2^3} \leq k_1 \|y_{1,\alpha}\|_{W_2^3}, \\ |L_2 y_{2,\alpha}(t)| &= |\langle y_{2,\alpha}(s), L_2 R_t(s) \rangle_{W_2^3}| \leq \|y_{2,\alpha}\|_{W_2^3} \|L_2 R_t(s)\|_{W_2^3} \leq k_2 \|y_{2,\alpha}\|_{W_2^3}, \\ \left| \frac{d}{dt} (L_1 y_{1,\alpha}(t)) \right| &= \left| \left\langle y_{1,\alpha}(s), \frac{d}{dt} (L_1 R_t(s)) \right\rangle_{W_2^3} \right| \leq \|y_{1,\alpha}\|_{W_2^3} \left\| \frac{d}{dt} (L_1 R_t(s)) \right\|_{W_2^3} \\ &\leq k_3 \|y_{1,\alpha}\|_{W_2^3}, \end{aligned}$$

$$\begin{aligned} \left| \frac{d}{dt} (L_2 y_{2,\alpha}(t)) \right| &= \left| \left\langle y_{2,\alpha}(s), \frac{d}{dt} (L_2 R_t(s)) \right\rangle_{W_2^3} \right| \leq \|y_{2,\alpha}\|_{W_2^3} \left\| \frac{d}{dt} (L_2 R_t(s)) \right\|_{W_2^3} \\ &\leq k_4 \|y_{2,\alpha}\|_{W_2^3}. \end{aligned}$$

Thus,

$$\begin{aligned} \|Ly(t)\|_{W_2^1 \oplus W_2^1}^2 &\leq k_1^2 \|y_{1,\alpha}\|_{W_2^3}^2 + (b-a)k_3^2 \|y_{1,\alpha}\|_{W_2^3}^2 + k_2^2 \|y_{2,\alpha}\|_{W_2^3}^2 + (b-a)k_4^2 \|y_{2,\alpha}\|_{W_2^3}^2 \\ &= (k_1^2 + (b-a)k_3^2) \|y_{1,\alpha}\|_{W_2^3}^2 + (k_2^2 + (b-a)k_4^2) \|y_{2,\alpha}\|_{W_2^3}^2 \\ &\leq \xi^2 (\|y_{1,\alpha}\|_{W_2^3}^2 + \|y_{2,\alpha}\|_{W_2^3}^2) \\ &= \xi^2 \sum_{i=1}^2 \|y_{i,\alpha}\|_{W_2^3}^2 \\ &= \xi^2 \|y\|_{W_2^3 \oplus W_2^3}^2, \end{aligned}$$

where  $\xi^2 = \max\{k_1^2 + (b-a)k_3^2, k_2^2 + (b-a)k_4^2\}$ . The proof is complete.  $\square$

Next, to construct an orthogonal system of functions  $\{\psi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$  of the space  $W_2^3[a, b] \oplus W_2^3[a, b]$ , we let  $\psi_{ij}(t) = L^* \Phi_{ij}(t)$ , where  $L^* = \begin{bmatrix} L_1^* & 0 \\ 0 & L_2^* \end{bmatrix}$  is the adjoint operator of  $L$  and  $\Phi_{ij}(t) = (\Phi_{i1}(t), \Phi_{i2}(t))^T$ . From now on the orthonormal system  $\{\bar{\psi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$  in the space  $W_2^3[a, b] \oplus W_2^3[a, b]$  can be derived from the Gram–Schmidt orthogonalization process of  $\{\psi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$  as follows:

$$\bar{\psi}_{ij}(t) = \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} \psi_{lk}(t), \quad (9)$$

where  $\beta_{lk}^{ij}$  are orthogonalization coefficients given by

$$\begin{aligned} \beta_{11}^{ij} &= \frac{1}{\|\psi_{11}\|}, \quad \beta_{lk}^{ij} = \left( \|\psi_{lk}\|^2 - \sum_{p=1}^{l-1} \langle \psi_{lk}(t), \bar{\psi}_{lp}(t) \rangle^2 \right)^{-\frac{1}{2}} \quad (l = k \neq 1), \\ \beta_{lk}^{ij} &= \frac{-\sum_{p=k}^{l-1} \langle \psi_{lk}(t), \bar{\psi}_{lp}(t) \rangle \beta_{pk}^{ij}}{\sqrt{\|\psi_{lk}\|^2 - \sum_{p=1}^{l-1} \langle \psi_{lk}(t), \bar{\psi}_{lp}(t) \rangle^2}} \quad (l > k). \end{aligned}$$

The main result of this article is the following theorems, which give the exact expression of the solution of FVIDE (7)–(8) in the space  $W_2^3[a, b] \oplus W_2^3[a, b]$  and the convergence of our method.

**Theorem 4.6** Suppose  $\{t_i\}_{i=1}^\infty$  is dense in  $[a, b]$ , if  $y(t) \in W_2^3[a, b] \oplus W_2^3[a, b]$  is the solution of FVIDE (7)–(8), then

$$y(t) = \sum_{i=1}^\infty \sum_{j=1}^2 \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} G_k(t_i, f(t_i), h(y(t_i))) \bar{\psi}_{ij}(t), \quad (10)$$

which is a convergent series in the sense of  $\|\cdot\|_{W_2^3 \oplus W_2^3}$ .

*Proof* Firstly, we need to prove that  $\{\psi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$  is the complete system in  $W_2^3[a, b] \oplus W_2^3[a, b]$  as follows:

$$\begin{aligned}\psi_{ij}(t) &= L^* \Phi_{ij}(t) = \langle L^* \Phi_{ij}(s), R_t(s) \rangle_{W_2^3 \oplus W_2^3} \\ &= \langle \Phi_{ij}(s), L_s R_t(s) \rangle_{W_2^1 \oplus W_2^1} \\ &= L_s R_t(s)|_{s=t_i} \in W_2^3[a, b] \oplus W_2^3[a, b].\end{aligned}$$

On the other hand, for each  $y \in W_2^3[a, b] \oplus W_2^3[a, b]$ , let  $\langle y(t), \psi_{ij}(t) \rangle_{W_2^3 \oplus W_2^3} = 0$ . One has

$$\begin{aligned}\langle y(t), \psi_{ij}(t) \rangle_{W_2^3 \oplus W_2^3} &= \langle y_{1,\alpha}(t), \psi_{i1}(t) \rangle_{W_2^3} + \langle y_{2,\alpha}(t), \psi_{i2}(t) \rangle_{W_2^3} \\ &= \langle y_{1,\alpha}(t), L_1^* \Phi_{i1}(t) \rangle_{W_2^3} + \langle y_{2,\alpha}(t), L_2^* \Phi_{i2}(t) \rangle_{W_2^3} \\ &= \langle L_1 y_{1,\alpha}(t), \Phi_{i1}(t) \rangle_{W_2^1} + \langle L_2 y_{2,\alpha}(t), \Phi_{i2}(t) \rangle_{W_2^1} \\ &= L_1 y_{1,\alpha}(t_i) + L_2 y_{2,\alpha}(t_i) \\ &= Ly(t_i).\end{aligned}$$

Thus,  $Ly(t) = 0$  because  $\{t_i\}_{i=1}^\infty$  is dense on  $[a, b]$ . Hence,  $y(t) = 0$  because of the existence of  $L^{-1}$ . Secondly, since  $\{\psi_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$  is the complete system in  $W_2^3[a, b] \oplus W_2^3[a, b]$  and from Eq. (9), then the sequence  $\{\bar{\psi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$  is the complete orthonormal system in  $W_2^3[a, b] \oplus W_2^3[a, b]$ . Thirdly, using Fourier series expansion about  $\{\bar{\psi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ , we have

$$\begin{aligned}y(t) &= \sum_{i=1}^\infty \sum_{j=1}^2 \langle y(t), \bar{\psi}_{ij}(t) \rangle_{W_2^3 \oplus W_2^3} \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^2 \left\langle y(t), \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} \psi_{lk}(t) \right\rangle_{W_2^3 \oplus W_2^3} \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^2 \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} \langle y(t), \psi_{lk}(t) \rangle_{W_2^3 \oplus W_2^3} \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^2 \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} \langle y(t), L^* \Phi_{lk}(t) \rangle_{W_2^3 \oplus W_2^3} \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^2 \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} \langle Ly(t), \Phi_{lk}(t) \rangle_{W_2^1 \oplus W_2^1} \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^2 \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} \langle G_k(t, f(t), h(y(t))), \Phi_{lk}(t) \rangle_{W_2^1 \oplus W_2^1} \bar{\psi}_{ij}(t) \\ &= \sum_{i=1}^\infty \sum_{j=1}^2 \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} G_k(t_i, f(t_i), h(y(t_i))) \bar{\psi}_{ij}(t).\end{aligned}$$



Since  $\sum_{i=1}^{\infty} \sum_{j=1}^2 \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} G_k(t_l, f(t_l), h(y(t_l))) \overline{\psi}_{ij}(t)$  is a Fourier series in  $W_2^3[a, b] \oplus W_2^3[a, b]$  and  $W_2^3[a, b] \oplus W_2^3[a, b]$  is a Hilbert space, then it is a convergent series in the sense of  $\|\cdot\|_{W_2^3 \oplus W_2^3}$ . The proof is complete.  $\square$

For numerical computations, we put the initial function  $y_0(t_l) = y(t_l)$  and define the  $n$ -term numerical solution of FVIDE (7)–(8) by truncating the series given in Eq. (10) as follows:

$$y_n(t) = \sum_{i=1}^n \sum_{j=1}^2 \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} G_k(t_l, f(t_l), h(y_{l-1}(t_l))) \overline{\psi}_{ij}(t). \quad (11)$$

**Theorem 4.7** Suppose that  $y(t)$  is the solution of FVIDE (7)–(8) and  $y_n(t)$  is the approximate solution of FVIDE (7)–(8), where  $y(t)$  and  $y_n(t)$  are given by Eqs. (10) and (11), respectively, then

- (i) if  $\|y_n\|_{W_2^3 \oplus W_2^3}$  is bounded and  $\{t_i\}_{i=1}^n$  is dense on  $[a, b]$ , then  $\|y_n - y\|_{W_2^3 \oplus W_2^3} \rightarrow 0$  as  $n \rightarrow \infty$ ;
- (ii)  $\|y_n - y\|_C \rightarrow 0$  as  $n \rightarrow \infty$ ;
- (iii) the second derivative of  $y_n$  converges to the second derivative of  $y$  uniformly as  $n \rightarrow \infty$ ;
- (iv) if  $\|y_{n-1} - y\|_{W_2^3 \oplus W_2^3} \rightarrow 0$  as  $n \rightarrow \infty$ ,  $\|y_{n-1}\|_{W_2^3 \oplus W_2^3}$  is bounded,  $t_n \rightarrow \tau$  as  $n \rightarrow \infty$ ,  $G(t, f(t), h(y(t)))$  is continuous for  $t \in [a, b]$  and  $f, h$  are continuous functions, then  $G(t_n, f(t_n), h(y_{n-1}(t_n))) \rightarrow G(\tau, f(\tau), h(y(\tau)))$ .

*Proof* (i) From Eq. (11) and the orthonormality of  $\{\overline{\psi}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ , we have  $\|y_{n+1}\|_{W_2^3 \oplus W_2^3}^2 = \|y_0\|_{W_2^3 \oplus W_2^3}^2 + \sum_{i=1}^{n+1} \sum_{j=1}^2 A_{ij}^2$ , where  $A_{ij} = \sum_{l=1}^i \sum_{k=1}^j \beta_{lk}^{ij} G_k(t_l, f(t_l), h(y_{l-1}(t_l))) \overline{\psi}_{ij}(t)$ . Since  $\|y_{n+1}\|_{W_2^3 \oplus W_2^3}^2 \geq \|y_n\|_{W_2^3 \oplus W_2^3}^2$  and  $\|y_n\|_{W_2^3 \oplus W_2^3}$  is bounded, then  $\|y_n\|_{W_2^3 \oplus W_2^3}$  is convergent and  $\{\sum_{j=1}^2 A_{ij}^2\}_{i=1}^{\infty} \in l^2$ . If  $m > n$ , then  $\|y_m - y_n\|_{W_2^3 \oplus W_2^3}^2 = \sum_{l=n+1}^m \sum_{j=1}^2 A_{lj}^2 \rightarrow 0$  as  $m, n \rightarrow \infty$ . But  $W_2^3[a, b] \oplus W_2^3[a, b]$  is a Hilbert space, then  $\|y_n - y\|_{W_2^3 \oplus W_2^3} \rightarrow 0$  as  $n \rightarrow \infty$ . More details can be found in [26].

(ii) Based on the reproducing property of  $R_t(s)$  and from (i), then

$$\begin{aligned} |y_n(t) - y(t)| &= |\langle y_n(s) - y(s), R_t(s) \rangle_{W_2^3 \oplus W_2^3}| \\ &\leq \|y_n(s) - y(s)\|_{W_2^3 \oplus W_2^3} \|R_t(s)\|_{W_2^3 \oplus W_2^3} \\ &\leq k_5 \|y_n - y\|_{W_2^3 \oplus W_2^3} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which means that  $\|y_n - y\|_C \rightarrow 0$  as  $n \rightarrow \infty$ .

(iii) Since  $y_n$  converges to  $y$  uniformly, then

$$\begin{aligned} |y_n''(t) - y''(t)| &= \left| \frac{d^2}{dt^2} (y_n(t) - y(t)) \right| \\ &= \left| \frac{d^2}{dt^2} \langle y_n(s) - y(s), R_t(s) \rangle_{W_2^3 \oplus W_2^3} \right| \\ &= \left| \left\langle y_n(s) - y(s), \frac{d^2}{dt^2} (R_t(s)) \right\rangle_{W_2^3 \oplus W_2^3} \right| \end{aligned}$$

$$\begin{aligned} &\leq \|y_n(s) - y(s)\|_{W_2^3 \oplus W_2^3} \left\| \frac{d^2}{dt^2} (R_t(s)) \right\|_{W_2^3 \oplus W_2^3} \\ &\leq k_6 \|y_n - y\|_{W_2^3 \oplus W_2^3} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus,  $\|y_n'' - y''\|_C \rightarrow 0$  as  $n \rightarrow \infty$ .

(iv) Firstly, we prove that  $y_{n-1}(t_n) \rightarrow y(\tau)$  ( $n \rightarrow \infty$ ) as follows:

$$\begin{aligned} |y_{n-1}(t_n) - y(\tau)| &= |y_{n-1}(t_n) - y_{n-1}(\tau) + y_{n-1}(\tau) - y(\tau)| \\ &\leq |y_{n-1}(t_n) - y_{n-1}(\tau)| + |y_{n-1}(\tau) - y(\tau)|. \end{aligned}$$

From  $\|y_{n-1} - y\|_{W_2^3 \oplus W_2^3} \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that  $|y_{n-1}(\tau) - y(\tau)| \rightarrow 0$  as  $n \rightarrow \infty$ . Based on the reproducing property of  $R_t(s)$ , then

$$\begin{aligned} |y_{n-1}(t_n) - y_{n-1}(\tau)| &= |\langle y_{n-1}(t), R_{t_n}(s) \rangle - \langle y_{n-1}(t), R_\tau(s) \rangle| \\ &= |\langle y_{n-1}(t), R_{t_n}(s) - R_\tau(s) \rangle| \\ &\leq \|y_{n-1}(t)\|_{W_2^3 \oplus W_2^3} \|R_{t_n}(s) - R_\tau(s)\|_{W_2^3 \oplus W_2^3}. \end{aligned}$$

Since the kernel function  $R_t(s)$  is symmetric,  $\|R_{t_n}(s) - R_\tau(s)\|_{W_2^3 \oplus W_2^3} \rightarrow 0$  as  $n \rightarrow \infty$ , which means that  $|y_{n-1}(t_n) - y_{n-1}(\tau)| \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,  $|y_{n-1}(t_n) - y(\tau)| \rightarrow 0$  as  $n \rightarrow \infty$ . Secondly, since  $f$  and  $h$  are continuous functions and  $t_n \rightarrow \tau$ , then  $f(t_n) \rightarrow f(\tau)$  as  $n \rightarrow \infty$  and  $h(y_{n-1}(t_n)) \rightarrow h(y(\tau))$  as  $n \rightarrow \infty$ . From the continuity of  $G$ , then  $G(t_n, f(t_n), h(y_{n-1}(t_n))) \rightarrow G(\tau, f(\tau), h(y(\tau)))$ . The proof is complete.  $\square$

## 5 Numerical results

In order to illustrate the accuracy of the method proposed in Sect. 4, we present three examples using Mathematica 7.0. The results obtained by our method are compared with the exact solution of each example and are found to be in good agreement with each other.

**Example 5.1** Consider the following second-order FVIDE:

$$y''(t) = [\alpha - 1, 1 - \alpha] + \int_0^t y(s) ds, \quad 0 \leq t \leq 1, \alpha \in [0, 1], \quad (12)$$

subject to the fuzzy initial conditions

$$y(0) = [\alpha - 1, 1 - \alpha], \quad y'(0) = [0, 0]. \quad (13)$$

Equations (12) and (13) are replaced by two systems of Volterra integro-differential equations as (1, 1)-system

$$\left. \begin{aligned} y_{1,\alpha}''(t) &= (\alpha - 1) + \int_0^t y_{1,\alpha}(s) ds, & 0 \leq t \leq 1, \\ y_{2,\alpha}''(t) &= (1 - \alpha) + \int_0^t y_{2,\alpha}(s) ds, & 0 \leq t \leq 1, \\ y_{1,\alpha}(0) &= \alpha - 1, & y_{2,\alpha}(0) = 1 - \alpha, \\ y_{1,\alpha}'(0) &= 0, & y_{2,\alpha}'(0) = 0. \end{aligned} \right\} \quad (14)$$

**Table 1** Numerical results for (1, 1)-system at  $t = 0.5$  and various  $\alpha$ 

$\alpha$	Exact solution	Approximate solution
0	$[-0.890245911500779, 0.890245911500779]$	$[-0.888487875540245, 0.888487875540245]$
0.2	$[-0.712196729200623, 0.712196729200623]$	$[-0.710790300432196, 0.710790300432196]$
0.4	$[-0.534147546900467, 0.534147546900467]$	$[-0.533092725324147, 0.533092725324147]$
0.6	$[-0.356098364600311, 0.356098364600311]$	$[-0.355395150216098, 0.355395150216098]$
0.8	$[-0.178049182300155, 0.178049182300155]$	$[-0.177697575108049, 0.177697575108049]$
1	$[0, 0]$	$[0, 0]$

**Table 2** Numerical results for (1, 2)-system at  $t = 0.5$  and various  $\alpha$ 

$\alpha$	Exact solution	Approximate solution
0	$[-0.8544486824714743, 0.8544486824714743]$	$[-0.8521702477360533, 0.8521702477360531]$
0.2	$[-0.6835589459771796, 0.6835589459771796]$	$[-0.6817361981888437, 0.681736198188843]$
0.4	$[-0.5126692094828845, 0.5126692094828845]$	$[-0.5113021486416323, 0.5113021486416318]$
0.6	$[-0.3417794729885898, 0.3417794729885898]$	$[-0.34086809909442184, 0.3408680990944215]$
0.8	$[-0.1708897364942948, 0.1708897364942948]$	$[-0.17043404954721061, 0.1704340495472106]$
1	$[0, 0]$	$[0, 0]$

(1, 2)-system

$$\left. \begin{aligned} y''_{1,\alpha}(t) &= (1 - \alpha) + \int_0^t y_{2,\alpha}(s) ds, & 0 \leq t \leq 1, \\ y''_{2,\alpha}(t) &= (\alpha - 1) + \int_0^t y_{1,\alpha}(s) ds, & 0 \leq t \leq 1, \\ y_{1,\alpha}(0) &= \alpha - 1, & y_{2,\alpha}(0) = 1 - \alpha, \\ y'_{1,\alpha}(0) &= 0, & y'_{2,\alpha}(0) = 0. \end{aligned} \right\} \quad (15)$$

The corresponding (1, 1)-system has the exact solution

$$\begin{aligned} y(t) &= \left[ (\alpha - 1)e^{0.5t} \left( \cos\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right) \right), \right. \\ &\quad \left. (1 - \alpha)e^{0.5t} \left( \cos\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}} \sin\left(\frac{t}{\sqrt{2}}\right) \right) \right], \end{aligned} \quad (16)$$

and the corresponding (1, 2)-system has the exact solution

$$\begin{aligned} y(t) &= \left[ (\alpha - 1)e^{0.5t} \left( \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \right), \right. \\ &\quad \left. (1 - \alpha)e^{0.5t} \left( \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \right) \right]. \end{aligned} \quad (17)$$

Using the RKHS method, taking  $\alpha = 0.2(i - 1)$ ,  $i = 1, 2, \dots, 6$ , the numerical results at  $t = 0.5$  for  $n = 10$  are presented for both the (1, 1)-system and the (1, 2)-system in Tables 1 and 2, respectively. In order to show the efficiency of the RKHS method in solving FVIDE (12)–(13), absolute errors are calculated at  $\alpha = 0.5$ ,  $n = 10$  and various  $t$  for the two branches of the (1, 1)-system in Tables 3 and 4 and for the two branches of the (1, 2)-system in Tables 5 and 6.

Without losing the general issue, we will solve the next two examples, unlike Example 5.1, either using the (1, 1)-system or the (1, 2)-system to avoid repetitions.

**Table 3** The absolute errors of approximating  $y_{1,\alpha}$  for (1, 1)-system at various  $t$  and  $\alpha = 0.5$ 

$t_i$	Exact solution	Approximate solution	Absolute error
0	-0.5	-0.5	0
0.2	-0.4919942732092149	-0.49155676396105286	$4.3750924816210457 \times 10^{-4}$
0.4	-0.4659175034022215	-0.46513632990261633	$7.811734996051589 \times 10^{-4}$
0.6	-0.4186325706793184	-0.4177266247210882	$9.059459582302365 \times 10^{-4}$
0.8	-0.3470103621084155	-0.34632653813195646	$6.838239764590526 \times 10^{-4}$
1	-0.2480352042594187	-0.24803520425941872	0

**Table 4** The absolute errors of approximating  $y_{2,\alpha}$  for (1, 1)-system at various  $t$  and  $\alpha = 0.5$ 

$t_i$	Exact solution	Approximate solution	Absolute error
0	0.5	0.5	0
0.2	0.4919942732092149	0.49155676396105263	$4.375092481623266 \times 10^{-4}$
0.4	0.4659175034022215	0.46513632990261655	$7.811734996049369 \times 10^{-4}$
0.6	0.4186325706793184	0.41772662472108835	$9.059459582300699 \times 10^{-4}$
0.8	0.3470103621084155	0.34632653813195679	$6.838239764587195 \times 10^{-4}$
1	0.2480352042594187	0.24803520425941916	$4.440892098500626 \times 10^{-16}$

**Table 5** The absolute errors of approximating  $y_{1,\alpha}$  for (1, 2)-system at various  $t$  and  $\alpha = 0.5$ 

$t_i$	Exact solution	Approximate solution	Absolute error
0	-0.5	-0.5000000000000001	$1.1102230246251565 \times 10^{-16}$
0.2	-0.4893347110786599	-0.4887536023044621	$5.811087741978138 \times 10^{-4}$
0.4	-0.45471216929013725	-0.45368890889731417	$1.0232603928230777 \times 10^{-3}$
0.6	-0.3923561778762831	-0.3911974694888205	$1.1587083874625703 \times 10^{-3}$
0.8	-0.298878446816582	-0.2980336644678224	$8.447823487595651 \times 10^{-4}$
1	-0.1715140126819416	-0.17151401268194177	$1.6653345369377348 \times 10^{-16}$

**Table 6** The absolute errors of approximating  $y_{2,\alpha}$  for (1, 2)-system at various  $t$  and  $\alpha = 0.5$ 

$t_i$	Exact solution	Approximate solution	Absolute error
0	0.5	0.5000000000000002	$2.220446049250313 \times 10^{-16}$
0.2	0.4893347110786599	0.4887536023044618	$5.811087741980914 \times 10^{-4}$
0.4	0.45471216929013725	0.45368890889731406	$1.0232603928231887 \times 10^{-3}$
0.6	0.3923561778762831	0.3911974694888207	$1.1587083874623483 \times 10^{-3}$
0.8	0.298878446816582	0.2980336644678221	$8.447823487598982 \times 10^{-4}$
1	0.1715140126819416	0.17151401268194189	$2.7755575615628914 \times 10^{-16}$

**Example 5.2** (see [9]) Consider the following second-order FVIDE:

$$\begin{aligned}
 y''(t) = & [(-10 + 5\alpha)e^{1-t} - (1 + 0.5\alpha)t, (-3 - 2\alpha)e^{1-t} - (0.6 + 0.1\alpha - 0.2\alpha^2)t] \\
 & + \int_0^t [-0.2 + 0.1\alpha, -0.1]e^{s-1}y(s)ds, \quad 0 \leq t \leq 1,
 \end{aligned} \quad (18)$$

subject to the fuzzy initial conditions

$$y(0) = [(-10 + 5\alpha)e, (-3 - 2\alpha)e], \quad y'(0) = [(3 + 2\alpha)e, (10 - 5\alpha)e]. \quad (19)$$

**Table 7** The absolute errors of approximating  $y_{1,\alpha}$  for (1, 1)-system at various  $t$  and  $\alpha$ 

$\alpha$	$y_{1,\alpha}(0)$	$y_{1,\alpha}(0.25)$	$y_{1,\alpha}(0.5)$	$y_{1,\alpha}(0.75)$	$y_{1,\alpha}(1)$
0.5	$3.55271 \times 10^{-15}$	$9.35382 \times 10^{-6}$	$9.70896 \times 10^{-6}$	$5.66864 \times 10^{-6}$	$7.10543 \times 10^{-14}$
0.75	$7.10543 \times 10^{-15}$	$7.79485 \times 10^{-6}$	$8.09080 \times 10^{-6}$	$4.72387 \times 10^{-6}$	$5.32907 \times 10^{-14}$
1	$1.24345 \times 10^{-14}$	$6.23588 \times 10^{-6}$	$6.47264 \times 10^{-6}$	$3.77909 \times 10^{-6}$	$4.61853 \times 10^{-14}$

**Table 8** The absolute errors of approximating  $y_{2,\alpha}$  for (1, 1)-system at various  $t$  and  $\alpha$ 

$\alpha$	$y_{2,\alpha}(0)$	$y_{2,\alpha}(0.25)$	$y_{2,\alpha}(0.5)$	$y_{2,\alpha}(0.75)$	$y_{2,\alpha}(1)$
0.5	$1.06581 \times 10^{-14}$	$4.98870 \times 10^{-6}$	$5.17811 \times 10^{-6}$	$3.02328 \times 10^{-6}$	$6.75016 \times 10^{-14}$
0.75	$1.06581 \times 10^{-14}$	$5.61229 \times 10^{-6}$	$5.82538 \times 10^{-6}$	$3.40119 \times 10^{-6}$	$4.97379 \times 10^{-14}$
1	$1.24345 \times 10^{-14}$	$6.23588 \times 10^{-6}$	$6.47264 \times 10^{-6}$	$3.77909 \times 10^{-6}$	$4.61853 \times 10^{-14}$

Here, the (1, 1)-system is

$$\left. \begin{aligned} y''_{1,\alpha}(t) &= (-10 + 5\alpha)e^{1-t} - (1 + 0.5\alpha)t + \int_0^t (-0.2 + 0.1\alpha)e^{s-1}y_{1,\alpha}(s) ds, \\ y''_{2,\alpha}(t) &= (-3 - 2\alpha)e^{1-t} - (0.6 + 0.1\alpha - 0.2\alpha^2)t - \int_0^t (0.1)e^{s-1}y_{2,\alpha}(s) ds, \\ y_{1,\alpha}(0) &= (-10 + 5\alpha)e, \quad y_{2,\alpha}(0) = (-3 - 2\alpha)e, \\ y'_{1,\alpha}(0) &= (3 + 2\alpha)e, \quad y'_{2,\alpha}(0) = (10 - 5\alpha)e, \end{aligned} \right\} \quad (20)$$

where  $0 \leq t \leq 1$ . The corresponding (1, 1)-system has the exact solution

$$y(t) = [(-10 + 5\alpha)e^{1-t}, (-3 - 2\alpha)e^{1-t}]. \quad (21)$$

Using the method presented in Sect. 4, taking  $n = 100$ ,  $t_i = 0.25(i - 1)$ ,  $i = 1, 2, \dots, 5$ , the numerical results of (1, 1)-system in Example 5.2 in different values of  $\alpha$  are shown in Tables 7 and 8.

**Example 5.3** (see [9]) Consider the following second-order FVIDE:

$$\begin{aligned} y''(t) &= \left[ (3\alpha - 3) \sin t + \frac{3}{20}(\alpha^2 - 6\alpha + 5)t \cos^2 t, \right. \\ &\quad \left. (9 - 9\alpha) \sin t - \frac{9}{20}(\alpha^2 - 6\alpha + 5)t \cos^2 t \right] \\ &\quad + \int_0^t [0.1 + 0.3\alpha, 0.5 - 0.1\alpha]t \cos(s)y(s) ds, \quad 0 \leq t \leq 1, \end{aligned} \quad (22)$$

subject to the fuzzy initial conditions

$$y(0) = [0, 0], \quad y'(0) = [-9 + 9\alpha, 3 - 3\alpha]. \quad (23)$$

Here, the (1, 2)-system is

$$\begin{aligned} y''_{2,\alpha}(t) &= (3\alpha - 3) \sin t + \frac{3}{20}(\alpha^2 - 6\alpha + 5)t \cos^2 t + \int_0^t (0.1 + 0.3\alpha)t \cos(s)y_{1,\alpha}(s) ds, \\ y''_{1,\alpha}(t) &= (9 - 9\alpha) \sin t - \frac{9}{20}(\alpha^2 - 6\alpha + 5)t \cos^2 t + \int_0^t (0.5 - 0.1\alpha)t \cos(s)y_{2,\alpha}(s) ds, \\ y_{1,\alpha}(0) &= 0, \quad y_{2,\alpha}(0) = 0, \\ y'_{1,\alpha}(0) &= -9 + 9\alpha, \quad y'_{2,\alpha}(0) = 3 - 3\alpha, \end{aligned} \quad (24)$$

**Table 9** The absolute errors of approximating (1,2)-system at various  $t$  and  $\alpha = 0.5$ 

$n$	$y_{1,\alpha}(0.16)$	$y_{2,\alpha}(0.16)$	$y_{1,\alpha}(0.96)$	$y_{2,\alpha}(0.96)$
5	$4.40853 \times 10^{-3}$	$1.46951 \times 10^{-3}$	$1.48218 \times 10^{-2}$	$4.94061 \times 10^{-3}$
10	$1.12017 \times 10^{-3}$	$3.73391 \times 10^{-4}$	$5.20311 \times 10^{-3}$	$1.73437 \times 10^{-3}$
100	$4.84584 \times 10^{-6}$	$1.61528 \times 10^{-6}$	$7.23713 \times 10^{-6}$	$2.41238 \times 10^{-6}$

where  $0 \leq t \leq 1$ . The corresponding (1,2)-system has the exact solution

$$y(t) = [(-9 + 9\alpha) \sin t, (3 - 3\alpha) \sin t]. \quad (25)$$

One can observe from Table 9 that as  $n$  gets larger, the physical behavior of the absolute errors approaches to zero very rapidly. Anyway, Table 9 also shows that our RKHS method gives reasonable results for small  $n$  as well.

## 6 Conclusion

We have introduced a new method for solving second-order FVIDEs based on reproducing kernel theory. The proposed technique gives solutions with good generalization and high exactness. The robustness of this technique has been proved by numerical simulations. As a future research line, this technique can be expanded to solving a class of fuzzy ordinary differential algebraic equations and fuzzy partial integro-differential equations.

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### Authors' contributions

All the authors worked jointly. All the authors read and approved the final manuscript.

### Author details

<sup>1</sup>Faculty of Engineering Technology, Al-Balqa Applied University, Amman, Jordan. <sup>2</sup>Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun, Jordan.

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